# POLYNOMIAL INEQUALITIES ON THE $\pi/4$–CIRCLE SECTOR

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Abstract. A number of sharp inequalities are proved for the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the polarization constant and the unconditional constant of the canonical basis of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

## 1. Preliminaries

The study of low dimensional spaces of polynomials can be an interesting source of examples and counterexamples related to more general questions. In this paper we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. The space of such polynomials is represented by $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. This paper can be seen as a continuation of [15] and [20]. Other publications in the same spirit can be found in [11, 12, 21, 22, 24, 25].

If $P(x, y) = ax^2 + by^2 + cxy$, we will often represent $P$ as the point $(a, b, c)$ in $\mathbb{R}^3$. Hence, the norm of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is in fact the norm in $\mathbb{R}^3$ given by

$$\|(a, b, c)\|_{D\left(\frac{\pi}{4}\right)} = \sup\left\{|ax^2 + by^2 + cxy| : (x, y) \in D\left(\frac{\pi}{4}\right)\right\}.$$

In Section 3, the notation $\mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ will be useful to represent the symmetric bilinear forms on $\mathbb{R}^2$ endowed with the supremum norm on $D\left(\frac{\pi}{4}\right)$.

In order to obtain sharp polynomial inequalities in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ we will use the so called Krein-Milman approach, which is based on the fact that norm attaining convex functions attain their norm at an extreme point of their domain. Hence, an explicit description of the norm $\|\cdot\|_{D\left(\frac{\pi}{4}\right)}$ and the extreme points of the unit ball $B_{D\left(\frac{\pi}{4}\right)}$, denoted by $\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)$, will be required. Both are presented below:

**Lemma 1.1.** [20, Theorem 3.1] *If $P(x, y) = ax^2 + by^2 + cxy$, then*

$$\|P\|_{D\left(\frac{\pi}{4}\right)} = \begin{cases} \max\left\{|a|, \frac{1}{2}|a+b+c|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\} & \text{if } c(a-b) \geq 0, \\ \max\{|a|, \frac{1}{2}|a+b+c|\} & \text{if } c(a-b) \leq 0, \end{cases}$$

**Lemma 1.2.** [20, Theorem 4.4] *The extreme points of the unit ball of $\mathcal{P}({}^2D(\frac{\pi}{4}))$ are given by*

$$\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right) = \left\{\pm P_t, \pm Q_s, \pm(1,1,0) : \ -1 \leq t \leq 1 \text{ and } 1 \leq s \leq 5 + 4\sqrt{2}\right\},$$

*where*

$$P_t := (t, 4 + t + 4\sqrt{1+t}, -2 - 2t - 4\sqrt{1+t}),$$
$$Q_s := (1, s, -2\sqrt{2(1+s)}).$$

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Let us describe now the three inequalities that will be studied in this paper. Section 2 is devoted to obtain a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Namely, for a fixed $(x,y)\in D\left(\frac{\pi}{4}\right)$, we find the best (smallest) constant $\Phi(x,y)$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $\|\cdot\|_2$ denotes the euclidean norm in $\mathbb{R}^2$. Similarly, we also obtain a Markov global estimate on the gradient of polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, or in other words, the smallest constant $M>0$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq M\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y)\in D\left(\frac{\pi}{4}\right)$. It is necessary to mention that the study of Bernstein and Markov type inequalities has a longstanding tradition. The interested reader can find further information on this classical topic in [2, 13, 14, 16, 18, 19, 23, 26, 28, 29, 30, 31].

In Section 3 we find the smallest constant $K>0$ in the inequality

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq K\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where $P$ is an arbitrary polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $L\in\mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is the polar of $P$. Observe that here $\|L\|_{D\left(\frac{\pi}{4}\right)}$ stands for the sup norm of $L$ over $D\left(\frac{\pi}{4}\right)^2$. Hence, what we do is to provide the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. The calculation of polarization constants in various polynomial spaces is largely motivated as the extensive, existing bibliography on the topic shows (see for instance [10, 13, 17, 27]).

Finally, in Section 4 we investigate the smallest constant $C>0$ in the inequality

$$\||P|\|_{D\left(\frac{\pi}{4}\right)} \leq C\|P\|_{D\left(\frac{\pi}{4}\right)}, \tag{1.1}$$

for all $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $|P|$ is the modulus of $P$, i.e., if $P(x,y)=ax^2+by^2+cxy$, then $|P|(x,y)=|a|x^2+|b|y^2+|c|xy$. The constant $C$ turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. It is interesting to note that already in 1914, H. Bohr [4] studied this type of inequalities for infinite complex power series. Actually, the study of Bohr radii is nowadays a fruitful field (see for instance [1, 3, 6, 7, 8, 9]). Observe that the relationship between unconditional constants in polynomial spaces and inequalities of the type (1.1) was already noticed in [7].

## 2. Bernstein and Markov-type inequalities for polynomials on sectors

In this section we provide sharp estimates on the Euclidean length of the gradient $\nabla P$ of a polynomial $P$ in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

**Theorem 2.1.** *For every $(x,y)\in D\left(\frac{\pi}{4}\right)$ and $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ we have*

$$\|\nabla P\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

*where*

$$\Phi(x,y) = \begin{cases} 4\left[\left(13+8\sqrt{2}\right)x^2+\left(69+48\sqrt{2}\right)y^2-2\left(28+20\sqrt{2}\right)xy\right] & \text{if } 0\leq y\leq \frac{\sqrt{2}-1}{2}x \text{ or } \left(4\sqrt{2}-5\right)x\leq y\leq x,\\ \frac{x^4}{y^2}+4(x^2+y^2) & \text{if } \frac{\sqrt{2}-1}{2}x\leq y\leq\left(\sqrt{2}-1\right)x,\\ \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2} & \text{if } \left(\sqrt{2}-1\right)x\leq y\leq\left(4\sqrt{2}-5\right)x. \end{cases}$$

*Proof.* In order to calculate $\Phi(x,y):=\sup\{\|\nabla P(x,y)\|_2 : \|P\|_{D\left(\frac{\pi}{4}\right)}\leq 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x,y)\|_2 : P\in \operatorname{ext}(B_{D\left(\frac{\pi}{4}\right)})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with $P_t(x,y)=tx^2+\left(4+t+4\sqrt{1+t}\right)y^2-2\left(1+t+2\sqrt{1+t}\right)xy$, $t\in[-1,1]$. Then,

$$\nabla P_t(x,y)=\left(2tx-2\left(1+t+2\sqrt{1+t}\right)y, 2\left(4+t+4\sqrt{1+t}\right)y-2\left(1+t+2\sqrt{1+t}\right)x\right),$$

so that

$$\begin{aligned}\|\nabla P_t(x,y)\|_2^2 =& 4t^2x^2 + 4\left(1+t+2\sqrt{1+t}\right)^2 y^2 - 8t\left(1+t+2\sqrt{1+t}\right)xy \\ &+ 4\left(4+t+4\sqrt{1+t}\right)^2 y^2 + 4\left(1+t+2\sqrt{1+t}\right)^2 x^2 \\ &- 8\left(4+t+4\sqrt{1+t}\right)\left(1+t+2\sqrt{1+t}\right)xy\end{aligned}$$

Make now the change $u=\sqrt{1+t}\in\left[0,\sqrt{2}\right]$, so that

$$\begin{aligned}\|\nabla P_u(x,y)\|_2^2 =& 8(x-y)^2u^4 + 16\left(x^2-4xy+3y^2\right)u^3 \\ &+ 8\left(x^2-10xy+13y^2\right)u^2 + 32\left(3y^2-xy\right)u + 4\left(x^2+9y^2\right).\end{aligned}$$

Since

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 = 16\Big(2\left(x-y\right)^2u^2 + \left(x^2-8xy+7y^2\right)u + 2y\left(3y-x\right)\Big)(u+1),$$

it follows that the critical points of $\|DP_u(x,y)\|_2^2$ are $u=\frac{2y}{x-y}$, $u=\frac{3y-x}{2(x-y)}$ and $u=-1$ if $x\neq y$ and $u=4$ and $u=-1$ if $x=y$. Since we need to consider $0\leq u\leq\sqrt{2}$, we can directly omit the case $x=y$.

