

Degree of coherence for vectorial electromagnetic fields as the distance between correlation matrices

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We assess the degree of coherence of vectorial electromagnetic fields in the space–frequency domain as the distance between the cross-spectral density matrix and the identity matrix representing completely incoherent light. This definition is compared with previous approaches. It is shown that this distance provides an upper bound for the degree of coherence and visibility for any pair of scalar waves obtained by linear combinations of the original fields. This same approach emerges when applying a previous definition of global coherence to a Young interferometer. © 2007 Optical Society of America
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1. INTRODUCTION

Most analyses of the coherence properties of light fields are carried out within a scalar representation. However, the complete picture of light requires the inclusion of the vectorial character of electromagnetic waves. In this regard different and controversial proposals have been introduced in recent times in order to include polarization in coherence phenomena. In particular, we can refer to two conflicting definitions,^{1–7} which have been scrutinized and discussed in the literature with the introduction of sound counterexamples showing their respective weak points.^{4–7}

These counterexamples show that, contrary to the scalar case,^{8–11} currently there is no satisfactory degree of coherence for vectorial fields. Thus there is room for an alternative approach meeting all the advantages of two previous definitions and improving their performance.

To this end, in this work we propose to assess the degree of coherence as the distance between the electric field correlation matrix in the space–frequency domain (the cross-spectral density tensor) and the identity matrix representing fully incoherent and fully unpolarized light. This is consistent with recent works where we have demonstrated the usefulness of formulating optical properties such as polarization, visibility, and field correlations as suitable distances to unpolarized, uniform, and uncorrelated fields, respectively.^{12–19} We will show that this alternative formalism suitably merges the concepts of polarization, coherence, and visibility, embracing previous conflicting approaches and providing satisfactory answers to all the consistency tests presented so far. This approach has already been applied satisfactorily to a Young interferometer comparing degree of polarization and degree of coherence in the interference region.²⁰

For definiteness we assume that the electric field along axis z can be neglected so that only two transversal field components $E_l(\mathbf{r})$, $l=x,y$, are necessary. We deal with

second-order correlation properties of stationary electric fields described in the space–frequency domain by the cross-spectral density tensor,

$$\langle E_l^*(\mathbf{r}_j, \omega) E_m(\mathbf{r}_k, \omega) \rangle = \int d\tau \langle E_l^*(\mathbf{r}_j, t) E_m(\mathbf{r}_k, t + \tau) \rangle \exp(i\omega \tau), \quad (1)$$

with $l, m=x,y$, and $j, k=1,2$, where the angle brackets denote ensemble average. For the sake of simplicity in the following formulas we will not specify the frequency ω .

Since we are comparing two electric field components at two different spatial points we actually have four statistical variables $E_l(\mathbf{r}_j)$ with $l=x,y$, and $j=1,2$, so that the proper matrix encompassing complete information about second-order coherence properties is the 4×4 Hermitian matrix \mathbf{M} made with the 16 matrix elements $\langle E_l^*(\mathbf{r}_j) E_m(\mathbf{r}_k) \rangle$, instead of the 2×2 matrix usually considered.^{2–7} Such a 4×4 matrix \mathbf{M} has already been used in the context of coherence in Ref. 21.

In Section 2 we propose that the amount of coherence conveyed by the four fields $E_l(\mathbf{r}_j)$ can be measured by the distance D between \mathbf{M} and the cross-spectral density tensor associated to fully incoherent and fully unpolarized light, which is proportional to the 4×4 identity matrix \mathbf{I} . We show that D embraces on an equal footing the degree of polarization as well as previously introduced degrees of coherence for vectorial fields.

In Section 3 we show that D determines the maximum interferometric visibility and maximum degree of coherence achievable by pairs of scalar waves obtained by linear combinations of the four original waves.

In Section 4 we recall previous approaches to the coherence of electromagnetic vectorial waves showing that the formalism presented in this work provides suitable answers to all consistency tests revealing the weak points of previous approaches.

In Section 5 we show that D can be alternatively derived as the local measure of coherence for vectorial electromagnetic waves emerging when the global definition of coherence introduced in Ref. 22 is applied to a Young interferometer with small enough apertures.

