S-Duality and the Calabi-Yau Interpretation of the $N\!=\!4$ to $N\!=\!2$ Flow.

César Gómez[a], **Rafael Hernández**[b] **and Esperanza López**[a][1]

[a]Instituto de Matemáticas y Física Fundamental, CSIC
Serrano 123, 28006 Madrid, Spain

[b]Departamento de Física Teórica, C-XI, Universidad Autónoma de Madrid
Cantoblanco, 28049 Madrid, Spain

Abstract

The action of the S-duality $Sl(2,Z)$ group on the moduli of the Calabi-Yau manifold $W\mathbb{P}^{12}_{11226}$ appearing in the rank two dual pair $(K^3 \times T^2/W\mathbb{P}^{12}_{11226})$ is defined by interpreting the $N\!=\!4$ to $N\!=\!2$ flow, for $SU(2)$ supersymmetric Yang-Mills, in terms of the Calabi-Yau moduli. The different singularity loci are mapped in a one to one way, and the ($N\!=\!2$ limit/point particle limit) is obtained in both cases by the same type of blow up. Moreover, it is shown that the S-duality group permutes the different singularity loci of the moduli of $W\mathbb{P}^{12}_{11226}$. We study the transformation under S-duality of the Calabi-Yau Yukawa couplings.

---

[1]Address after January 1996, Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030.

1   Introduction.

The discovery of heterotic-type II dual pairs [1]-[5] opens the possibility to enter into the realm of string non perturbative effects. The first direct application of heterotic-type II dual pairs will be its use to derive the exact quantum moduli space of quantum field theories defined as the low energy limit of heterotic string compactifications. This has been done for the dual pair defined by $(K^3 \times T^2/W\mathbb{P}^{12}_{11226})$ [5] by analizing the point particle limit [6]-[9] of the moduli of complex structures of $W\mathbb{P}^{12}_{11226}$, reproducing in [9] the exact results of the Seiberg-Witten solution for pure $N = 2$ $SU(2)$ supersymmetric Yang-Mills [10]. In this letter, we will proceed in a somewhat opposite way: instead of seeking the field theory point particle limit of the string, we will try to read off the string theory directly from the field theory. To do so, we will focus on pure $N = 2$ $SU(2)$ supersymmetric Yang-Mills theory from the point of view of the $N = 4$ to $N = 2$ flow [11], i.e., we will work with the $N = 2$ theory possesing $N = 4$ matter content. In this way, we start with an *extended* moduli space, parametrized by $\hat{u}$ ($\hat{u} \equiv \frac{u}{f}$, with u the Seiberg-Witten moduli variable, and $f = \frac{1}{4}m^2$ the soft breaking mass term) and τ (the $N = 4$ moduli). It will be this space the one we will put in correspondence with the moduli of complex structures of $W\mathbb{P}^{12}_{11226}$ [12, 13], mapping, in a one to one way, the different singular loci of the two spaces. In both cases, the pure $N = 2$ theory can be derived as the blow up of a weak coupling ($\tau \to \infty/S \to \infty$) singular point.

The $Sl(2, Z)$ S-duality group, acting on the $N = 4$ moduli τ, induces duality transformations on a double covering of the moduli of complex structures of $W\mathbb{P}^{12}_{11226}$. We study the transformations of the Yukawa couplings with respect to S-duality.

2 $N = 4$ to $N = 2$ Flow and Duality.

Let us consider $N = 2$ $SU(2)$ supersymmetric Yang-Mills with one hypermultiplet in the adjoint representation. The curve describing this model is given by [11][2]:

$$y^2 = (x - e_1(\tau)\tilde{u} - e_1^2(\tau)f)(x - e_2(\tau)\tilde{u} - e_2^2(\tau)f)(x - e_3(\tau)\tilde{u} - e_3^2(\tau)f), \qquad (2.1)$$

where $f = \frac{1}{4}m^2$, with m the mass of the hypermultiplet. In the massless limit we recover the $N = 4$ curve of the elliptic moduli τ. After the finite renormalization $u = \tilde{u} + \frac{1}{2}e_1(\tau)f$

---

[2]The Weierstrass invariants e_i can be defined in the terms of Jacobi theta functions:

$$e_1 = \frac{1}{3}(\theta_2^4(0, \tau) + \theta_3^4(0, \tau)), e_2 = -\frac{1}{3}(\theta_1^4(0, \tau) + \theta_3^4(0, \tau)), e_3 = \frac{1}{3}(\theta_1^4(0, \tau) - \theta_2^4(0, \tau)).$$

[11], and the *double scaling* limit defined by

$$\lim_{\substack{\tau \to i\infty \\ m \to \infty}} 2q^{1/2}m^2 = \Lambda^2, \tag{2.2}$$

where $q \equiv e^{2\pi i\tau}$, we obtain, from (2.1), the Seiberg-Witten solution for pure $N=2$ $SU(2)$ supersymmetric Yang-Mills with $u = \langle Tr\phi^2 \rangle$. By an affine transformation we can put the curve (2.1) in the form

$$y^2 = x(x-1)(x - \lambda(\tilde{u}, \tau, f)), \tag{2.3}$$

where

$$\lambda(\tilde{u}, \tau, f) = \frac{(e_3 - e_1)(\tilde{u} + f(e_3 + e_1))}{(e_2 - e_1)(\tilde{u} + f(e_2 + e_1))}. \tag{2.4}$$

The elliptic curve (2.3) is characterized by the j-invariant

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}, \tag{2.5}$$

which satisfies the relations $j(\lambda) = j(\frac{1}{\lambda}) = j(1 - \lambda)$.

