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Lineability within Probability Theory settings

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Dedicated to Prof. Manuel Maestre on the occasion of his 60th birthday

Abstract The search of lineability consists on finding large vector spaces of mathematical objects with special properties. Such examples have arisen in the last years in a wide range of settings such as in real and complex analysis, sequence spaces, linear dynamics, norm-attaining functionals, zeros of polynomials in Banach spaces, Dirichlet series, and non-convergent Fourier series, among others.

In this paper we present the novelty of linking this notion of lineability to the area of Probability Theory by providing positive (and negative) results within the framework of martingales, random variables, and certain stochastic processes.

Keywords lineability · spaceability · probability theory · random variable · stochastic process · martingale.

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1 Introduction

Since the beginning of the 21st century many authors have become interested in the study of linearity within non linear settings or, in other words, the search for linear structures of mathematical objects enjoying certain special or *unexpected* properties. Vector spaces and linear algebras are elegant mathematical structures which, at first glance, seem to be "forbidden" to families of "strange" objects. In other words, take a function with some special or (as sometimes it is called) "pathological" property (for example, the classical nowhere differentiable function, also known as Weierstrass' monster). Coming up with a concrete example of such a function might be difficult. In fact, it may seem so difficult that if you succeed, you think that there cannot be too many functions of that kind. Probably one cannot find infinite dimensional vector spaces or infinitely generated algebras of such functions. This is, however, exactly what has been happening in the last years in many fields of mathematics, from Linear Chaos to Real and Complex Analysis [6,2,15], passing through Set Theory [17] and Linear and Multilinear Algebra, or even Operator Theory [9,11], Topology, Measure Theory [6,5,13], and Abstract Algebra.

Recall that, as it nowadays is common terminology, a subset M of a topological vector space X is called *lineable* (respectively, *spaceable*) in X if there exists an infinite dimensional linear space (respectively, infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$. Moreover, given an algebra $\mathcal{A}$, a subset $\mathcal{B} \subset \mathcal{A}$ is said to be *algebrable* if there is a subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any generator of $\mathcal{C}$ is infinite (see, e.g., [2,7,3]).

As we mentioned above, there have recently been many results regarding the linear structure of certain *special* subsets. One of the earliest results in this direction was provided by Gurariy, who showed that the set of Weierstrass' monsters is lineable [18]. Also, and more recently, Enflo et al. [15] proved that, for every infinite dimensional closed subspace X of $\mathcal{C}[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X (see, also, [12,16]). A vast literature on this topic have been built during the last decade, and we refer the interested reader to the survey paper [7] or, for a much detailed and thorough study, to the forthcoming monograph [3].

In this paper, we relate for the first time, the topic of lineability with Probability Theory and Stochastic Processes. However one needs to be careful when trying to find linear structures within certain sets of objects in this setting. Indeed, the set of probability density functions cannot contain any linear space since any non-trivial multiple of one already fails to be a probability density function or, in a deeper level, if we had two martingales $\{X_n\}_n$, $\{Y_n\}_n$, with their corresponding filtrations $\{\mathcal{F}_n\}_n$ and $\{\mathcal{G}_n\}_n$, the sequence of random variables $\{X_n + Y_n\}_n$ is not, in general, a martingale unless we had a "universal" filtration that would comply with both simultaneously. Nevertheless, we shall consider some classical (counter)examples in probability theory and study up to what level it is possible to obtain lineability-related results. In this paper we shall consider lineability and algebrability problems related to the following concepts:

i) Convergent martingales that are not L_1 bounded,
ii) pointwise convergence of random variables,

iii) stochastic processes being L_2 bounded, converging in L_2, and not converging for any point off a null set, and
iv) zero-mean sequences of mutually independent random variables with divergent sample mean.
v) unbounded random variables with finite expected value.

2 Preliminaries and notation

In this section, we recall some results that will be needed throughout the paper (for more details see, e.g., [10]).

Let Ω be a non-empty space and let $\mathcal{F}$ be a σ-algebra over Ω. We say that the pair $(\Omega, \mathcal{F})$ is a *probabilizable (measurable) space.* Given $(\Omega, \mathcal{F})$, a *filtration* of σ-algebras of $\mathcal{F}$ is an increasing sequence of σ-algebras, such that $\mathcal{F}_n \subset \mathcal{F}$ for every $n \in \mathbb{N}$.