2. DEGREE OF COHERENCE FOR VECTORIAL ELECTROMAGNETIC FIELDS

The 4×4 Hermitian cross-spectral density matrix \mathbf{M} can be expressed as

$$\mathbf{M}(\mathbf{r}_1, \mathbf{r}_2) = \begin{bmatrix} \mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_1) & \mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) \\ \mathbf{\Gamma}^\dagger(\mathbf{r}_1, \mathbf{r}_2) & \mathbf{\Gamma}(\mathbf{r}_2, \mathbf{r}_2) \end{bmatrix}, \quad (2)$$

where $\mathbf{\Gamma}$ are 2×2 matrices:

$$\mathbf{\Gamma}(\mathbf{r}_i, \mathbf{r}_j) = \begin{bmatrix} \langle E_x^*(\mathbf{r}_i) E_x(\mathbf{r}_j) \rangle & \langle E_x^*(\mathbf{r}_i) E_y(\mathbf{r}_j) \rangle \\ \langle E_y^*(\mathbf{r}_i) E_x(\mathbf{r}_j) \rangle & \langle E_y^*(\mathbf{r}_i) E_y(\mathbf{r}_j) \rangle \end{bmatrix}. \quad (3)$$

The amount of coherence between the four field variables $E_l(\mathbf{r}_j)$ for $l=x,y$, $j=1,2$ can be measured by the distance D between \mathbf{M} and the 4×4 identity matrix \mathbf{I} associated with fully incoherent and fully unpolarized light,

$$D = \frac{4}{3} \operatorname{tr} \left[\left(\frac{1}{4} \mathbf{I} - \frac{1}{\operatorname{tr} \mathbf{M}} \mathbf{M} \right)^2 \right] = \frac{4}{3} \left[\frac{\operatorname{tr}(\mathbf{M}^2)}{(\operatorname{tr} \mathbf{M})^2} - \frac{1}{4} \right], \quad (4)$$

where the numerical factors are introduced for normalization. We can appreciate that D coincides with the generalized degree of polarization in Refs. 23 and 24. We recall that both \mathbf{M} and D are functions of position $D(\mathbf{r}_1, \mathbf{r}_2)$, $M(\mathbf{r}_1, \mathbf{r}_2)$. For the sake of simplicity when there is no risk of confusion the spatial arguments will be omitted.

The maximum $D=1$ is obtained when \mathbf{M} has only one nonvanishing eigenvalue [which is equivalent to the factorization $\langle E_l^*(\mathbf{r}_j) E_m(\mathbf{r}_k) \rangle = \mathcal{E}_l^*(\mathbf{r}_j) \mathcal{E}_m(\mathbf{r}_k)$], while the minimum $D=0$ is obtained when \mathbf{M} is proportional to the 4×4 identity.

Since traces are invariant under unitary transformations $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ we have that D is invariant under arbitrary linear unitary transformations of the four fields, which are implemented by arbitrary combinations of lossless beam splitters and transparent plates.²⁵ In particular this includes arbitrary changes of the polarization basis, which transform the matrices $\mathbf{\Gamma}$ in the form $\mathbf{\Gamma}(\mathbf{r}_i, \mathbf{r}_j) \rightarrow \mathbf{U}_i^\dagger \mathbf{\Gamma}(\mathbf{r}_i, \mathbf{r}_j) \mathbf{U}_j$, where \mathbf{U}_j , $j=1,2$ are 2×2 unitary matrices, and we have considered that the transformation can be different at each point \mathbf{r}_j . This invariance demonstrates that the final result does not depend on the polarization basis chosen to express the fields and compute the coherence. From an active perspective, this invariance means that polarization transformations do not alter the amount of coherence.