Using the transformation laws of the $e_i(\tau)$ as modular forms of weight two with respect to Γ_2, we easily get the following set of duality relations:

$$1 - \lambda(\tilde{u}, \tau, f) = \lambda(\tilde{u}^M, \frac{-1}{\tau}, f) \equiv \lambda(\tilde{u}', \tau, f), \tag{2.6}$$

$$\frac{1}{\lambda(\tilde{u}, \tau, f)} = \lambda(\tilde{u}, \tau + 1, f) \equiv \lambda(\tilde{u}'', \tau, f) \tag{2.7}$$

where we have defined

$$\tilde{u}^M = \tau^2 \tilde{u}, \qquad \tilde{u}' = \frac{a - f_1(\tilde{u})b}{(e_2 - e_1)f_1(\tilde{u}) - (e_3 - e_1)}$$

$$\tilde{u}'' = \frac{a - f_2(\tilde{u})b}{(e_2 - e_1)f_2(\tilde{u}) - (e_3 - e_1)}, \tag{2.8}$$

with

$$a = f(e_3^2 - e_1^2), \qquad b = f(e_2^2 - e_1^2)$$

$$f_1(\tilde{u}) = \frac{1}{\lambda(\tilde{u}, \tau, f)}, \qquad f_2(\tilde{u}) = 1 - \lambda(\tilde{u}, \tau, f). \tag{2.9}$$

Notice from the transformation rule for $\tilde{u}$ in (2.8) that $\tilde{u}$ transforms as a modular form of weight two.

In the double scaling limit defined by (2.2) we get

$$\lim_{\substack{\tau \to i\infty \\ m^2 \to \infty}} \lambda(\tilde{u}, \tau, f) = \frac{u + \Lambda^2}{u - \Lambda^2} \equiv \lambda_{SW}(u), \tag{2.10}$$

2

with $\lambda_{SW}(u)$ the value of λ obtained from the Seiberg-Witten curve $y^2 = (x + \Lambda^2)(x - \Lambda^2)(x - u)$ for pure $N=2$ supersymmetric Yang-Mills. Moreover, in this limit we notice, using relations (2.6) and (2.7), that the $Sl(2, Z)$ duality transformations $T : \tau \to \tau+1$ and $S : \tau \to \frac{-1}{\tau}$ induce, on the Seiberg-Witten plane, the transformations $u \to -u$ and $u \to \frac{3\Lambda^4 + \Lambda^2 u}{\Lambda^2 - u}$, respectively. These transformations, which generate the group $\Gamma_W \equiv Sl(2, Z)/\Gamma_2$ [14], interchange the singularities of the pure $N=2$ theory. In other words, we observe how the $Sl(2, Z)$ duality transformations on the $N=4$ moduli τ permute, in the double scaling limit, the phases of pure $N=2$ theory [15].

Next, let us study the singularity locus for the curve (2.1). We will work in the $(\hat{u}, \tau)$-plane, with $\hat{u} \equiv \frac{u}{f}$; the use of this dimensionless variable will prove important later on, in the correspondence with the Calabi-Yau moduli space. The following set of regions can then be differentiated (see Figure 1):

$$
\begin{aligned}
\mathcal{C}_\infty &\equiv \{\tau = i\infty\}, \\
\mathcal{C}_0 &\equiv \{\hat{u}(\tau) = \tfrac{3}{2}e_1(\tau)\}, \\
\mathcal{C}_c^{(1)} &\equiv \{\hat{u}(\tau) = (e_3 + \tfrac{1}{2}e_1)(\tau)\}, \\
\mathcal{C}_c^{(2)} &\equiv \{\hat{u}(\tau) = (e_2 + \tfrac{1}{2}e_1)(\tau)\}, \\
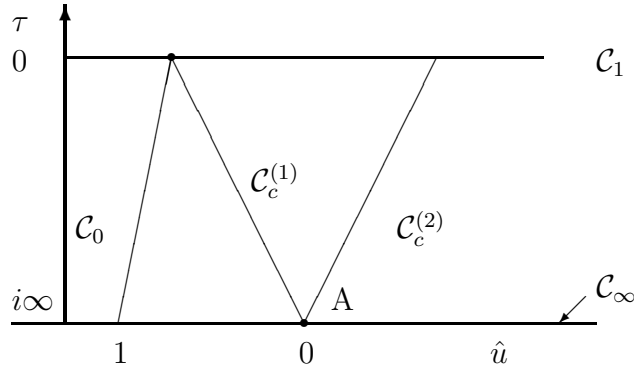\mathcal{C}_1 &\equiv \{\tau = 0\}.
\end{aligned}
\tag{2.11}
$$

Figure 1: The different loci described in (2.11). Notice that $e_2 = e_3$ at $\tau = i\infty$, and $e_1 = e_3$ at $\tau = 0$.

