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Integrability and non-perturbative effects in the AdS/CFT correspondence

César Gómez [a,b], Rafael Hernández [a,b,*]

[a] Instituto de Física Teórica UAM/CSIC, Universidad Autónoma de Madrid, Cantoblanco, E-28049 Madrid, Spain
[b] Theory Group, Physics Department, CERN, CH-1211 Geneva 23, Switzerland

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Abstract

We present a non-perturbative resummation of the asymptotic strong-coupling expansion for the dressing phase factor of the $AdS_5 \times S^5$ string S-matrix. The non-perturbative resummation provides a general form for the coefficients in the weak-coupling expansion, in agreement with crossing symmetry and transcendentality. The ambiguities of the non-perturbative prescription are discussed together with the similarities with the non-perturbative definition of the $c = 1$ matrix model.

1. Introduction

The uncovering of integrable structures on both sides of the AdS/CFT correspondence [1] has suggested a path toward a complete formulation of the duality. On the gauge theory side, focusing on operators with large quantum numbers [2] lead to the identification of the planar dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang–Mills with the Hamiltonian of an integrable spin chain [3,4]. Assuming that integrability holds at higher orders a long-range Bethe ansatz was then proposed to describe the spectrum of Yang–Mills operators [5]. Classical integrability of type IIB string theory on $AdS_5 \times S^5$ [6] allowed a resolution of the sigma model spectrum in terms of spectral curves [7], and suggested a discrete set of Bethe equations for the quantum string sigma model [8,9]. Integrability on each side of the correspondence is thus encoded in an asymptotic factorisable S-matrix satisfying the Yang–Baxter equation. However the Yang–Baxter relations do not completely constrain the S-matrix, and it can only be fixed up to a scalar dressing phase factor [10]. The dressing phase factor of the S-matrix could be determined by requiring some sort of crossing invariance [11]. The structure of this dressing phase factor modifies in such a way the long-range Bethe ansatz equations that if it remained non-trivial in the weak-coupling regime it would induce perturbative violations of the BMN-scaling limit. One interesting feature of the long-range Bethe ansatz for high twist operators is that, assuming a trivial dressing factor, it agrees [12] with the Kotikov–Lipatov transcendentality principle [13]. Moreover, it is possible to have non-trivial dressing phase factors that violate perturbative BMN-scaling but still preserve the transcendentality structure [14].

The dressing phase factor has been argued to have the general form [8,15]

$$\sigma_{12} = \exp i[\theta_{12}] = \exp\left[i \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} \left(q_r(x_1^{\pm}) q_s(x_2^{\pm}) - q_s(x_1^{\pm}) q_r(x_2^{\pm}) \right) \right], \tag{1}$$

with $q_r(x)$ the conserved magnon charges, defined through

$$q_r(x^{\pm}) = \frac{i}{r-1} \left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right), \tag{2}$$

and $c_{r,s}$ some coefficients depending on the coupling constant $g = \sqrt{\lambda}/4\pi$, with λ the 't Hooft coupling. A strong-coupling expansion for the phase θ_{12} has been proposed in [16],

$$c_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}, \tag{3}$$

* Corresponding author.
E-mail addresses: cesar.gomez@uam.es (C. Gómez), rafael.hernandez@cern.ch (R. Hernández).

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with the coefficients given by

$$c^{(n)}_{r,s} = (r-1)(s-1)\mathrm{B}_n\mathcal{A}(r,s,n), \tag{4}$$

where B_n denotes the nth Bernoulli number, and

$$\mathcal{A}(r,s,n) = \frac{((-1)^{r+s}-1)}{4\cos(\frac{1}{2}\pi n)\Gamma[n+1]\Gamma[n-1]} \times \frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]}\frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]}, \tag{5}$$

which vanishes when $r+s$ is even or if $n \geqslant s-r+3$. This expression agrees with the perturbative expansion for strings in $AdS_5 \times S^5$ at leading order [8], and includes the first quantum correction [17]. Recently an educated guess was suggested in [14] for the weak-coupling expansion coefficients,

