An eight–dimensional approach to G_2 manifolds

Rafael Hernández[1]   and   **Konstadinos Sfetsos**[2]

[1] Institut de Physique, Université de Neuchâtel
Breguet 1, CH-2000 Neuchâtel, Switzerland
`rafael.hernandez@unine.ch`

[2] Department of Engineering Sciences, University of Patras
26110 Patras, Greece
`sfetsos@mail.cern.ch, des.upatras.gr`

Abstract

We develop a systematic approach to G_2 holonomy manifolds with an $SU(2) \times SU(2)$ isometry using maximal eight-dimensional gauged supergravity to describe D6-branes wrapped on deformed three-spheres. A quite general metric ansatz that generalizes the celebrated Bryant–Salamon metric involves nine functions. We show that only six of them are the independent ones and derive the general first order system of differential equations that they obey. As a byproduct of our analysis, we generalize the notion of the twist that relates the spin and gauge connections in a way that involves non-trivially the scalar fields.

Compactifications of M-theory on manifolds of exceptional holonomy have recently attracted great attention, mostly as a consequence of their relation to minimally supersymmetric gauge theories. Four dimensional $\mathcal{N} = 1$ supersymmetry (in Minkowski space) requires the internal seven manifold to have G_2 holonomy. But G_2 holonomy also appears in the geometric dual description of the large N limit of four dimensional gauge theories with four supercharges: the conjectured duality between D6-branes on the deformed conifold and a type IIA geometry with RR flux on the resolved conifold in [1] was better understood in terms of M-theory on a seven manifold of G_2 holonomy [2], where it corresponds to a flop transition [3]. Extensions of this duality and construction of new metrics from diverse approaches have revived the study of compactifications on manifolds of exceptional holonomy [4]-[31].

The number of known complete metrics of G_2 holonomy is still quite reduced. It is therefore of great interest to obtain new metrics of G_2 holonomy in order to improve our understanding of the above dualities and compactifications. The aim of this letter is to elaborate on a gauged supergravity approach to the systematic construction of manifolds of G_2 holonomy.

Branes wrapped on supersymmetric cycles have also been lately quite extensively studied within the framework of gauged supergravity as a promising candidate to gravity duals of field theories with low supersymmetry [32]-[48]. In [37] a configuration of D6-branes wrapping special Lagrangian 3-spheres was considered as a gravity dual of four dimensional field theories with $\mathcal{N} = 1$ supersymmetry. The lift to eleven dimensions of the eight dimensional solution describing the deformation on the worldvolume of the wrapped branes was there shown to correspond to one of the known metrics of G_2 holonomy [49]. In this letter we will show how gauged supergravity in eight dimensions provides a natural framework to construct general metrics of G_2 holonomy by allowing deformations on the 3-cycle. We will derive the conditions to guarantee G_2 holonomy on a seven manifold metric of the form

$$ds_7^2 = dr^2 + \sum_{i=1}^{3} a_i^2 \sigma_i^2 + \sum_{i=1}^{3} b_i^2 (\Sigma_i + c_i \sigma_i)^2 \ ,$$

where, as it will become clear from our analysis below, only six of the nine functions involved in this general metric are independent.

In what follows we will briefly review some relevant facts about eight dimensional supergravity. We will then construct the equations describing a supersymmetric configuration

1

corresponding to a set of D6-branes wrapped on a deformed 3-sphere. The lift to eleven dimensions of this configuration will prove to be a seven manifold of G_2 holonomy with $SU(2) \times SU(2)$ isometry which includes some of the proposed ansatzs in the literature.

Maximal gauged supergravity in eight dimensions was constructed by Salam and Sezgin [50] through Scherk–Schwarz compactification of eleven dimensional supergravity on an $SU(2)$ group manifold. The field content in the gravity sector of the theory consists of the metric $g_{\mu\nu}$, a dilaton Φ, five scalars given by a unimodular 3×3 matrix L_α^i in the coset $SL(3, \mathbf{R})/SO(3)$ and an $SU(2)$ gauge potential A_μ.[1] In addition, on the fermion side we have the pseudo–Majorana spinors ψ_μ and χ_i.

The Lagrangian density for the bosonic fields is given, in $\kappa = 1$ units, by

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{4}e^{2\Phi}F_{\mu\nu}^\alpha F^{\mu\nu\,\beta}g_{\alpha\beta} - \frac{1}{4}P_{\mu\,ij}P^{\mu\,ij} - \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{g^2}{16}e^{-2\Phi}(T_{ij}T^{ij} - \frac{1}{2}T^2),\qquad(1)$$

where $F_{\mu\nu}^\alpha$ is the Yang–Mills field strength.

Supersymmetry is preserved by bosonic solutions to the equations of motion of eight dimensional supergravity if the supersymmetry variations for the gaugino and the gravitino vanish. These are, respectively, given by

$$\delta\chi_i = \frac{1}{2}(P_{\mu\,ij} + \frac{2}{3}\delta_{ij}\partial_\mu\Phi)\hat\Gamma^j\Gamma^\mu\epsilon - \frac{1}{4}e^\Phi F_{\mu\nu\,i}\Gamma^{\mu\nu}\epsilon - \frac{g}{8}e^{-\Phi}(T_{ij} - \frac{1}{2}\delta_{ij}T)\epsilon^{jkl}\hat\Gamma_{kl}\epsilon = 0 \qquad(2)$$

and

$$\delta\psi_\gamma = \mathcal{D}_\gamma\epsilon + \frac{1}{24}e^\Phi F_{\mu\nu}^i\hat\Gamma_i(\Gamma_\gamma{}^{\mu\nu} - 10\delta_\gamma^\mu\Gamma^\nu)\epsilon - \frac{g}{288}e^{-\Phi}\epsilon_{ijk}\hat\Gamma^{ijk}\Gamma_\gamma T\epsilon = 0 \ . \qquad(3)$$