This approach properly merges the concepts of polarization and interference, which can be regarded as different realizations of the same idea: the coherent superposition of two fields.² This is clearly revealed when computing the trace of the square of \mathbf{M} ,

$$\operatorname{tr}(\mathbf{M}^2) = \frac{1}{2} \{ (1 + P_1^2) \mathcal{I}_1^2 + (1 + P_2^2) \mathcal{I}_2^2 + 4 \mathcal{I}_1 \mathcal{I}_2 \tilde{\mu}^2 \}, \quad (5)$$

where P_j , $j=1,2$, are the degrees of polarization of the field at each point \mathbf{r}_j ,^{23,24,26-28}

$$P^2(\mathbf{r}_j) = 2 \frac{\operatorname{tr} \mathbf{\Gamma}^2(\mathbf{r}_j, \mathbf{r}_j)}{[\operatorname{tr} \mathbf{\Gamma}(\mathbf{r}_j, \mathbf{r}_j)]^2} - 1, \quad (6)$$

$\mathcal{I}_j = \operatorname{tr} \mathbf{\Gamma}(\mathbf{r}_j, \mathbf{r}_j)$ represent the field intensities at points \mathbf{r}_j , and $\tilde{\mu}$ is the degree of coherence for vectorial electromagnetic waves introduced in Refs. 4, 5, and 7,

$$\tilde{\mu}^2(\mathbf{r}_1, \mathbf{r}_2) = \frac{\operatorname{tr}[\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) \mathbf{\Gamma}^\dagger(\mathbf{r}_1, \mathbf{r}_2)]}{\operatorname{tr} \mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_1) \operatorname{tr} \mathbf{\Gamma}(\mathbf{r}_2, \mathbf{r}_2)}. \quad (7)$$

Moreover, D also includes the modulus of the complex degree of mutual polarization introduced in Ref. 29, since $\operatorname{tr}[\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) \mathbf{\Gamma}^\dagger(\mathbf{r}_1, \mathbf{r}_2)] \propto |\nu(\mathbf{r}_1, \mathbf{r}_2)|^2$ where $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$ with

$$\begin{aligned} \nu_0(\mathbf{r}_i, \mathbf{r}_j) &= \Gamma_{x,x}(\mathbf{r}_i, \mathbf{r}_j) + \Gamma_{y,y}(\mathbf{r}_i, \mathbf{r}_j), \\ \nu_1(\mathbf{r}_i, \mathbf{r}_j) &= \Gamma_{x,y}(\mathbf{r}_i, \mathbf{r}_j) + \Gamma_{y,x}(\mathbf{r}_i, \mathbf{r}_j), \\ \nu_2(\mathbf{r}_i, \mathbf{r}_j) &= i[\Gamma_{x,y}(\mathbf{r}_i, \mathbf{r}_j) - \Gamma_{y,x}(\mathbf{r}_i, \mathbf{r}_j)], \\ \nu_3(\mathbf{r}_i, \mathbf{r}_j) &= \Gamma_{x,x}(\mathbf{r}_i, \mathbf{r}_j) - \Gamma_{y,y}(\mathbf{r}_i, \mathbf{r}_j). \end{aligned} \quad (8)$$

All this means that D represents a measure of the overall interfering capabilities of the four fields $E_l(\mathbf{r}_j)$, with $l=x,y$, and $j=1,2$. This is because D includes the coherence between the field components at different points \mathbf{r}_1 and \mathbf{r}_2 , as well as the coherence between the polarization components at each point. Whether one type of coherence or the other, or a combination of both, manifests in a given experiment depends on the particular arrangement considered. Unlike the scalar case, in the vectorial case one and the same fields $E_l(\mathbf{r}_j)$ can lead to many different inequivalent interferometric experiments with the help of polarizers and phase plates, including interference between polarization components of the same wave, as vividly illustrated by the interference effects observed in crystal plates.¹⁰

3. D AS THE MAXIMUM DEGREE OF COHERENCE AND VISIBILITY FOR PAIRS OF SCALAR FIELDS

The amount of coherence quantifies the degree of statistical dependence between field components, which manifests in the visibility of interference fringes produced by the superposition of particular combinations of these fields. In the scalar case the visibility is always less than or equal to the degree of coherence. The question is whether this idea holds in the vectorial case.

The aim of this section is to show that D provides an upper bound for the degree of coherence μ and visibility v of interference fringes for pairs of scalar waves obtained by energy conserving linear combinations of the original field components E_j , $j=1,2,3,4$, with

$$E_1 = E_x(\mathbf{r}_1), \quad E_2 = E_y(\mathbf{r}_1), \quad E_3 = E_x(\mathbf{r}_2), \quad E_4 = E_y(\mathbf{r}_2). \quad (9)$$

This idea is better illustrated if we first consider the relation between D and μ for the simpler case of two instead of four components.

