

# **$K3$ -Fibrations and Softly Broken $N=4$ Supersymmetric Gauge Theories.**

César Gómez<sup>a</sup>, Rafael Hernández<sup>a,b</sup> and Esperanza López<sup>c</sup>

<sup>a</sup>*Instituto de Matemáticas y Física Fundamental, CSIC,  
Serrano 123, 28006 Madrid, Spain*

<sup>b</sup>*Departamento de Física Teórica, C-XI, Universidad Autónoma de Madrid,  
Cantoblanco, 28049 Madrid, Spain*

<sup>c</sup>*Institute for Theoretical Physics, University of California,  
Santa Barbara, CA 93106-4030*

Global geometry of  $K3$ -fibration Calabi-Yau threefolds, with Hodge number  $h_{2,1} = r+1$ , is used to define  $N=4$  softly broken  $SU(r+1)$  gauge theories, with the bare coupling constant given by the dual heterotic dilaton, and the mass of the adjoint hypermultiplet given by the heterotic string tension. The  $U(r+1)$  Donagi-Witten integrable model is also derived from the  $K3$ -fibration structure, with the extra  $U(1)$  associated to the heterotic dilaton. The case of  $SU(2)$  gauge group is analyzed in detail. String physics beyond the heterotic point particle limit is partially described by the  $N=4$  softly broken theory.

# 1 Introduction

The Seiberg-Witten solution [1]-[11] of  $N=2$  supersymmetric gauge theories turns out to be intimately related to both integrable models [12]-[22] and string theory [23]-[26]. The mass formula for BPS states

$$M = \sum_{i=1}^r \|n_i^e a_i(\vec{u}) + n_i^m a_i^D(\vec{u})\|, \quad (1.1)$$

where  $r$  is the rank of the gauge group of the theory, admits a geometrical representation in terms of the periods,

$$a_i(\vec{u}) = \oint_{\alpha_i} \lambda, \quad a_i^D(\vec{u}) = \oint_{\beta_i} \lambda, \quad (1.2)$$

of a meromorphic form  $\lambda$  on an hyperelliptic curve of genus  $r$ ,  $\Sigma_{\vec{u}}$ , with  $\vec{u} = u_1, \dots, u_r$  and  $u_i$  the Casimir expectation values.

The solution given by (1.2) defines a family of abelian varieties, i.e. the Jacobian of  $\Sigma_{\vec{u}}$ , parameterized by the quantum moduli manifold. This is the generic structure underlying algebraic integrable models [27]. As it was shown in [15], an integrable model describing  $N=2$  gauge theories can be directly derived, through Hitchin's construction [28], from a two dimensional Higgs system defined on an elliptic Riemann surface,  $E_\tau$ . This approach leads to the Seiberg-Witten solution for  $N=2$  gauge theories with one massive hypermultiplet in the adjoint representation. The pure gauge theory appears as a double scaling limit in which hypermultiplet is decoupled ( $m \rightarrow \infty$ ), and the bare coupling constant is sent to  $\infty$ .

In addition, the geometrical representation in terms of periods (1.2) of the BPS mass formula, calls for a stringy interpretation of Seiberg-Witten geometry based on some non critical string, with effective string tension  $\lambda$ , winding around an "internal" space described by the Riemann surface  $\Sigma_{\vec{u}}$  [26]. This string interpretation nicely combines two ingredients, namely the string derivation of Seiberg-Witten solution, and its integrable model representation. The first one arises from the discovery of heterotic-type II dual pairs [29, 30, 23, 24], with the heterotic string compactified on  $K3 \times T^2$ . Higgsing the gauge group leads to different heterotic compactifications having  $r+2$  vector excitations and  $s$  neutral hypermultiplets, therefore the corresponding type IIA dual should be defined on a Calabi-Yau threefold with  $h_{1,1} = r+1$ , and  $h_{2,1} = s-1$ . These Calabi-Yau threefolds provide the string theory extension of Seiberg-Witten quantum moduli for a gauge group of rank  $r$ . An important step in the construction of dual pairs is the use of threefolds which are  $K3$ -fibrations [31]. The reason for this requirement goes back to the identification of the Calabi-Yau modulus that we could put in correspondence with the heterotic dilaton. This moduli is singled out, for  $K3$ -fibrations, as the one associated with the size of the  $\mathbb{P}^1$  base space [32].

On the other hand, the integrable model representation of the Seiberg-Witten curve for pure gauge theories, can be defined as a fibration of the spectral cover set given by the

vanishing locus of the Landau-Ginzburg potential associated to the Dynkin diagram of the corresponding gauge group [13]. The relation between this 0-dimensional fibration, and the  $K3$ -fibration defining the Calabi-Yau threefold can be now obtained by fixing the  $K3$ -fiber at an orbifold point, described by the corresponding Dynkin diagram, and blowing it up to an ALE space [26]. This geometrical manipulation can be formally undertaken by turning gravity off ( $S \rightarrow \infty$ ) and simultaneously going to the point particle regime by sending the string tension to zero [25]. Within this approach, a string interpretation of the periods (1.2) is obtained as the wrapping of a 3-brane on a 3-cycle of the Calabi-Yau threefold. To put it more precisely, the meromorphic form  $\lambda$  appears as the string tension of the self-dual string obtained when wrapping a 3-brane on 2-cycles of the ALE-space.

In this paper, we will try to go one step further in the study of the deep interplay between  $K3$ -fibered Calabi-Yau threefolds and the integrability underlying the Seiberg-Witten solution. In order to do that we will choose, continuing the line in reference [33], the approach to integrability used by Donagi and Witten [15], based on Hitchin's gauge model on  $E_\tau$ , where one naturally lands onto the Seiberg-Witten solution for  $N=4$  softly broken gauge theories. The physical reason for such a choice is related to the existence, in addition to the Higgs moduli, of two extra parameters, namely, the bare coupling constant  $\tau$  and the bare mass for the hypermultiplet in the adjoint representation. Our main goal will be to find a string derivation of this theory. Moreover we will claim that  $K3$ -fibered threefolds naturally lead to the  $N=4$  softly broken version of  $N=2$  gauge theories, with the bare coupling constant  $\tau$  and the hypermultiplet mass mapped in a concrete way into the heterotic dilaton and the string scale  $\alpha'$  respectively.

The main difference between our case and the string derivation of pure  $N=2$  gauge theory is that, in order to obtain the  $N=4$  softly broken version, we should work beyond the point particle limit, at generic values of the heterotic dilaton field. Thus, we will be using the global information on the  $K3$ -fibration structure. The important physical question raised by this analysis is, of course, to unravel the type of global string dynamics that we are capturing with the  $N=4$  softly broken quantum field theory. We will concentrate mainly in the case of gauge group  $SU(2)$ . What we observe is that the gauge theory that we are associating with the threefold fibration captures the features which are universal for a certain set of  $K3$ -fibrations, with identical Hodge number  $h_{2,1} = 2$ , but differing in the modular properties of the mirror map and in Hodge number  $h_{1,1}$ . In particular, following [34], we will relate the different modular properties of the mirror map to the Kac-Moody level at which the gauge symmetry is realized in the heterotic dual string. The  $N=4$  softly broken theory will then determine the relation between the Kac-Moody level and the genus of a curve of singularities developed by the Calabi-Yau manifolds at the locus corresponding to vanishing heterotic dilaton [35, 36].

The  $N=4$  softly broken version of the Calabi-Yau threefold singles out a particular singular locus on the moduli of complex structures of the threefold, namely that in correspondence with the field theory locus where some component of the adjoint hypermultiplet becomes massless. If the mass of the hypermultiplet is correctly capturing part

of the dynamics of the string scale,  $\alpha'$ , we should expect the corresponding singular locus of the Calabi-Yau threefold to be somehow related to the existence of the string scale, in very much a similar way as the self dual point for compactifications on  $S^1$  is given by a radius  $R$  equal to  $\sqrt{\alpha'}$ . We will present some evidence in this direction.

Concerning Donagi-Witten version of integrability as based on Hitchin's gauge model on an elliptic curve  $E_\tau$ , we will observe that this "reference Riemann surface" can also be recovered from the geometrical data of the  $K3$ -fibration.

## 2 The Calabi-Yau Curve and its Quantum Field Theory Analogue.

### 2.1 $K3$ -Fibrations.

Let us consider the string embedding of  $SU(2)$   $N = 2$  supersymmetric Yang-Mills theory according to [23]. The Higgs mechanism that produces the desired gauge group originates 129 neutral hypermultiplets, therefore we must choose as the type IIA dual a  $K3$ -fibered threefold with Hodge numbers  $h_{1,1} = 2$ ,  $h_{2,1} = 128$ , i.e.  $W = \mathbb{P}^4_{\{1,1,2,2,6\}}$  [12]. The mirror manifold  $W^*$  can be obtained from the orbifold construction [37], and has defining polynomial

$$W^* = \frac{1}{12}x_1^{12} + \frac{1}{12}x_2^{12} + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 - \psi x_1 x_2 x_3 x_4 x_5 - \frac{1}{6}\phi(x_1 x_2)^6. \quad (2.1)$$

The moduli space of complex deformations of  $W^*$  is parameterized by  $(\psi, \phi)$ , subject to the global symmetry

$$\mathcal{A} : (\psi, \phi) \rightarrow (\beta\psi, -\phi), \quad \beta^{12} = 1. \quad (2.2)$$

This symmetry forces to introduce invariant quantities; we will use  $b = 1/\phi^2$  and  $c = -\phi/\psi^6$ . The  $K3$ -fibration structure of (2.1) becomes manifest by the change of variables  $x_1/x_2 \equiv z^{1/6}b^{-1/12}$ ,  $x_1^2 \equiv x_0 z^{1/6}$  [26]:

$$W^* = \frac{1}{12}\left(z + \frac{b}{z} + 2\right)x_0^6 + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 + c^{-1/6}x_0 x_3 x_4 x_5, \quad (2.3)$$

with the variable  $z$  acting as coordinate on the  $\mathbb{P}^1$  base space. It is convenient to define

$$d(z; b) = \frac{1}{2}\left(z + \frac{b}{z} + 2\right), \quad \hat{c}(z; b, c) = c d(z; b). \quad (2.4)$$

Substituting this into (2.3) and rescaling  $x_0$ ,  $W^*$  acquires the explicit form of a  $K3$ -surface

$$W^* = \frac{1}{6}x_0^6 + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 + \hat{c}(z; b, c)^{-1/6}x_0 x_3 x_4 x_5. \quad (2.5)$$