The covariant derivative is

$$\mathcal{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon + \frac{1}{4}Q_{\mu\,ij}\hat\Gamma^{ij}\epsilon\,, \qquad(4)$$

where $P_{\mu\,ij}$ and $Q_{\mu\,ij}$ are, respectively, the symmetric and antisymmetric quantities entering the Cartan decomposition of the $SL(3, \mathbf{R})/SO(3)$ coset, defined through

$$P_{\mu\,ij} + Q_{\mu\,ij} \equiv L_i^\alpha(\partial_\mu\delta_\alpha^\beta - g\,\epsilon_{\alpha\beta\gamma}A_\mu^\gamma)L_{\beta\,j}\,, \qquad(5)$$

and T_{ij} is the T-tensor defining the potential energy associated to the scalar fields,

$$T^{ij} \equiv L_\alpha^i L_\beta^j \delta^{\alpha\beta}\,, \qquad(6)$$

---

[1]The fields arising from reduction of the eleven dimensional three-form are a scalar, three vector fields, three two-forms and a three-form. However, we will only consider pure gravitational solutions of the eleven dimensional theory, so that all these fields can be set to zero.

with $T \equiv T_{ij}\delta^{ij}$, and

$$L^i_\alpha L^\alpha_j = \delta^i_j \ , \quad L^i_\alpha L^j_\beta \delta_{ij} = g_{\alpha\beta} \ , \quad L^i_\alpha L^j_\beta g^{\alpha\beta} = \delta^{ij} \ . \tag{7}$$

As usual, curved directions are labeled by greek indices, while flat ones are labeled by latin, and $\mu, a = 0, 1, \ldots, 7$ are spacetime coordinates, while $\alpha, i = 8, 9, 10$ are in the group manifold. Note also that upper indices in the gauge field, A^α_μ, are always curved.

We will turn on scalars in the diagonal

$$L^i_\alpha = \text{diag}(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}) \ , \quad \lambda_1 + \lambda_2 + \lambda_3 = 0 \ , \tag{8}$$

and in order to describe the worldvolume of the wrapped D6-branes on the deformed 3-cycle we will choose a metric ansatz of the form[2]

$$ds^2_8 = \alpha^2_1 \sigma^2_1 + \alpha^2_2 \sigma^2_2 + \alpha^2_3 \sigma^2_3 + e^{2f} ds^2_{1,3} + d\rho^2 \ . \tag{9}$$

All four functions α_i, f as well as the scalars λ_i and the dilaton Φ depend only on ρ, and the left-invariant Maurer–Cartan $SU(2)$ 1-forms satisfy

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k \ . \tag{10}$$

In this basis we also expand the gauge field 1-forms as $A^\alpha = A^\alpha_i \sigma^i$, with components A^α_i that depend only on the variable ρ. For simplicity, the four-dimensional metric $ds^2_{1,3}$ will be taken to be the Minkowski metric, but our analysis can be easily extended to Ricci-flat metrics.

We will represent the 32×32 gamma matrices in 11 dimensions as

$$\Gamma^a = \gamma^a \times \mathbf{1}_2, \quad \hat{\Gamma}^i = \gamma_9 \times \tau^i , \tag{11}$$

where the γ_a's denote the 16×16 gamma matrices in 8 dimensions and as usual $\gamma_9 = i\gamma^0\gamma^1 \ldots \gamma^7$, so that $\gamma^2_9 = \mathbf{1}$. Also τ^i are Pauli matrices. It will prove useful to introduce $\Gamma_9 \equiv \frac{1}{6i}\epsilon_{ijk}\hat{\Gamma}^{ijk} = -i\hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3 = \gamma_9 \times \mathbf{1}_2$.

Within this ansatz, the only consistent way to obtain non-trivial solutions to the Killing spinor equations is to impose on the spinor ϵ the projections

$$\Gamma_{ij}\epsilon = -\hat{\Gamma}_{ij}\epsilon \ , \quad \Gamma^7\epsilon = -i\Gamma^9\epsilon \ . \tag{12}$$

---

[2]We should note that deformation of the 3-cycle requires the existence of non-trivial scalars on the coset manifold.

The first of these projections relates the two $SU(2)$ algebras obeyed separately by the sets of generators $\{\Gamma_{ij}\}$ and $\{\hat{\Gamma}_{ij}\}$ and consequently the "spacetime" and internal deformed 3-spheres. It also states that only singlets of the diagonal $SU(2)_D$ of the tensor product of the two $SU(2)$'s are allowed. We emphasize that simple algebraic considerations reveal that the only allowed coefficient in a relation of the form $\Gamma_{ij}\epsilon = \lambda \hat{\Gamma}_{ij}\epsilon$ is $\lambda = -1$. We also note that among the possible pairs $\{ij\} = \{12, 23, 31\}$ only two are independent. Therefore the projections (12) represent three conditions in total, thus reducing the number of supersymmetries to $32/2^3 = 4$.[3] In the forthcoming derivation of the equations, the relations

$$\Gamma_i \hat{\Gamma}_j \epsilon = \hat{\Gamma}_i \Gamma_j \epsilon \ , \qquad i \neq j \ ,$$

$$\Gamma_1 \hat{\Gamma}_1 \epsilon = \Gamma_2 \hat{\Gamma}_2 \epsilon = \Gamma_3 \hat{\Gamma}_3 \epsilon \ , \tag{13}$$

which can be readily derived from (12), will also be useful.

When we wrap the D6-branes on the 3-cycle the $SO(1,6) \times SO(3)_R$ symmetry group of the unwrapped branes is broken to $SO(1,3) \times SO(3) \times SO(3)_R$. The worldvolume of the brane will support covariantly constant spinors after some twisting or mixing of the spin and gauge connections. In the presence of scalars this twisting can not be simply performed through a direct identification of the spin and gauge connections. As detailed in the appendix the gauge field is defined through the generalized twist[4]

$$\frac{1}{g}\frac{\omega_1^{23}}{\alpha_1} + \frac{A_1^1}{\alpha_1}\cosh\lambda_{23} + \frac{A_2^2}{\alpha_2}\sinh\lambda_{31} - \frac{A_3^3}{\alpha_3}\sinh\lambda_{12} = 0 \tag{14}$$

and cyclic in $1, 2, 3$, so that the solution is

$$A_1^1 = \frac{\alpha_1}{g}\left[-\frac{\omega_1^{23}}{\alpha_1}\cosh\lambda_{23} + e^{\lambda_{21}}\sinh\lambda_{31}\frac{\omega_2^{31}}{\alpha_2} - e^{\lambda_{31}}\sinh\lambda_{12}\frac{\omega_3^{12}}{\alpha_3}\right] , \tag{15}$$

and so on for A_2^2 and A_3^3. We have used the notation

$$\omega_i^{jk} = \epsilon_{ijk}\frac{\alpha_j^2 + \alpha_k^2 - \alpha_i^2}{2\alpha_j\alpha_k} \ , \tag{16}$$

---

[3] All conditions in (12) can be cast in the form

$$(\Gamma_7\hat{\Gamma}_i + \frac{1}{2}\epsilon_{ijk}\Gamma_{jk})\epsilon = 0 \ .$$

[4] In the absence of scalar fields, with $L_\alpha^i = \delta_\alpha^i$, and with no deformation of the 3-sphere, the gauge field reduces simply to $A_i^i = -\frac{1}{2g}$ [37].