Therefore, we can write

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 = 32(x-y)^2\left(u-\frac{2y}{x-y}\right)\left(u-\frac{3y-x}{2(x-y)}\right)(u+1).$$

Let $u_1=\frac{2y}{x-y}$ and $u_2=\frac{3y-x}{2(x-y)}$ (Again, since we need to consider $0\leq u\leq\sqrt{2}$, we can omit the solution $u=-1$). Also, we have the extra conditions $u_1\in[0,\sqrt{2}]$ whenever $0\leq y\leq\left(\sqrt{2}-1\right)x$ and $u_2\in[0,\sqrt{2}]$ whenever $\frac{1}{3}x\leq y\leq\left(4\sqrt{2}-5\right)x$. Considering all these facts, we need to compare the quantities

$$\begin{aligned}C_1(x,y) := \|\nabla P_{u_1}(x,y)\|_2^2 = \|\nabla P_{t_1}\|_2^2 &= 4\frac{x^6-4x^5y+7x^4y^2-8x^3y^3+7x^2y^4-4xy^5+y^6}{(x-y)^4} \\ &= 4\left(x^2+y^2\right),\end{aligned}$$

for $0\leq y\leq\left(\sqrt{2}-1\right)x$ and $t_1=\frac{3y^2+2xy-x^2}{(x-y)^2}$,

$$\begin{aligned}C_2(x,y) := \|\nabla P_{u_2}(x,y)\|_2^2 = \|\nabla P_{t_2}\|_2^2 &= \frac{9x^6-30x^5y+55x^4y^2-68x^3y^3+55x^2y^4-30xy^5+9y^6}{2(x-y)^4} \\ &= \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2},\end{aligned}$$

for $\frac{1}{3}x\leq y\leq\left(4\sqrt{2}-5\right)x$ and $t_2=\frac{5y^2+2xy-3x^2}{4(x-y)^2}$,

$$C_3(x,y) := \|\nabla P_{t_3=-1}\|_2^2 = 4\left(x^2+9y^2\right),$$

and

$$C_4(x,y) := \|\nabla P_{t_4=1}\|_2^2 = 4\left[\left(13+8\sqrt{2}\right)x^2+\left(69+48\sqrt{2}\right)y^2-2\left(28+20\sqrt{2}\right)xy\right].$$

Let us focus now on $Q_s=\left(1,s,-2\sqrt{2(1+s)}\right)$, $1\leq s\leq 5+4\sqrt{2}$. Then, we have

$$\|\nabla Q_s(x,y)\|_2^2 = 4x^2+4s^2y^2+8(1+s)(x^2+y^2)-8(1+s)\sqrt{2(1+s)}xy.$$

Making the change $v=\sqrt{2(1+s)}\in\left[2,2+2\sqrt{2}\right]$, we need to study the function

$$\|\nabla Q_v(x,y)\|_2^2 = v^2\left(y^2v^2-4xyv+4x^2\right)+4\left(x^2+y^2\right).$$

If $x=y=0$ we have $\|\nabla Q_v(x,y)\|_2^2=0$, so we will assume both $x\neq 0$ and $y\neq 0$. The critical points of $\|\nabla Q_v(x,y)\|_2^2$ are $v=\frac{x}{y}$, $v=\frac{2x}{y}$ and $v=0$ (but $0\notin[2,2+2\sqrt{2}]$). Observe that $v_1=\frac{x}{y}\in\left[2,2+2\sqrt{2}\right]$ whenever $\frac{\sqrt{2}-1}{2}x\leq y\leq\frac{1}{2}x$ and $v_2=\frac{2x}{y}\in\left[2,2+2\sqrt{2}\right]$ whenever $y\geq\left(\sqrt{2}-1\right)x$. Thus, we also need to compare the quantities

$$C_5(x,y) := \|\nabla Q_{v_1}(x,y)\|_2^2 = \|\nabla Q_{s_1}(x,y)\|_2^2 = \frac{x^4}{y^2}+4\left(x^2+y^2\right),$$

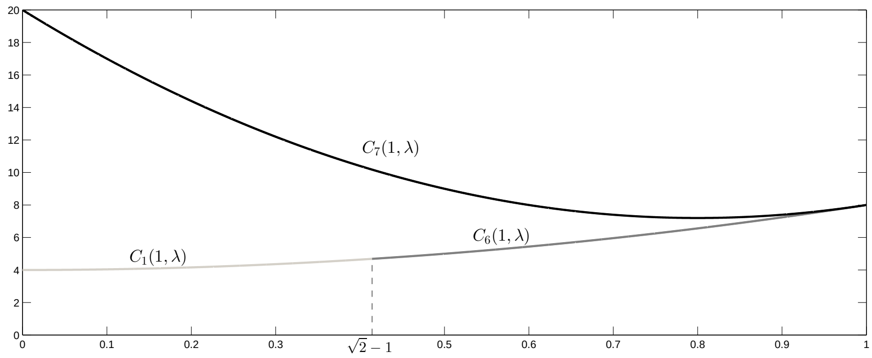


FIGURE 1. Graphs of the mappings $C_1(1,\lambda)$, $C_6(1,\lambda)$, $C_7(1,\lambda)$.

for $\frac{\sqrt{2}-1}{2}x \leq y \leq \frac{1}{2}x$ and $s_1 = \frac{x^2-2y^2}{2y^2}$,

$$C_6(x,y) := \|\nabla Q_{v_2}(x,y)\|_2^2 = \|\nabla Q_{s_2}(x,y)\|_2^2 = 4\left(x^2+y^2\right),$$

for $\left(\sqrt{2}-1\right)x \leq y \leq x$ and $s_2 = \frac{2x^2-y^2}{y^2}$, and also

$$C_7(x,y) := \|\nabla Q_{s_3=1}\|_2^2 = 4\left(x^2+y^2\right) + 16(x-y)^2,$$

and

$$\begin{aligned} C_8(x,y) &:= \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2 \\ &= \left(12+8\sqrt{2}\right)\left[4x^2 + \left(12+8\sqrt{2}\right)y^2 - \left(8+8\sqrt{2}\right)xy\right] + 4\left(x^2+y^2\right) \\ &= 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right]. \end{aligned}$$

Note that (the reader can take a look at Figures 1, 2 and 3)

$$\begin{aligned} &C_1(x,y), C_6(x,y) \leq C_7(x,y) \leq \begin{cases} C_4(x,y) & \text{if } 0 \leq y \leq \frac{2-\sqrt{2}}{2}x \text{ or } \frac{1}{2}x \leq y \leq x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \leq y \leq \frac{1}{2}x, \end{cases} \\ &C_3(x,y) \leq \begin{cases} C_2(x,y) & \text{if } \frac{1}{3}x \leq y \leq \left(4\sqrt{2}-5\right)x, \\ C_4(x,y) & \text{if } 0 \leq y \leq \frac{1}{3}x \text{ or } (4\sqrt{2}-5)x \leq y \leq x, \end{cases} \\ &C_8(x,y) = C_4(x,y). \end{aligned}$$

Hence, for $(x,y) \in D\left(\frac{\pi}{4}\right)$,

$$\begin{aligned} \Phi(x,y) &= \sup\left\{\|\nabla P(x,y)\|_2 : P \in \text{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\} \\ &= \begin{cases} C_4(x,y) & \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2}x \text{ or } (4\sqrt{2}-5)x \leq y \leq x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \leq y \leq (\sqrt{2}-1)x, \\ C_2(x,y) & \text{if } (\sqrt{2}-1)x \leq y \leq (4\sqrt{2}-5)x. \end{cases} \end{aligned}$$

In order to illustrate the previous step, the reader can take a look at Figure 4. □

**Corollary 2.2.** *If $P \in \mathcal{P}\left(D\left(\frac{\pi}{4}\right)\right)$, then*

$$\sup\left\{\|\nabla P(x,y)\|_2 : (x,y) \in D\left(\frac{\pi}{4}\right)\right\} \leq 4(13+8\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

*with equality for the polynomials $P_1(x,y) = \pm\left(x^2 + (5+4\sqrt{2})y^2 - 2(2+2\sqrt{2})xy\right)$.*

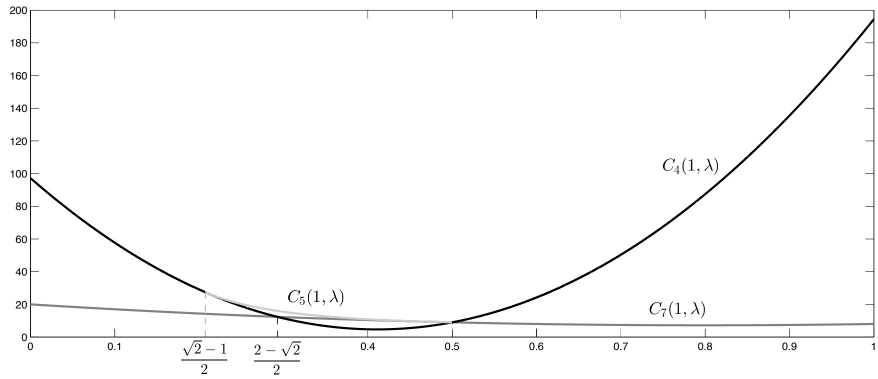


FIGURE 2. Graphs of the mappings $C_4(1,\lambda)$, $C_5(1,\lambda)$, $C_7(1,\lambda)$.