As we move in  $\mathbb{P}^1$ , the  $K3$ -fiber can become singular. From (2.5) it is easy to deduce that this occurs for the  $K3$  modulus values  $\hat{c}(z; b, c) = 0, 1$ . These values of  $\hat{c}$  are acquired at the following  $\mathbb{P}^1$  points,  $z = e_i^\pm$ :

$$\begin{aligned}\hat{c} = 0 &\rightarrow e_0^\pm = -1 \pm \sqrt{1-b}, \\ \hat{c} = 1 &\rightarrow e_1^\pm = \frac{1-c \pm \sqrt{(1-c)^2 - bc^2}}{c}.\end{aligned}\tag{2.6}$$

The discriminant of (2.5) is therefore given by  $\Delta(z; b, c) = \prod_{i=0}^1 (z - e_i^+(b, c))(z - e_i^-(b, c))$ . There is an additional singularity at  $\hat{c}(z; b, c) = \infty$ , which is originated in the quotient by discrete reparameterizations of (2.5) inherited from the orbifold construction of  $W^*$ . It corresponds to the points

$$\hat{c} = \infty \rightarrow z = 0, \infty \quad (b \neq \infty).\tag{2.7}$$

The Calabi-Yau manifold becomes singular when some of the points (2.6)-(2.7) coalesce. We will now analyze the regions in moduli space where this situation happens [38, 39] (we will follow notation in [38]). The loci

$$\begin{aligned}\mathcal{C}_1 &= \{b = 1\}, \\ \mathcal{C}_C &= \{(1-c)^2 - bc^2 = 0\},\end{aligned}\tag{2.8}$$

are respectively obtained from the identifications  $e_0^+ = e_0^-$  and  $e_1^+ = e_1^-$ .  $\mathcal{C}_C$  is the conifold locus, where 3-cycles of the threefold degenerate to points, while  $\mathcal{C}_1$  corresponds to the appearance of a genus two curve of  $A_1$  singularities. We can also consider

$$\begin{aligned}\mathcal{C}_0 &= \{c = \infty\}, \\ \mathcal{C}_\infty &= \{b = 0\},\end{aligned}\tag{2.9}$$

which are defined, respectively, by the identifications  $e_1^\pm = e_0^\pm$  and  $e_0^+ = e_1^+ = 0$ .  $\mathcal{C}_0$  is an orbifold locus, given by the fixed points under  $\mathcal{A}^2$ .  $\mathcal{C}_\infty$  corresponds to the weak coupling limit locus, once we identify the heterotic dilaton with the size of the base space [32]. In addition we have

$$\mathcal{D}_{(0,-1)} = \{c = 0\},\tag{2.10}$$

implying  $e_1^+ = 0$ ,  $e_1^- = \infty$ .  $\mathcal{D}_{(0,-1)}$  is an exceptional divisor in moduli space whose intersection with  $\mathcal{C}_\infty$  identifies the large complex structures limit of (2.1). Finally, let us notice that at  $b = \infty$  the points (2.7) are ill defined, giving raise to a very degenerate situation. We can put in correspondence

$$\mathcal{D}_{(-1,0)} = \{b = \infty\},\tag{2.11}$$

with the exceptional divisor introduced to resolve a conical singularity in the moduli space generated from quotienting by the discrete transformation  $\mathcal{A}^1$ . In section 2.3 we will return again to this point, in relation with a double covering of the moduli space<sup>2</sup>.

More in general, we can consider  $K3$ -fibered Calabi-Yau threefolds whose mirror  $W^*$  is also a  $K3$ -fibration that can be written as [26]

$$W^* = \frac{1}{2n} \left( z + \frac{b}{z} + 2 \right) x_0^n + W_{K3}^*(x_0, x_3, x_4, x_5; c_i). \quad (2.12)$$

The discriminant of the fiber will be given by a polynomial  $\Delta(z; b, c_i)$ , depending on the point  $z$  in the base and the moduli parameters, with zeroes at  $z = e_i^\pm$  where the  $K3$ -fiber becomes singular. The number of singular points is given by  $2h_{2,1} = 2(r+1)$ .

## 2.2 The Calabi-Yau Curve.

From the above  $K3$ -fibration structure it is possible to recover [26], in the heterotic point particle limit  $b \rightarrow 0$ ,  $\alpha' \rightarrow 0$  [25], the Seiberg-Witten curves for  $N=2$   $SU(r+1)$  Yang-Mills theory [3, 4, 13]. Namely for the  $SU(2)$  case, using the map

$$\begin{aligned} b &= \alpha'^2 \Lambda^4, \\ c &= 1 + \alpha' u, \end{aligned} \quad (2.13)$$

with  $\Lambda$  the  $SU(2)$  dynamical scale, and rescaling  $z \rightarrow \alpha' z$ , the points (2.6) become the branch points of the associated Seiberg-Witten curve [13]

$$\begin{aligned} e_0^\pm &= 0, \infty, \\ e_1^\pm &= -u \pm \sqrt{u^2 - \Lambda^4}. \end{aligned} \quad (2.14)$$

In general, the geometrical meaning of the point particle limit amounts to replacing the  $K3$ -fiber by an ALE space that blows up an  $A_r$  orbifold singular  $K3$ . Let us denote by  $G = SU(r+1)$  the gauge group whose Dynkin diagram describes the orbifold singularity. In this situation, the branched cover of the  $\mathbb{P}^1$  base space that defines the Seiberg-Witten curve for the pure gauge theory in the integrable model formulation of [13] (whose branch points are at the generalization of (2.14)),  $\mathcal{C}_G$ , can be directly derived from the ALE space homology.

---

<sup>1</sup>Defining the  $\mathcal{A}$ -invariant quantities  $\xi \equiv \psi^8$ ,  $\eta \equiv \psi^4 \phi$  and  $\zeta \equiv \phi^2$ , the quotiented moduli space is given by the projective cone  $\xi \zeta = \eta^2$ .

<sup>2</sup>This will explain why the analysis of singular loci provided by the fibration structure is missing the  $\mathcal{D}_{(-1,-1)}$  divisor, which, together with  $\mathcal{D}_{(0,-1)}$ , is associated with the resolution of a tangency point between  $\mathcal{C}_\infty$  and  $\mathcal{C}_1$  originated in the  $\mathcal{A}$  quotient. This completes the set of toric divisors in the compactification of the moduli space worked out in [38].

A different approach consists in using the global structure of the  $K3$ -fibration in order to build an hyperelliptic curve such that its moduli space is isomorphic to the Calabi-Yau moduli. We can define

$$\mathcal{C}^{CY} : \quad y^2 = \Delta(z; b, c_i) = \prod_{i=0}^r (z - e_i^+(b, c_i))(z - e_i^-(b, c_i)). \quad (2.15)$$

By construction, this curve becomes singular at the moduli values where the threefold  $W^*$  acquires a singularity, i.e. those where two roots coalesce. Performing the point particle limit [25],  $\mathcal{C}^{CY}$  reduces to the Seiberg-Witten curve  $\mathcal{C}_G$ . However, it is important to stress the following. In order to define  $\mathcal{C}^{CY}$  we have interpreted the singular points of the  $K3$ -fibration as branch points of an associated hyperelliptic curve. But contrary to the point particle limit, in which the  $K3$ -fiber is substituted by an ALE space, we are not using the  $K3$  periods for the direct construction of the curve.

Our first aim in analyzing (2.15) will be, following previous work in reference [33], to identify the  $N = 2$  supersymmetric gauge theory represented by  $\mathcal{C}^{CY}$ . Being defined for arbitrary values of the moduli parameters, the first difference between  $\mathcal{C}_G$  and  $\mathcal{C}^{CY}$  is that the second depends on the “dilaton modulus”  $b$ . In the heterotic string framework, the expectation value of the dilaton field determines the bare gauge coupling constant. Therefore we should look for a gauge theory in which the coupling constant behaves as a modulus, i.e. an ultraviolet finite theory.

The simplest candidate for  $\mathcal{C}^{CY}$  is a theory with the field content of  $N = 4$ , namely Yang-Mills plus a matter hypermultiplet in the adjoint representation of  $G$ , where in general the adjoint hypermultiplet can be massive. Let us consider the case where the adjoint hypermultiplet has a bare mass  $m$ . It was shown in [2] that in a double scaling limit that sends  $m$  and the  $N = 4$  coupling constant to  $\infty$ , we can recover the pure Yang-Mills theory. In this process the mass plays a role formally analogous to that of the string coupling constant in the double limit that takes  $\mathcal{C}^{CY}$  to  $\mathcal{C}_G$ . Therefore, in the proposed interpretation of string notions in terms of gauge theory quantities, we should identify  $(\alpha')^{-1} \sim m^2$ .

Let us now review briefly some results in reference [33] for the case  $G = SU(2)$ . The curve for  $SU(2)$   $N = 2$  gauge theory with one massive adjoint hypermultiplet [2], is given by

$$y^2 = (x - a_1 \hat{u} + a_2^2)(x + a_2(\hat{u} - a_1))(x - a_2(\hat{u} - a_1)), \quad (2.16)$$

with  $u = \text{Tr}\phi^2$  the quadratic Casimir, and  $\hat{u}$  defined by the convenient normalization  $\frac{1}{4}m^2\hat{u} = u$ . The quantities  $a_1$  and  $a_2$  depend on the asymptotic value of the gauge coupling constant of the theory,  $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ , according to<sup>3</sup>

$$a_1 = \frac{3}{2}e_1(\tau) \quad , \quad a_2 = \frac{1}{2}(e_3(\tau) - e_2(\tau)). \quad (2.17)$$

---

<sup>3</sup>The Weierstrass invariants  $e_i$  can be defined in the terms of Jacobi theta functions:  $e_1 = \frac{1}{3}(\theta_2^4(0, \tau) + \theta_3^4(0, \tau))$ ,  $e_2 = -\frac{1}{3}(\theta_1^4(0, \tau) + \theta_3^4(0, \tau))$ ,  $e_3 = \frac{1}{3}(\theta_1^4(0, \tau) - \theta_2^4(0, \tau))$ .