Adding a function $\mu : \mathcal{F} \to [0, 1]$, we say that the triplet $(\Omega, \mathcal{F}, P)$ is a *probability space.* A *random variable* X on $(\Omega, \mathcal{F}, P)$ is a real-valued function defined on Ω, such that for every open subset $B \subset \mathbb{R}$ we have $X^{-1}(B) \in \mathcal{F}$. The *expected value* of the random variable X, namely $E(X)$, is computed as

$$E[X] = \int_\Omega X dP. \tag{1}$$

A collection of random variables indexed by a totally ordered set, representing the evolution of some system of random variables is said to be a stochastic process.

We now introduce the notion of a conditional expectation of a random variable X.

Definition 1 Let $(\Omega, \mathcal{F}, P)$ be a probability space, let X be a random variable on this probability space, and let $\mathcal{H} \subseteq \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. The *conditional expectation* of X, denoted as $E[X \mid \mathcal{H}]$, is any $\mathcal{H}$-measurable function $\Omega \to \mathbb{R}$ which satisfies

$$\int_H E[X \mid \mathcal{H}] \, dP = \int_H X \, dP \quad \text{for every} \quad H \in \mathcal{H}. \tag{2}$$

A sequence of random variables $\{X_n\}_n$ defined on $(\Omega, \mathcal{F}, \mu)$ is said to be a *Markov chain* if for every $n \geq 1$, the variable X_{n+1} only depends upon the state of X_n. Given a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ and a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of σ-algebras of $\mathcal{F}$, we say that $\{X_n\}_{n\in\mathbb{N}}$ is a *martingale* if X_n is integrable and $E[X_{n+1}|\mathcal{F}_n] = X_n$ almost surely (a.s. from now one) for all $n \in \mathbb{N}$.

Finally, let us recall the following definition that will be necessary in order to introduce the notion of a *martingale indexed by a directed set* (see, e.g., [14]).

Definition 2 **(*directed set)*** A directed set is a nonempty set D with a relation $\sim_R$ such that:

i) $a \sim_R a$ for every $a \in D$.
ii) If $a, b, c \in D$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
iii) If $a, b \in D$ then there exists $c \in D$ with $a \sim_R c$ and $b \sim_R c$.

We point out that $a \sim_R b$ is (usually) denoted by $a \leq b$.

Let D be a directed set and let $\{X_d : d \in D\}$ be an indexed family of random variables. Let $\{\mathcal{F}_d : d \in D\}$ be a family of σ-algebras such that for $d_1 \leq d_2$, we have $\mathcal{F}_{d_1} \subset \mathcal{F}_{d_2}$. We also say that $\{X_d\}$ is a *martingale indexed by a directed set* D if for every $d \in D$ we have $E[|X_d|] < \infty$, X_d is $\mathcal{F}_d$-measurable, and for every $d_1 \leq d_2$ we have $E[X_{d_2}|\mathcal{F}_{d_1}] = X_{d_1}$ almost surely.

3 Lineability of special sequences of random variables

The motivation for our first result is the fact that many martingale convergence theorems require the martingale to be L_1-bounded (for instance, in the famous Doob's martingale convergence theorems or in Lévy's zero–one law, [10]). However, this condition (although sufficient) is not necessary. Indeed, there is a classical and well known example due to Ash (see [4], or [21, Example 9.15] for a more modern reference), in which (briefly) the author constructed a martingale via a Markov chain $\{X_n : n \in \mathbb{N}\}$, properly defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $(X_n)_n$ converges for every $\omega \in \Omega$, and with $E[|X_n|] \overset{n\to\infty}{\longrightarrow} \infty$.

Here, and although (as we mentioned in the Introduction) one cannot consider lineability within martingales, we shall show that one can construct an infinite dimensional vector space every non-zero element of which, $\{X_n : n \in \mathbb{N}\}$, is a sequence of convergent random variables with $E[|X_n|] \overset{n\to\infty}{\longrightarrow} \infty$. That is, the main tool in Ash's example is, actually, "not as uncommon" as one might expect. The proof is a little bit technical, although constructive.

Theorem 1 *The set of convergent sequences of random variables* $\{X_n : n \in \mathbb{N}\}$ *with* $E[|X_n|] \overset{n\to\infty}{\longrightarrow} \infty$ *is lineable.*