Notice that the loci $\mathcal{C}_0$, $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ correspond to the singularities of the curve (2.1) when written in $\hat{u}$ variables. It can be easily proved that the duality transformations (2.6) and (2.7) permute among themselves the singular loci (2.11). Namely, the transformation S permutes $\mathcal{C}_1$ with $\mathcal{C}_\infty$, and $\mathcal{C}_0$ with $\mathcal{C}_c^{(2)}$, while the locus $\mathcal{C}_c^{(1)}$ is mapped into itself; the transformation T permutes the locus $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$, and maps into itself the locus $\mathcal{C}_0$.

The loci $\mathcal{C}_0$, $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ can be described using the language of double ramified coverings introduced in [15] in connection with integrable models. In fact, for the double

3

ramified covering defined by

$$\begin{cases} y^2 &=& (x-e_1)(x-e_2)(x-e_3) \\ 0 &=& t^2 - x + \tilde{u}, \end{cases} \qquad (2.12)$$

the loci $\mathcal{C}_0$, $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ are defined by the relation $\tilde{u}(\tau)$, characterizing the values of $\tilde{u}$ for which (2.12) becomes a double *unramified* covering[3].

3 Blowing up and the $N=2$ Limit.

Let us now consider more carefully the neighbourhood of point $A \equiv (\hat{u}=0, \tau=i\infty)$ in the $(\hat{u}, \tau)$-plane depicted in Figure 1. Introducing a new coordinate $\epsilon \equiv 8q^{1/2}$, the loci $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ in the neighbourhood of the point A can be described in $(\epsilon^2, \hat{u})$ coordinates by the parabole $\epsilon^2 = \hat{u}^2$. This parabole is tangent at the point A to the locus $\mathcal{C}_\infty = \{\tau=i\infty\}$. Using standard techniques [16] we can blow up this tangency point. To do it a double blow up is needed: in the first blow up we introduce the coordinate $v = \frac{\epsilon^2}{\hat{u}}$, which transforms the tangency into a crossing between the curve $v = \hat{u}$ and $\mathcal{C}_\infty$; in the second step, we blow up this crossing introducing the coordinate $w = \frac{v}{\hat{u}}$, mapping the parabole to the line $w = 1$. The exceptional divisors E_1 and E_2, and coordinates introduced by this blow up are described in Figure 2.
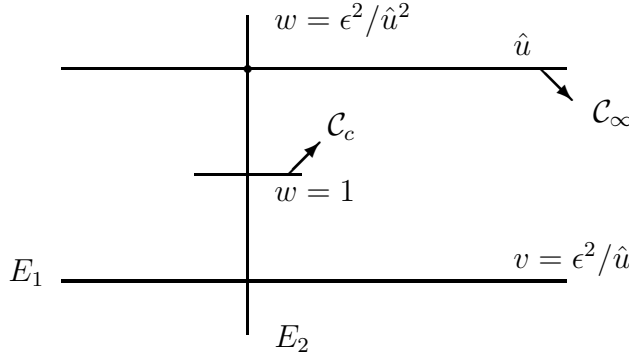

Figure 2: The double blow up of the tangency point A of Figure 1.

Defining $\Lambda^2 = 8q^{1/2}f$ [10] we observe that the coordinate on E_1 is given by $\Lambda^4/u^2 \equiv 1/\tilde{u}^2$.

Some comments are now necessary for a proper understanding of the physical meaning of the above construction. First of all, the point A in the $(\hat{u}, \tau)$-plane we are blowing up

---

[3]A massive vacuum, in the notation of reference [15].

4

is a point where the value of $\lambda(\hat{u}, \tau)$ is undetermined (see equation (2.4)). In order to give a precise meaning to the double scaling limit in the $(\hat{u}, \tau)$-plane, we need to blow up the point A, and to define λ_{SW} (see equation (2.10)) as a function on the exceptional divisor parametrized by w. Notice that in the double scaling limit $\tau \to i\infty, m^2 \to \infty$ $\hat{u}$ goes to $\hat{u} = 0$ for any value of $u = \langle Tr\phi^2 \rangle$. In other words, the only point in the $(\hat{u}, \tau)$-plane of the $N{=}2$ theory with $N{=}4$ matter content that can have a pure $N{=}2$ interpretation is the enhancement of symmetry singular point $(\hat{u} = 0, \tau = i\infty)$. By means of the blow up of this singular point we create an extra divisor which represents the quantum moduli of the pure $N{=}2$ theory.