4

for the components of the spin connection along the 3-sphere expanded as $\omega^{jk} = \omega_i^{jk} \sigma_i$, and $\lambda_{ij} = \lambda_i - \lambda_j$. We see that, in general, the relation between the spin connection and the gauge field involves in a rather complicated way the scalar fields.

A detailed account of the computations required to derive the equations obeyed by the various fields is given in the appendix. Here we just collect the results. From the gaugino variation in (2) one obtains the equations obeyed by the dilaton

$$\frac{d\Phi}{d\rho} = \frac{1}{2} e^{\Phi} \left(\frac{e^{\lambda_1}}{\alpha_2 \alpha_3} F_{23}^1 + \frac{e^{\lambda_2}}{\alpha_3 \alpha_1} F_{31}^2 + \frac{e^{\lambda_3}}{\alpha_1 \alpha_2} F_{12}^3 \right) + \frac{g}{8} e^{-\Phi} (e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}) . \tag{17}$$

and by the scalars,

$$\frac{d\lambda_1}{d\rho} = \frac{e^{\Phi}}{3} \left(2 \frac{e^{\lambda_1}}{\alpha_2 \alpha_3} F_{23}^1 - \frac{e^{\lambda_2}}{\alpha_3 \alpha_1} F_{31}^2 - \frac{e^{\lambda_3}}{\alpha_1 \alpha_2} F_{12}^3 \right) - \frac{g}{6} e^{-\Phi} (2e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3}) , \tag{18}$$

and cyclic in the $1, 2, 3$ indices for the other two equations. Also we have denoted the field strength components by F_{jk}^i in the $\sigma^j \wedge \sigma^k$ basis. In terms of the gauge field components they read

$$F_{23}^1 = A_1^1 + g A_2^2 A_3^3 , \qquad \text{and cyclic permutations .} \tag{19}$$

From the gravitino equation one determines the warp factor f in terms of the dilaton Φ as

$$f = \frac{\Phi}{3} , \tag{20}$$

as well as the differential equation

$$\frac{1}{\alpha_1} \frac{d\alpha_1}{d\rho} = \frac{e^{\Phi}}{6} \left(e^{\lambda_1} \frac{F_{23}^1}{\alpha_2 \alpha_3} - 5 e^{\lambda_2} \frac{F_{31}^2}{\alpha_3 \alpha_1} - 5 e^{\lambda_3} \frac{F_{12}^3}{\alpha_1 \alpha_2} \right) + \frac{g}{24} e^{-\Phi} (e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}) , \tag{21}$$

together with two more equations obtained by cyclic permutations of the $1, 2, 3$ indices. Furthermore, from $\delta \psi_\rho$ we can obtain the radial dependence of the spinor ϵ, which is simply given by

$$\epsilon = e^{f/2} \epsilon_0 = e^{\Phi/6} \epsilon_0 , \tag{22}$$

for ϵ_0 a constant spinor obeying the projection conditions (12). This radial dependence is of the general form $\epsilon = g_{00}^{1/4} \epsilon_0$, which can be proved using general arguments based on the supersymmetric algebra. The dependence on the particular model is only via the projections imposed on the constant spinor ϵ_0, which reduce the number of its independent components (see for instance [51]).

Using the appropriate formulae in [50] we may lift our 8-dimensional background into a full solution of 11-dimensional supergravity with only the metric turned on. The result

5

is of the form $ds_{11}^2 = ds_{1,3}^2 + ds_7^2$ where the 7-dimensional part is

$$ds_7^2 = e^{-2\Phi/3}d\rho^2 + e^{-2\Phi/3}\sum_{i=1}^3 \alpha_i^2\sigma_i^2 + e^{4\Phi/3}\sum_{i=1}^3 e^{2\lambda_i}(2/g\Sigma_i + 2A_i^i\sigma_i)^2 \ . \qquad (23)$$

This metric, when the various functions are subject to the conditions (15) and (17)-(21), describes G_2 holonomy manifolds with an $SU(2) \times SU(2)$ isometry.

It is worth examining what the Killing spinor in (22) represents from an eleven dimensional point of view. Recall that, in general, when a supersymmetry variation parameter ϵ is lifted from eight to eleven dimensions, it is multiplied by a factor, i.e. $\epsilon_{11} = e^{-\Phi/6}\epsilon$.[5] Using, in our case, the expression (22) we see that the constant spinor ϵ_0 is indeed the 11-dimensional spinor which, being subject to the projections (12), has 4 independent components. We will next show that it splits into the form $\epsilon_0 = \epsilon_{1,3} \times \epsilon_7$ in such a way that the spinor ϵ_7 in seven dimensions has only one independent component, in agreement with the correct amount of independent supercharges preserved by a G_2 holonomy manifold. In order to proceed we specialize the index μ to represent only the flat directions, i.e. $\mu = 0, 1, 2, 3$ and we denote by $\bar{\mu} = 4, 5, 6, 7$ the rest. Then we may represent the gamma matrices in 11-dimensions as

$$\Gamma^\mu = \gamma^\mu \times \mathbf{1_4} \times \mathbf{1_2} \ ,$$

$$\Gamma^{\bar{\mu}} = \gamma_5 \times \gamma^{\bar{\mu}} \times \mathbf{1_2} \ , \qquad (24)$$

$$\hat{\Gamma}^i = \gamma_5 \times \bar{\gamma}_5 \times \tau_i \ ,$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\bar{\gamma}_5 = \gamma^4\gamma^5\gamma^6\gamma^7$ and where we have used that $\epsilon^{0123} = \epsilon^{4567} = 1$. Using the split $\epsilon_0 = \epsilon_{1,3} \times \epsilon_7$ we see that the projections (12) imply

$$(\gamma_{ij} \times \mathbf{1_2})\epsilon_7 = -(\mathbf{1_4} \times \tau_{ij})\epsilon_7 \ , \qquad (\gamma^7 \times \mathbf{1_2})\epsilon_7 = -i(\bar{\gamma}_5 \times \mathbf{1_2})\epsilon_7 \ . \qquad (25)$$