Proof First let us denote by $\mathcal{S} = \{s_j\}_{j\in\mathbb{N}}$ the (increasing) sequence of odd prime numbers. Next, for every $s \in \mathcal{S}$ we consider the Markov chain defined as follows. Let $X_1^{(s)} = 0$. Also, if $X_n^{(s)} = 0$ let

$$X_{n+1}^{(s)} = \begin{cases} s^{n+1} \cdot (n+1)^s & \text{with probability } 1/s^{n+1}, \\ -s^{n+1} \cdot (n+1)^s & \text{with probability } 1/s^{n+1}, \\ 0 & \text{with probability } 1 - 2/s^{n+1}, \end{cases} \tag{3}$$

and, if $X_n^{(s)} \neq 0$, we let $X_{n+1}^{(s)} = X_n^{(s)}$. Notice that, if $X_n^{(s)} \neq 0$, then $X_j^{(s)} = X_n^{(s)}$ for every $j \geq n$. Let us consider $A = \{\omega : X_n^{(s)}(\omega) \neq 0 \quad \text{for} \quad \text{some} \quad n \in \mathbb{N}\}$. If $\omega \in A$, then $X_j^{(s)}(\omega) = X_n^{(s)}(\omega)$ for every $j \geq n$. In contrast, if $\omega \in \Omega \setminus A$ then $X_{n+1}^{(s)}$ is defined following equation (3). Moreover, note that for every $n \in \mathbb{N}$,

$$E\left[X_{n+1}^{(s)}|X_n^{(s)} = 0\right] = (s^{n+1}\cdot(n+1)^s)\cdot\frac{1}{s^{n+1}} - (s^{n+1}\cdot(n+1)^s)\cdot\frac{1}{s^{n+1}} + 0\cdot(1-\frac{2}{s^{n+1}}) = 0, \tag{4}$$

$$E\left[X_{n+1}^{(s)}|X_n^{(s)} = s^{n+1}\cdot(n+1)^s\right] = s^{n+1}\cdot(n+1)^s, \text{ and} \tag{5}$$

$$E\left[X_{n+1}^{(s)}|X_n^{(s)} = -s^{n+1}\cdot(n+1)^s\right] = -s^{n+1}\cdot(n+1)^s. \tag{6}$$

Therefore, for every $s \in \mathcal{S}$, the Markov chain $\{X_n^{(s)} : n \in \mathbb{N}\}$ is a martingale respect on the natural filtration, that is, $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$[1] for all n. Furthermore, given $s \in \mathcal{S}$, and assuming all of the above random variables are properly defined on a probability space $(\Omega, \mathcal{F}, P)$, we have that either $X_n^{(s)}(\omega) = 0$ for every $n \in \mathbb{N}$ or even in the case that there is some $m \in \mathbb{N}$ such that $X_n^{(s)} \neq 0$ for all $n \geq m$, we can conclude that $\{X_n^{(s)}\}_n$ is a convergent sequence on $(\Omega, \mathcal{F}, P)$.

Before carrying on with the main construction, let us recall that it can be assumed, without loss of generality, that the set $\{X_n^{(s)} : s \in \mathcal{S}\}$ is linearly independent, just taking, for instance, disjoint supports in the construction of the random variables.

Our aim now is to show that any non-zero element in the linear span of $\{X_n^{(s)} : s \in \mathcal{S}\}$ is convergent and not L_1-bounded. The convergence is straightforward from the fact that $\{X_n^{(s)}\}_n$ converges for every $\omega \in \Omega$ and any element in the linear span of $\{X_n^{(s)} : s \in \mathcal{S}\}$ is a finite linear combination of these random variables in the sequence $\{X_n^{(s)}\}_n$.

We still need a couple of estimates in order to achieve our goal. For every $I \in \mathcal{F}$, let us define I_A as the characteristic function on the set A. Let $s \in \mathcal{S}$ and $k \in \mathbb{N}$, we have that

$$\begin{aligned} X_k^{(s)} &= X_2^{(s)} \cdot I_{\{X_2^{(s)} \neq 0\}} + X_3^{(s)} \cdot I_{\{X_2^{(s)}=0, X_3^{(s)} \neq 0\}} + \\ &+ X_4^{(s)} \cdot I_{\{X_2^{(s)}=X_3^{(s)}=0, X_4^{(s)} \neq 0\}} + \ldots + \\ &+ X_k^{(s)} \cdot I_{\{X_1^{(s)}=\ldots=X_{k-1}^{(s)}=0, X_k^{(s)} \neq 0\}} + 0 \cdot I_{\{X_1^{(s)}=\ldots=X_k^{(s)}=0\}}, \end{aligned} \tag{7}$$

from which we obtain that

$$\begin{aligned} E\left[|X_k^{(s)}|\right] &= 2a_2p_2 + (1-2p_2) \cdot 2a_3p_3 + (1-2p_2)(1-2p_3) \cdot 2a_4p_4 + \ldots + \\ &+ (1-2p_2)(1-2p_3) \cdot \ldots \cdot (1-2p_{k-1}) \cdot 2a_kp_k, \end{aligned} \tag{8}$$

where, for the sake of simplicity, we have denoted $a_n := s^n n^s$ and $p_n := 1/s^n$. Applying the definition of $X_n^{(s)}$, making some simple calculations, and keeping in mind that for every $j \in \{1, \ldots, k-1\}$, we have $0 < 1 - 2p_j < 1$, and

$$1 > (1-2p_2) \geq (1-2p_2)(1-2p_3) \geq \ldots \geq (1-2p_2)(1-2p_3) \cdot \ldots \cdot (1-2p_{k-1}). \tag{9}$$