Secondly, it should be noticed that in $(\epsilon, \hat{u})$ coordinates the curve defined by the locus $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ in the neighbourhood of the point A can be described by $\hat{u} = \pm\epsilon$. This crossing can be regularized by a single blow up, with the coordinate on the exceptional divisor given by $w = \frac{\epsilon}{\hat{u}} = \frac{1}{\hat{u}}$, and the loci $\mathcal{C}_c^{(1)}$, $\mathcal{C}_c^{(2)}$ mapped into the lines $w = \pm 1$, which represents a more natural description of the Seiberg-Witten plane. When we use $(\epsilon^2, \hat{u})$ coordinates we effectively quotient by the R-symmetry $\tilde{u} \to -\tilde{u}^4$, paying the price of creating, by the double blow up, the exceptional divisor (at zero) parametrized by the coordinate v. In order to match the monodromies when we work with $\tilde{u}^2$-coordinates, we need to take into account, as we move $\tilde{u} \to e^{2\pi i}\tilde{u}$ around the intersection points, the contribution coming from moving in the w-divisor, and the extra piece arising from moving in the "orthogonal" divisor. As we will see in next section, the reason for considering the blow up in $(\epsilon^2, \hat{u})$-coordinates comes from the fact that the $(\hat{u}, \tau)$-plane defines a double covering of the Calabi-Yau moduli space.

In the same way as the point $(\hat{u} = 0, \tau = i\infty)$ is used to recover the pure $N{=}2$ theory $(f \to \infty)$, the line $\{\hat{u} = \infty\}$ can be interpreted as corresponding to the pure $N{=}4$ theory $(f \to 0)$. In this sense, there exits an interesting similarity between the singular locus $\mathcal{C}_0$ and the "$N{=}4$" line $\{\hat{u} = \infty\}$, namely on the line $\mathcal{C}_0$ a component of the elementary hypermultiplet becomes massless.

4 Calabi-Yau Interpretation.

The interpretation above of the pure $N{=}2$ theory as the blow up of a singular (enhancement of symmetry) point in the extended moduli of the $N{=}2$ theory with $N{=}4$ matter content, and the results in reference [9] concerning the point particle limit, motivate us to compare, in more detail, the $(\hat{u}, \tau)$-plane to the moduli of complex structures of the Calabi-Yau weighted projective space $W\mathbb{P}^{12}_{11226}$. The defining polynomial [12, 13] is

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6. \tag{4.1}$$

---

[4]See comment iii) in next section.

Introducing the variables

$$x \equiv \frac{-1}{864} \frac{\phi}{\psi^6}, \qquad y \equiv \frac{1}{\phi^2}, \tag{4.2}$$

the singular loci are given [12] by[5]:

$$
\begin{aligned}
(1)\, \mathcal{C}_{con} &\equiv \{(1-x)^2 - x^2 y = 0\}, \\
(2)\, \mathcal{C}_{\infty} &\equiv \{y = 0\}, \\
(3)\, \mathcal{C}_1 &\equiv \{y = 1\}, \\
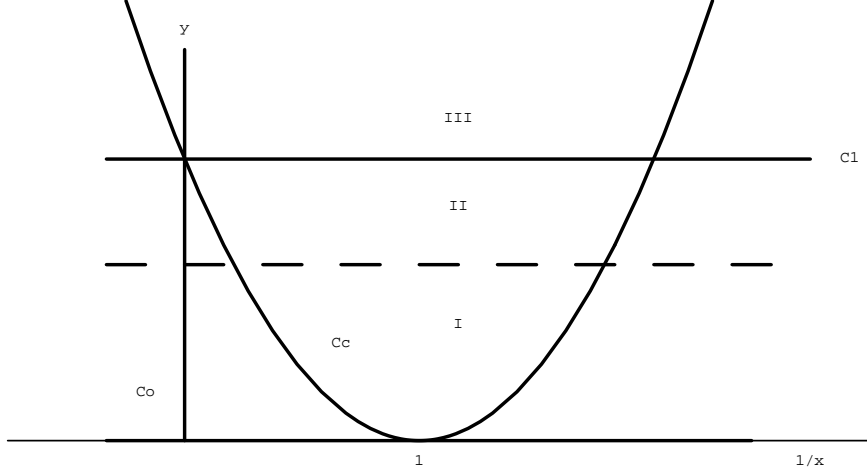(4)\, \mathcal{C}_0 &\equiv \{x = \infty\}.
\end{aligned} \tag{4.3}
$$

Figure 3: Singular loci of $W\mathbb{P}^{12}_{11226}$ moduli space.

Before entering into a more detailed study of the moduli of $W\mathbb{P}^{12}_{11226}$, let us simply consider the blow up of the tangency point $(x = 1, y = 0)$ between $\mathcal{C}_{con}$ and $\mathcal{C}_{\infty}$. By means of a two step blow up [9, 12], we obtain two exceptional divisors E_1 and E_2 (see Figure 4), with coordinates $\frac{yx^2}{(1-x)^2}$ and $\frac{yx}{(1-x)}$, respectively. If we are now bold enough to identify the blow up in the $(\hat{u}, \tau)$-plane (Figure 2) with the Calabi-Yau blow up of Figure 4, we will get the following relation[6]

$$\frac{yx^2}{(1-x)^2} = \frac{\epsilon^2}{\hat{u}^2} \equiv \frac{1}{\tilde{u}^2}, \tag{4.4}$$

---

[5]In ϕ, ψ variables, the four singularity loci of $W\mathbb{P}^{12}_{11226}$ are (1) $\{864\psi^6 + \phi = \pm 1\}$, (2) $\{\phi = \pm 1\}$, (3) $\{\phi, \psi = \infty\}$ and (4) $\{\psi = 0\}$. In equation (4.3) we give these four loci in the chart with coordinates (x^{-1}, y) (see Figure 3).