These are 8 conditions in total on the 8-component spinor ϵ_7 and therefore the latter has indeed only one independent component, as advertised. Moreover, as shown in footnote 6 below the spinor ϵ_7 is G_2 invariant. The spinor $\epsilon_{1,3}$ is subject to no conditions at all and therefore the $\mathcal{N} = 1$ supersymmetry in four dimensions is intact we may have reduced supersymmetry if the Minkowski space is replaced by a Ricci flat manifold which admits less Killing spinors that Minkowski space).

---

[5]This corrects an apparent typo in equation (34) of [50].

For completeness we also construct the 3-form which is closed and co-closed and whose existence implies that the manifold has G_2 holonomy. On general grounds its components in the 7-bein basis $e^a \wedge e^b \wedge e^c$ are of the form $\Phi^{(3)}_{abc} = i\bar{\epsilon}_7 \Gamma_{abc}\epsilon_7$, where $a, b, c = 1, 2, \ldots, 7$ and the gamma matrices in seven dimensions are the corresponding part of the decomposition (24). Using the split $a = (i, \hat{i}, 7)$, where $i = 1, 2, 3$ and $\hat{i} = i+3$, as well as the normalization choice $i\bar{\epsilon}_7 \Gamma_{123}\epsilon_7 = 1$, we find that

$$\Phi^{(3)} = \frac{1}{6}\psi_{abc}\, e^a \wedge e^b \wedge e^c \; , \tag{26}$$

where ψ_{abc} are the octonionic structure constants with non-vanishing components in our basis being given by[6]

$$\psi_{ijk} = \epsilon_{ijk} \; , \qquad \psi_{i\hat{j}\hat{k}} = -\epsilon_{ijk} \; , \qquad \psi_{7i\hat{j}} = \delta_{ij} \; . \tag{27}$$

It is convenient to cast the metric and the equations in the different form

$$ds_7^2 = dr^2 + \sum_{i=1}^{3} a_i^2 \sigma_i^2 + \sum_{i=1}^{3} b_i^2 (\Sigma_i + c_i\sigma_i)^2 \; , \tag{28}$$

where $c_i = 2A_i^i$ and

$$a_i = e^{-\Phi/3}\alpha_i \; , \qquad b_i = e^{2\Phi/3}e^{\lambda_i} \; , \qquad e^{2\Phi} = b_1 b_2 b_3 \; , \qquad dr = e^{-\Phi/3}d\rho \; . \tag{29}$$

Then the equations (17), (18) and (21) become

$$\frac{da_1}{dr} = -\frac{b_2}{a_3}F_{31}^2 - \frac{b_3}{a_2}F_{12}^3 \; ,$$

$$\frac{db_1}{dr} = \frac{b_1^2}{a_2 a_3}F_{23}^1 - \frac{g}{4b_2 b_3}(b_1^2 - b_2^2 - b_3^2) \; , \tag{30}$$

and cyclic in the $1, 2, 3$ indices, where the field strength components in (19) are computed using

$$A_1^1 = \frac{a_1}{g}\left[-\frac{a_2^2 + a_3^2 - a_1^2}{2a_1 a_2 a_3}\frac{b_2^2 + b_3^2}{2b_2 b_3} + \frac{b_3^2 - b_1^2}{2b_3 b_1}\frac{b_2}{b_1}\frac{a_3^2 + a_1^2 - a_2^2}{2a_1 a_2 a_3} - \frac{b_1^2 - b_2^2}{2b_1 b_2}\frac{b_3}{b_1}\frac{a_1^2 + a_2^2 - a_3^2}{2a_1 a_2 a_3}\right]$$

$$= -\frac{1}{g}\frac{d_2^2 + d_3^2 - d_1^2}{2d_2 d_3} \equiv -\frac{1}{g}\Omega_1^{23} \; , \tag{31}$$

---

[6]In the same basis the non-vanishing components of the G_2 invariant 4-index tensor ψ_{abcd} are

$$\psi_{7ij\hat{k}} = \epsilon_{ijk} \; , \qquad \psi_{7\hat{i}\hat{j}\hat{k}} = -\epsilon_{ijk} \; , \qquad \psi_{ij\hat{m}\hat{n}} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \; .$$

Using these, one can show that the projectors (25) imply that

$$(\Gamma_{ab} + \frac{1}{4}\psi_{abcd}\Gamma_{cd})\epsilon_7 = 0 \; ,$$

which is precisely the condition for a G_2 invariant Killing spinor.

7

where $d_i \equiv \frac{a_i}{b_i}$ and cyclic in $1, 2, 3$. We see that the generalized twist condition (15) takes the form of the ordinary twist, but for an auxiliary 3-sphere deformed metric obtained by replacing the a_i's in the metric (28) by the d_i's defined above. In the rest of this paper we will set the parameter $g = 2$ which is equivalent to the rescaling $b_i \to b_i g/2$. This does not apply for the various formulae in the appendix.