$$c_{r,s} = \sum_{n=0}^{\infty} \tilde{c}^{(n)}_{r,s} g^{n+1}, \tag{6}$$

that leads to a violation of BMN-scaling at four-loop order, in remarkable agreement with the results of [18]. Moreover, the conjecture in [14] still preserves the Kotikov–Lipatov transcendentality principle. The aim of this note we will be to derive the weak-coupling coefficients in (6) and the pattern of transcendentality by a *non-perturbative prescription* for resummation of the asymptotic series defining the dressing phase factor in the strong coupling regime. The non-perturbative prescription reproducing the result in [14] and [18] is formally the same used to define non-perturbatively the $c=1$ matrix model [19]. Moreover, the dressing phase factors at leading order can be interpreted in terms of a modified $c=1$ matrix model.

2. Weak-coupling expansion

Let us start writing a convenient symmetrization for the strong-coupling expansion of the dressing phase factor [20],

$$\theta_{12} = +\chi(x_1^+,x_2^+) - \chi(x_1^+,x_2^-) - \chi(x_1^-,x_2^+) + \chi(x_1^-,x_2^-) - \chi(x_2^+,x_1^+) + \chi(x_2^-,x_1^+) + \chi(x_2^+,x_1^-) - \chi(x_2^-,x_1^-), \tag{7}$$

where

$$\chi(x_1,x_2) = -\sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} \frac{c_{r,s}}{(r-1)(s-1)} \frac{1}{x_1^{r-1}x_2^{s-1}}. \tag{8}$$

At strong-coupling we get, from (3),

$$\chi(x_1,x_2) = \sum_{n=0}^{\infty} \chi^{(n)}(x_1,x_2)\frac{\mathrm{B}_n}{g^{n-1}}, \tag{9}$$

with

$$\chi^{(n)}(x_1,x_2) = -\sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} \frac{\mathcal{A}(r,s,n)}{x_1^{r-1}x_2^{s-1}}. \tag{10}$$

The strong-coupling expansion (9) is an asymptotic expansion. As it contains the Bernoulli numbers B_n, which grow like $n!$, it is highly divergent. However, it can still be defined non-perturbatively in a similar way to the one used in [19] for the non-perturbative definition of the $c=1$ matrix model. In order to show this, we will first introduce some new variables

$$\mu_i \equiv x_i g. \tag{11}$$

In terms of these variables

$$\chi(\mu_1,\mu_2) = \sum_{\alpha} g^{\alpha}\left(\sum_{r,s} \frac{\mathrm{B}_{r+s-1-\alpha}\mathcal{A}(r,s,r+s-1-\alpha)}{\mu_1^{r-1}\mu_2^{s-1}}\right). \tag{12}$$

The leading order term in (12) is

$$\chi(\mu_1,\mu_2)^{\mathrm{LO}} = g^2\sum_s \frac{\mathrm{B}_{s-1}\mathcal{A}(2,s,s-1)}{\mu_1\mu_2^{s-1}} \equiv \frac{g^2}{2\mu_1}\left(\sum_{n=2} \frac{i^n\mathrm{B}_n}{n\mu_2^n}\right). \tag{13}$$

The procedure we will now apply is as follows: we first will try to evaluate the sum in (13) performing a Borel transform. However the Borel transform contains an infinite number of poles on the real axis and the series is thus non-summable, unless a non-perturbative prescription is chosen in order to evaluate the integral. This prescription introduces an infinite number of parameters. Following a principal value prescription as in the $c=1$ matrix model, the non-perturbative definition of (13) will provide a perfectly convergent weak-coupling expansion in powers on μ_2 of the form $\sum \tilde{c}_n\mu_2^n$, for some coefficients $\tilde{c}_n$. The final step will be the derivation of this weak-coupling expansion from the general expression for $\chi(x_1,x_2)$, but now using the weak-coupling expansion (6) for the coefficients in the dressing phase. This provides an explicit expression for the weak-coupling coefficients $\tilde{c}^{(n)}_{r,s}$ in (6) in terms of the coefficients $\tilde{c}_n$ derived from the non-perturbative prescription. Let us now perform all these steps.