As a consequence, we obtain the following lower bound for $E\left[|X_k^{(s)}|\right]$:

$$E\left[|X_k^{(s)}|\right] \geq 2\left[\prod_{j=1}^{k-1}(1-2p_j)\right] \cdot \left[\sum_{j=2}^{k} a_jp_j\right] = 2\left[\prod_{j=1}^{k-1}\left(1-\frac{2}{s^j}\right)\right] \cdot \left[\sum_{j=2}^{k} j^s\right]. \tag{10}$$

In the previous expression, let us recall that the amount $\prod_{j=1}^{\infty}\left(1-\frac{2}{s^j}\right)$ is known, in Number Theory, as the *q-Pochhammer symbol* (also known as *q-shifted factorial*, see [8]) $(2; s)_\infty$, which verifies

$$0 < (2; s)_\infty < 1$$

[1] By $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ we mean the smallest σ-algebra in which $\{X_i : i \leq n\}$ are measurable.

if $s > 2$ (which complies with our hypotheses). We, thus, have

$$E[|X_k^{(s)}|] \geq 2\left[\prod_{j=1}^{k-1}\left(1-\frac{2}{s^j}\right)\right]\cdot\left(\sum_{j=2}^{k} j^s\right) \overset{k\to\infty}{\longrightarrow} \quad 2\cdot(2;s)_\infty\cdot\lim_{k\to\infty}\sum_{j=2}^{k} j^s = \infty,$$

and $\left\{X_k^{(s)}\right\}_k$ is not L_1-bounded. However, our aim is to show that any non-zero element in the linear span of $\{X_n^{(s)} : s \in \mathcal{S}\}$ is not L_1-bounded and, in order to obtain this, we shall need another estimate for $E\left[|X_k^{(s)}|\right]$. Recall that, since $(2;s)_\infty \in (0,1)$, we also have

$$E\left[|X_k^{(s)}|\right] \leq R_{s+1}(k) := 2\sum_{j=2}^{k} j^s \tag{11}$$

and it can be easily checked that the expression $R_{s+1}(k)$ is a polynomial of degree $s+1$ with

$$\lim_{k\to\infty} R_{s+1}(k) = +\infty. \tag{12}$$

Now, let $X_k \in \operatorname{span}\left\{X_k^{(s)} : s \in \mathcal{S}\right\}$, then:

$$X_k = \alpha_1 X_k^{(s_1)} + \alpha_2 X_k^{(s_2)} + \ldots + \alpha_m X_k^{(s_m)}, \tag{13}$$

where $s_1 < s_2 < \ldots < s_m$ are elements from $\mathcal{S}$, $\{\alpha_n\}_n \subset \mathbb{R}$, and (without loss of generality) $\alpha_m \neq 0$. Let us now show that X_k is not L_1-bounded. Indeed, using the linearity of $E[\cdot]$, the reverse triangle inequality, and equations (10) and (11), we have:

$$\begin{aligned} E\left[|X_k|\right] &= E\left[|\alpha_1 X_k^{(s_1)} + \alpha_2 X_k^{(s_2)} + \ldots + \alpha_m X_k^{(s_m)}|\right] \geq \\ &\geq |\alpha_m|\cdot E\left[|X_k^{(s_m)}\right] - |\alpha_1|\cdot E\left[|X_k^{(s_1)}\right] - \ldots - |\alpha_{m-1}|\cdot E[|X_k^{(s_{m-1})}] \geq \\ &\geq |\alpha_m|\cdot(2;s_m)_\infty\cdot R_{s_m+1}(k) - 2|\alpha_1|\sum_{j=2}^{k} j^{s_1} - \ldots - 2|\alpha_{m-1}|\sum_{j=2}^{k} j^{s_{m-1}} = \\ &= |\alpha_m|\cdot(2;s_m)_\infty\cdot R_{s_m+1}(k) - 2\sum_{i=1}^{m-1}\left(\sum_{j=2}^{k} j^{s_i}\right) \overset{k\to\infty}{\longrightarrow} \infty, \end{aligned} \tag{14}$$

since the expression $2|\alpha_m|\cdot(2;s_m)_\infty\cdot R_{s_m+1}(k)$ is a polynomial of degree s_m+1 with

$$\lim_{k\to\infty} R_{s_m+1}(k) = +\infty, \tag{15}$$

the expression

$$\sum_{i=1}^{m-1}\left(\sum_{j=2}^{k} j^{s_i}\right) \tag{16}$$

is a polynomial of degree $s_{m-1}+1$, and $s_{m-1} < s_m$. Therefore, X_k is not L_1-bounded, and the result is proved.