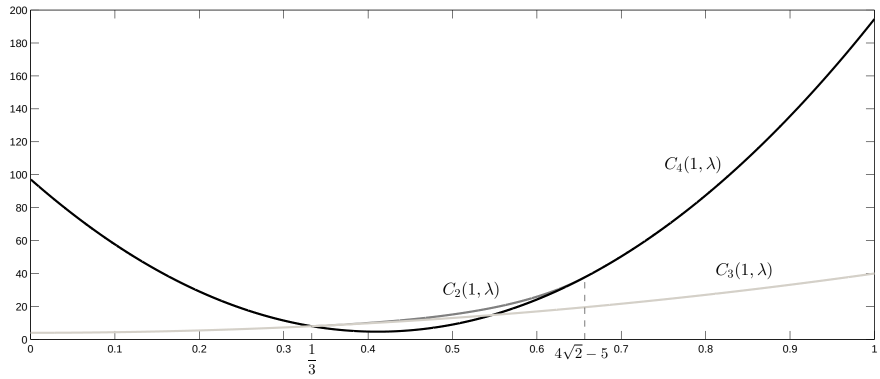


FIGURE 3. Graphs of the mappings $C_2(1,\lambda)$, $C_3(1,\lambda)$, $C_4(1,\lambda)$.

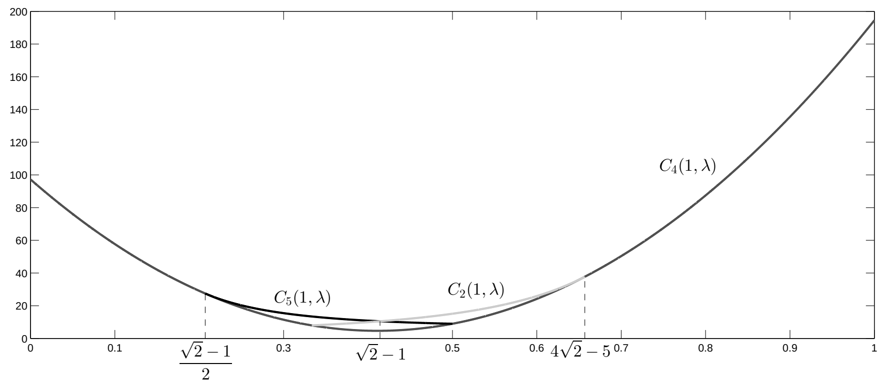


FIGURE 4. Graphs of the mappings $C_2(1,\lambda)$, $C_4(1,\lambda)$, $C_5(1,\lambda)$.

## 3. POLARIZATION CONSTANTS FOR POLYNOMIALS ON SECTORS

In this section we find the exact value of the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. In order to do that, we prove a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Observe that if $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y) \in D\left(\frac{\pi}{4}\right)$ then the differential $DP(x,y)$ of $P$ at $(x,y)$ can be viewed as a linear form. What we shall do is to find the best estimate for $\|DP(x,y)\|_{D(\frac{\pi}{4})}$ (the sup norm of $DP(x,y)$ over the sector $D\left(\frac{\pi}{4}\right)$) in terms of $(x,y)$ and $\|P\|_{D(\frac{\pi}{4})}$. First, we state a lemma that will be useful in the future:

**Lemma 3.1.** *Let $a, b \in \mathbb{R}$. Then,*

$$\sup_{\theta\in\left[0,\frac{\pi}{4}\right]} |a\cos\theta + b\sin\theta| = \begin{cases} \max\left\{|a|, \frac{\sqrt{2}}{2}|a+b|\right\} & \text{if } \frac{b}{a} > 1 \text{ or } \frac{b}{a} < 0, \\ \sqrt{a^2+b^2} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sqrt{a^2+b^2} & \text{if } 0 < \frac{b}{a} < 1, \\ \frac{\sqrt{2}}{2}|a+b| & \text{if } \left(1-\sqrt{2}\right)b < a < b \text{ or } b < a < \left(1-\sqrt{2}\right)b, \\ |a| & \text{if } -\left(1+\sqrt{2}\right)a < b < 0 \text{ or } 0 < b < -\left(1+\sqrt{2}\right)a. \end{cases}$$

**Theorem 3.2.** *For every $(x,y) \in D(\frac{\pi}{4})$ and $P \in \mathcal{P}(^2D(\frac{\pi}{4}))$ we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{4})} \leq \Psi(x,y)\|P\|_{D(\frac{\pi}{4})}, \tag{3.1}$$

*where*

$$\Psi(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < (\sqrt{2}-1)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } (\sqrt{2}-1)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x \end{cases}$$

*Moreover, inequality* (3.1) *is optimal for each $(x,y) \in D(\frac{\pi}{4})$.*

*Proof.* In order to calculate $\Psi(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : \|P\|_{D(\frac{\pi}{4})}) \leq 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : P \in \text{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with

$$P_t(x,y) = tx^2 + \left(4+t+4\sqrt{1+t}\right)y^2 - \left(2+2t+4\sqrt{1+t}\right)xy.$$

So we may write

$$\nabla P_t(x,y) = \left(2tx - \left(2+2t+4\sqrt{1+t}\right)y,\ 2\left(4+t+4\sqrt{1+t}\right)y - \left(2+2t+4\sqrt{1+t}\right)x\right),$$

from which

$$\begin{aligned} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} &= \sup_{0\leq\theta\leq\frac{\pi}{4}} \Big|2\left[tx - \left(1+t+2\sqrt{1+t}\right)y\right]\cos\theta \\ &\quad + 2\left[\left(4+t+4\sqrt{1+t}\right)y - \left(1+t+2\sqrt{1+t}\right)x\right]\sin\theta\Big| \\ &= 2x \sup_{0\leq\theta\leq\frac{\pi}{4}} |f_\lambda(t,\theta)|, \\ &\quad \text{for } f_\lambda(t,\theta) = \left[t - \left(1+t+2\sqrt{1+t}\right)\lambda\right]\cos\theta \\ &\qquad\qquad + \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right]\sin\theta, \end{aligned}$$

where $\lambda = \frac{y}{x}$, $x \neq 0$ (the case $x = 0$ is trivial, since the only point in $D(\frac{\pi}{4})$ where $x = 0$ is $(0,0)$, in which case $P_t(0,0) = \|DP_t(0,0)\|_{D(\frac{\pi}{4})} = 0$).

We need to calculate

$$\sup_{-1\leq t\leq 1} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{\substack{0\leq\theta\leq\frac{\pi}{4} \\ -1\leq t\leq 1}} |f_\lambda(t,\theta)|.$$

Let us define $C_1 = [-1,1] \times [0, \frac{\pi}{4}]$. We will analyze 5 cases.

(1) $(t,\theta) \in (-1,1) \times (0, \frac{\pi}{4})$.