In the moduli space  $(\hat{u}, \tau)$ , we can differentiate the following loci where (2.16) becomes singular

$$\begin{aligned}\hat{\mathcal{C}}_0 &= \{\hat{u}(\tau) = a_1(\tau)\}, & \hat{\mathcal{C}}_\infty &= \{\tau = i\infty\}, \\ \hat{\mathcal{C}}_C^{(1)} &= \{\hat{u}(\tau) = a_2(\tau)\}, & \hat{\mathcal{C}}_1^+ &= \{\tau = 0\}, \\ \hat{\mathcal{C}}_C^{(2)} &= \{\hat{u}(\tau) = -a_2(\tau)\}, & \hat{\mathcal{C}}_1^- &= \{\tau = 1\}, \\ \hat{\mathcal{D}} &= \{\hat{u} = \infty\}.\end{aligned}\tag{2.18}$$

We can now try to put in correspondence the moduli space of Kähler deformations of  $\mathbb{P}_{\{1,1,2,2,6\}}^4$  [12] with the moduli space of the  $N = 4$  softly broken theory (2.16). The basic idea will be to map the singular loci described by the fibration structure of the mirror  $W^*$ , (2.8)-(2.10), with the set (2.18). This is achieved by the map [33]

$$c = \frac{a_1(\tau)}{a_1(\tau) - \hat{u}} \quad , \quad b = \left(\frac{a_2(\tau)}{a_1(\tau)}\right)^2.\tag{2.19}$$

It is important to notice, using the modular properties of  $a_i(\tau)$ <sup>4</sup>, that (2.19) is effectively quotienting by the  $(\hat{u}, \tau)$ -plane transformation

$$T : (\hat{u}, \tau) \rightarrow (\hat{u}, \tau + 1).\tag{2.20}$$

Indeed the proposed map sends  $\hat{\mathcal{C}}_C^{(1,2)} \rightarrow \mathcal{C}_C$  and  $\hat{\mathcal{C}}_1^\pm \rightarrow \mathcal{C}_1$ , while  $\hat{\mathcal{C}}_\infty$ ,  $\hat{\mathcal{C}}_0$  and  $\hat{\mathcal{D}}$ , which are fixed under (2.20), are mapped respectively into  $\mathcal{C}_\infty$ ,  $\mathcal{C}_0$  and  $\mathcal{D}_{(0,-1)}$ <sup>5</sup>. Therefore we observe that the  $(\hat{u}, \tau)$ -plane behaves as a double cover of the Calabi-Yau moduli space.

In the weak coupling limit  $\tau \rightarrow i\infty$ , the map (2.19) becomes

$$c = \frac{1}{1 - \hat{u}} \quad , \quad b = 64e^{2\pi i\tau}.\tag{2.21}$$

From the heterotic-type II identification at leading order  $b = e^{-S}$  [23], with  $S$  the heterotic dilaton, the above expression explicitly shows that we are associating  $\tau$  with  $S$ . Setting  $(\alpha')^{-1} = \frac{1}{4}m^2$ , as we proposed, (2.21) reproduces the relation  $\frac{c^2 b}{(1-c)^2} = \frac{\Lambda^4}{u^2}$  used in [25] for defining the point particle limit of the string (of which (2.13) is the first order).

We would like now to prove that, by the map (2.19), the curve  $\mathcal{C}^{CY}$  is in fact the Seiberg-Witten curve for  $N=2$   $SU(2)$  Yang-Mills theory with one massive hypermultiplet in the adjoint representation. In order to do so, we rewrite the Calabi-Yau curve (2.15) for the  $SU(2)$  case as

$$y^2 = \prod_{i=0,1} (z - e_i^+(\hat{u}, \tau))(z - e_i^-(\hat{u}, \tau)),\tag{2.22}$$

<sup>4</sup>We have  $a_1(\tau + 1) = a_1(\tau)$  and  $a_2(\tau + 1) = -a_2(\tau)$ .

<sup>5</sup>A similar map between the singular loci of the hyperelliptic curve describing  $SU(3)$   $N=2$  supersymmetric Yang-Mills theory with adjoint matter and the moduli space of the  $SU(3)$  gauge group Calabi-Yau manifold [23],  $\mathbb{P}_{\{1,1,2,8,12\}}^4$  [24], is proposed in Appendix B.



with

$$\begin{aligned}
e_0^\pm &= \frac{-a_1(\tau) \pm \sqrt{a_1(\tau)^2 - a_2(\tau)^2}}{a_1(\tau)}, \\
e_1^\pm &= \frac{-\hat{u} \pm \sqrt{\hat{u}^2 - a_2(\tau)^2}}{a_1(\tau)}.
\end{aligned}
\tag{2.23}$$

It is convenient now to transform the Calabi-Yau curve (2.22) into the standard cubic form  $y^2 = z(z-1)(z-\lambda)$ , where

$$\lambda = \frac{(e_1^- - e_0^+)(e_0^- - e_1^+)}{(e_1^+ - e_0^+)(e_0^- - e_1^-)}.
\tag{2.24}$$

Using results in Appendix A, we observe that the field theory curve (2.16) is isogenic to the quartic

$$y^2 = (x^2 + a_1\hat{u} - a_2^2)^2 - (a_2(\hat{u} - a_1))^2.
\tag{2.25}$$

The  $\lambda$  parameter that characterizes the standard cubic form of (2.25) is given by

$$\lambda' = \frac{a_1\hat{u} - a_2^2 + \sqrt{(a_1^2 - a_2^2)(\hat{u}^2 - a_2^2)}}{a_1\hat{u} - a_2^2 - \sqrt{(a_1^2 - a_2^2)(\hat{u}^2 - a_2^2)}}.
\tag{2.26}$$

Substituting (2.23), we can see that  $\lambda$  and  $\lambda'$  precisely coincide.

This concludes the proof that  $\mathcal{C}^{CY}$  for the Calabi-Yau mirror of  $\mathbb{P}_{\{1,1,2,2,6\}}^4$  [12] is, by the map (2.19), the curve describing  $SU(2)$   $N = 2$  supersymmetric Yang-Mills with one massive hypermultiplet in the adjoint representation. The necessity of introducing an isogeny transformation is originated in different conventions for the Higgs field normalization. The curve (2.16) follows the convention in [1], which is appropriate when only integer charges for BPS states can appear. However the Seiberg-Witten curve  $\mathcal{C}_G$ , obtained from the point particle limit of the string, adopts a normalization adequate to gauge theories that can include fundamental matter [2]. Since  $\mathcal{C}^{CY}$  flows to  $\mathcal{C}_G$  in the point particle limit, it shares with it the same normalization of the Higgs field, differing, up to an isogenic transformation (see (A.5)), of that in (2.16).

### 2.3 Double Covering and $K3$ -Fibrations.

We have pointed out that the  $(\hat{u}, \tau)$ -moduli space of the  $SU(2)$  theory with adjoint matter, using the map (2.19), acts as a double covering of the Calabi-Yau  $\mathbb{P}_{\{1,1,2,2,6\}}^4$  [12] moduli space  $(b, c)$ . At the same time, we have seen that the Seiberg-Witten curve for this gauge theory can be explicitly constructed from string theory compactification. This raises the following puzzle: if the string moduli space is correctly labeled by  $(b, c)$ , how can the double covering variables  $(\hat{u}, \tau)$  be naturally derived from the Calabi-Yau moduli?

The answer comes from the way we define the  $K_3$ -fibration. In (2.3) the choice  $\sqrt{b} = -\frac{1}{\phi}$  has been implicitly done. If we instead choose  $\sqrt{b} = \frac{1}{\phi}$ , the fibration structure (2.4) changes into

$$\hat{c}' = \frac{1}{2} \left( -z - \frac{b}{z} + 2 \right) c. \quad (2.27)$$

We see that the choice of one or another branch of  $\sqrt{b}$  amounts to changing  $z \rightarrow -z$ . This is an effect that can be absorbed in a trivial redefinition of  $z$ , and therefore does not, in essence, affect  $\mathcal{C}^{CY}$ . However it shows how the  $K_3$ -fibration structure, and any notion based on it, leads to a double covering of the moduli space determined by the two branches of  $\sqrt{b}$ . Moreover in terms of the map (2.19), written as

$$\sqrt{b} = \frac{a_2(\tau)}{a_1(\tau)}, \quad (2.28)$$

the change of branch becomes equivalent to the transformation  $T : \tau \rightarrow \tau + 1$ .

This problem is related to a subtlety in the derivation of the Seiberg-Witten solution for  $SU(2)$  Yang-Mills in reference [25], accomplished by performing the blow up of the tangency point ( $b = 0, c = 1$ ) between the conifold locus  $\mathcal{C}_C$  and the weak coupling locus  $\mathcal{C}_\infty$ . The variable parameterizing the second exceptional divisor arising from the blow up of the tangency,  $\frac{bc^2}{(1-c)^2}$ , is identified with the field theory  $\mathbf{Z}_2$  invariant quantity  $\frac{\Lambda^4}{u^2}$ . The reason why the blow up approach recovers  $(\Lambda^2/u)^2$  is that, from (2.13), the change of branch  $\sqrt{b} \rightarrow -\sqrt{b}$  implies  $\Lambda^2 \rightarrow -\Lambda^2$ , and the Calabi-Yau moduli space naturally quotients by this transformation. On the contrary, from the  $K3$ -fibration in the ALE limit we get in a direct way the Seiberg-Witten curve, parameterized by  $u$ , and not its  $\mathbf{Z}_2$  quotient.

The transformation  $T$  has indeed an string analogue. Using the map (2.19),  $T$  is equivalent to the discrete symmetry (2.2) of the Calabi-Yau moduli space,  $\mathcal{A}$ . After performing the  $\mathcal{A}$ -quotient, the Calabi-Yau moduli space becomes isomorphic [38] to the space  $\mathbb{P}^2_{\{1,1,2\}}$ , which has a conical singularity at the origin (see footnote 1). This is precisely the geometry of the  $T$ -quotiented  $(\hat{u}, \tau)$ -plane<sup>6</sup>. We stress that the  $(\hat{u}, \tau)$  variables, although undo the  $\mathcal{A}$ -quotient, preserve an  $\mathcal{A}^2$ -quotient. This can be seen from the fact that  $\mathcal{A}^2$  fixes the locus  $\mathcal{C}_0$ , which has a counterpart in the  $(\hat{u}, \tau)$  moduli space.