Remark 1 We recall that the previous result could certainly be stated in terms of martingales assuming, of course, that the martingales adapted to the same filtration form a vector space (the proof would follow the same ideas as in that of Theorem 1).

Now, let us continue focusing on obtaining lineability-related results of certain subsets of random variables enjoying "unexpected" properties. For instance, in [21, Example 9.2], the authors provide (given any $b > 0$) a sequence of integrable random variables $\{X_n\}_{n\in\mathbb{N}}$ and an integrable random variable X such that X_n converges to X pointwise and, yet, $E[X_n] = -b$ and $E[X] = b$ (the important point here is that one has, under the previous hipotheses, $E[X_n] \neq E[X]$ for every $n \in \mathbb{N}$). This construction can be generalized in order to construct a *positive cone* (see, e.g., [1]) of such elements since, in general, linearity of elements enjoying such properties might get lost.

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be sequences of integrable random variables converging, pointwise, to the integrable random variables X, Y (respectively). Let $b, c > 0$, and X_n, X, Y_n, Y random variables such that $E[X_n] = -b, E[X] = b, E[Y_n] = -c, E[Y] = c$. Now, let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha b + \beta c = 0$, then $E[\alpha X_n + \beta Y_b] = 0 = E[\alpha X + \beta Y]$, which does not fall into the class of examples we are working with. Thus, the above property is "not a lineable one". However, one could try to find a positive cone of such objects, as it was done in [1] when certain sets failed to be lineable (calling these sets *coneable*). More precisely, a subset M of a topological vector space X is called *positively coneable* in X if there exists an infinite dimensional set M such that $\alpha M \subset M$ for every $\alpha > 0$.

Theorem 2 *Let us consider the probability space* $([0,1], \mathbb{B}([0,1]), \lambda)$*, where* λ *denotes the Lebesgue measure. The set of sequences of integrable random variables* $\{X_n\}_n$ *converging to an integrable random variable* X *such that* $\lim_{n\to\infty} E[X_n] \neq E[X]$ *is positively coneable.*

Proof For every $m \in \mathbb{N}$, let us take $B^{(m)}, C^{(m)} > 0$ and let us define the following random variables for every $\omega \in [0,1]$

$$X^{(m)}(\omega) = \frac{a_m}{a_m - 1} \cdot B^{(m)} \cdot I_{[1/a_m,1]}(\omega) \quad \text{for every } w \in \omega \in [0,1] \text{and} \tag{17}$$

$$X_n^{(m)}(\omega) = \begin{cases} B^{(m)} + C^{(m)} & \text{if } n \leq a_m, \\ n \cdot C^{(m)} \cdot I_{[1/a_m - 1/n, 1/a_m]}(\omega) + X^{(m)}(\omega) & \text{if } n > a_m, \end{cases} \tag{18}$$

where $\{a_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$ is defined, recursively, as follows:

$$a_1 = 2 \quad \text{and} \quad a_{m+1} = (a_m + 1) \cdot a_m \quad \text{for } m > 1. \tag{19}$$

This permits us to state that the set of sequences $\{X_n^{(m)} : m \in \mathbb{N}\}$ are linearly independent when seen as regular functions in $\mathbb{R}^{[0,1]}$ (due to the choice of the a_m's in order to avoid *major* overlappings). The sequence $X_n^{(m)}$ converges to $X^{(m)}$ pointwisely when n tends to infinity. It can be easily seen that $\{X_n^{(m)}\}_n$ is a sequence of integrable random variables for every $m \in \mathbb{N}$ and that $X^{(m)}$ is an integrable random variable, too.

