Abstracts

Waves in nonlinear discrete systems

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The damped Frenkel-Kontorova model of dislocations [1, 2], the spatially discrete FitzHugh-Nagumo (FHN) model of nerve conduction in myelinated neurons [3] or the discrete drift-diffusion model of electron transport in doped semiconductor superlattices (SL) [4] are examples of nonlinear discrete systems. These models are described by systems of coupled autonomous differential-difference equations having nonlinear N-shaped source terms and their dynamical behavior can be understood in terms of fronts, pulses or wave trains.

Wave fronts have monotone profiles joining two different constant solutions as the discrete index i goes to $-\infty$ or $+\infty$. These fronts are either traveling wave solutions moving at a constant velocity or stationary solutions (in whose case we say that the fronts are pinned by the lattice). Typically stationary fronts exist when a control parameter (the load in the FK model or the current J in the superlattice model) takes values on an open interval. The transition between moving and pinned fronts (pinning-depinning transition) depends on the dynamics of the system. For the overdamped FK model or the SL model, it is a global saddle-node bifurcation such that the front velocity vanishes as $c(J) \propto |J - J_c|^{1/2}$ as J goes to one of the extremes of the pinning interval [4, 5]. To be precise, consider the SL case:

$$(1) \quad \frac{dE_i}{dt} + v(E_i)\,\frac{E_i - E_{i-1}}{\nu} - D(E_i)\,\frac{E_{i+1} + E_{i-1} - 2E_i}{\nu} = J - v(E_i),$$

in which the pinning interval is (J_{c1}, J_{c2}). $E_i(t)$ are electric field values at the SL quantum wells, $v(E)$ and $D(E)$ are the electron drift velocity and diffusivity, respectively [4]. The critical currents J_{c1} and J_{c2} depend on another parameter, the dimensionless doping density ν. For $J > J_{c2}$, the wave fronts move towards the left, with negative velocity, whereas they move to the right if $J < J_{c1}$. As $J \to J_{c1}-$ or $J \to J_{c2}+$, the front profile develops steps (and it loses smoothness at the critical currents). For large values of ν there is one prominent step in the front profile: most $E_i(t) = E(i - ct)$ are either $E^{(1)}(J)$ or $E^{(3)}(J)$, with $v(E^{(n)}(J)) = J$, except for a single *active* point $E_0(t)$ which is in between these two values. The evolution of $E_0(t)$ is given approximately by (1) with $E_i = E^{(1)}(J)$ for $i < 0$ and $E_i = E^{(3)}(J)$ for $i > 0$. The front profile $E(z)$ can be reconstructed from the motion of the active point by using $E(z) = E_0(-z/c)$ [1, 4, 5]. For J in the pinning interval, the equation for E_0 has one unstable and two stable stationary solutions. The unstable and one of the stable solutions merge in a saddle-node bifurcation as $J \to J_{cn}$, $n = 1, 2$. As $J \to J_{c1}-$ or $J \to J_{c2}+$, the corresponding normal form $d\varphi/dt = \alpha(J - J_{cn}) + \beta^2\varphi^2$, has the solution $\varphi = (-1)^n \sqrt{\alpha(J - J_{cn})/\beta}\, \tan[\sqrt{\alpha\beta(J - J_{cn})}\,(t - t_0)]$, which blows up at $(t - t_0) = \pm 1/(2c)$, with $c = \sqrt{\alpha\beta(J - J_{cn})}/\pi$. c is the approximate front velocity. At the blow-up times, $E_0(t)$ solves (1) with $J = J_{cn}$, $E_i = E^{(1)}(J)$ for $i < 0$ and $E_i = E^{(3)}(J)$ for $i > 0$, and the matching conditions $E_0 \to E^{(3)}(J_{cn})$ as $(t - t_0) \to +\infty$ for $(t - t_0) = 1/(2c)$ (resp. $E_0 \to E^{(1)}(J_{cn})$ as $(t - t_0) \to -\infty$ for $(t - t_0) = -1/(2c)$). As ν decreases, there are more active points between $E^{(1)}(J)$ and $E^{(3)}(J)$ and finitely many equations need to be kept to approximate (1). See Ref. [5] for a detailed description and results. In the continuum limit, $\nu \to 0$, the pinning interval disappears and (1) may be approximated by a first-order hyperbolic equation together with shock and entropy conditions that yield approximate wave front velocities [4].