We are interested just in critical points. Hence,

$$\frac{\partial f_\lambda}{\partial t}(t,\theta) = \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right]\sin\theta + \left[1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right]\cos\theta = 0, \tag{3.2}$$

$$\frac{\partial f_\lambda}{\partial \theta}(t,\theta) = \left[\left(1+t+2\sqrt{1+t}\right)\lambda - t\right]\sin\theta + \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right]\cos\theta = 0 \tag{3.3}$$

Equation (3.3) tells us that

$$\sin\theta = \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t-\left(1+t+2\sqrt{1+t}\right)\lambda}\cos\theta. \tag{3.4}$$

If we now plug (3.4) in equation (3.2), we obtain

$$0 = \left\{\left[1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right] \times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t-\left(1+t+2\sqrt{1+t}\right)\lambda}\right\}\cos\theta.$$

Using that $0<\theta<\frac{\pi}{4}$, we can conclude

$$0 = \left[1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right] \times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t-\left(1+t+2\sqrt{1+t}\right)\lambda}$$

and thus

$$\begin{aligned}
0 &= \left[1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right]\cdot\left[t-\left(1+t+2\sqrt{1+t}\right)\lambda\right] \\
&\quad + \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right]\cdot\left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right] \\
&= t - \left(1+t+2\sqrt{1+t}\right)\lambda - t\lambda + \left(1+t+2\sqrt{1+t}\right)\lambda^2 - \frac{\lambda t}{\sqrt{1+t}} \\
&\quad + \frac{\lambda^2}{\sqrt{1+t}}\left(1+t+2\sqrt{1+t}\right) + \left(1+\frac{2}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda^2 \\
&\quad - \left(1+\frac{2}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda \\
&\quad + \left(1+\frac{1}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right) \\
&= t\left(1-2\lambda+2\lambda^2-2\lambda+1\right) + \left(-2\lambda+2\lambda^2+4\lambda^2-2\lambda-4\lambda+2\right)\sqrt{1+t} \\
&\quad + \frac{t}{\sqrt{1+t}}\left(-\lambda+\lambda^2+2\lambda^2-2\lambda-\lambda+1\right) + \frac{1}{\sqrt{1+t}}\left(\lambda^2+8\lambda^2-2\lambda-4\lambda+1\right) \\
&\quad + \left(-\lambda+\lambda^2+2\lambda^2+4\lambda^2-\lambda-4\lambda+1+2+8\lambda^2-8\lambda\right) \\
&= 2t(\lambda-1)^2 + 6\sqrt{1+t}(\lambda-1)\left(\lambda-\frac{1}{3}\right) + 3\frac{t}{\sqrt{1+t}}(\lambda-1)\left(\lambda-\frac{1}{3}\right) \\
&\quad + \frac{1}{\sqrt{1+t}}(3\lambda-1)^2 + 15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right).
\end{aligned}$$

Working with this last expression, we get

$$\begin{aligned} 0 &= 2t\sqrt{1+t}(\lambda-1)^2 + 6(1+t)(\lambda-1)\left(\lambda-\frac{1}{3}\right) + 3t(\lambda-1)\left(\lambda-\frac{1}{3}\right) \\ &\quad +(3\lambda-1)^2 + 15\sqrt{1+t}\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right) \end{aligned}$$

and hence, rearranging terms,

$$\sqrt{1+t}\left[15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right)+2t(\lambda-1)^2\right] = -9t(\lambda-1)\left(\lambda-\frac{1}{3}\right) - 15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right). \tag{3.5}$$

If $\lambda = 1$, we obtain

$$\sqrt{1+t}+1=0$$

and so, in particular, we have $\lambda \neq 1$. Equation (3.5) has two solutions,

$$t_1(\lambda) = \frac{-1+2\lambda+3\lambda^2}{(\lambda-1)^2} \quad \text{and} \quad t_2(\lambda) = \frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2}.$$

Using equation (3.2), we may see

$$\tan\theta = \frac{\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda - 1}{\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)}.$$

In particular, evaluating in $t_1(\lambda)$ we obtain

$$\tan\theta_1 = \frac{\left(1+\frac{1-\lambda}{2\lambda}\right)\lambda - 1}{\left(1+\frac{1-\lambda}{\lambda}\right)\lambda - \left(1+\frac{1-\lambda}{2\lambda}\right)} = \lambda,$$

in which case we have

$$D_{1,1}(\lambda) := |f_\lambda(t_1,\theta_1)| = \left|-\sqrt{1+\lambda^2}\right| = \sqrt{1+\lambda^2}.$$

Regarding $t_2(\lambda)$, we obtain

$$\tan\theta_2 = \frac{\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda - 1}{\left(1+2\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda - \left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)}.$$

Since $\theta_2 \in \left(0,\frac{\pi}{4}\right)$, we need to guarantee $0<\tan\theta_2<1$, and for this we need $0<\lambda<\frac{1}{5}$. Therefore

$$\tan\theta_2 = \frac{5\lambda-1}{7\lambda-3}$$

and in this case,

$$\begin{aligned} D_{1,2}(\lambda) &:= |f_\lambda(t_2,\theta_2)| \\ &= \Bigg|\left[\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2} - \left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{3\lambda-1}{\lambda-1}\right)\lambda\right]\frac{3-7\lambda}{\sqrt{74\lambda^2-52\lambda+10}} \\ &\quad + \left[\left(3+\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{6\lambda-2}{\lambda-1}\right)\lambda - \left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{3\lambda-1}{\lambda-1}\right)\right]\frac{1-5\lambda}{\sqrt{74\lambda^2-52\lambda+10}}\Bigg| \\ &= \left|-\frac{78\lambda^4-208\lambda^3+196\lambda^2-80\lambda+14}{4(\lambda-1)^2\sqrt{74\lambda^2-52\lambda+10}}\right| \\ &= \left|-\frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}\right| \\ &= \frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}. \end{aligned}$$

(2) $\theta = 0, -1 \le t \le 1$.

We have

$$f_\lambda(t,0)=t-\left(1+t+2\sqrt{1+t}\right)\lambda.$$

Then,

$$f_\lambda(-1,0)=-1,$$
$$f_\lambda(1,0)=1-2\left(1+\sqrt{2}\right)\lambda,$$

and hence

$$|f_\lambda(1,0)|=\begin{cases}1-2(1+\sqrt{2})\lambda & \text{if } 0\le\lambda<\frac{\sqrt{2}-1}{2},\\ 2\left(1+\sqrt{2}\right)\lambda-1 & \text{if } \frac{\sqrt{2}-1}{2}\le\lambda\le 1.\end{cases}$$

Working now on $(-1,1)$, since

$$f'_\lambda(t,0)=1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda,$$

the critical point of $f_\lambda(t,0)$ is

$$t=\frac{\lambda^2}{(1-\lambda)^2}-1.$$

Recall that we need to make sure that $-1<t<1$. Therefore, in this case we also need to ask

$$\lambda<\frac{\sqrt{2}}{1+\sqrt{2}}=2-\sqrt{2}.$$

Plugging the critical point of $f_\lambda(t,0)$ into $f_\lambda(t,0)$, we obtain

$$f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2}-1,0\right)=\frac{\lambda^2}{(\lambda-1)^2}-1-\left[\frac{\lambda^2}{(\lambda-1)^2}+\frac{2\lambda}{1-\lambda}\right]\lambda=\frac{\lambda^2}{\lambda-1}-1,$$

and hence

$$\left|f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2}-1,0\right)\right|=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume first $0\le\lambda<\frac{\sqrt{2}-1}{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,1-2\left(1+\sqrt{2}\right)\lambda,\,1+\frac{\lambda^2}{1-\lambda}\right\}=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume now $\frac{\sqrt{2}-1}{2}\le\lambda<2-\sqrt{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,2\left(1+\sqrt{2}\right)\lambda-1,\,1+\frac{\lambda^2}{1-\lambda}\right\}=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume finally $2-\sqrt{2}\le\lambda\le 1$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,2\left(1+\sqrt{2}\right)\lambda-1\right\}=2\left(1+\sqrt{2}\right)\lambda-1.$$

So, in conclusion,

$$\begin{aligned}\sup_{-1\le t\le 1}|f_\lambda(t,0)| &= \begin{cases}1+\frac{\lambda^2}{1-\lambda} & \text{if } 0\le\lambda<2-\sqrt{2},\\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\\ &=: \begin{cases}D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2},\\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1.\end{cases}\end{aligned}$$

(3) $\theta=\frac{\pi}{4}$ and $-1\le t\le 1$.

We have

$$\begin{aligned}f_\lambda\left(t,\frac{\pi}{4}\right)&=\frac{\sqrt{2}}{2}\left[t-\left(1+t+2\sqrt{1+t}\right)\lambda+\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)\right]\\&=\frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{1+t}\right)\lambda-\left(1+2\sqrt{1+t}\right)\right].\end{aligned}$$

Again, we have

$$\begin{aligned} f_\lambda\left(-1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left(3\lambda-1\right), \\ f_\lambda\left(1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right], \\ f'_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\frac{\lambda}{\sqrt{1+t}}-\frac{1}{\sqrt{1+t}}\right]. \end{aligned}$$

and $f'_\lambda(t,\frac{\pi}{4})=0$ implies $\lambda=1$ (in which case $f_\lambda(t,\frac{\pi}{4})=\sqrt{2}$ for every $t$).