Finally we notice that  $(\hat{u}, \tau)$  points are not in a one-to-one correspondence with  $(\sqrt{b}, c)$ . The divisor  $\mathcal{D}_{(-1,0)} = \{b = \infty\}$ , which appears in the blow up of the conical singularity created by the  $\mathcal{A}$ -quotient, presents non-trivial monodromy. However in the  $(\hat{u}, \tau)$ -plane, having undone the quotient by  $\mathcal{A}$ , the value  $b = \infty$  should not imply an additional monodromy locus. This is in fact the case. Since  $b = \infty$  corresponds to  $a_1 = 0$ , from (2.19) we see that  $\mathcal{D}_{(-1,0)}$  is blown down to the point  $\hat{u} = 0$ . The fibration structure is also reflecting this remark through its dependence on the  $K3$ -fiber modulus  $\hat{c} = cd$ . The

---

<sup>6</sup>Defining the invariants with respect to (2.20)  $\xi \equiv \tilde{u}^2$ ,  $\eta \equiv \frac{\tilde{u}}{\epsilon}$ ,  $\zeta \equiv \frac{1}{\epsilon^2}$ , where  $\epsilon \equiv 8e^{\pi i \tau}$ ,  $\tilde{u} \equiv \frac{\hat{u}}{\epsilon}$ , the quotiented  $(\hat{u}, \tau)$ -moduli space is also given by the projective cone  $\xi\zeta = \eta^2$ .

value  $b = \infty$  implies  $d = \infty$ , so that the combination  $\hat{c}$  blows down the line  $(b = \infty, c)$  to a point.

### 3 Integrability.

In the previous section we have compared  $\mathcal{C}^{CY}$  with the curve for  $SU(2)$   $N = 2$  Yang-Mills with one massive hypermultiplet in the adjoint representation. In this section we will recover, from the Calabi-Yau manifold, the basic building elements used in the Donagi-Witten formulation [15] of the integrability of gauge theories with adjoint matter.

To start with, let us just briefly recall the construction in [15] for the simple case of  $SU(2)$  gauge theory. According to Hitchin’s construction, we start with the elliptic curve

$$E_\tau : y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)), \quad (3.1)$$

on which a two dimensional Higgs field  $\phi$  is defined as an holomorphic 1-form transforming in the adjoint representation of the  $SU(2)$  gauge group. In terms of  $\phi$ , an spectral cover curve is defined through (3.1) and

$$0 = t^2 - x + A_2, \quad (3.2)$$

with  $A_2$  related to the Higgs expectation value of the  $N = 2$  theory by  $A_2 = u - \frac{1}{2}e_1(\tau)$ . Equations (3.1) and (3.2) define a genus 2 curve, symmetric under the  $\mathbf{Z}_2 \times \mathbf{Z}_2$  transformation  $t \rightarrow -t, y \rightarrow -y$ . The curve (2.16) we have employed in the previous section corresponds to the  $\mathbf{Z}_2 \times \mathbf{Z}_2$  invariant part.

The integrable model version of pure gauge theory curves [12, 13], which is recovered from string theory in the point particle limit [26], is given by

$$z + \frac{\Lambda^4}{z} + 2P_{A_1}(t, u) = 0. \quad (3.3)$$

This can be derived from the “classical” expression (3.2)<sup>7</sup> by the “quantization” procedure  $x \rightarrow -\frac{1}{2}(z + \frac{\Lambda^4}{z})$ . Instead, in [15] the quantization of (3.2) is implemented by forcing  $x$  to live on the elliptic curve (3.1), parameterized by the  $N = 4$  coupling constant  $\tau$ .

In order to show that the Donagi-Witten integrability construction is the natural extension of (3.3) when we move off the point particle limit, we have still to determine the elliptic curve  $E_\tau$  from Calabi-Yau data. Associated to a  $K3$ -fibration threefold

$$W^* = \frac{1}{n}d(z; b) x_0^n + W_{K3}^*(x_0, x_3, x_4, x_5; c_i) \quad (3.4)$$

with  $d(z; b)$  given by (2.4), we can always consider the following four points on the  $\mathbb{P}^1$  base space where the  $K3$ -fiber becomes singular

$$\begin{aligned} d(z; b) = 0 &\rightarrow e_0^\pm = -1 \pm \sqrt{1 - b}, \\ d(z; b) = \infty &\rightarrow z = 0, \infty. \end{aligned} \quad (3.5)$$

---

<sup>7</sup>The Landau-Ginzburg potential  $P_{A_1}(t, u) = t^2 + u$  is being used.

In the spirit of the previous section, we can define the elliptic curve associated to these data

$$y^2 = z(z - e_0^+)(z - e_0^-). \quad (3.6)$$

This curve is independent of the particular Calabi-Yau we are working with, as far as the  $K3$ -fibration structure is defined by  $d = \frac{1}{2}(z + \frac{b}{z} + 2)$ . The curve (3.6) is characterized by the  $\lambda$  parameter

$$\lambda = \frac{e_0^-}{e_0^+}, \quad (3.7)$$

which only depends on the dilaton modulus  $b$ .

We will follow the same path as in section 2.2 for comparing (3.6) with  $E_\tau$ . Namely, after shifting  $x \rightarrow x + \frac{1}{2}e_1$ , we apply the isogeny described in Appendix A. This transforms  $E_\tau$  into the quartic

$$y^2 = (x^2 + a_1(\tau))^2 - a_2(\tau)^2. \quad (3.8)$$

Using the map between  $b$  and  $\tau$  proposed in (2.19), the  $\lambda$  parameter of this curve can be easily seen to coincide with (3.7), implying that (3.6) is isogenic to  $E_\tau$ . In this sense we notice that, according to (A.5), the elliptic parameter of the curve (3.6) is  $2\tau$ .

Let us stress the meaning of the extra singularity at  $d = \infty$  used in (3.5), which corresponds to  $z = 0, \infty$ . The Seiberg-Witten differential for  $\mathcal{C}^{CY}$ , which we will describe in the beginning of the next section (see (4.2)), has poles with residue at the points  $z = 0, \infty$ . The residue at these poles is defining the mass of the hypermultiplet in the adjoint [2]. (In the point particle limit [25] the structure (3.6) disappears, as  $e_0^\pm \rightarrow 0, \infty$ ). In the context of the Donagi-Witten formulation, the singularities at  $z = 0, \infty$  have a similar role to the point  $x = \infty$  in  $E_\tau$  which, by equation (3.2), corresponds to a degenerate spectral set  $t = \pm\infty$ . Recalling the underlying Hitchin model, we observe that  $x = \infty$  is the pole of the associated two dimensional Higgs field, whose residue also defines the mass of the adjoint hypermultiplet [15].

In summary, from the Calabi-Yau geometry we get the integrability structure of the Seiberg-Witten model, as it is described in the Donagi-Witten construction. This integrability structure only shows up when we keep alive both the string scale,  $\alpha'$ , and the gravitational effects due to the existence of the dilaton. However, we can not expect the Donagi-Witten model to be equivalent to full fledged string theory, an issue that will be addressed in the next section.

## 4 Picard-Fuchs Equations.

It was shown in reference [26] that in the point particle limit, where the  $K3$  degenerates to an ALE space, we can effectively map the second homology group of  $K3$  into the homology group of 0-cycles defined by the spectral set  $\{t \mid P_{A_1}(t; u_i) = 0\}$ . Alternatively, the integration of the holomorphic top form of the Calabi-Yau manifold on a 2-cycle of

the  $K3$ , in its ALE limit, defines the meromorphic Seiberg-Witten form  $\lambda$  for the curve (3.3). This form is [13]

$$\lambda = \sqrt{u + \frac{1}{2}\left(z + \frac{\Lambda^4}{z}\right)} \frac{dz}{z}. \quad (4.1)$$

Using the Calabi-Yau curve defined in the previous sections we can propose a generalization of (4.1) to the case gravity is turned on. Namely, we can consider the meromorphic form  $\lambda$  derived from the  $\mathcal{C}^{CY}$  by means of the map (2.19). The result is

$$\tilde{\lambda} = \sqrt{1 - \frac{1}{\hat{c}} \frac{dz}{z}}, \quad (4.2)$$

with  $\hat{c}$  defined in (2.4). Our aim in this section is to understand the meaning of (4.2), and equivalently  $\mathcal{C}^{CY}$ , in the string context. In order to do so, we will analyze in what way  $\tilde{\lambda}$  is related with the periods of  $K3$  and with the associated Picard-Fuchs equation.

The information used to construct  $\mathcal{C}^{CY}$  reduces to the discriminant of the  $K3$ -fiber  $\Delta(z; b, c)$  and, derived from it, the discriminant of the Calabi-Yau threefold. However these data do not determine in a unique way the threefold. There exist different Calabi-Yau spaces whose moduli of Kähler deformations share common features. As an example the manifold  $\mathbb{P}^4_{\{1,1,2,2,6\}}$  [12] we have been considering up to now, exhibits the same singular loci and Yukawa couplings structure that the hypersurface  $\mathbb{P}^4_{\{1,1,2,2,2\}}$  [8] and the complete intersections  $\mathbb{P}^5_{\{1,1,2,2,2,2\}}$  [4, 6] and  $\mathbb{P}^6_{\{1,1,2,2,2,2,2\}}$  [4, 4, 4] [38, 39, 40]. These four manifolds are  $K3$ -fibrations and from any of them, in the point particle limit (2.13), can be derived the exact physics of  $SU(2)$   $N = 2$  Yang-Mills theory. For simplicity we will denote them as  $A : \mathbb{P}^4_{\{1,1,2,2,6\}}$  [12],  $B : \mathbb{P}^4_{\{1,1,2,2,2\}}$  [8],  $C : \mathbb{P}^5_{\{1,1,2,2,2,2\}}$  [4, 6] and  $D : \mathbb{P}^6_{\{1,1,2,2,2,2,2\}}$  [4, 4, 4] (see Appendix C for a brief description of these spaces).