[6]The variable $\tilde{u}$ entering the following definition should not be confused with that used in (2.1).

which implies, in the neighbourhoood of the point $(x = 1, y = 0)$,

$$x = \frac{1}{1 \pm \hat{u}} + \cdots \qquad y = \epsilon^2 + \cdots \qquad (4.5)$$

Notice that relation (4.4) is nothing but the one used in reference [9] to define the point particle limit. Here this relation is derived by simply identifying the blow ups of Figures 2 and 4. Moreover, using the mirror map of [12], it was suggested in [5] the possibility to interpret y in terms of the heterotic dilaton S through $y = e^{-S} + \cdots$. This identification, together with equation (4.5), strongly suggests the following string interpretation of the $N=2$ theory with $N=4$ matter content:

$$\begin{array}{rcl} (\alpha')^{-1} & \leftrightarrow & f = \frac{1}{4}m^2, \\ e^{-S} & \leftrightarrow & 64q = \epsilon^2. \end{array} \qquad (4.6)$$

Namely, the $N=4$ bare coupling constant τ can be thought of as the dilaton, and the soft breaking mass term f for the hypermultiplets in the adjoint representation as the inverse of the string tension[7].
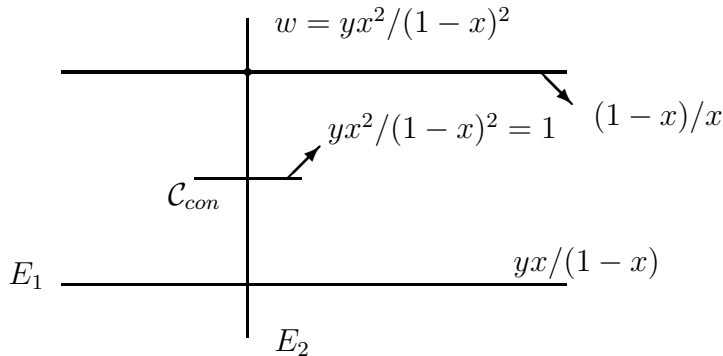

Figure 4: The double blow up as seen in the Calabi-Yau variables.

The formal similarity between the singular loci in $(\hat{u}, \tau)$-plane and those for $W\mathbb{P}^{12}_{11226}$ seems to be more than a coincidence. We can in fact define a one to one map between the two moduli spaces such that in the weak coupling limit we recover relations (4.5). This can be done using the following correspondences:

$$x = \frac{3/2e_1(\tau)}{3/2e_1(\tau) - \hat{u}}, \qquad \sqrt{y} = -\frac{e_2(\tau) - e_3(\tau)}{3e_1(\tau)}, \qquad (4.7)$$

[7]The relation (4.5) between x and $\hat{u}$ was derived for $\alpha' = 1$ in reference [6]. The crucial role of α' in the whole blow up analysis was first pointed out in reference [9].

which map the loci (2.11) into the loci (4.3). Some comments are now necessary concerning (4.7):

i) To fix the correspondence (4.7) we are forced to work with $\sqrt{\bar{y}}$, i.e., in a double covering. The reason, already clear in the $\pm$ ambiguity appearing in (4.5), is that we identify the loci $\mathcal{C}_c^{(1)}$ and $\mathcal{C}_c^{(2)}$ of the $(\hat{u}, \tau)$-plane with the two conifold branches $\sqrt{\bar{y}} = \pm \left(\frac{1-x}{x}\right)$. It is important to stress that the "monopole" and "dyon" singularities, in the $N{=}2$ field theory language, are identified in the (x, y)-plane.

ii) In the coordinate chart II $(x, x^2 y)$ of the corresponding toric diagram [12], the strong coupling loci $\mathcal{C}_1$ is tangent to $\mathcal{C}_\infty$ at the origin. This tangency can be again blown up in two steps. We get, in this way, the extra divisor $\{x = 0\}$. By (4.7) this divisor corresponds to the loci $\{\hat{u} = \infty\}$ in the $(\hat{u}, \tau)$-plane. Recalling now that $\hat{u}$ was defined as u/f, this loci corresponds to the pure $N{=}4$ limit $f \to 0$.

The two step blow up in the coordinate chart II produces, as mentioned above, two exceptional divisors, $\{x = 0\}$ and $D_{(-1,-1)}$ (in the notation of reference [12]). This second divisor has no analog in the $(\hat{u}, \tau)$-plane. The reason is again that the $(\hat{u}, \tau)$-plane is properly speaking in correspondence with the plane $(x, \sqrt{\bar{y}})$, where the tangency we are working out becomes a crossing with a single step blow up, and one exceptional divisor, which is the divisor $\{x = 0\}$. As mentioned above, the $(\hat{u}, \tau)$-plane defines a double covering; thus, the second divisor $D_{(-1,-1)}$ in the Calabi-Yau moduli will only appear when we quotient by the covering transformation.