- Assume first $0\le\lambda<\frac{1}{3}$. Then,

$$\begin{aligned} \sup_{-1\le t\le 1}|f_\lambda\left(t,\frac{\pi}{4}\right)| &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\, 1-3\lambda\right\} \\ &= \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] \end{aligned}$$

- Assume now $\frac{1}{3}\le\lambda<4\sqrt{2}-5$. Then,

$$\begin{aligned} \sup_{-1\le t\le 1}|f_\lambda\left(t,\frac{\pi}{4}\right)| &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\, 3\lambda-1\right\} \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } \frac{1}{3}\le\lambda<\frac{2\sqrt{2}+1}{7}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda<4\sqrt{2}-5. \end{cases} \end{aligned}$$

- Assume finally $4\sqrt{2}-5\le\lambda\le 1$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda\left(t,\frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2}\max\left\{3\lambda-1,\, \left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right\} = \frac{\sqrt{2}}{2}(3\lambda-1).$$

Hence, we can say that

$$\begin{aligned} \sup_{-1\le t\le 1}|f_\lambda\left(t,\tfrac{\pi}{4}\right)| &= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7} \\ \frac{\sqrt{2}}{2}\left(3\lambda-1\right) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases} \\ &=: \begin{cases} D_{3,1}(\lambda) & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7} \\ D_{3,2}(\lambda) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases} \end{aligned}$$

(4) $t=-1$, $0\le\theta\le\frac{\pi}{4}$.

Applying lemma 3.1, we obtain

$$\begin{aligned} \sup_{0\le\theta\le\frac{\pi}{4}} f_\lambda(-1,\theta) &= \begin{cases} 1 & \text{if } 0\le\lambda<\frac{1+\sqrt{2}}{3}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{1+\sqrt{2}}{3}\le\lambda\le 1. \end{cases} \\ &=: \begin{cases} D_{4,1}(\lambda) & \text{if } 0\le\lambda<\frac{1+\sqrt{2}}{3}, \\ D_{4,2}(\lambda) & \text{if } \frac{1+\sqrt{2}}{3}\le\lambda\le 1. \end{cases} \end{aligned}$$

(5) $t=1$, $0\le\theta\le\frac{\pi}{4}$.

We use again lemma 3.1, with $a=1-\left(2+2\sqrt{2}\right)\lambda$ and $b=\left(5+4\sqrt{2}\right)\lambda-\left(2+2\sqrt{2}\right)$. Through standard calculations, we see that $\frac{b}{a}<0$ if and only if $\lambda\in\left[0,\frac{\sqrt{2}-1}{2}\right)\cup\left(\frac{6-2\sqrt{2}}{7},1\right]$ and $\frac{b}{a}>1$ if and only if $\frac{\sqrt{2}-1}{2} < \lambda < \frac{3+4\sqrt{2}}{23}$. Therefore,

$$\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)|$$
$$= \begin{cases} \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23}, \\ \sqrt{\left(1-\left(2+2\sqrt{2}\right)\lambda\right)^2+\left(\left(5+4\sqrt{2}\right)\lambda-\left(2+2\sqrt{2}\right)\right)^2} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7}, \\ \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}$$

Since $0\le\lambda<\sqrt{2}-1$ implies $\left|1-\left(2+2\sqrt{2}\right)\lambda\right| < \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|$, it follows that

$$\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)|$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right| & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left|1-\left(2+2\sqrt{2}\right)\lambda\right| & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1 \end{cases}$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}$$
$$=: \begin{cases} D_{5,1}(\lambda) & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ D_{5,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ D_{5,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}$$

Since (see Figures 5 and 6)

$$D_{1,1}(\lambda) \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1, \end{cases}$$
$$D_{1,2}(\lambda) \le D_{3,1}(\lambda) \text{ for } 0<\lambda<\tfrac{1}{5},$$

we can rule out case (1). Since

$$D_{3,1}(\lambda) = D_{5,1}(\lambda) \quad \text{for } 0\le\lambda\le\tfrac{3+4\sqrt{2}}{23},$$
$$D_{3,2}(\lambda) = D_{4,2}(\lambda) \quad \text{for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,$$

we can directly rule out case (3). Since (see Figures 5 and 7)

$$D_{4,1}(\lambda) = 1 \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda<\frac{1+\sqrt{2}}{3}, \end{cases}$$
$$D_{4,2}(\lambda) \le D_{2,2} \text{ for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,$$

we can rule out case (4). Finally, since (see Figure 8)

$$D_{5,2}(\lambda) \le D_{2,1}(\lambda) \quad \text{for } \tfrac{3+4\sqrt{2}}{23}\le\lambda<\tfrac{6-2\sqrt{2}}{7},$$
$$D_{5,3}(\lambda) = D_{2,2}(\lambda) \quad \text{for } 2-\sqrt{2}\le\lambda\le 1,$$

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of case (5).

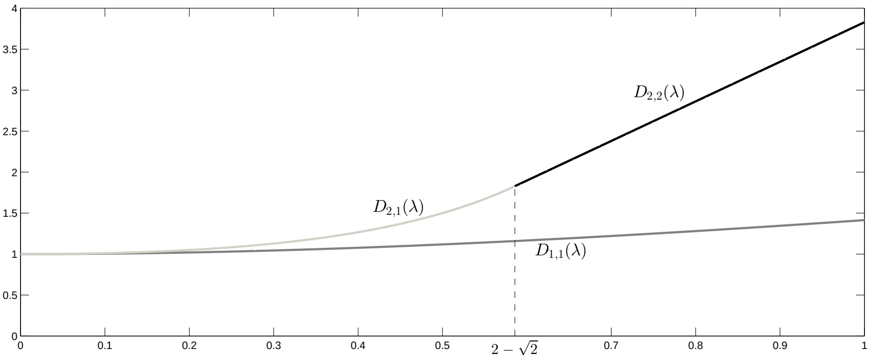


FIGURE 5. Graphs of the mappings $D_{1,1}(\lambda)$, $D_{2,1}(\lambda)$ and $D_{2,2}(\lambda)$.

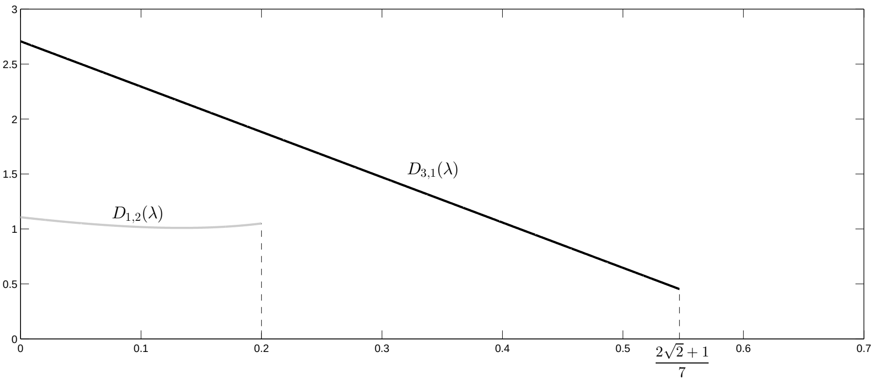


FIGURE 6. Graphs of the mappings $D_{1,2}(\lambda)$ and $D_{3,1}(\lambda)$.

Thus, putting all the above cases together, we may reach the conclusion

$$\begin{aligned}\sup_{(t,\theta)\in C_1}|f_\lambda(t,\theta)| &= \begin{cases} D_{5,1}(\lambda) & \text{if } 0\le\lambda<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14},\\ D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}\le\lambda<2-\sqrt{2},\\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14},\\ 1+\frac{\lambda^2}{1-\lambda} & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}\le\lambda<2-\sqrt{2},\\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\end{aligned}$$

and hence

$$\begin{aligned}&\sup_{-1\le t\le 1}\|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x\sup_{(t,\theta)\in C_1}|f_\lambda(t,\theta)|\\ &= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0\le y<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x,\\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x\le y<\left(2-\sqrt{2}\right)x,\\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x\le y\le x,\end{cases}\end{aligned}$$

assuming in every moment $x\neq 0$ (in order to illustrate the previous step, the reader can take a look at Figure 9).