Furthermore, for every $n, m \in \mathbb{N}$ we have

$$E[X^{(m)}] = \int_{[0,1]} \frac{a_m}{a_m - 1} \cdot B^{(m)} \cdot I_{[1/a_m,1]}(\omega) d\omega = B^{(m)}, \tag{20}$$

and

$$E[X_n^{(m)}] = \int_{[0,1]} X^{(m)}(\omega) + n \cdot C^{(m)} \cdot I_{[1/a_m - 1/n, 1/a_m]}(\omega) d\omega = B^{(m)} + C^{(m)}. \quad (21)$$

We then consider the positive cone given by $\mathcal{C}_n = \{\alpha X_n^{(m)} : m \in \mathbb{N}, \alpha > 0\}$ where any element $Y_n \in \mathcal{C}_n$ can be written as $Y_n = \sum_{i=1}^k \alpha_i X_n^{(m_i)}$, where $\alpha_i > 0$ and $m_i \in \mathbb{N}$ for every $i \in \{1, \ldots, k\}$. By linearity of $E[\cdot]$ we have

$$E[Y_n] = E\left[\sum_{i=1}^k \alpha_i X_n^{(m_i)}\right] = \sum_{i=1}^k \alpha_i E\left[X_n^{(m_i)}\right] = \sum_{i=1}^k \alpha_i (B^{(m_i)} + C^{(m_i)}), \quad (22)$$

and given $Y = \sum_{i=1}^k \alpha_i X^{(m_i)}$, one obtains

$$E[Y] = \sum_{i=1}^k \alpha_i E\left[X^{(m_i)}\right] = \sum_{i=1}^k \alpha_i B^{(m_i)}, \quad (23)$$

which gives that, although by linearity, Y_n converges pointwise to Y with $E[Y_n] \neq E[Y]$ (actually, and more precisely, $E[Y_n] > E[Y]$) for every $n \in \mathbb{N}$.

The following result shows the algebrability of the set of unbounded random variables with a finite expected value. The example used for the construction is inspired in [21, Example 5.2].

Theorem 3 *Let us consider the probability space* $(\mathbb{R}^+, \mathbb{B}(\mathbb{R}^+), \lambda)$*, where* λ *denotes the Lebesgue measure. The set of unbounded random variables* $f : \mathbb{R} \to \mathbb{R}$ *that have a finite expected value is algebrable.*

Proof Let us consider the function

$$T(x) := \begin{cases} 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

For each $n \in \mathbb{N}$, we define:

$$f_n(x) := nT(n^3(x - n)) \quad (25)$$

Each function f_n is null except in the interval $J_n := \left[n, n + \frac{1}{n^3}\right]$. Moreover,

$$\int_{J_n} f_n(x) dx = \frac{1}{2n^2}. \quad (26)$$

and then, the random variable defined as

$$X(x) := \sum_{n=1}^{\infty} f_n(x) \quad (27)$$

has an expected value $E[X] = \frac{\pi^2}{12}$.

Let us consider a Cantor set on the unit interval obtained as $C = \cup_{n=1}^{\infty} I_n$, where $I_0 = [0, 1]$ and I_n is obtained from I_{n-1} removing the inner third of each of its subintervals. Let us define $L_n := J_n \cap (n + I_n)$. Then, we have

$$\int_{L_n} f_n(x)dx = \frac{1}{2n^2}\left(\frac{2}{3}\right)^n \qquad \text{for every } n \in \mathbb{N}. \tag{28}$$

Let $\{\alpha_l\}_{l\in\Lambda}$ be a non-numerable set of irrational numbers on $(0, 1)$ which are not $\mathbb{Q}$ linearly dependent, then for every $\alpha \in \{\alpha_l\}_{l\in\Lambda}$ we define the functions:

$$X_n^{(\alpha)}(x) = \begin{cases} f_n(x-\alpha) & \text{if } x \in \alpha + L_n, \\ 0 & \text{elsewhere.} \end{cases} \tag{29}$$

and then, we consider the random variable

$$X_\alpha(x) := \sum_{n=1}^{\infty} X_n^{\alpha}(x)dx. \tag{30}$$

Consider the algebra generated by these functions $\mathcal{A}(\{X_\alpha\}_{\alpha\in\Lambda})$. It is clear that for every $\alpha \in \{\alpha_l\}_{l\in\Lambda}$ the random variable X_α has a finite expected value and it is an unbounded random variable. Besides, this algebra is uncountably generated.

Given an arbitrary function

$$X(x) := \sum_{m=1}^{m_0} \lambda_m X_{\alpha_m}(x), \qquad \text{with } \alpha_m \in \Lambda, \lambda_m \in \mathbb{K} \text{ for all } m \in \mathbb{N}. \tag{31}$$

On the one hand, these random variables are unbounded, too. Indeed, let $\alpha_{min} := \min\{\alpha_m : 1 \leq m \leq m_0\}$ and we get $X(n + \alpha_{min}) = n$. Additionally, this random variable has a finite expected value as well.