iii) By correspondence (4.7), the $(\hat{u}, \tau)$-plane becomes a double covering of the (x, y)-plane. The covering transformation is precisely given by the T element in the $Sl(2, Z)$ duality group, $T : (\hat{u}, \tau) \to (\hat{u}, \tau{+}1)$. Moreover, the action of T corresponds to the map $A : (\phi, \psi) \to (-\phi, \alpha\psi)$, with $\alpha^{12} = 1$, i.e., to the transformation between the two branches $\sqrt{\bar{y}} = \pm \left(\frac{1-x}{x}\right)$ of the conifold locus. This explains why the point particle limit defined as the blow up of the tangency point produces a quantum moduli parametrized in terms of $\tilde{u}^2$ instead of $\tilde{u}$. More precisely the parametrization in terms of $\tilde{u}^2$ does not mean that we can quotient in the rigid theory by the global R-symmetry $u \to -u$. To arrive to $\tilde{u}^2$ we need first to go to the extended moduli $(\hat{u}, \tau)$, secondly to notice that the $(\hat{u}, \tau)$-plane is a double covering of the (x, y)-moduli space and third to quotient by the "stringy" symmetry A which corresponds to the covering map $T : (\hat{u}, \tau) \to (\hat{u}, \tau + 1)$.

iv) Taking into account the transformation rules of the roots e_i we can use the correspondences (4.7) to map different regions of H^+/Γ_2 (see Figure 5) into the (x, y) moduli space. So, the domain I:$[\tau = i, \tau = i\infty]$ goes into the domain $[y = 0, y = 1/3]$. In just the same way, the interval $[i, 0]$ goes into the domain $[y = 1/3, y = 1]$. For the domain III of Figure 5, we consider the line going from $\tau = 0$ to the point $\tau = \frac{1}{2}(i - 1)$; this point is the ST transformed of $\tau = i$. Using (4.7) we see that

8

$y(\tau = \frac{1}{2}(i - 1)) = \infty$, and therefore we map the domain III of Figure 5 into the region $[y = 1, \infty]$.
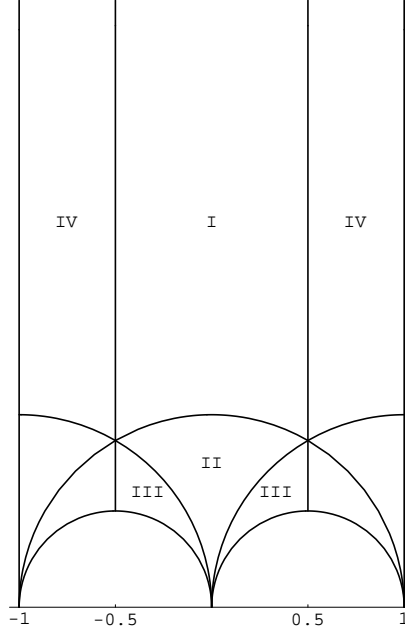

Figure 5: Modular domain of Γ_2.

v) The previous discussion can be repeated "mutatis mutandis" for the Calabi-Yau weighted projective space $W\mathbb{P}^8_{11222}$. The relevant difference between these two spaces is the modular group in the limit $y \to 0$, which is $Sl(2, Z)$ for $W\mathbb{P}^{12}_{11226}$, and $\Gamma_0(2)_+$ for $W\mathbb{P}^8_{11222}$ [17]. In both cases we can recover the rigid $SU(2)$ Seiberg-Witten solution [9, 18]. The enhancement of symmetry point for $W\mathbb{P}^{12}_{11226}$ is given by $T_{heterotic} = i$, while for $W\mathbb{P}^8_{11222}$ is $T_{heterotic} = i/\sqrt{2}$, reflecting the difference in the mirror map: Jacobi's j function for $W\mathbb{P}^{12}_{11226}$, and the Haupmodul for $\Gamma_0(2)_+$ in the case of $W\mathbb{P}^8_{11222}$. In the above construction, leading to the correspondence (4.7), these differences between $W\mathbb{P}^{12}_{11226}$ and $W\mathbb{P}^8_{11222}$ are not taken into account; we will come back to this point in next section.

vi) As a last comment on the correspondence (4.7), we consider the behaviour at $\tau = \frac{1}{2}(i - 1)$. Since $e_1(\frac{1}{2}(1 - i)) = 0$, the point $(\hat{u} = 0, \tau = \frac{1}{2}(1 - i))$, which is in the locus $\mathcal{C}_0$, blows up by (4.7) into the whole line $(x, y = \infty)$. This fact has indeed its counterpart in the Calabi-Yau moduli space. Namely, at the point $y = \infty$ ($\phi = 0$), the locus $\mathcal{C}_0 = \{\psi = 0\}$ corresponds to an undetermined value of $x = -\frac{\phi}{864\psi^6}$.

5 S-Duality and Yukawa Couplings.

We will use the map (4.7) to induce the action of the S-duality group $Sl(2, Z)$, acting on the $N{=}4$ moduli τ, on the Calabi-Yau space.