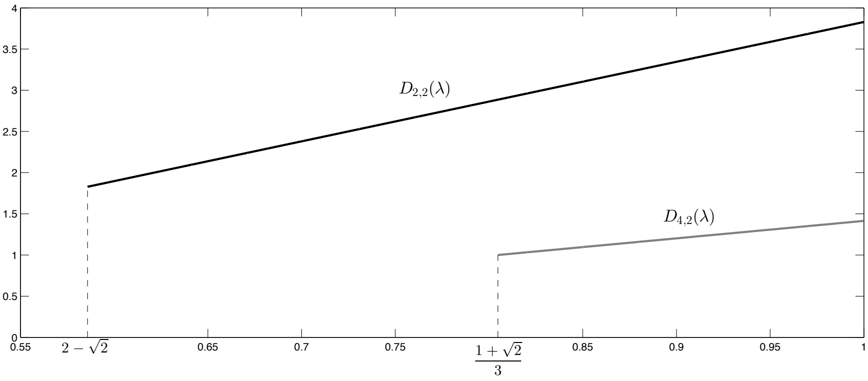


FIGURE 7. Graphs of the mappings $D_{2,2}(\lambda)$ and $D_{4,2}(\lambda)$.

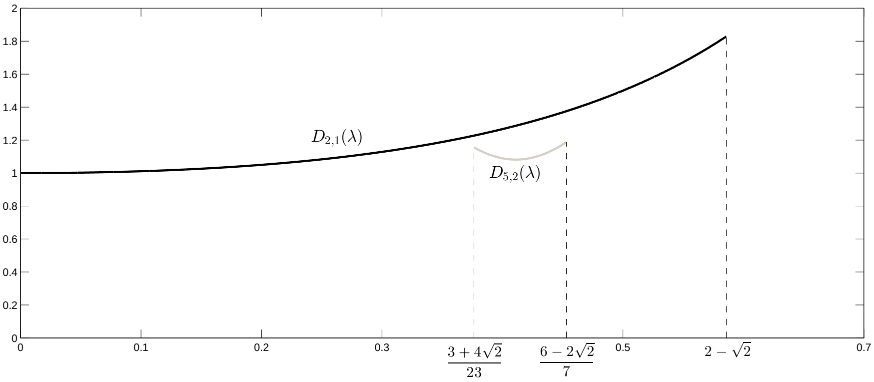


FIGURE 8. Graphs of the mappings $D_{2,1}(\lambda)$ and $D_{5,2}(\lambda)$.

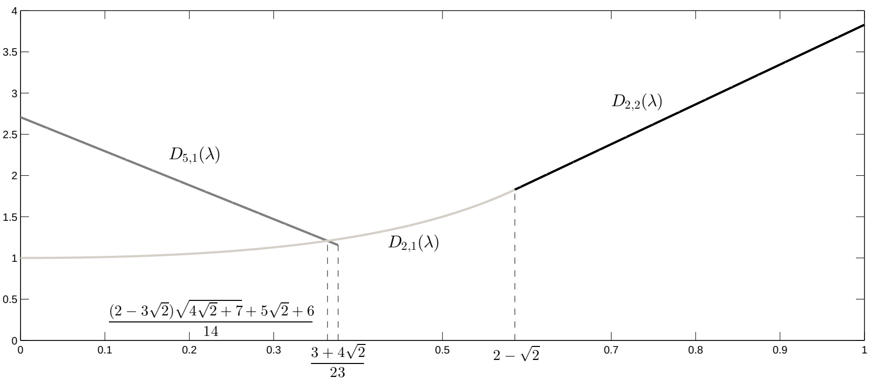


FIGURE 9. Graphs of the mappings $D_{2,1}(\lambda)$, $D_{2,2}(\lambda)$ and $D_{5,1}(\lambda)$.

Let us deal now with the polynomials

$$Q_s(x,y) = x^2 + sy^2 - 2\sqrt{2(1+s)}xy, \quad 1 \le s \le 5 + 4\sqrt{2}.$$

Then,

$$\nabla Q_s(x,y) = \left(2x - 2\sqrt{2(1+s)}y,\, 2sy - 2\sqrt{2(1+s)}x\right),$$
$$\|DQ_s(x,y)\|_{D(\frac{\pi}{4})} = \sup_{0\le\theta\le\frac{\pi}{4}} \left|2x\left[\left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta + \left(s\lambda - \sqrt{2(1+s)}\right)\sin\theta\right]\right|,$$

and thus

$$\sup_{1\le s\le 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(s,\theta)\in C_2} |g_\lambda(s,\theta)|,$$

with

$$g_\lambda(s,\theta) = \left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta + \left(s\lambda - \sqrt{2(1+s)}\right)\sin\theta$$

and $C_2 = [1, 5+4\sqrt{2}] \times [0, \frac{\pi}{4}]$. Again, we have several cases:

(6) $(s,\theta) \in (1, 5+4\sqrt{2}) \times (0, \frac{\pi}{4})$.

Let us first calculate the critical points of $g_\lambda$ over $C_2$.

$$\frac{\partial g_\lambda}{\partial s}(s_0,\theta_0) = \frac{-\lambda}{\sqrt{2(1+s_0)}}\cos\theta_0 + \left(\lambda - \frac{1}{\sqrt{2(1+s_0)}}\right)\sin\theta_0,$$
$$\frac{\partial g_\lambda}{\partial \theta}(s_0,\theta_0) = \left(s_0\lambda - \sqrt{2(1+s_0)}\right)\cos\theta_0 - \left(1-\sqrt{2(1+s_0)}\lambda\right)\sin\theta_0,$$

so, if $Dg_\lambda(s_0,\theta_0)=0$, using the first expression, we obtain $\tan\theta_0 = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1}$, and, using the second one, we obtain $\tan\theta_0 = \frac{s_0\lambda-\sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda}$.
Hence, we may say

$$\frac{s_0\lambda-\sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1}$$

and thus

$$s_0 = \frac{2-\lambda^2}{\lambda^2}.$$

Then, $\tan\theta_0 = \lambda$ and also, if we want to guarantee that $1 < s_0 < 5+4\sqrt{2}$, we need $\sqrt{2}-1<\lambda<1$.
In that case, $\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}$, and then

$$g_\lambda(s_0,\theta_0) = \frac{-1}{\sqrt{1+\lambda^2}} + \frac{-\lambda^2}{\sqrt{1+\lambda^2}} = -\sqrt{1+\lambda^2},$$

so

$$D_6(\lambda) := |g_\lambda(s_0,\theta_0)| = \sqrt{1+\lambda^2}.$$

(7) $s=1$, $0\le\theta\le\frac{\pi}{4}$.

Apply lemma 3.1 with $a = 1-2\lambda$ and $b = \lambda-2$. Using $0\le\lambda\le 1$, observe that we always have $b<0$ and $b\le a$. Also, $a < \left(1-\sqrt{2}\right)b$ if and only if $\lambda > \frac{5-3\sqrt{2}}{7}$.
Putting everything together, we can say

$$\begin{aligned}\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(1,\theta)| &= \begin{cases} 1-2\lambda & \text{if } 0\le\lambda<\frac{5-3\sqrt{2}}{7},\\ \frac{\sqrt{2}}{2}(1+\lambda) & \text{if } \frac{5-3\sqrt{2}}{7}\le\lambda\le 1,\end{cases}\\ &=: \begin{cases} D_{7,1}(\lambda) & \text{if } 0\le\lambda<\frac{5-3\sqrt{2}}{7},\\ D_{7,2}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7}\le\lambda\le 1.\end{cases}\end{aligned}$$

(8) $s = 5+4\sqrt{2}$, $0\le\theta\le\frac{\pi}{4}$.

Apply again lemma 3.1, this time to $a = 1-2\left(1+\sqrt{2}\right)\lambda$ and $b = \left(5+4\sqrt{2}\right)\lambda - 2\left(1+\sqrt{2}\right)$. As usual, we notice that $a<0$ if and only if $\lambda > \frac{\sqrt{2}-1}{2}$, $b<0$ if and only if $\lambda < \frac{6-2\sqrt{2}}{7}$ and $a<b$ if and only if $\lambda > \frac{3+4\sqrt{2}}{23}$. All together, we can say that, for $\frac{3+4\sqrt{2}}{23} < \lambda < \frac{6-2\sqrt{2}}{7}$, we have

$$\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| = \sqrt{a^2+b^2} = \sqrt{13+8\sqrt{2}-\left(56+40\sqrt{2}\right)\lambda+\left(69+48\sqrt{2}\right)\lambda^2}.$$

Also, notice that, for any $\lambda\in[0,1]$, we are going to have $b < -\left(1+\sqrt{2}\right)a$ and $a < \left(1-\sqrt{2}\right)b$. Hence,

$$\begin{aligned}
&\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23}, \\ \sqrt{13+8\sqrt{2}-\left(56+40\sqrt{2}\right)\lambda+\left(69+48\sqrt{2}\right)\lambda^2} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda-1 & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1, \end{cases} \\
&=: \begin{cases} D_{8,1}(\lambda) & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23}, \\ D_{8,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}
\end{aligned}$$

(9) $\theta=0$, $1\le s\le 5+4\sqrt{2}$.