Remark 2 Let us recall that, in the previous result, the unboundedness holds outside every interval of finite length, which adds an extra pathology to the considered property.

For the final part of this paper, let us recall the work [20] (see, also, [21, Example 9.17]), in which Walsh provided an example of a martingale (indexed by a directed set) that is L_2 bounded and converging in L_2 and that, also, does not converge for any point off a null set. Our aim here shall be to generalize this example in order to build an infinite dimensional linear space such that every non zero element of which is a martingale enjoying the previous property. Before starting its proof, we need to recall the following lemma (due to Muñoz, Palmberg, Puglisi, and the second author), which is a particular case of [19, Theorem 3.5]. In what follows $(\ell_p, \|\cdot\|_p)$ denotes the Banach space of real valued sequences with the usual p-norm.

Lemma 1 *The set $\ell_2 \setminus \ell_1$ is lineable.*

Theorem 4 *The set of stochastic processes that are L_2 bounded, converging in L_2 and that, also, do not converge for any point of a null set, is lineable.*

Proof By lemma 1, let V be any (countably generated) linear space contained in $(\ell_2 \setminus \ell_1) \cup \{0\}$ and let $\left\{\{h_n^{(m)}\}_n : m \in \mathbb{N}\right\}$ be a basis for V. For instance, and in order to be more clear in the coming construction, we can take (see [19, Theorem 3.5])

$$V = \operatorname{span}\left\{h_n^{(m)} := \left\{\frac{1}{n^m}\right\}_{n\in\mathbb{N}} : m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)\right\}. \tag{32}$$

For every $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$, let $\{X_n^{(m)}\}_n$ be an linearly independent (and infinite) set, every element of which is a sequence of mutually independent random variables such that provided $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$

$$P\left(X_n^{(m)} = -1\right) = P\left(X_n^{(m)} = 1\right) = 1/2. \tag{33}$$

for every $n \in \mathbb{N}$.

By construction, one has that $\sum_{n\in\mathbb{N}} h_n^{(m)} X_n^{(m)}$ converges almost surely for every $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$.

Let D be the family of all finite subsets of $\mathbb{N}$, partially ordered by set inclusion, which is a directed set. For every $d \in D, m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$ we define:

$$M_d^{(m)} = \sum_{n\in d} h_n^{(m)} X_n^{(m)}. \tag{34}$$

Therefore, for every $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$, and with respect to its own filtration, it can be easily checked that $\{M_d^{(m)} : d \in D\}$ is a martingale and it converges in probability. By construction, we also have that the set $\{(M_d^{(m)})_{d\in D} : m \in \mathbb{N}\}$ is linearly independent and, by linearity, any non-zero element in $W := \operatorname{span}\{(M_d^{(m)})_{d\in D} : m \in \mathbb{N}\}$ also converges in probability.

However, we will see $\{(M_d^{(m)})_{d\in D} : m \in \mathbb{N}\}$ as, simply, stochastic processes (dropping the filtration). Moreover, any element in W is, also, L_2-bounded, since (for every $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$) we have

$$E\left[(M_d^{(m)})^2\right] = \sum_{n=1}^{\infty} \left(h_n^{(m)}\right)^2 < \infty,$$

since the set $\left\{\{h_k^{(m)}\}_{k\in\mathbb{N}} : m \in \mathbb{N}\right\}$ is contained in $\ell_2 \setminus \ell_1$. However, notice that (for every $m \in \mathbb{Q} \cap \left(\frac{1}{2}, 1\right)$), $M_d^{(m)}(\omega)$ converges only if it converges regardless of the order of summation (that is, absolutely), but

$$\sum_{n=1}^{\infty} \left|h_n^{(m)} X_n^{(m)}\right| = \sum_{n=1}^{\infty} \left|h_n^{(m)}\right| = \sum_{n=1}^{\infty} 1/n^m = \infty.$$

It only remains to show that, for any $m_1, m_2, \ldots, m_q \in \left(\frac{1}{2}, 1\right)$ and $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$,

$$\lim_{s\to\infty} \sum_{n=1}^{s} \left|\alpha_1 h_n^{(m_1)} X_n^{(m_1)} + \alpha_2 h_n^{(m_2)} X_n^{(m_2)} + \ldots + \alpha_q h_n^{(m_q)} X_n^{(m_q)}\right| = \infty.$$

Indeed, if we apply the reverse triangle inequality to the above expression assuming, without loss of generality, that $m_1 < m_2 < \ldots < m_q$, and $\alpha_1 \neq 0$, we obtain

$$\sum_{n=1}^{s} \left| \alpha_1 h_n^{(m_1)} X_n^{(m_1)} + \alpha_2 h_n^{(m_2)} X_n^{(m_2)} + \ldots + \alpha_q h_n^{(m_q)} X_n^{(m_q)} \right| \geq$$

$$\geq \sum_{n=1}^{s} \left(|\alpha_1| \frac{1}{n^{m_1}} - |\alpha_2| \frac{1}{n^{m_2}} - \ldots - |\alpha_q| \frac{1}{n^{m_q}} \right)$$

and this last sum is divergent to $+\infty$, by construction and by the above definition of V.