   As already mentioned in the previous paragraph, points related by the T-transformation $(\hat{u}, \tau) \rightarrow (\hat{u}, \tau + 1)$ map into the same (x, y) point. The T action is in fact non trivial only when we work in the double covering space with coordinates $(x, \sqrt{y})$. On this space the T transformation is interchanging the two branches of the conifold locus, and the two branches $\sqrt{y} = \pm 1$ of the singular locus $\mathcal{C}_1$.

   The non perturbative generator $S : \tau \rightarrow -\frac{1}{\tau}$ induces the change

$$
\begin{aligned}
x(\hat{u}, \tau) &\rightarrow x(\hat{u}^M, -\tfrac{1}{\tau}) \equiv x'(\hat{u}, \tau), \\
y(\tau) &\rightarrow y(-\tfrac{1}{\tau}) \equiv y'(\tau).
\end{aligned}
\tag{5.1}
$$

where

$$
\hat{u}^M = \tau^2 \left(\hat{u} + \frac{1}{2}(e_2(\tau) - e_1(\tau)) \right)
\tag{5.2}
$$

has been defined using the transformation of $\tilde{u}$ $(u = \tilde{u} + \frac{1}{2}e_1 f)$ as a modular form of weight two. From the map (4.7) it is now easy to derive

$$
x' = \frac{1}{2} \frac{1 + 3\sqrt{y}}{1 + \sqrt{y} - \frac{1}{x}}, \qquad \sqrt{y'} = \frac{1 - \sqrt{y}}{1 + 3\sqrt{y}}.
\tag{5.3}
$$

   The transformations of the different loci under S, as defined by (5.3), are given as follows. The positive branch $\sqrt{y} = + \left(\frac{1-x}{x} \right)$ of the conifold locus is mapped by S into $\mathcal{C}_0$, while the negative branch $\sqrt{y} = - \left(\frac{1-x}{x} \right)$ is mapped into itself. In a similar way, the negative branch $\sqrt{y} = -1$ of the locus $\mathcal{C}_1$ is mapped into itself, while the positive branch $\sqrt{y} = +1$ is mapped into $\mathcal{C}_\infty$. Moreover, the enhancement of symmetry point $(x = 1, y = 0)$ is mapped by S into the point of crossing between $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_{con}$.

   Now we pass to study the action of the S duality group $Sl(2, Z)$ on some geometrical objects. In what follows, we will reduce ourselves to the Yukawa couplings; this check will again be unable to distinguish between $W\mathbb{P}^{12}_{11226}$ and $W\mathbb{P}^{8}_{11222}$, since the couplings of both spaces are identical up to a global factor of four [12, 13]. From reference [13] we take[8]

$$
Y_{xxx} = \frac{1}{x^3 \Delta}, \qquad Y_{xxy} = \frac{1 - x}{2x^2 y \Delta}, \qquad Y_{xyy} = \frac{2x - 1}{4xy(1 - y)\Delta}.
\tag{5.4}
$$

where $\Delta {=} (1 - x)^2 - x^2 y$ is the conifold locus discriminant. Under the S-transformations

---

[8]These couplings are obviously invariant under the T transformation.

(5.3) the Yukawa couplings are transformed as follows:

$$Y_{x'x'x'} = \left(\frac{\partial x}{\partial x'}\right)^3 Y_{xxx} = \frac{1}{x'^3 \Delta'},$$

$$Y_{x'x'y'} = \left(\frac{\partial x}{\partial x'}\right)^2 \left(\frac{\partial x}{\partial y'}\right) Y_{xxx} + \left(\frac{\partial x}{\partial x'}\right)^2 \left(\frac{\partial y}{\partial y'}\right) Y_{xxy} = \frac{1-x'}{2x'^2 y' \Delta'} \left(\frac{-\sqrt{y'}}{(1-x')(1-\sqrt{y'})}\right),$$

$$Y_{x'y'y'} = \left(\frac{\partial x}{\partial x'}\right) \left(\frac{\partial x}{\partial y'}\right)^2 Y_{xxx} + 2 \left(\frac{\partial x}{\partial x'}\right) \left(\frac{\partial x}{\partial y'}\right) \left(\frac{\partial y}{\partial y'}\right) Y_{xxy} + \left(\frac{\partial x}{\partial x'}\right) \left(\frac{\partial y}{\partial y'}\right)^2 Y_{xyy} =$$

$$= \frac{2x'-1}{4x'y'(1-y')} \frac{1}{\Delta'} \left(\frac{2x'-1-(1+\sqrt{y'})}{(2x'-1)\sqrt{y'}}\right)$$

$$(5.5)$$

In the above calculations we have made use of the following property, derived from the transformations (5.1):

$$\Delta(x', y') = \frac{\Delta(x, y)}{(1 - x(1 + \sqrt{y}))^3} \tag{5.6}$$

As an example, using $(\frac{dx}{dx'}) = (1 - x(1 + \sqrt{y}))x/x'$ and (5.6), it is immediate to obtain $Y_{x'x'x'}$ as given in (5.5).