The Picard-Fuchs equation for their  $K3$ -fiber is [41]

$$L = \theta_{\hat{c}}^3 - \hat{c} \left( \theta_{\hat{c}}^3 + \frac{3}{2}\theta_{\hat{c}}^2 + \frac{1}{2}\theta_{\hat{c}} + 2r\left(\theta_{\hat{c}} + \frac{1}{2}\right) \right), \quad (4.3)$$

with  $r = \frac{N-1}{2N^2}$  and  $N = 6, 4, 3, 2$  respectively. Since by construction  $\mathcal{C}^{CY}$  can not distinguish between the mentioned four spaces, it is natural to expect that (4.2) is related to the common part of their Picard-Fuchs equation. Indeed, it is easy to see that (4.2) is solution of

$$L_1 = \theta_{\hat{c}}^3 - \hat{c} \left( \theta_{\hat{c}}^3 + \frac{3}{2}\theta_{\hat{c}}^2 + \frac{1}{2}\theta_{\hat{c}} \right) - \frac{1}{4}\theta_{\hat{c}}. \quad (4.4)$$

The operators  $L$  and  $L_1$  differ in the first order differential operator

$$L_2 = L - L_1 = \frac{1}{4}\theta_{\hat{c}} - 2r\hat{c} \left( \theta_{\hat{c}} + \frac{1}{2} \right). \quad (4.5)$$

The singular points of the  $K3$  Picard-Fuchs equation (4.3) are at  $\hat{c} = 0, 1, \infty$ . At  $\hat{c} = 1$  the  $K3$  develops an  $A_1$  singularity, and its associated monodromy reproduces the

Weyl group for  $SU(2)$ . However the point  $\hat{c} = 0$  presents logarithmic monodromy, as can be deduced from the corresponding indicial equations. This fact does not allow to use the solutions of (4.3), in particular the period carrying Weyl monodromy around  $\hat{c} = 1$ , for defining the Seiberg-Witten differential of a Riemann surface. Definition (4.4) is the necessary modification of  $L$  in order to achieve this. The singular points of  $L_1$  are still at  $\hat{c} = 0, 1, \infty$ , but the asymptotic behavior for both  $0, \infty$  have been modified. The indicial equations of  $L$  and  $L_1$  at  $\hat{c} = 1$  coincide, therefore the leading behavior of the solutions at this singularity is not affected.

We want now to analyze how working with  $L_1$  instead of  $L$  affects the physics that we obtain. Namely, we will compare the physics associated to  $\mathcal{C}^{CY}$  with that of a type IIA string compactified in manifolds  $A, B, C$  or  $D$ , and their corresponding heterotic dual strings.

Let us begin considering the conifold locus of these Calabi-Yau manifolds. This locus corresponds to the melting of the  $K3$ -fiber singular points  $e_1^\pm$ , given in (2.6). At these points  $\hat{c} = 1$  and  $L$  and  $L_1$  coincide. Therefore, in a neighborhood of the conifold locus,  $\mathcal{C}^{CY}$  (equivalently an  $N = 4$  softly broken gauge theory) should describe essentially the same physics as the threefold. Indeed in both cases the singularity is interpreted as due to BPS dyons becoming massless [1, 2], [42].

The singular locus  $\hat{\mathcal{C}}_0$  of the  $N = 4$  softly broken theory corresponds to an electric hypermultiplet acquiring zero mass [2]. This multiplet derives from components of the initial massive adjoint hypermultiplet, and the singularity occurs for Higgs expectation values of order  $m$ , with  $m$  the adjoint hypermultiplet bare mass. The gravity counterpart of this locus is  $\mathcal{C}_0$  in (2.9), which we observed can be represented by the melting of the singular points  $e_0^\pm = e_1^\pm$ . Since over  $e_0^\pm$  the  $K3$  develops the  $\hat{c} = 0$  singularity, at which the operators  $L$  and  $L_1$  strongly differ, the interpretation of field theory and string loci,  $\hat{\mathcal{C}}_0$  and  $\mathcal{C}_0$ , can naturally be different. However we will propose that both share a common origin in the presence of an additional (non-moduli) scale in the theory. For the  $N = 4$  softly broken theory, this scale is of course provided by the mass of the adjoint hypermultiplet. In the case of the Calabi-Yau space we want to argue that the scale behind  $\mathcal{C}_0$  is the string tension  $\alpha'$ . Let us stress that the identification between  $m^2$  and  $(\alpha')^{-1}$  was one of the main consequences of the map (2.19) between gauge theory variables  $(\hat{u}, \tau)$  and string moduli  $(b, c)$ .

We will concentrate in the Calabi-Yau model  $A : \mathbb{P}_{\{1,1,2,2,6\}}^4$  [12]. The locus  $\mathcal{C}_0$  was defined as the set of points invariant under  $\mathcal{A}^2$ , with  $\mathcal{A}$  given by the moduli space symmetry (2.20). This transformation satisfies  $\mathcal{A}^6 = -1$ , discarding its origin in an additional massless particle. The mirror map relates complex structures of  $A^*$ , parameterized by  $(b, c)$ , to Kähler structures of  $A$ , parameterized by special coordinates  $(t_1, t_2)$  ( $t_1 \sim \log c$ ,  $t_2 \sim \log b$ , for  $c, b \rightarrow 0$  [43]). Heterotic-type II duality further relates the Kähler coordinates  $(t_1, t_2)$  to the heterotic moduli [23, 25]

$$t_1 = T \quad , \quad t_2 = -\frac{S}{2\pi i} \quad , \quad (4.6)$$

with  $T$  an space-time modulus and  $S$  the heterotic dilaton. In [38, 40, 25] the monodromy group for the manifold  $A$  was worked out, from where it can be deduced

$$\mathcal{A}^2 : T \rightarrow -\frac{T+1}{T}, \quad S \rightarrow S. \quad (4.7)$$

We see that the monodromy around  $\mathcal{C}_0$  produces a  $T$ -duality transformation for the heterotic dual string<sup>8</sup>.

More in general, and without doing reference to string-string dualities, we can relate  $\mathcal{C}_0$ -type loci, i.e. set of fixed points under global symmetries in moduli space, to self-dual regions with respect to the Calabi-Yau generalization of the target-space duality  $R \rightarrow \frac{1}{R}$ , where  $R$  stands for a Kähler modulus. This can already be observed in the simpler case of the quintic [44]. In that case we have a single moduli,  $\psi$ ; the conifold singularity is located at  $\psi = 1$ . The symmetry transformation  $\mathcal{A} : \psi \rightarrow \beta\psi$ , with  $\beta^5 = 1$ , lets fixed the point  $\psi = 0$ , which is the analog in this simple example of the  $\mathcal{C}_0$  locus. At this point the Zamolodchikov's metric is regular, while the Kähler potential becomes singular. However, this singularity in the Kähler potential is not producing any singular contribution to the partition function, as can be seen from the topological analysis of reference [45]. Thus, we should not expect new massless particles at  $\psi = 0$ <sup>9</sup>. Alternatively, the  $\mathcal{A}$  transformation acts on the Calabi-Yau radius in a similar (but more complicated) way to  $R \rightarrow \frac{1}{R}$  [44].

$T$ -duality transformations always imply the introduction of an additional scale. Having identified  $\mathcal{C}_0$  as the Calabi-Yau generalization of a self-dual locus, we should determine the scale that is associated to it. Expression (4.7), together with the fact that the Calabi-Yau weak coupling limit  $b \rightarrow 0$  gets mapped by string-string duality to heterotic perturbative effects [23], indicate that this scale is given by the heterotic-dual string tension  $\alpha'$ . Restoring unities, the self-dual point of (4.7) is  $T = \rho \sqrt{\alpha'}$ , with  $\rho^3 = 1$ . Therefore perturbative string excitations will have, at that point, a typical mass  $\alpha'^{-1/2}$ . This agrees with the field theory interpretation  $\alpha'^{-1/2} \sim m$ , for  $m$  the adjoint hypermultiplet bare mass. Values  $b > 0$  will imply corrections of the characteristic mass of string excitations at  $\mathcal{C}_0$ . In this case, the natural candidate for comparing square masses should be the Casimir expectation value  $u \sim m^2 e_1(\tau)$ , that determines the field theory locus  $\hat{\mathcal{C}}_0$ .

From the type II perspective, we should obtain the same result for the typical mass of excitations in a neighborhood of  $\mathcal{C}_0$ , by looking at BPS states associated to Ramond-Ramond charged branes. Notice that, although  $\mathcal{C}_0$  is equally a self-dual locus on the type

---

<sup>8</sup>The monodromy around the locus  $\mathcal{D}_{(0,-1)} = \{c = 0\}$  corresponds to  $T \rightarrow T + 1$ . Combining this and (4.7), or equivalently surrounding the two conifold branches, we can obtain the standard  $T \rightarrow -1/T$ .

<sup>9</sup>It must be recalled that the Kähler potential enters always into the special geometry equations in the form  $e^{-K} \|f\|^2$ , with  $f$  a meromorphic section of the special geometry Hodge line bundle. A zero of  $f$  at the origin ( $\psi = 0$  for the quintic) regularizes the singularity of  $K$ . The meromorphic section  $f$  contributes to the definition of the topological propagator from which the topological version of the partition function is built up. At the conifold locus, this is the propagator reproducing the logarithmic singularity at one loop. At the  $\mathcal{C}_0$  locus, this propagator is smoothed out with the help of special geometry structure, namely, the freedom to normalize the vacuum section of the Hodge line bundle.

IIA side, the mass of these excitations will not be governed by the type II string tension, since Ramond-Ramond charged branes are non-perturbative objects [46].

The same considerations apply to models  $B$  and  $C$ . The case  $D$  is however different, as the monodromy around  $\mathcal{C}_0$  involves logarithms<sup>10</sup> and therefore it does not admit to be interpreted as a self-dual locus. In this case we could think in the appearance of a massless particle to explain the singularity. If underlying scale is still  $\alpha'$ , it would indicate an electrically charged particle. Physically, the  $D$  model can be the one more closely related to  $\mathcal{C}^{CY}$ . It is worth noticing that the singular points of the first order differential operator  $L_2$  are at  $\hat{c} = 0, 8r, \infty$ . Only for the model  $D$  we have  $8r = 1$ , and the singular points of  $L_2$  coincide with those of  $L$  and  $L_1$ . This fact could be directly connected with passing from a logarithmic monodromy transformation around the hypermultiplet locus  $\hat{\mathcal{C}}_0$  in the field theory approach, to an smoothed out (non-logarithmic) monodromy  $\mathcal{A}^2$  for the models  $A$ ,  $B$  and  $C$ , while not for the Calabi-Yau  $D$ .

Finally we consider the locus  $\mathcal{D}_{(0,-1)} = \{c = 0\}$ , which correspond to the degenerate situation  $e_1^+ = 0$  and  $e_1^- = \infty$ . The operators  $L$  and  $L_1$  also differ at  $z = 0, \infty$ , implying that close to this locus  $\mathcal{C}^{CY}$  will not reproduce the string dynamics. This is indeed as expected, since the mirror map [38] fixes  $t_1 = i\infty$  at  $\mathcal{D}_{(0,-1)}$ , therefore representing a decompactification limit. Using (2.19), this locus corresponds to the ultraviolet regime  $u = \infty$  for the  $N = 4$  softly broken theory, where string and field theory should strongly differ.