In order to evaluate the sum in (13) we rewrite

$$\frac{i^n\mathrm{B}_n}{n\mu_2^n} = \int_0^{\infty} dt\, e^{-i\mu_2 t}\frac{\mathrm{B}_n}{n!}t^{n-1}, \tag{14}$$

so that the Borel transform is

$$\int_0^{\infty} dt\, e^{-\mu_2 t}\sum_{k=1}^{\infty}(-1)^k\frac{\mathrm{B}_{2k}}{(2k)!}t^{2k-1} = \int_0^{\infty} dt\, e^{-\mu_2 t}\left(\frac{1}{2}\cot(t/2) - \frac{1}{t}\right). \tag{15}$$

The Borel transform does not exist because the integrand has an infinite number of poles on the real axis. Therefore in order to find the sum some integration prescription around each pole needs to be specified. Such a prescription is interpreted as a non-perturbative definition of the sum. In particular, in order to evaluate the integral we will use a principal value prescription.

From the sum of residues we then get

$$\pi\sum_{n=1}^{\infty}e^{-2\pi\mu_2 n}=\pi\left(\frac{1}{2\pi\mu_2}-\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\mathrm{B}_{2k}}{(2k)!}(2\pi\mu_2)^{2k}\right)=\frac{\pi}{2}\big(\coth(\pi\mu_2)-1\big). \tag{16}$$

Using now that

$$\Psi(z)=-\gamma-\frac{\pi}{2}\cot(\pi z), \tag{17}$$

we finally get[1]

$$\chi(\mu_1,\mu_2)^{\mathrm{LO}}=-\frac{g^2}{2\mu_1}\left(\gamma+\Re\Psi(i\mu_2)+\frac{\pi}{2}\right), \tag{18}$$

where $\Re\Psi(i\mu)$ denotes the real part of $\Psi(i\mu)$. From the non-perturbative expression (18) we get the following weak-coupling expansion, convergent for $|\mu_2|<1$,

$$\begin{aligned}\chi(\mu_1,\mu_2)^{\mathrm{LO}}&=-\frac{g^2}{2\mu_1}(\gamma+\pi/2)\\&\quad-\frac{g^2}{2\mu_1}\left(\gamma-\sum_{k=1}(-1)^k\zeta(1+2k)\mu_2^{2k}\right)\\&=\frac{g^2}{2\mu_1}\sum_{k=1}^{\infty}(-1)^k\zeta(1+2k)\mu_2^{2k}-\frac{g^2}{2\mu_1}\frac{\pi}{2}.\end{aligned} \tag{19}$$

In order to get the weak-coupling coefficients in the dressing factor we will write $\chi(x_1,x_2)$ in the weak-coupling regime (6) for the unknown coefficients $\tilde{c}_{r,s}^{(n)}$,

$$\chi(x_1,x_2)=\sum_{r,s,n}g^{n+1}\frac{\tilde{c}_{r,s}^{(n)}}{(r-1)(s-1)}\frac{1}{x_1^{r-1}x_2^{s-1}}. \tag{20}$$

Comparison with (19) requires considering the $r=2$, $n=s-1$ piece, which leads to

$$\frac{g}{x_1}\sum_s\tilde{c}_{2,s}^{(s-1)}\frac{g^{s-1}}{x_2^{s-1}}=\frac{g}{x_1}\sum_s\frac{\tilde{c}_{2,s}^{(s-1)}}{(s-1)}(gy_2)^{s-1}, \tag{21}$$

that we can compare with (19) for $gy_2=\mu_2$. From this we get the result

$$\tilde{c}_{2,s}^{(s-1)}=(-1)^{(s-1)/2}\zeta(s)\frac{(s-1)}{2} \tag{22}$$

for s odd, and

$$\tilde{c}_{2,s}^{(s-1)}=0 \tag{23}$$

for s even. This result is in complete agreement with the Kotikov–Lipatov transcendentality principle.