We have

$$\begin{aligned}
g_\lambda(s,0) &= 1-\sqrt{2(1+s)}\lambda, \\
g_\lambda(1,0) &= 1-2\lambda, \\
g_\lambda(5+4\sqrt{2},0) &= 1-2\left(1+\sqrt{2}\right)\lambda, \\
g'_\lambda(s,0) &= -\frac{\lambda}{\sqrt{2(1+s)}} \ne 0 \text{ for } \lambda\ne 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_{1\le s\le 5+4\sqrt{2}} |g_\lambda(s,0)| &= \max\left\{|1-2\lambda|,\ |1-2(1+\sqrt{2})\lambda|\right\} \\
&= \begin{cases} 1-2\lambda & \text{if } 0\le\lambda<\frac{2-\sqrt{2}}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda-1 & \text{if } \frac{2-\sqrt{2}}{2}\le\lambda\le 1, \end{cases} \\
&=: \begin{cases} D_{9,1}(\lambda) & \text{if } 0\le\lambda<\frac{2-\sqrt{2}}{2}, \\ D_{9,2}(\lambda) & \text{if } \frac{2-\sqrt{2}}{2}\le\lambda\le 1. \end{cases}
\end{aligned}$$

(10) $\theta=\frac{\pi}{4}$, $1\le s\le 5+4\sqrt{2}$.

We have

$$g_\lambda\left(s,\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left[1+s\lambda-\sqrt{2(1+s)}(1+\lambda)\right].$$

Then

$$\begin{aligned}
g_\lambda\left(1,\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}(1+\lambda), \\
g_\lambda\left(5+4\sqrt{2},\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right], \\
g'_\lambda\left(s_0,\frac{\pi}{4}\right) &= 0 \text{ if and only if } s_0 = \frac{(1+\lambda)^2}{2\lambda^2}-1
\end{aligned}$$

and since we need to ensure that $1<s_0<5+4\sqrt{2}$, we need $\frac{2\sqrt{2}-1}{7}<\lambda<1$. In that case,

$$g_\lambda\left(s_0,\frac{\pi}{4}\right) = -\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda}.$$

Hence,

$$\begin{aligned}\sup_{1\le s\le 5+4\sqrt{2}}\left|g_\lambda\left(s.\frac{\pi}{4}\right)\right| &= \begin{cases}\frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{2\sqrt{2}-1}{7},\\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7}\le\lambda\le 1,\end{cases}\\ &=: \begin{cases} D_{10,1}(\lambda) & \text{if } 0\le\lambda<\frac{2\sqrt{2}-1}{7},\\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7}\le\lambda\le 1.\end{cases}\end{aligned}$$

Since (the reader can take a look at Figure 10)

$$D_6(\lambda)\le\begin{cases} D_{8,2}(\lambda) & \text{if } \sqrt{2}-1<\lambda<\frac{6-2\sqrt{2}}{7},\\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda<1,\end{cases}$$

we can rule out case (6). Since (see Figures 11 and 12)

$$D_{7,1}(\lambda)\le D_{10,1}(\lambda) \text{ for } 0\le\lambda<\frac{5-3\sqrt{2}}{7}$$

$$D_{7,2}(\lambda)\le\begin{cases} D_{10,1}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7}\le\lambda<\frac{2\sqrt{2}-1}{7},\\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7}\le\lambda\le 1,\end{cases}$$

we can rule out case (7). Since

$$D_{8,1}(\lambda)=D_{10,1}(\lambda) \text{ for } 0\le\lambda<\frac{2\sqrt{2}-1}{7}$$

we can rule out the expression $D_{8,1}(\lambda)$ of case (8). Since

$$\begin{aligned} D_{9,1}(\lambda)&=D_{7,1}(\lambda) \quad \text{for } 0\le\lambda<\tfrac{5-3\sqrt{2}}{7},\\ D_{9,2}(\lambda)&=D_{8,3}(\lambda) \quad \text{for } \tfrac{6-2\sqrt{2}}{7}\le\lambda\le 1,\end{aligned}$$

we can directly rule out case (9). Furthermore, since (see Figure 13)

$$\begin{aligned} D_{8,2}(\lambda)&\le D_{10,2}(\lambda) \text{ for } \tfrac{3+4\sqrt{2}}{23}\le\lambda<\tfrac{6-2\sqrt{2}}{7},\\ D_{8,3}(\lambda)&\le D_{10,2}(\lambda) \text{ for } \tfrac{6-2\sqrt{2}}{7}\le\lambda\le\tfrac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},\end{aligned}$$

we can conclude that

$$\begin{aligned}\sup_{(s,\theta)\in C_2}|g_\lambda(s,\theta)| &= \begin{cases} D_{10,1}(\lambda) & \text{if } 0\le\lambda<\frac{2\sqrt{2}-1}{7},\\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7}\le\lambda<\frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},\\ D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}\le\lambda\le 1.\end{cases}\\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{2\sqrt{2}-1}{7},\\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7}\le\lambda<\frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},\\ 2\left(1+\sqrt{2}\right)\lambda-1 & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}\le\lambda\le 1,\end{cases}\end{aligned}$$

and hence

$$\begin{aligned}&\sup_{1\le s\le 5+4\sqrt{2}}\|DQ_s(x,y)\|_{D(\frac{\pi}{4})}\\ &=\begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0\le y<\frac{2\sqrt{2}-1}{7}x,\\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x\le y<\frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x,\\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x\le y\le x.\end{cases}\end{aligned}$$

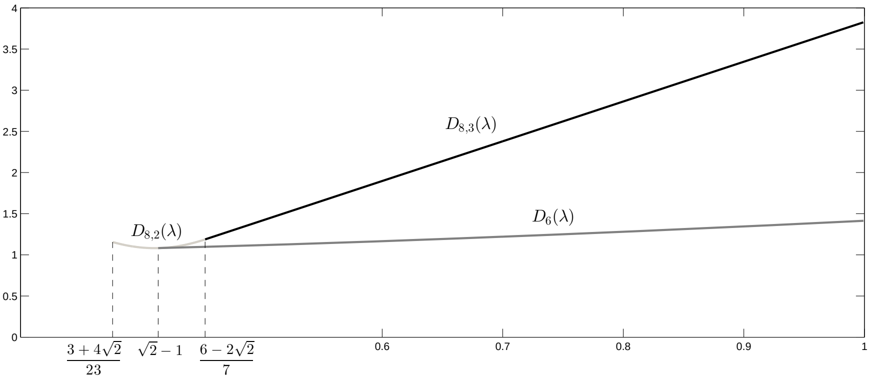


FIGURE 10. Graphs of the mappings $D_6(\lambda)$, $D_{8,2}(\lambda)$ and $D_{8,3}(\lambda)$.

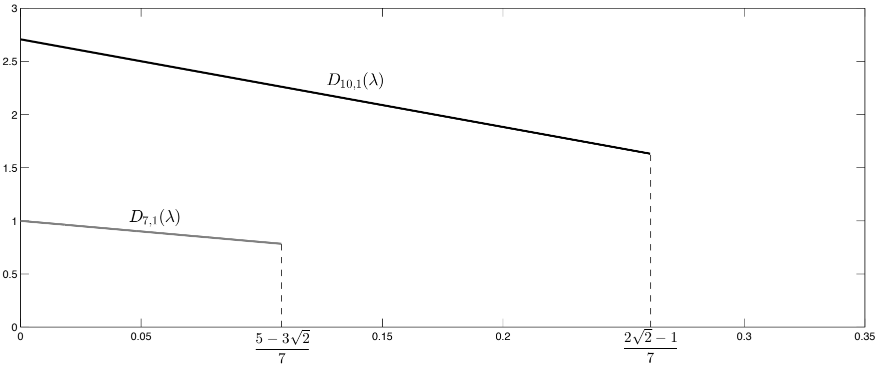


FIGURE 11. Graphs of the mappings $D_{7,1}(\lambda)$ and $D_{10,1}(\lambda)$.