Summarizing, we have seen that  $\mathcal{C}_{CY}$  can provide a good description of string phenomena only in a neighborhood of the conifold locus. However,  $\mathcal{C}_{CY}$  proved useful in interpreting the locus  $\mathcal{C}_0$ . In the next section, following this path, we will use  $\mathcal{C}_{CY}$  as a tool for understanding further differences in the coupling to gravity of  $SU(2)$   $N = 2$  Yang-Mills provided by models  $A$ ,  $B$ ,  $C$  and  $D$ .

## 5 Higher Kac-Moody Level String Models.

The differences between the four Calabi-Yau manifolds we are considering can be resumed in the properties of the mirror map for their  $K3$ -fiber, or equivalently, the mirror map between  $c$  and  $t_1$  for the dilaton modulus value  $b = 0$ . Using the string-string identification  $t_1 = T$ , we have

$$c = \frac{h_k(T_0^{(k)})}{h_k(T)}, \quad (5.1)$$

with  $h_k$  the Hauptmodul function of  $\Gamma_0(k)_+$  (shifted by a constant), and  $k = 1, 2, 3, 4$  for spaces  $A$ ,  $B$ ,  $C$  and  $D$  respectively [38, 41]. The value  $T_0^{(k)}$  is given by the self-dual point

---

<sup>10</sup>The indicial equations derived from the third order Picard-Fuchs operator (C.2) of model  $D$ , around  $c = \infty$ , posses a triple solution.



of one distinguished element of the group  $\Gamma_0(k)_+$ , the Atkin-Lehner involution

$$T \rightarrow -\frac{1}{kT}. \quad (5.2)$$

From (5.1) we see that the  $\Gamma_0(k)_+$  determines the  $T$ -duality group of the heterotic dual string. The dual for the  $A$  model is a rank three heterotic string compactified in  $K3 \times T^2$  [23], where  $T$  corresponds to a  $T^2$  modulus. Since  $\Gamma_0(1)_+ = PSL(2, Z)$  and  $h_1$  coincides with the  $j$ -invariant, we obtain the usual  $T$ -duality group for this model.

In [34] the case  $B : \mathbb{P}_{\{1,1,2,2,2\}}^4[8]$  was considered. It was proposed to be dual to an heterotic string compactification with  $SU(2)$  perturbative enhancement of symmetry realized at Kac-Moody level 2. The mirror map (5.1) fixes the heterotic enhancement of symmetry point at  $T = i/\sqrt{2}$ . The right and left moving momenta for the rank three models are given [47] by

$$\begin{aligned} p_L &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2\bar{T}^2 + 2m\bar{T}), \\ p_R &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2T\bar{T} + m(T + \bar{T})). \end{aligned} \quad (5.3)$$

The condition for new massless states,  $p_L = 0$ ,  $|p_R|^2 \leq 2$ , is satisfied at  $T = i/\sqrt{2}$  for  $n_1 = 2n_2 = \pm 1/2$ ,  $m = 0$ . This implies the value  $p_R^2 = 1$ . If we denote the Kac-Moody level at which the enhanced gauge group is realized by  $k_G$ , the following relation holds

$$k_G |p_R|^2 = 2. \quad (5.4)$$

Therefore  $k_G = 2$  for the heterotic dual associated to the manifold  $B$ . In an analogous way we can analyze the heterotic duals that could correspond to models  $C$ ,  $D$ . The mirror map fixes the perturbative enhancement of symmetry at  $T = i/\sqrt{k}$ , with  $k = 3, 4$  respectively. At this value, using (5.3), the conditions for two additional massless states are again verified, satisfying (5.4) for  $k_G = k$ . Although further checks along the lines of [48, 31, 34] have to be done for the cases  $C$ ,  $D$ , and the explicit construction of the heterotic dual [23] accomplished for models  $B$ ,  $C$  and  $D$ , we will assume that a main difference between the coupling to gravity of  $N = 2$   $SU(2)$  Yang-Mills that the four manifolds can provide is given by the Kac-Moody level at which they realize the gauge symmetry, being  $k_G = 1, 2, 3, 4$  respectively.

The Kac-Moody level also affects the relation between gauge group bare coupling constant and heterotic dilaton [49]. With a convenient normalization we can set  $\tau_G = -\frac{kS^{inv}}{2\pi i}$ , where now it is important to distinguish between the special coordinate dilaton  $S$ , with modular properties under  $T$ -duality transformations, and the invariant dilaton  $S^{inv}$  [50, 51]. Since  $\mathcal{C}^{CY}$  captures the Calabi-Yau information in a neighborhood of the conifold locus, and in the limit  $b \rightarrow 0$  this information reproduces the heterotic enhancement of

symmetry, we can, in this limit, identify  $\tau$  and  $\tau_G$ . Using (2.21), the relation between  $\tau$  and  $S^{inv}$  translates into

$$t_2 = -\frac{kS}{2\pi i} \quad (5.5)$$

for the special geometry coordinates.

We have analyzed properties of the models  $A$ ,  $B$ ,  $C$  and  $D$  which correspond to the heterotic-dual weak coupling regime. We can now use both pieces of information, namely  $\mathcal{C}^{CY}$  and the level  $k$ , in trying to reproduce properties associated to the heterotic strong coupling limit. We will concentrate therefore on the locus  $\mathcal{C}_1 = \{b = 1\}$ , which by the mirror map implies  $t_2 = 0$  [35]. On the type IIA side this locus is associated to the appearance of a curve of  $\mathbf{Z}_2$  quotient singularities,  $\mathcal{C}_{Z_2}$ , corresponding to blowing down the exceptional divisor whose size controls the dilaton modulus  $b$ . In [36, 35] this singularity was interpreted as a non-perturbative enhancement of symmetry in the  $U(1)$  vector field associated to the dilaton,  $U(1) \rightarrow SU(2)$ , together with the appearance of massless hypermultiplets in the adjoint representation of  $SU(2)$ . The number of massless adjoint hypermultiplets is given by the genus  $g$  of the curve of singularities  $\mathcal{C}_{Z_2}$ . The monodromy around  $\mathcal{C}_1$  can be resummed in the  $2 \times 2$  matrix

$$\begin{pmatrix} -1 & 2(g-1) \\ 0 & -1 \end{pmatrix}, \quad (5.6)$$

acting on a column vector whose first entry we denote by  $t_2^D$ , and the second is given by the special coordinate  $t_2$ .

Since  $t_2$  behaves as the special coordinate version of the  $N = 4$  coupling constant  $\tau$ , and  $\tau$  is an intrinsic parameter of  $\mathcal{C}^{CY}$ , it should be possible to describe the different monodromy matrices (5.6) for the spaces  $A$ ,  $B$ ,  $C$  and  $D$  in an unified way. In order to do this, we will allow the freedom to change the normalization of  $t_2^D$ , while preserving that of  $t_2$ . Inspired by (5.5) we consider the change  $t_2^D \rightarrow t_2^D/k$ , transforming (5.6) into

$$\begin{pmatrix} -1 & 2(g-1)/k \\ 0 & -1 \end{pmatrix}. \quad (5.7)$$

For the four models we are considering the following relation between the genus  $g$  of the curve of singularities at  $\mathcal{C}_1$ , and the level  $k$  dictated by the modular properties of the mirror map, is verified (see Appendix C)

$$\frac{g-1}{k} = 1. \quad (5.8)$$

Substituting this into (5.7) we obtain a single representation of the  $\mathcal{C}_1$  monodromy for the four Calabi-Yau spaces. It is given by  $PT^{-2}$ , where now  $T$  denote the  $Sl(2; Z)$  generator and  $P = -1$ .

It must be noticed that a  $T^{-2}$  monodromy is the one we will get for the Donagi-Witten curve at the singular locus  $\tau = 0$ <sup>11</sup>, which by the map between field theory and string variables (2.19) corresponds to  $\mathcal{C}_1$ . Namely, encoding the monodromy around  $\tau = 0$  through its action on a column vector with entries 1 and  $\tau$ , the matrix  $T^{-2}$  implies  $\tau^D \rightarrow \tau^D + 2$ , with  $\tau^D = -1/\tau$ . We observe that, although the  $\mathcal{C}^{CY}$  curve can not describe the physics at  $\mathcal{C}_1$  because is missing the parameter  $g$  and the Weyl generator  $P$ , it can however explain its underlying structure. Indeed, using  $\mathcal{C}^{CY}$  and the Kac-Moody level information, it is possible to reproduce in a natural way (5.6). We can also put in correspondence the monodromies for the Calabi-Yau and field theory loci  $\mathcal{C}_\infty$  and  $\tau = i\infty$ . Taking into account the double covering of the string moduli space that the field theory is doing, the monodromy  $ST^{-2}S^{-1}$  around  $\tau = i\infty$ , producing  $\tau \rightarrow \tau + 2$ , becomes the  $\mathcal{C}_\infty$  monodromy [38, 40]  $t_2 \rightarrow t_2 + 1$ .

## 6 Final Comments.

Ultraviolet finite  $N = 2$  supersymmetric gauge theories are described [2] in terms of the Higgs vacuum expectation values, the bare coupling constant  $\tau$  and the masses of their matter content. In order to relate these theories with a string compactification on a Calabi-Yau threefold, we can not follow the standard path of performing the heterotic point particle limit. In this paper we have shown that  $N = 2$  gauge theories with  $N = 4$  matter content are associated with the global structure of  $K3$ -fibration threefolds. Moreover the global geometrical information on  $K3$ -fibrations reproduces all the ingredients used in the characterization [15] of the integrability of these ultraviolet finite  $N = 2$  theories. For the case of  $SU(2)$ , our philosophy has been to consider the gauge theory with  $N = 4$  matter as encoding the common geometrical structure of  $K3$ -fibrations sharing the same  $h_{2,1}$  Hodge number, and structure of the complex deformations moduli space.

The special role of a ultraviolet finite  $N = 2$   $SU(2)$  gauge theory, in particular the case  $2N_c = N_f$ , appears in a different context [52] as proves [53] of F-theory [54] backgrounds. The prove approach is related to duality properties of type II strings, while our approach should be related to strong-weak coupling duality of the heterotic string.

### Acknowledgments

We thank conversations with K. Landsteiner, W. Lerche, A. Sen, C. Vafa and N. Warner. This work is partially supported under grant OFES contract number 930083 by European Community grant ERBCHRXCT920069 and by PB92-1092. The work of E. L. is supported by C.A.P.V. fellowship. The work of R. H. is supported by U.A.M. fellowship.