3. Non-perturbative prescription

Let us now briefly elaborate on the non-perturbative ambiguity. As we have already discussed the Borel transform does not properly exist due to the infinite number of poles along the integration range. The non-perturbative prescription that we have employed above is based on the Cauchy principal part. However, we must recall that the general procedure in order to give meaning to an infinite integral, in the distribution sense, is to first define a regularized distribution on the space of test functions, with support away from the singularities. Then an extension to the whole space of test functions is constructed. This extension in general does exist, but it is not unique. In our case a simple way to parameterize the intrinsic ambiguity in the definition of the infinite integral as a distribution is including a distribution of the type

$$2\pi\sum_i c_i\delta(x-x_i), \tag{24}$$

with x_i the location of the (simple) poles. The coefficients c_i are thus the non-perturbative parameters that we should fix from some alternative non-perturbative definition of the theory. In the absence of such an alternative definition we are unfortunately forced to deal with all these free constants. An economic possibility is to have all the c_i equal to some arbitrary constant α, with $\alpha=1$ corresponding to the Cauchy principal part. This implies a modification of (22) to

$$\tilde{c}_{2,s}^{(s-1)}=(-1)^{(s-1)/2}\alpha\zeta(s)\frac{(s-1)}{2}, \tag{25}$$

and it therefore looks that at least to fourth order $\alpha=4$ is the right non-perturbative prescription [18]. However, we should still keep in mind that any violation of this guess at higher orders will only force a different choice of the arbitrary parameters c_i.

4. Discussion

One interesting aspect of the result (18) for the phase factor is the very strong analogy with the $c=1$ matrix model. Using the relation of the digamma function and Hurwitz zeta function,

$$\lim_{s\to1}\left[\zeta(s,z)-\frac{1}{s-1}\right]=-\Psi(z), \tag{26}$$

with

$$\zeta(s,z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^s}, \tag{27}$$

we can map the dressing phase factor (18) with the density of states $\rho(\mu)$ of a matrix model, with the only *important* difference that instead of using the harmonic oscillator energy spectrum, $(n+1/2)\omega\hbar$, we now have $n\omega\hbar$. In matrix models the density $\rho(\mu)$ is related to the phase shift introduced in the wave function by the matrix potential through $\rho(\mu)=\partial\delta(\mu)/\partial\mu$. In this sense it looks like some parts of the dressing phase factor entering the integrable spin chain description of planar $\mathcal{N}=4$ supersymmetric Yang–Mills could be related to a phase shift

[1] The non-perturbative result (18) slightly differs from the resummation in [16], where the integration was performed with a rotation, $\int_0^\infty dt\,e^{-i\mu t}(1/2\coth(t/2)-1/t)=-\gamma+i/2\mu+\log(i\mu)-\Psi(i\mu)$. This difference is crucial in order to recover the Yang–Mills phase factor at second order. In fact (18) is the exact analogue of the matrix model solution [19].

in some matrix model through a formal relation of the form $\delta_{\mathcal{N}=4} = \partial\delta_{\text{matrix}}$.

The analogy with the matrix model goes a bit further if we consider the strong and weak-coupling expansions of the density $\rho(\mu)$. In fact both map, respectively, into the weak and strong-coupling expansions of the dressing phase factor. Moreover, as it seems to be the case for the dressing phase factor, the $c = 1$ weak-coupling expansion is only asymptotic, while the strong-coupling one is perfectly convergent. This potential connection with $c = 1$ matrix models certainly deserves future research.[2]

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[2] An appealing probe for the matrix analogy could be the relation of a finite-temperature matrix model to the finite-size exponential corrections in the quantum string energy shift [21].