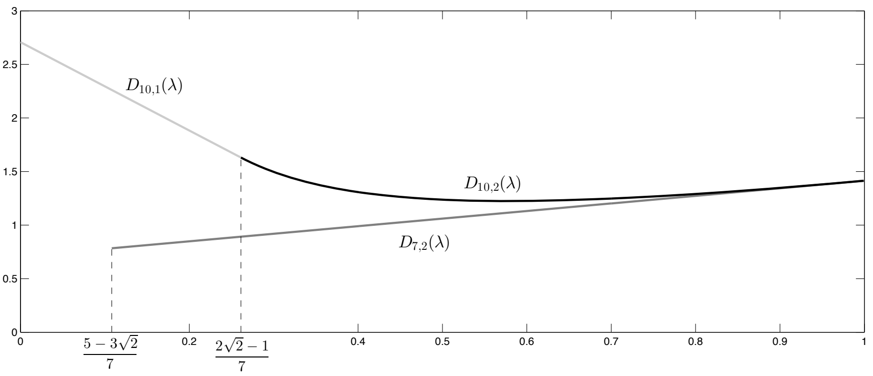


FIGURE 12. Graphs of the mappings $D_{7,2}(\lambda)$, $D_{10,1}(\lambda)$ and $D_{10,2}(\lambda)$.

Finally, if we compare the results obtained with $P_t$ and $Q_s$, since $\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} \geq 1 + \frac{\lambda^2}{1-\lambda}$ whenever $\lambda \leq \sqrt{2}-1$, we obtain

$$\Phi(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \left(\sqrt{2}-1\right)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } \left(\sqrt{2}-1\right)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x. \end{cases}$$

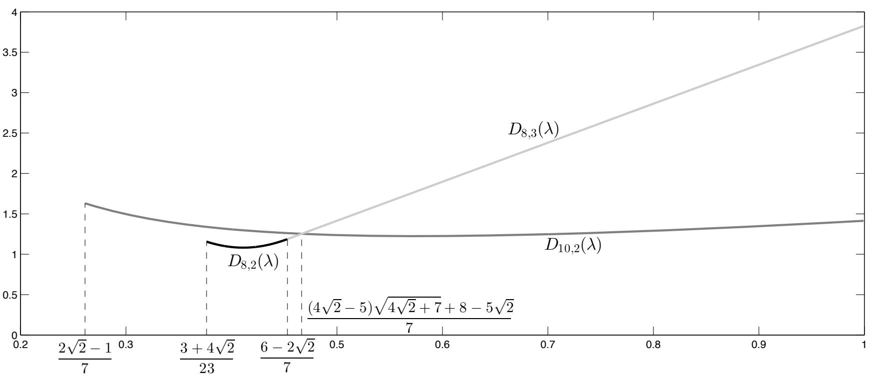


FIGURE 13. Graphs of the mappings $D_{8,2}(\lambda)$, $D_{8,3}(\lambda)$ and $D_{10,2}(\lambda)$.

□

We can see that $\Phi(x, y) \leq 4 + \sqrt{2}$, for all $(x, y) \in D\left(\frac{\pi}{4}\right)$. Furthermore, the maximum is attained by the polynomials

$$P_1(x, y) = x^2 + \left(5 + 4\sqrt{2}\right) y^2 - \left(4 + 4\sqrt{2}\right) xy = Q_{5+4\sqrt{2}}(x, y).$$

**Corollary 3.3.** *Let $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and assume $L \in \mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is the polar of $P$. Then*

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq \left(2 + \frac{\sqrt{2}}{2}\right) \|P\|_{D\left(\frac{\pi}{4}\right)}.$$

*Moreover, equality is achieved for $P_1(x, y) = Q_{5+4\sqrt{2}}(x, y) = x^2 + \left(5 + 4\sqrt{2}\right) y^2 - \left(4 + 4\sqrt{2}\right) xy$. Hence, the polarization constant of the polynomial space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $2 + \frac{\sqrt{2}}{2}$.*

## 4. Unconditional constants for polynomials on sectors

Here, we obtain a sharp estimate on the norm of the modulus of a polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ in terms of it norm. That sharp estimate turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

**Theorem 4.1.** *The unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $5 + 4\sqrt{2}$. In other words, the inequality*

$$|||P|||_{D\left(\frac{\pi}{4}\right)} \leq (5 + 4\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

*for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Furthermore, the previous inequality is sharp and equality is attained for the polynomials $\pm P_1(x, y) = \pm Q_{5+4\sqrt{2}}(x, y) = \pm\left[x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy\right]$.*

*Proof.* We just need to calculate

$$\sup\left\{|||P|||_{D\left(\frac{\pi}{4}\right)} : \ P \in \operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\}.$$

In order to calculate the above supremum we use the extreme polynomials described in Lemma 1.2. If we consider first the polynomials $P_t$, then $|P_t| = \left(|t|, 4 + t + 4\sqrt{1+t}, 2 + 2t + 4\sqrt{1+t}\right)$. Now, using Lemma 1.1 we have

$$\begin{aligned} \sup_{-1\leq t\leq 1} |||P_t|||_{D\left(\frac{\pi}{4}\right)} &= \sup_{-1\leq t\leq 1} \max\left\{|t|, \frac{1}{2}\left(|t| + 4 + t + 4\sqrt{1+t} + 2 + 2t + 4\sqrt{1+t}\right)\right\} \\ &= \sup_{-1\leq t\leq 1} \frac{1}{2}\left(|t| + 6 + 3t + 8\sqrt{1+t}\right) = 5 + 4\sqrt{2}. \end{aligned}$$

Notice that the above supremum is attained at $t = 1$. On the other hand, if we consider the polynomials $Q_s$, we have $|Q_s| = \left(1, s, 2\sqrt{2(1+s)}\right)$. Now, using Lemma 1.1 we have

$$\begin{aligned}\sup_{1\leq s\leq 5+4\sqrt{2}} |||Q_s|||_{D\left(\frac{\pi}{4}\right)} &= \sup_{1\leq s\leq 5+4\sqrt{2}} \max\left\{1, \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right)\right\} \\ &= \sup_{1\leq s\leq 5+4\sqrt{2}} \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right) = 5+4\sqrt{2}.\end{aligned}$$

Observe that the last supremum is now attained at $s = 5 + 4\sqrt{2}$. □

## 5. Conclusions

Comparing the results obtained in [11] and [25] for polynomials on the simplex $\Delta$, in [12] for polynomials on the unit square □, in [15] for polynomials on the sector $D\left(\frac{\pi}{2}\right)$ and the results obtained in the previous sections, we have the following:

| | $\mathcal{P}(^2\Delta)$ | $\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$ | $\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$ | $\mathcal{P}(^2\square)$ |
|---|---|---|---|---|
| Markov constants | $2\sqrt{10}$ | $2\sqrt{5}$ | $4(13+8\sqrt{2})$ | $\sqrt{13}$ |
| Polarization constants | 3 | 2 | $2+\frac{\sqrt{2}}{2}$ | $\frac{3}{2}$ |
| Unconditional Constants | 2 | 3 | $5+4\sqrt{2}$ | 5 |

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

(1) $\pm(x^2 + y^2 - 6xy)$ for the simplex.
(2) $\pm(x^2 + y^2 - 4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
(3) $\pm\left(x^2 + (5+4\sqrt{2})y^2 - (4+4\sqrt{2})xy\right)$ for the sector $D\left(\frac{\pi}{4}\right)$.
(4) $\pm(x^2 + y^2 - 3xy)$ for the unit square.

Compare the previous table with similar results that hold for 2-homogeneous polynomials on the Banach spaces $\ell_1^2$, $\ell_2^2$ and $\ell_\infty^2$:

| | $\mathcal{P}(^2\ell_1^2)$ | $\mathcal{P}\left(^2\ell_2^2\right)$ | $\mathcal{P}(^2\ell_\infty^2)$ |
|---|---|---|---|
| Markov constants | 4 | 2 | $2\sqrt{2}$ |
| Polarization constants | 2 | 1 | 2 |
| Unconditional Constants | $\frac{1+\sqrt{2}}{2}$ | $\sqrt{2}$ | $1+\sqrt{2}$ |

Observe that the Markov constants of the spaces $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ can be calculated taking into consideration the description of the geometry of those spaces given in [5]. Also, the Markov constant of $\mathcal{P}(^2\ell_2^2)$ is twice its polarization constant, or in other words, 2.
On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [27]).
Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [11].

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