Now, we wold like to consider a new interesting property of random variables. In [21], the authors show that there exists a sequence $\{X_n : n \in \mathbb{N}\}$ of mutually independent random variables, having zero mean, and such that $\left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|$ diverges to ∞ almost surely. However, this example can be extended in order to obtain lineability, as our following result states.

Theorem 5 *Given a common probability space $(\Omega, \mathcal{F}, P)$, the set of sequences $\{X_n : n \in \mathbb{N}\}$ of mutually independent random variables having zero mean and such that $\left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|$ diverges to ∞ (almost surely) is lineable.*

Proof Given $s \in \mathbb{N}, s \geq 2$, let $\{X_n^{(s)} : n \in \mathbb{N}\}$ be a set of linearly independent sequences, each of which is formed by mutually independent random variables, and such that

$$P\left(X_n^{(s)} = -n^s\right) = 1 - \frac{1}{n^{2s}} \quad \text{and}$$

$$P\left(X_n^{(s)} = n^{3s} - n^s\right) = \frac{1}{n^{2s}}.$$

It is easy to check that, by construction,

$$E[X_n^{(s)}] = 0 \quad \text{and} \tag{35}$$

$$\frac{X_n^{(s)}}{n} \longrightarrow -\infty \qquad \text{almost surely} \tag{36}$$

for every $s, n \in \mathbb{N}$ with $s \geq 2$. From equation (36) it follows that (for every $s \geq 2, s \in \mathbb{N}$)

$$\frac{1}{n} \sum_{i=1}^{n} X_i^{(s)} \longrightarrow -\infty \quad \text{a.s.} \tag{37}$$

Indeed, take $s \geq 2, s \in \mathbb{N}$, and let $\Omega_1 = \{\omega \in \Omega : X_n^{(s)}(\omega) = -n^s\}$ and $\Omega_2 = \Omega \backslash \Omega_1$. Now, if $\omega \in \Omega_1$, we have $X_k^{(s)}(\omega) = -k^s$, obtaining that, as $n \to \infty$, equation (37) holds.

Let now $V = \text{span}\{X_n^{(s)} : s \geq 2, s \in \mathbb{N}\}$ and let $Y_n \in V$. Thus, Y_n can be written as

$$Y_n = \sum_{i=1}^{N} \alpha_i X_n^{(s_i)},$$

for some $N \in \mathbb{N}$, $s_i \in \mathbb{N}$, $2 \leq s_1 < s_2 < \ldots < s_N$, and $\alpha_i \in \mathbb{R}$ for every $i \in \{1, 2, \ldots, N\}$ with $\alpha_N \neq 0$. By the linearity of $E[\cdot]$, and equation (35), we have that

$$E[Y_n] = \sum_{i=1}^{N} \alpha_i E[X_n^{(s_i)}] = \sum_{i=1}^{N} \alpha_i 0 = 0.$$

Also, notice that

$$\left|\frac{1}{n}\sum_{k=1}^{n} Y_k\right| = \left|\frac{\alpha_1}{n} \cdot \sum_{k=1}^{n} X_k^{(s_1)} + \ldots + \frac{\alpha_N}{n} \cdot \sum_{k=1}^{n} X_k^{(s_N)}\right| \geq$$
$$\geq \frac{|\alpha_N|}{n}\left|\sum_{k=1}^{n} X_k^{(s_N)}\right| - \frac{|\alpha_{N-1}|}{n}\left|\sum_{k=1}^{n} X_k^{(s_{N-1})}\right| - \ldots - \frac{|\alpha_1|}{n}\left|\sum_{k=1}^{n} X_k^{(s_1)}\right|.$$

From the previous inequality, the fact that $s_N > s_{N-1} > \ldots > s_1 \geq 2$, and equations (36) and (37) it can be seen that $\left|\frac{1}{n}\sum_{k=1}^{n} Y_k\right| \to \infty$ a.s. and the claim holds for $\omega \in \Omega_1$. The case $w \in \Omega_2$ also holds in a similar fashion and, thus, we spare the details of the calculations involved in it.

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