From (5.4) and (5.5) we observe that Y_{xxx} is invariant, while the others pick up, by the S-duality transformation, an extra factor. It is important to observe the non trivial fact that in all cases the extra factor becomes *one on the negative branch of the conifold locus*. This branch is precisely the locus which is mapped into itself by the action of the S-duality transformation. In this sense, we can interpret results (5.5) as reflecting the different "modular weights" of the Yukawa couplings with respect to the S-duality group $Sl(2, Z)$ [19], a difference that should certainly vanish on the locus that is mapped into itself by the S-duality transformation. Moreover, it should be noticed that the extra factors in (5.5) are due to the fact that the transformations (5.3) are defined on the double covering.

6   Comments.

To conclude this letter we reduce ourselves to mentioning some aspects of our analysis that deserve a deeper understanding.

i) The physical picture, as described in the (x, y) plane, differs from the one we will obtain in its double cover in many aspects. In particular, the S-duality action, as defined in this letter, can only be implemented on the double covering; namely the T part of the $Sl(2, Z)$ duality group is precisely the transformation interchanging the two branches of the double covering. A second aspect related with the double covering goes to the more technical point on the blow up of the tangency between C_1 and C_∞. If we work in the double covering, we only need one exceptional divisor, which is precisely the one describing the pure $N=4$ theory.

ii) It would also be interesting to work out the physics of the singular locus $\mathcal{C}_0$, which in our approach is the S-dual of the negative branch of the conifold, a fact again hidden by working in (x, y) variables.

iii) The $N=4$ to $N=2$ flow framework we have used in our study is, as was pointed out in reference [15], intimately connected with the relation between integrable models and $N=2$ theories [20, 21]. A natural question that appears from our study will be the connection between these integrable models and the Calabi-Yau manifold used to reinterpret the $N=4$ to $N=2$ flow. Moreover, it would be important to understand whether the connection between $N=2$ gauge theories and integrable models can be reinterpreted from the point of view of the underlying string theory.

iv) At a more speculative level, we can wonder whether the stringy interpretation of the $N=4$ to $N=2$ flow described in this letter can be extended to a flow from $N=2$ to $N=0$ or $N=1$.

v) The role of $N=4$ in our approach, and in this sense the physics of the double covering, can be perhaps understood if we think of the $N=4$ theory as the dimensional reduction of $N=2$ in $D=5$, in the spirit of reference [2]. The strong coupling spectrum is interpreted ,in Kaluza-Klein terms, in parallel to the $D=11$ interpretation of the strong coupling regime in string theory. It would also be interesting to have a D-brane interpretation [22, 23] of the hypermultiplets in the adjoint that control the strong coupling loci in a similar way to Strominger's interpretation [24] of the conifold locus.

References

[1] C. Hull and P. Townsend, Nucl. Phys. **B438** (1995), 109, hep-th/9410167.

[2] E. Witten, Nucl. Phys. **B443** (1995), 85, hep-th/9503124.

[3] A. Sen, Nucl. Phys. **B450** (1995), 103, hep-th/9504027.

[4] J. Harvey and A. Strominger, Nucl. Phys. **B449** (1995), 535, hep-th/9504047.

[5] S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995), 69, hep-th/9505105.

[6] C. Gómez and E. López, Phys. Lett **B356** (1995), 487, hep-th/9506024.

[7] M. Billó, A. Ceresole, R. D'Auria, S.Ferrara, P. Fré, T. Regge, P. Soriani and A. van Proeyen, "A Search for Non-Perturbative Dualities of Local $N = 2$ Yang-Mills Theories from Calabi-Yau Threefolds", hep-th/9506075.

[8] G.-L. Cardoso, D. Lüst and T. Mohaupt, "Non-Perturbative Monodromies in $N = 2$ Heterotic String Vacua", hep-th/9507113.

[9] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, "Nonperturbative Results on the Point Particle Limit of $N = 2$ Heterotic String Compactifications", hep-th/9508155.

[10] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994), 19, hep-th/9407087.

[11] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994), 484, hep-th/9408099.

[12] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. **B416** (1994), 481, hep-th/9308083.

[13] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, Comm. Math. Phys. **167** (1995), 301, hep-th/9308122.

[14] C. Gómez and E. López, Phys. Lett. **B357** (1995), 558, hep-th/9505135.

[15] R. Donagi and E. Witten, "Supersymmetric Yang-Mills and Integrable Systems", hep-th/9510101.

[16] See for instance, J. Harris, Algebraic Geometry, Graduate Text in Mathematics, Springer-Verlag.

[17] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **B357** (1995), 313, hep-th/9506112.

[18] I. Antoniadis and H. Partouche, "Exact Monodromy Group of $N = 2$ Heterotic Superstring", hep-th/9509009.

[19] C. Gómez, R. Hernández and E. López, work in progress.

[20] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, et al., "Integrability and Exact Seiberg-Witten Solution", Phys. Lett. **B355** (1995), 466, hep-th/9505035.

[21] E. Martinec and N. Warner, "Integrable Systems and Supersymmetric Gauge Theories", hep-th/9509161.

[22] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges", hep-th/9510017.

[23] J. Polchinski and E. Witten, "Evidence For Heterotic-Type I String Duality", hep-th/9510169.

[24] A. Strominger, Nucl. Phys. **B451** (1995), 96, hep-th/9504090.