It is worth examining the limit where the radius of the "spacetime" 3-sphere becomes very large so that it can be approximated by $\mathbb{R}^3$. This means that effectively the D6-branes are unwrapped. This limit can be taken systematically as follows: consider the rescaling $\sigma_i \to \epsilon dx_i$, $b_i \to \epsilon b_i$ and $r \to \epsilon r$ in the limit $\epsilon \to 0$. Then, since the functions $c_i = 2A_i^i$ do not scale, the metric (28) takes the form $ds_7^2 = dx_i^2 + ds_4^2$, where the four-dimensional non-trivial part of the metric is

$$ds_4^2 = dr^2 + \sum_{i=1}^{3} b_i^2 \Sigma_i^2 \ . \tag{32}$$

The coefficients b_i as functions of r obey a set of differential equations that also follow from the above mentioned limiting procedure from (30). Indeed the first equation in (30) reduces to the statement that the coefficients $a_i = $ constant and therefore they can be absorbed into a rescaling of the new coordinates x_i, as we have already done above. The result is

$$\frac{db_1}{dr} = \frac{1}{2b_2 b_3}(b_2^2 + b_3^2 - b_1^2) \ , \qquad \text{and cyclic permutations} \ . \tag{33}$$

This is nothing but the Lagrange system or, equivalently, the Euclidean version of the Euler spinning top system. The four-dimensional metrics (32) governed by that system correspond to a class of hyperkähler metrics with $SU(2)$ isometry with famous example, when an extra $U(1)$ symmetry develops (i.e., for instance when $b_2 = b_3$), the Eguchi–Hanson metric which is the first non-trivial ALE four-manifold.[7] This is in agreement with the fact that the near horizon limit of D6-branes of type IIA when uplifted to M-theory, contains, besides the D6-brane worldvolume, the Eguchi–Hanson metric.

Returning back to the generic case, it is obvious that integrating the system of first order non-linear equations (30) is a difficult task in general. Nevertheless one can show

---

[7]In fact, the Eguchi–Hanson metric is the only regular metric in the family described by (33). As it was shown in [52] a generalization of it with $b_1 \neq b_2 \neq b_3 \neq b_1$ leads to singular metrics. It can be shown that, from a string theoretical view point, this corresponds to continuous distributions of D6-branes in type IIA with physically unacceptable densities.

that

$$I = a_1 a_2 a_3 - a_1 b_2 b_3 c_2 c_3 - a_2 b_3 b_1 c_3 c_1 - a_3 b_1 b_2 c_1 c_2 \ , \tag{34}$$

is a constant of motion. The existence of this constant of motion fits well with the fact that the 3-form in (26), after using the explicit basis (37) below in terms of the $SU(2)$ Maurer–Cartan 1-forms, can be written as

$$\Phi^{(3)} = I \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + d\Lambda \ , \tag{35}$$

where I is the conserved quantity in (34) and Λ is some 2-form. Hence the conservation of I is a direct consequence of the closure of the 3-form $\Phi^{(3)}$, and appears as the coefficient of the volume form of the "spacetime" 3-sphere. Notice that there is no conserved quantity associated with the internal 3-sphere.[8] A promising avenue towards finding explicit new solutions will arise if the system (30) can be related to well studied in the literature spinning top-like systems which in many cases are integrable. This is the line of approach advocated in [53], but will leave this and related investigations for future research.

Let us now consider the consistent truncation $a_2 = a_3$ and $b_2 = b_3$, where an extra $U(1)$ symmetry develops. Then after some algebra we conclude that the remaining four independent functions obey the system[9]

$$
\begin{aligned}
\dot{a}_1 &= \frac{1}{4} \frac{a_1^3 b_2^4}{a_2^4 b_1^3} \ , \\
\dot{a}_2 &= \frac{1}{2} \frac{b_1}{a_2} - \frac{3}{8} \frac{a_1^2 b_2^2}{a_2^3 b_1} + \frac{1}{8} \frac{a_1^2 b_2^4}{a_2^3 b_1^3} \ , \\
\dot{b}_1 &= -\frac{1}{2} \frac{b_1^2}{a_2^2} + \frac{3}{8} \frac{a_1^2 b_2^2}{a_2^4} - \frac{1}{2} \left(\frac{b_1^2}{b_2^2} - 2 \right) \ , \\
\dot{b}_2 &= \frac{1}{2} \frac{b_1}{b_2} - \frac{1}{8} \frac{a_1^2 b_2^5}{a_2^4 b_1^3} \ ,
\end{aligned}
\tag{36}
$$

---

[8]In the notation of [28] $p = I$ and $q = 0$. In principle, the information contained into our system (30) for the metric (28) is also encoded into equations (80)-(81) of [28] for the metric (78)-(79) of the same reference. These equations are highly non-linear second order equations for three functions. In our approach they would arise upon eliminating three among our six unknown functions. A simple counting argument shows that in both cases the number of integration constants is the same. We note here that it does not seem possible to investigate metrics with both $p \neq 0$ and $q \neq 0$ using eight-dimensional gauged supergravity. The reason is that, in the original metric ansatz (9) there cannot be by definition any dependence on the internal $SU(2)$ coordinates that parameterize the Maurer–Cartan 1-forms Σ_i.

[9]It is straightforward to verify that the further consistent truncation with $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$ gives a system which is trivially solved, leading to the metric of [49].

where we have used that in this case $c_2 = -\frac{a_1 b_2}{2a_2 b_1}$ and $c_1 = 2c_2^2 - 1$. This system coincides (after we let $r \to -r$) with that in equation (23) of [29] and in the limit of $a_1 = 0$ it is just the system corresponding to the resolved conifold.

We will finally show how the system of equations (30) can also be derived from self-duality of the spin connection for the seven manifold. In order to do so, we will split the indices in (28) as before, namely as $a = (i, \hat{i}, 7)$, and use the 7-bein basis

$$e^7 = dr , \qquad e^i = a_i \sigma_i , \qquad e^{\hat{i}} = b_i (\Sigma_i + c_i \sigma_i) , \qquad i = 1, 2, 3 , \quad \hat{i} = i + 3 . \qquad (37)$$