The pinning-depinning transition of wave fronts is modified by disorder. For example, fluctuations in the SL doping density result in adding a term $\gamma D(E_i)(\xi_{i+1} - \xi_i) - \gamma v(E_i)\xi_i$ to the right hand side of (1), where ξ_i is a random variable taking values on $(-1, 1)$ with equal probability and $\gamma \to 0$. An extension of the active point theory has been used to show that the effect of disorder is to shift the critical currents and to change the critical exponent from 1/2 to 3/2 [6]. The effect of inertia may be even more dramatic. In the underdampled FK model with a piecewise linear source, the pinning-depinning transition may become subcritical: the stable branch of moving fronts is connected to the stationary solution by branches having infinitely many turning points that accumulate at the static critical J [2].

Wave fronts are stable solutions of the differential-difference equations considered here. We can use their profiles and velocities to describe more complex dynamical behaviors. Two examples. A voltage biased SL is described by (1) for $i = 1, \ldots, N$, the bias condition $\sum_{i=0}^{N} E_i = (N + 1)\,\phi$ (for a given constant voltage ϕ) and boundary conditions at $i = 0$ and N. The unknowns in this problem are $E_i(t)$ and $J(t)$. Depending on the values of ν and ϕ, this problem has stable stationary or time-periodic solutions which can be visualized in a bifurcation diagram of J versus ϕ (current-voltage diagram) [7]. For large ν, the only stable solutions are stationary and there may be several stable solution branches for a given value of ϕ. The field profile of each solution branch is a stationary wave front pinned at a given SL period i. For lower doping densities, there are intervals of ϕ for which the stable solutions are self-sustained oscillations having a periodic $J(t)$. The corresponding field profiles are pulses moving from $i = 0$ to $i = N$. These pulses are regions with $E_i = E^{(3)}(J)$ bounded by monotonically increasing and monotonically decreasing wave fronts. Between pulses or between pulses and contacts, $E_i = E^{(1)}(J)$. During a self-oscillation, $J(t)$ varies slowly whereas the field (either at wave fronts or at flat regions with $E_i = E^{(n)}(J)$) adapts rapidly to the instantaneous value of J. To find an equation for $J(t)$, we simply time-differentiate the voltage bias condition, use the known functions $c_\pm(J)$ (velocities of a monotone increasing or decreasing wave front in terms of J) and that $v'(E^{(n)}(J))\, dE^{(n)}/dt = dJ/dt$. The result is an equation $dJ/dt = A(J)\,[n_+c_+(J) - n_-c_-(J)]/N$, where $n_\pm$ is the number of increasing (+) or decreasing (-) fronts and $A(J, \phi) > 0$ is a known function. If we include stages of wave front formation and annihilation at contacts, this equation is the basis of an asymptotic description of self-oscillations in the limit of large N (long SL) [7].

The other example of reducing pulse dynamics to wave front dynamics is provided by the FHN system consisting of an overdamped FK equation for the excitatory unknown and a linear ODE for the recovery unknown (the load). Their respective time scales are widely separated. In the fast time scale, the recovery variable is frozen and there are monotone increasing and decreasing fronts bounding a pulse of the excitatory variable. In the slow time scale, the excitatory variable moves over the stable branches of the N-shaped source term for the overdamped FK equation following the evolution of the recovery variable. It is possible to find a reduced system of equations for the time lag between wave fronts, the length of the region between fronts and the values of the recovery variable at the fronts [3]. The solution of this reduced system describe the evolution of the pulse.

Recently, we have considered a model for electron and hole transport in an undoped SL. In addition to a discrete drift-diffusion equation similar to (1), this system has an additional equation for the hole density plus bias and boundary conditions. If the electron-hole recombination is calculated as a function of electric field, the resulting system may have excitable or oscillatory dynamics with only one stable constant stationary solution, a situation reminiscent of the FHN system [8]. The voltage bias condition gives rise to a large variety of oscillations mediated by wave fronts, pulses and wave trains. Different from the case of doped SL, a pulse may be created inside the SL (not at the contact region), split into two, and each resulting pulse then moves towards the closest contact. Repetition of this process produces chaotic current oscillations.

References

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