---

<sup>11</sup>The dependence on  $\tau$  of the Donagi-Witten construction is built using the Weierstrass invariants  $e_i(\tau)$  (see footnote 3), which are modular functions of  $\Gamma(2)$ . Therefore the monodromy matrices for the singular loci  $\tau = 0, 1, i\infty$  in (2.18), have to be conjugate to  $T^2$  or  $T^{-2}$ .

## Appendices.

### A Isogeny Between Cubic and Quartic Curves.

We consider an elliptic curve, given by the quartic

$$y^2 = (x^2 + a)^2 - b^2, \quad (\text{A.1})$$

for certain constants  $a$  and  $b$ . By defining  $x' = x^2 + a$ ,  $y' = yx$  we can convert (A.1) into an isogenic cubic

$$y'^2 = (x' - a)(x'^2 - b^2). \quad (\text{A.2})$$

This process amounts to quotienting by the symmetry  $x \rightarrow -x$  of (A.1), and blowing down the line  $(x = 0, y)$  in order to get again an elliptic curve.

It is immediate to see that the abelian differential for quartic and cubic are related by

$$\frac{dx'}{y'} = 2 \frac{dx}{y}. \quad (\text{A.3})$$

Let us choose a basis of cycles of (A.2) in the following form:  $\gamma'_1$  surrounding the branch points  $\pm b$ , and  $\gamma'_2$  surrounding the branch points  $b$  and  $-a$ . With this choice the cycles  $\gamma_1$  and  $\gamma_2$ , transformed of  $\gamma'_1$  and  $\gamma'_2$  respectively, satisfy

$$\gamma_1 = \gamma'_1, \quad \gamma_2 = 2\gamma'_2. \quad (\text{A.4})$$

Therefore the difference between quartic and cubic, from the point of view of the periods, reduces to a change in normalization

$$\int_{\gamma_1} \omega = \frac{1}{2} \int_{\gamma'_1} \omega', \quad \int_{\gamma_2} \omega = \int_{\gamma'_2} \omega', \quad (\text{A.5})$$

for  $\omega, \omega'$  the respective abelian differentials. This is precisely the difference between the two curves considered in [1], [2] for solving  $SU(2)$   $N = 2$  Yang-Mills.

The quartic form (A.1) reproduces the structure, for the particular case  $SU(2)$ , of the hyperelliptic curves for higher rank gauge groups proposed in [3, 4]

$$y^2 = P_{SU(N)}(x; u_i)^2 - \Lambda^{2N}, \quad (\text{A.6})$$

with  $u_i$  the  $SU(N)$  Casimirs.

We should also note that the Jacobi invariants for (A.1) and (A.2) differ. This can be seen by transforming both curves to the standard form

$$y^2 = x(x-1)(x-\lambda), \quad (\text{A.7})$$

and comparing the  $\lambda$ -parameters obtained. We have

$$\lambda = \frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}, \quad \lambda' = \frac{a + b}{a - b}. \quad (\text{A.8})$$

## B The Map Between $SU(3)$ Moduli Spaces.

In [23] the Calabi-Yau  $\mathbb{P}_{\{1,1,2,8,12\}}^4$  [24] was proposed as the type II dual for the string embedding of  $N = 2$   $SU(3)$  Yang-Mills theory. In this Appendix, extending part of the analysis done in previous sections for  $SU(2)$ , we will build a map between the moduli space of  $N=2$   $SU(3)$  Yang-Mills theory with adjoint matter and the moduli space of the hypersurface  $\mathbb{P}_{\{1,1,2,8,12\}}^4$  [24]. The analysis will be restricted to the weak coupling sections  $\tau = i\infty$  and  $b = 0$  of the respective moduli spaces.

The discriminant locus of the genus two curve describing  $SU(3)$  with adjoint matter [15], in the limit  $\tau = i\infty$ , is given by

$$\Delta_{SU(3)} = (4\hat{u}^3 - 27\hat{v}^2)(4(\hat{u} - 1)(\hat{u} - 4)^2 - 27\hat{v}^2), \quad (\text{B.1})$$

where  $\hat{u}$  and  $\hat{v}$  are the quadratic and cubic Casimir expectation values respectively, normalized by the adjoint hypermultiplet mass. The first factor corresponds to a classical enhancement of  $SU(2)$  gauge symmetry. We will denote its two branches by

$$\mathcal{C}^\pm = \{3\sqrt{3}\hat{v} = \pm 2\hat{u}^{3/2}\}. \quad (\text{B.2})$$

The second factor goes to zero when some component of the  $N=4$  hypermultiplet become massless; we will also divide it into

$$\mathcal{C}_h^\pm = \{3\sqrt{3}\hat{v} = \pm 2(\hat{u} - 4)\sqrt{\hat{u} - 1}\}. \quad (\text{B.3})$$

If we now concentrate on the  $K3$ -fibration structure of the  $\mathbb{P}_{\{1,1,2,8,12\}}^4$  [24] Calabi-Yau manifold [26]

$$\begin{aligned} W = & \frac{1}{24}(z + \frac{b}{z} + 2) + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 + \\ & + \frac{1}{6\sqrt{c}}(x_0x_3)^6 + \left(\frac{a}{\sqrt{c}}\right)^{1/6} x_0x_3x_4x_5 = 0, \end{aligned} \quad (\text{B.4})$$

we notice that the  $K3$ -fiber becomes singular at six points ( $z = e_i^\pm$ ,  $i = 0, 1, 2$ ) of the  $\mathbb{P}^1$  base space

$$\begin{aligned} e_0^\pm &= -1 \pm \sqrt{1 - b}, \\ e_1^\pm &= \frac{1 - c \pm \sqrt{(1 - c)^2 - bc^2}}{c}, \\ e_2^\pm &= \frac{(1 - a)^2 - c \pm \sqrt{((1 - a)^2 - c)^2 - bc^2}}{c}. \end{aligned} \quad (\text{B.5})$$

Merging of these points leads to two conifold-type loci, when  $e_i^+ = e_i^-$  for  $i = 1, 2$ , and a strong dilaton locus  $\mathcal{C}_1$ , when  $e_0^+ = e_0^-$ :

$$\begin{aligned}\mathcal{C}_C^{(1)} &= \{(1-c)^2 - bc^2 = 0\}, \\ \mathcal{C}_C^{(2)} &= \{((1-a)^2 - c)^2 - bc^2 = 0\}, \\ \mathcal{C}_1 &= \{b-1 = 0\}.\end{aligned}\tag{B.6}$$

The weak gravity locus

$$\mathcal{C}_\infty = \{b = 0\},\tag{B.7}$$

corresponds to  $0 = e_0^+ = e_1^+ = e_2^+$ . We can also consider the merging  $e_0^\pm = e_i^\pm$ ,  $i = 1, 2$ , which happens respectively at

$$\begin{aligned}\mathcal{C}_0^+ &= \{c = \infty \quad (\forall a/\sqrt{c})\}, \\ \mathcal{C}_0^- &= \{a - 1 = 0\},\end{aligned}\tag{B.8}$$

We will restrict now to the  $(a, c)$  section of the Calabi-Yau moduli space at  $b = 0$ . In the spirit of (2.19), we can propose the following map between the  $(u, v)$ -plane at  $\tau = i\infty$ , and the Calabi-Yau moduli space at  $b = 0$

$$1 - c = \frac{-2\hat{u}^{3/2} + 3\sqrt{3}\hat{v}}{2(\hat{u} - 4)\sqrt{\hat{u} - 1} + 3\sqrt{3}\hat{v}}, \quad 1 - \frac{c}{(1-a)^2} = \frac{2\hat{u}^{3/2} + 3\sqrt{3}\hat{v}}{2(\hat{u} - 4)\sqrt{\hat{u} - 1} - 3\sqrt{3}\hat{v}}.\tag{B.9}$$

This map is built by requiring that the loci  $\mathcal{C}^\pm$  get mapped into the two conifold  $\mathcal{C}_C^{(1,2)}$ , and the hypermultiplet loci  $\mathcal{C}_h^\pm$  go into  $\mathcal{C}_0^\pm$ .

The point  $(a = 0, c = 1)$ , one of the intersections between  $\mathcal{C}_C^{(1)}$  and  $\mathcal{C}_C^{(2)}$  at  $b = 0$ , corresponds by heterotic-type II duality to a perturbative  $SU(3)$  enhancement of symmetry point [23]. A first check of (B.9) is that, sending the mass of the adjoint hypermultiplet  $m \rightarrow \infty$  and identifying again  $m^2$  with the string tension  $(\alpha')^{-1}$ , it reproduces the point particle limit map for  $SU(3)$  [25, 26]

$$\begin{aligned}a &= -2(\alpha u)^{3/2}, \\ c &= 1 - \alpha^{3/2}(-2u^{3/2} + 3\sqrt{3}v).\end{aligned}\tag{B.10}$$

The scale governing the strong coupling effects of the  $N = 2$   $SU(3)$  Yang-Mills theory with adjoint matter is given by  $\Lambda^6 \sim e^{2\pi i\tau} m^6$  [15]. Assuming the relation (2.21) between the dilaton modulus  $b$  and the  $N = 4$  coupling constant  $\tau$ , this implies the double scaling identification required to perform the point particle limit [25]

$$b = \alpha^3 \Lambda^6.\tag{B.11}$$

It would be interesting to see if the map (B.9) can help to locate, by comparison with the field theory, Argyres-Douglas points [55] in string theory.