We then compute

$$de^i = \frac{\dot{a}_i}{a_i} e^7 \wedge e^i + \frac{1}{2} \frac{a_i}{a_j a_k} \epsilon_{ijk} e^j \wedge e^k ,$$

$$de^{\hat{i}} = \frac{\dot{b}_i}{b_i} e^7 \wedge e^{\hat{i}} + \frac{b_i \dot{c}_i}{a_i} e^7 \wedge e^i$$

$$+ \frac{1}{2} b_i \epsilon_{ijk} \left(\frac{1}{b_j b_k} e^{\hat{j}} \wedge e^{\hat{k}} + \frac{c_i + c_j c_k}{a_j a_k} e^j \wedge e^k - 2 \frac{c_j}{a_j b_k} e^j \wedge e^{\hat{k}} \right) , \qquad (38)$$

where the dot stands for $\frac{d}{dr}$. Using then the Cartan's structure equations $de^a + \omega^{ab} \wedge e^b = 0$ we compute the spin connection

$$\omega^{i7} = \frac{\dot{a}_i}{a_i} e^i + \frac{b_i \dot{c}_i}{2a_i} e^{\hat{i}} ,$$

$$\omega^{\hat{i}7} = \frac{\dot{b}_i}{b_i} e^{\hat{i}} + \frac{b_i \dot{c}_i}{2a_i} e^i ,$$

$$\omega^{ij} = \frac{1}{2} \epsilon_{ijk} \left(\frac{a_i}{a_j a_k} + \frac{a_j}{a_i a_k} - \frac{a_k}{a_i a_j} \right) e^k - \frac{1}{2} \epsilon_{ijk} \frac{b_k}{a_i a_j} (c_k + c_i c_j) e^{\hat{k}} , \qquad (39)$$

$$\omega^{\hat{i}\hat{j}} = \frac{1}{2} \epsilon_{ijk} \left(\frac{b_i}{b_j b_k} + \frac{b_j}{b_i b_k} - \frac{b_k}{b_i b_j} \right) e^{\hat{k}} - \frac{1}{2} \epsilon_{ijk} \frac{c_k}{a_k} \left(\frac{b_i}{b_j} + \frac{b_j}{b_i} \right) e^k ,$$

$$\omega^{i\hat{j}} = \frac{b_i \dot{c}_i}{2a_i} \delta_{ij} e^7 + \frac{1}{2} \epsilon_{ijk} \frac{b_j}{a_i a_k} (c_j + c_k c_i) e^k - \frac{1}{2} \epsilon_{ijk} \frac{c_i}{a_i} \left(\frac{b_j}{b_k} - \frac{b_k}{b_j} \right) e^{\hat{k}} .$$

Then, let us recall that imposing the self-duality condition on the spin connection, i.e. $\omega^{ab} = \frac{1}{2} \psi^{abcd} \omega^{cd}$, where ψ^{abcd} is the G_2 invariant 4-index tensor, is equivalent in our basis to the following seven equations

$$\omega^{7i} = \epsilon_{ijk} \omega^{j\hat{k}} ,$$

$$\omega^{7\hat{i}} = \frac{1}{2} \epsilon_{ijk} (\omega^{jk} - \omega^{\hat{j}\hat{k}}) , \qquad (40)$$

$$\omega^{i\hat{i}} = 0 .$$

10

Applying these to our case we obtain the differential equations (30) and the generalized twist condition (31), plus the condition $\sum_{i=1}^{3} \frac{b_i}{a_i} \dot{c}_i = 0$, which is equivalent to (A.11) in the appendix and is satisfied automatically once (30) and (31) are. Since self-duality of the spin connection in seven dimensions implies that the 3-form defined in (26) is closed and co-closed and, therefore, G_2 holonomy (noted in [54, 55], proved explicitly in [25] and used to rederive the metric of [49] in [56]) we have shown that our equations (30) (or equivalently (17), (18) and (21)) indeed describe a manifold of G_2 holonomy.

It will be interesting to extend the eight-dimensional gauged supergravity approach to G_2 manifolds in order to find general conditions for manifolds with weak G_2 holonomy [57] having an $SU(2) \times SU(2)$ isometry. The main difference in this case is that the three form is no longer closed, i.e. it obeys $d\Phi^{(3)} \sim *\Phi^{(3)}$ and consequently the Minkowski metric $ds_{1,3}^2$ has to be replaced by an Einstein space with negative cosmological constant. Nevertheless, supersymmetry can be preserved and a generalization of the self-duality condition on the spin connection (40) leading to manifolds with weak G_2 holonomy also exists [25]. We also believe that the eight-dimensional approach to G_2 manifolds will also prove useful in the investigation of Spin(7) manifolds. We hope to report work along these lines in the future.

A   Appendix

In this appendix we will provide some details on the derivation of the Killing spinor equations and conditions (14)-(22).

As already noted, the only consistent way to obtain a consistent set of differential equations from the supersymmetry variations (2) and (3) is to impose projections (12) on the spinors. We also provide for convenience the expressions for $P_{\mu i j}$ and $Q_{\mu i j}$ defined in

(5). For the diagonal matrix L_α^i in (8), it is convenient to represent them as forms in the index μ,

$$P_{ij} = \begin{pmatrix} \partial\lambda_1 & gA^3\sinh\lambda_{12} & gA^2\sinh\lambda_{31} \\ gA^3\sinh\lambda_{12} & \partial\lambda_2 & gA^1\sinh\lambda_{23} \\ gA^2\sinh\lambda_{31} & gA^1\sinh\lambda_{23} & \partial\lambda_3 \end{pmatrix} \tag{A.1}$$

and

$$Q_{ij} = \begin{pmatrix} 0 & -gA^3\cosh\lambda_{12} & gA^2\cosh\lambda_{31} \\ gA^3\cosh\lambda_{12} & 0 & -gA^1\cosh\lambda_{23} \\ -gA^2\cosh\lambda_{31} & gA^1\cosh\lambda_{23} & 0 \end{pmatrix} . \tag{A.2}$$

We start with the $i = 1$ case in the gaugino equation, $\delta\chi_1 = 0$, which implies two different equations: factorizing $\hat{\Gamma}_2\hat{\Gamma}_3\,\epsilon$ we get

$$\frac{d\lambda_1}{d\rho} + \frac{2}{3}\frac{d\Phi}{d\rho} = e^{\Phi+\lambda_1}\frac{F_{23}^1}{\alpha_2\alpha_3} - \frac{g}{4}e^{-\Phi}(e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3}) , \tag{A.3}$$

where F_{23}^1 is defined in (19). In addition, from terms proportional to $\hat{\Gamma}_2\Gamma_3\,\epsilon$ we get

$$e^{\Phi+\lambda_1}\frac{F_{\rho1}^1}{\alpha_1} + g\left(\frac{A_3^3}{\alpha_3}\sinh\lambda_{12} - \frac{A_2^2}{\alpha_2}\sinh\lambda_{31}\right) = 0 , \tag{A.4}$$

where $F_{\rho1}^1 = \partial_\rho A_1^1$, or equivalently, after the change of variables (29) (and setting $g = 2$)

$$\frac{b_1\dot{c}_1}{a_1} + \frac{c_3}{a_3}\left(\frac{b_1}{b_2} - \frac{b_2}{b_1}\right) - \frac{c_2}{a_2}\left(\frac{b_3}{b_1} - \frac{b_1}{b_3}\right) = 0 . \tag{A.5}$$

The four additional equations corresponding to $\delta\chi_i = 0$, for $i = 2, 3$ can be obtained from (A.3) and (A.4) by cyclic permutations in the indices $1, 2, 3$. Using then the constraint $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we get

$$\frac{d\lambda_1}{d\rho} = \frac{e^\Phi}{3}\left(2\frac{e^{\lambda_1}}{\alpha_2\alpha_3}F_{23}^1 - \frac{e^{\lambda_2}}{\alpha_3\alpha_1}F_{31}^2 - \frac{e^{\lambda_3}}{\alpha_1\alpha_2}F_{12}^3\right) - \frac{g}{6}e^{-\Phi}(2e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3}) \tag{A.6}$$

and

$$\frac{d\Phi}{d\rho} = \frac{1}{2}e^\Phi\left(\frac{e^{\lambda_1}}{\alpha_2\alpha_3}F_{23}^1 + \frac{e^{\lambda_2}}{\alpha_3\alpha_1}F_{31}^2 + \frac{e^{\lambda_3}}{\alpha_1\alpha_2}F_{12}^3\right) + \frac{g}{8}e^{-\Phi}(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}) . \tag{A.7}$$

Next we turn to the gravitino variation, $\delta\psi_\mu = 0$, where we should distinguish two possibilities, according to whether μ is a coordinate on the wrapped 3-sphere or on the unwrapped directions. Thus, if $\mu = \sigma_1$, from $\hat{\Gamma}_2\hat{\Gamma}_3\,\epsilon$ we get

$$\omega_1^{23} + gA_1^1\cosh\lambda_{23} - \alpha_1\frac{e^\Phi}{6}\left(\frac{e^{\lambda_2}}{\alpha_2}F_{\rho2}^2 + \frac{e^{\lambda_3}}{\alpha_3}F_{\rho3}^3 - 5\frac{e^{\lambda_1}}{\alpha_1}F_{\rho1}^1\right) = 0 \tag{A.8}$$

and, from terms proportional to $\hat{\Gamma}_2\Gamma_3\,\epsilon$

$$\frac{1}{\alpha_1}\frac{d\alpha_1}{d\rho} - \frac{1}{6}e^\Phi\left(\frac{e^{\lambda_1}}{\alpha_2\alpha_3}F_{23}^1 - 5\frac{e^{\lambda_2}}{\alpha_3\alpha_1}F_{31}^2 - 5\frac{e^{\lambda_3}}{\alpha_1\alpha_2}F_{12}^3\right) - \frac{g}{24}e^{-\Phi}(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}) = 0 , \tag{A.9}$$

12

where we have used the spin connection component $\omega_i^{i\rho} = \frac{d\alpha_i}{d\rho}$ for the metric ansatz (9). The equations obtained from cyclicity of (A.8) and (A.9) correspond to the choices $\mu = \sigma_2$, σ_3. The generalized twist (14) is then derived from (A.4) and (A.8). It amounts to turning on a gauge field given by (15).

It is important to verify that substituting back (15) into (A.4) and after using (A.6), (A.7) and (A.9), gives no new constraints for the various functions. After a straightforward but lengthly computation, one can show that this is indeed the case.

If μ is a coordinate on the unwrapped part of the worldvolume, using $\omega_\mu^{\mu\rho} = e^f \frac{df}{d\rho}$ and comparing terms proportional to $\Gamma_\mu \Gamma_7 \epsilon$ we get and equation relating the warp factor and the dilaton as

$$\frac{df}{d\rho} = \frac{1}{3}\frac{d\Phi}{d\rho} \ . \tag{A.10}$$

This then leads to (20) in the main text (a possible constant of integration can be absorbed into a rescaling of the corresponding unwrapped coordinates). Also from comparison of terms proportional to $\Gamma_\mu \Gamma_7 \Gamma_1 \hat{\Gamma}_1 \epsilon$ we obtain a constraint for the field strength

$$e^{\lambda_1}\frac{F_{\rho 1}^1}{\alpha_1} + e^{\lambda_2}\frac{F_{\rho 2}^2}{\alpha_2} + e^{\lambda_3}\frac{F_{\rho 3}^3}{\alpha_3} = 0 \ , \tag{A.11}$$

which holds identically from (A.4). Finally, considering the gravitino equation for $x^\mu = \rho$ gives after some algebra a simple differential equation for the spinor ϵ with solution given by (22).

References

[1] C. Vafa, J. Math. Phys. **42** (2001) 2798, `hep-th/0008142`.

[2] B.S. Acharya, *On Realizing $N{=}1$ Super Yang-Mills in M-Theory*, `hep-th/0011089`.

[3] M. Atiyah, J. Maldacena and C. Vafa, J. Math. Phys. **42** (2001), 3209, `hep-th/0011256`.

[4] F. Cachazo, K.A. Intriligator and C. Vafa, Nucl. Phys. **B603** (2001) 3, `hep-th/0103067`.

[5] J. Gomis, Nucl. Phys. **B606** (2001) 3, `hep-th/0103115`.

[6] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, Nucl. Phys. **B620** (2002) 29, `hep-th/0103155`.

[7] S. Kachru and J. McGreevy, JHEP **0106** (2001) 027, `hep-th/0103223`.

[8] J.D. Edelstein, K. Oh and R. Tatar, JHEP **0105** (2001) 009, `hep-th/0104037`.

[9] J. Gutowski and G. Papadopoulos, Nucl. Phys. **B615** (2001) 237, `hep-th/0104105`.

[10] P. Kaste, A. Kehagias and H. Partouche, JHEP **0105** (2001) 058, `hep-th/0104124`.

[11] K. Dasgupta, K. Oh and R. Tatar, Nucl. Phys. **B610** (2001) 331, hep-th/0105066.

[12] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, Nucl. Phys. **B611** (2001) 179, `hep-th/0106034`.

[13] M. Atiyah and E. Witten, *M-Theory Dynamics on a Manifold of G_2 Holonomy*, `hep-th/0107177`.