## C Summary of $A$ , $B$ , $C$ and $D$ models.

The Picard-Fuchs equations that govern the Kähler structure deformations of the Calabi-Yau manifolds  $A : \mathbb{P}^4_{\{1,1,2,2,6\}}$  [12],  $B : \mathbb{P}^4_{\{1,1,2,2,2\}}$  [8],  $C : \mathbb{P}^5_{\{1,1,2,2,2,2\}}$  [4, 6] and  $D : \mathbb{P}^6_{\{1,1,2,2,2,2,2\}}$  [4, 4, 4], consist of two differential operators of second and third order [39, 40]. The second order operator is common to the four manifolds

$$L^{(1)} = 4\theta_b^2 - b(2\theta_b - \theta_c + 1)(2\theta_b - \theta_c), \quad (\text{C.1})$$

while they differ in the third order one

$$\begin{aligned} A : \quad L^{(2)} &= \theta_c^2 (\theta_c - 2\theta_b) - c(\theta_c + 5/6)(\theta_c + 1/2)(\theta_c + 1/6), \\ B : \quad L^{(2)} &= \theta_c^2 (\theta_c - 2\theta_b) - c(\theta_c + 3/4)(\theta_c + 1/2)(\theta_c + 1/4), \\ C : \quad L^{(2)} &= \theta_c^2 (\theta_c - 2\theta_b) - c(\theta_c + 2/3)(\theta_c + 1/2)(\theta_c + 1/3), \\ D : \quad L^{(2)} &= \theta_c^2 (\theta_c - 2\theta_b) - c(\theta_c + 1/2)^3. \end{aligned} \quad (\text{C.2})$$

We are using the normalization conventions for the moduli parameters  $(b, c)$  implied in (2.1) and (2.3). The second order operator (C.1), in the point particle limit defined by (2.13), becomes precisely the Picard-Fuchs operator for  $N = 2$   $SU(2)$  Yang-Mills theory [25].

These four spaces develop a curve of  $\mathbf{Z}_2$  quotient singularities,  $\mathcal{C}_{Z_2}$ , at  $\mathcal{C}_1 = \{b = 1\}$ . This singularity comes from an ambient singularity in the weighted projective spaces in which they are immersed. Namely, in our case points  $(x_1, x_2, x_3, x_4, \dots)$  and  $(\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_4, \dots)$  should be identified, fixing, for  $\lambda = -1$ , the hyperplane of orbifold points  $x_1 = x_2 = 0$ . The intersection of this hyperplane with the defining hypersurface, or complete intersection, of the Calabi-Yau manifolds determines  $\mathcal{C}_{Z_2}$  in each case. The genus of  $\mathcal{C}_{Z_2}$  is  $g = 2, 3, 4, 5$  for the spaces  $A, B, C$  and  $D$  respectively [38, 36].

The exceptional divisor whose size controls the dilaton modulus  $b$ , is introduced in order to resolve the mentioned singularity. However for the value  $b = 1$ , the manifolds develop the original curve of  $\mathbf{Z}_2$  quotient singularities. This structure can be read from the following piece in the defining expression of their mirror manifolds

$$W^* = \frac{1}{2n}x_1^{2n} + \frac{1}{2n}x_2^{2n} + \frac{1}{\sqrt{b}n}(x_1x_2)^n + \dots \quad (\text{C.3})$$

Changing variables to  $x_1/x_2 = z^{1/n}b^{-1/2n}$  and  $x_1^2 = x_0z^{1/n}$  [26], the  $K3$ -fibration structure (2.12) for manifolds  $A, B, C$  and  $D$  becomes manifest. The  $K3$ -fiber of their mirror, which for model  $A$  is given in (2.5), is given by

$$\begin{aligned} B : \quad & x_0^4/4 + x_3^4/4 + x_4^4/4 + x_5^4/4 + \hat{c}^{-1/4}x_0x_3x_4x_5 = 0, \\ C : \quad & x_0^2/2 + x_3^2/2 + x_4^2/2 + \hat{c}^{-1/5}x_5x_6 = 0 \\ & x_5^3/3 + x_6^3/3 + \hat{c}^{-1/5}x_0x_3x_4 = 0, \end{aligned}$$

$$\begin{aligned}
D : \quad & x_0^2/2 + x_3^2/2 + \hat{c}^{-1/6}x_4x_5 = 0 \\
& x_4^2/2 + x_5^2/2 + \hat{c}^{-1/6}x_6x_7 = 0 \\
& x_6^2/2 + x_7^2/2 + \hat{c}^{-1/6}x_0x_3 = 0,
\end{aligned} \tag{C.4}$$

with  $\hat{c}(z; b, c)$  defined in (2.4). The Picard-Fuchs equation governing the above  $K3$  surfaces can be obtained from the third order differential operator of the threefold (C.2), by sending  $b \rightarrow 0$ .

| Model | Manifold                                    | $h_{2,1}$ | Genus $g$ | Mirror modular group |
|-------|---|-----------|-----------|----------------------|
| $A$   | $\mathbb{P}^4_{\{1,1,2,2,6\}}[12]$          | 128       | 2         | $\Gamma$             |
| $B$   | $\mathbb{P}^4_{\{1,1,2,2,2\}}[8]$           | 86        | 3         | $\Gamma_0(2)_+$      |
| $C$   | $\mathbb{P}^5_{\{1,1,2,2,2,2\}}[4, 6]$      | 68        | 4         | $\Gamma_0(3)_+$      |
| $D$   | $\mathbb{P}^6_{\{1,1,2,2,2,2,2\}}[4, 4, 4]$ | 58        | 5         | $\Gamma_0(4)_+$      |

The four manifolds we are considering have common Yukawa couplings [39, 40]

$$K_{ccc} = \frac{1}{c^3 \Delta_C}, \quad K_{ccb} = \frac{1-c}{2c^2 b \Delta_C}, \quad K_{cbb} = \frac{2c-1}{4cb \Delta_C \Delta_1}, \quad K_{bbb} = \frac{1-c+b-3cb}{8b^2 \Delta_C \Delta_1^2}, \tag{C.5}$$

where  $\Delta_C = (1-c)^2 - c^2 b$  is the conifold factor in the discriminant, and  $\Delta_1 = 1-b$ . To finish, let us notice that the Yukawa couplings of Calabi-Yau three-folds are determined by the Picard-Fuchs equations coefficients  $f_l^{k_1, \dots, k_n}$

$$L^{(l)} = \sum_{k_1, \dots, k_n} f_l^{k_1, \dots, k_n} \partial_{c_1}^{k_1} \dots \partial_{c_n}^{k_n}, \tag{C.6}$$

with  $\sum k_i \geq 2$  [39] ( $c_i, i = 1, \dots, n$  includes all Calabi-Yau moduli parameters and  $l$  labels the set of Picard-Fuchs differential operators). This information is retained in the modified Picard-Fuchs operator (4.4) that governs the Seiberg-Witten differential for  $\mathcal{C}^{CY}$ .



## References

- [1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087.
- [2] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484, hep-th/9408099.
- [3] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048.
- [4] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931, hep-th/9411057.
- [5] U. H. Danielsson and B. Sundborg, Phys. Lett. B358 (1995) 273, hep-th/9504102.
- [6] A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 283, hep-th/9505075.
- [7] P. Argyres, R. Plesser and A. Shapere, Phys. Rev. Lett. 75 (1995) 1699, hep-th/9505100.
- [8] A. Brandhuber and K. Landsteiner, Phys. Lett. B358 (1995) 73, hep-th/9507008.
- [9] J. A. Minahan and D. Nemeschansky, hep-th/9507032.
- [10] P. Argyres and A. Shapere, hep-th/9509175.
- [11] A. Hanany, hep-th/9509176.
- [12] A. Gorskii, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466, hep-th/9505035.
- [13] E. Martinec and N. P. Warner, Nucl. Phys. B459 (1996) 97, hep-th/9509161.
- [14] T. Nakatsu and K. Takasaki, Mod. Phys. Lett. A11 (1996) 157, hep-th/9509162.
- [15] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299, hep-th/9510101.
- [16] E. Martinec, Phys. Lett. B367 (1996) 91, hep-th/9510204.
- [17] E. Martinec and N. P. Warner, hep-th/9511052.
- [18] H. Itoyama and A. Morozov, hep-th/9511126 and hep-th/9512161.
- [19] C. Ahn and S. Nam, hep-th/9603028.
- [20] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, hep-th/9603140.
- [21] C. Gómez, R. Hernández and E. López, hep-th/9604057.
- [22] I. M. Krichever and D. H. Phong, hep-th/9604199.

- [23] S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505105.
- [24] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, hep-th/9505162.
- [25] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B459 (1996) 537, hep-th/9508155.
- [26] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, hep-th/9604034.
- [27] R. Donagi and E. Markman, alg-geom/9408004.
- [28] N. Hitchin, Duke Math. J. 54 (1987) 91.
- [29] C. Hull and P. Townsend, Nucl. Phys. B438 (1995) 109, hep-yh/9410167.
- [30] E. Witten, Nucl. Phys. B443 (1995) 85, hep-th/9503124.
- [31] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B357 (1995) 313, hep-th/9506112.
- [32] P. S. Aspinwall and J. Louis, Phys. Lett. B369 (1996) 233, hep-th/9510234.
- [33] C. Gómez, R. Hernández and E. López, hep-th/9512017.
- [34] I. Antoniadis and H. Partouche, Nucl. Phys. B460 (1996) 470, hep-th/9509009.
- [35] A. Klemm and P. Mayr, hep-th/9601014.
- [36] S. Katz, D. R. Morrison and M. R. Plesser, hep-th/9601108.
- [37] B. R. Greene and M. R. Plesser, Nucl. Phys. B338 (1990) 15.
- [38] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B416 (1994) 481, hep-th/9308083.
- [39] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, Comm. Math. Phys. 167 (1995) 301, hep-th/9308122.
- [40] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, Nucl. Phys. B433 (1995) 501-554, hep-th/940605.
- [41] B. H. Lian and S. T. Yau, hep-th/9411234.
- [42] A. Strominger, Nucl. Phys. B451 (1995) 96, hep-th/9504090.
- [43] P. S. Aspinwall, B. R. Greene and D. R. Morrison, Nucl. Phys. B420 (1994) 184-242, hep-th/9311042.
- [44] P. Candelas, X. de la Ossa, P. Green and L. Parkes, Nucl. Phys. B359 (1991), 21.

- [45] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Commun. Math. Phys. 165 (1994) 311, hep-th/9309140.
- [46] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724, hep-th/9510017.
- [47] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, hep-th/9507115.
- [48] V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. B357 (1995) 71-75, hep-th/9506110.
- [49] P. Ginsparg, Phys. Lett. B197 (1987) 139.
- [50] B. de Wit, V. Kaplunovsky, J. Louis and D. Luest, Nucl. Phys. B451 (1995) 53-95, hep-th/9504006.
- [51] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B447 (1995) 35-61, hep-th/9504034.
- [52] A. Sen, hep-th/9605150.
- [53] T. Banks, M. R. Douglas and N. Seiberg, hep-th/9605199.
- [54] C. Vafa, hep-th/9602022.
- [55] P. C. Argyres and M. R. Douglas, Nucl. Phys. B448 (1995) 93, hep-th/9505062.