[14] E. Witten, *Anomaly Cancellation on G_2 Manifolds*, `hep-th/0108165`.

[15] H. Kanno and Y. Yasui, *On Spin(7) holonomy metric based on $SU(3)/U(1)$*, `hep-th/0108226` and *On Spin(7) Holonomy Metric Based on $SU(3)/U(1)$. II*, `hep-th/0111198`.

[16] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, *Cohomogeneity One Manifolds of Spin(7) and G_2 Holonomy*, `hep-th/0108245`.

[17] S. Gukov and J. Sparks, Nucl. Phys. B **625** (2002) 3, `hep-th/0109025`.

[18] J. Majumder, JHEP **0201** (2002) 048, `hep-th/0109076`.

[19] B. Acharya and E. Witten, *Chiral Fermions from Manifolds of G_2 Holonomy*, `hep-th/0109152`.

[20] A. Giveon, A. Kehagias and H. Partouche, JHEP **0112** (2001) 021, `hep-th/0110115`.

[21] R. Blumenhagen and V. Braun, JHEP **0112** (2001) 006, `hep-th/0110232`. JHEP **0112** (2001) 013, `hep-th/0111048`.

[22] G. Ferretti, P. Salomonson and D. Tsimpis, JHEP **0203** (2002) 004, `hep-th/0111050`.

[23] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, *Orientifolds and Slumps in G_2 and Spin(7) Metrics*, `hep-th/0111096`.

[24] G. Curio, B. Kors and D. Lust, *Fluxes and Branes in Type II Vacua and M-theory Geometry with G_2 and Spin(7) Holonomy*, `hep-th/0111165`.

[25] A. Bilal, J.P. Derendinger and K. Sfetsos, Nucl. Phys. **B628** (2002) 112, `hep-th/0111274`.

[26] K. Oh and R. Tatar, *Duality and Confinement in N = 1 Supersymmetric Theories from Geometric Transitions*, `hep-th/0112040`.

[27] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Phys. Rev. Lett. **88** (2002) 121602, `hep-th/0112098`.

[28] A. Brandhuber, *G_2 Holonomy Spaces from Invariant Three-Forms*, `hep-th/0112113`.

[29] M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, *A G_2 Unification of the Deformed and Resolved Conifolds*, `hep-th/0112138`.

[30] S.A. Hartnoll, *Axisymmetric non-abelian BPS monopoles from G_2 metrics*, `hep-th/0112235`.

[31] B. Acharya, X. de la Ossa and S. Gukov, *G-flux, Supersymmetry and Spin(7) Manifolds*, `hep-th/0201227`.

[32] J. Maldacena and C. Núñez, Int. J. Mod. Phys. **A16** (2001) 822, `hep-th/0007018`. Phys. Rev. Lett. **86** (2001) 588, `hep-th/0008001`.

[33] B. Acharya, J.P. Gauntlett and N. Kim, Phys. Rev. **D63** (2001) 106003, `hep-th/0011190`.

[34] H. Nieder and Y. Oz, JHEP **0103** (2001) 008, `hep-th/0011288`.

[35] J.P. Gauntlett, N. Kim and D. Waldram, Phys. Rev. **D63** (2001) 126001, `hep-th/0012195`.

[36] C. Núñez, I.Y. Park, M. Schvellinger and T.A. Tran, JHEP **0104** (2001) 025, `hep-th/0103080`.

[37] J.D. Edelstein and C. Núñez, JHEP **0104** (2001) 028, `hep-th/0103167`.

[38] M. Schvellinger and T.A. Tran, JHEP **0106** (2001) 025, `hep-th/0105019`.

[39] J. Maldacena and H. Nastase, JHEP **0109** (2001) 024, `hep-th/0105049`.

[40] J.P. Gauntlett, N. Kim, S. Pakis and D. Waldram, Phys. Rev. **D65** (2002) 026003, `hep-th/0105250`.

[41] R. Hernández, Phys. Lett. **B521** (2001) 371, `hep-th/0106055`.

[42] J.P. Gauntlett, N. Kim, D. Martelli and D. Waldram, Phys. Rev. **D64** (2001) 106008, `hep-th/0106117`.
F. Bigazzi, A.L. Cotrone and A. Zaffaroni, Phys. Lett. **B519** (2001) 269, `hep-th/0106160`.

[43] J. Gomis and T. Mateos, Phys. Lett. **B524** (2002) 170, `hep-th/0108080`.

[44] J.P. Gauntlett and N. Kim, Phys. Rev. D **65** (2002) 086003, `hep-th/0109039`.

[45] J. Gomis and J.G. Russo, JHEP **0110** (2001) 028, `hep-th/0109177`.

[46] J. P. Gauntlett, N. W. Kim, D. Martelli and D. Waldram, JHEP **0111** (2001) 018, `hep-th/0110034`.

[47] J. Gomis, Nucl. Phys. **B624** (2002) 181, `hep-th/0111060`.

[48] P. Di Vecchia, H. Enger, E. Imeroni and E. Lozano-Tellechea, *Gauge theories from wrapped and fractional branes*, `hep-th/0112126`.

[49] R. Bryant and S. Salamon, Duke Math. J. **58** (1989) 829.

G. W. Gibbons, D. N. Page and C. N. Pope, Commun. Math. Phys. **127** (1990) 529.

[50] A. Salam and E. Sezgin, Nucl. Phys. **B258** (1985), 284.

[51] R. Kallosh and J. Kumar, Phys. Rev. **D56** (1997) 4934, `hep-th/9704189`.

[52] V. A. Belinsky, G. W. Gibbons, D. N. Page and C. N. Pope, Phys. Lett. B **76** (1978) 433.

[53] K. Sfetsos, *Integrable reductions of Spin(7) and G_2 invariant self-dual Yang-Mills equations and gravity*, hep-th/0112117.

[54] B.S. Acharya and M. O'Loughlin, Phys. Rev. **D55** (1997) 4521, `hep-th/9612182`.

[55] E.G. Floratos and A. Kehagias, Phys. Lett. **B427** (1998) 283, `hep-th/9802107`.

[56] D. Brecher and M.J. Perry, Nucl. Phys. **B566** (2000) 151, `hep-th/9908018`.

[57] A. Gray, Math. Z **123** (1971) 290.