A. Two-Dimensional Case

For two scalar waves the matrix \mathbf{M} is

$$\mathbf{M}_2 = \begin{bmatrix} \langle E_1^* E_1 \rangle & \langle E_1^* E_2 \rangle \\ \langle E_2^* E_1 \rangle & \langle E_2^* E_2 \rangle \end{bmatrix} = q_0 I_2 + \mathbf{q} \cdot \boldsymbol{\sigma}, \quad (10)$$

where I_2 is the 2×2 identity matrix, $\boldsymbol{\sigma}$ are the three Pauli matrices, and q_0 and the three-dimensional real vector \mathbf{q} are

$$q_0 = \frac{1}{2}(\langle |E_1|^2 \rangle + \langle |E_2|^2 \rangle), \quad q_1 = \frac{1}{2}(\langle E_1^* E_2 \rangle + \langle E_2^* E_1 \rangle),$$

$$q_2 = \frac{i}{2}(\langle E_1^* E_2 \rangle - \langle E_2^* E_1 \rangle), \quad q_3 = \frac{1}{2}(\langle |E_1|^2 \rangle - \langle |E_2|^2 \rangle), \quad (11)$$

where positivity of \mathbf{M}_2 requires $\mathbf{q}^2 \leq q_0^2$. The properly normalized two-dimensional version of D is

$$D_2 = 2 \left[\frac{\text{tr}(\mathbf{M}_2^2)}{(\text{tr} \mathbf{M}_2)^2} - \frac{1}{2} \right] = \frac{\mathbf{q}^2}{q_0^2}, \quad (12)$$

with $1 \geq D_2 \geq 0$. Let us consider energy-conserving linear combinations $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2)$, of the original fields $\mathbf{E} = (E_1, E_2)$, in the form $\tilde{\mathbf{E}} = \mathbf{U}\mathbf{E}$ where \mathbf{U} is a 2×2 unitary matrix. The key point is that the coherence μ and the visibility v depend on \mathbf{U} , so that $\mu(\tilde{\mathbf{E}}) \neq \mu(\mathbf{E})$ and $v(\tilde{\mathbf{E}}) \neq v(\mathbf{E})$ with

$$\mu^2(\tilde{\mathbf{E}}) = \frac{|\langle \tilde{E}_1^* \tilde{E}_2 \rangle|^2}{\langle |\tilde{E}_1|^2 \rangle \langle |\tilde{E}_2|^2 \rangle} = \frac{\tilde{q}_1^2 + \tilde{q}_2^2}{\tilde{q}_0^2 - \tilde{q}_3^2} \leq \frac{\tilde{\mathbf{q}}^2}{\tilde{q}_0^2},$$

$$v^2(\tilde{\mathbf{E}}) = \frac{4|\langle \tilde{E}_1^* \tilde{E}_2 \rangle|^2}{(\langle |\tilde{E}_1|^2 \rangle + \langle |\tilde{E}_2|^2 \rangle)^2} = \frac{\tilde{q}_1^2 + \tilde{q}_2^2}{\tilde{q}_0^2} \leq \frac{\tilde{\mathbf{q}}^2}{\tilde{q}_0^2}, \quad (13)$$

where $\tilde{q}_0, \tilde{\mathbf{q}}$ are the parameters [Eq. (11)] associated with $\tilde{\mathbf{E}}$. The question to be addressed is, which are the largest values of μ and v that can be extracted from some given original fields \mathbf{E} ?

The transformation \mathbf{U} produces just a rotation of the vector \mathbf{q} , so that $\tilde{q}_0 = q_0$ and $\tilde{\mathbf{q}}^2 = \mathbf{q}^2$. This and the invariance of traces under unitary transformations implies that $D_2(\tilde{\mathbf{E}}) = D_2(\mathbf{E})$. On the other hand, from Eqs. (13) and (12) we get $\mu^2(\tilde{\mathbf{E}}) \leq D_2$, and $v^2(\tilde{\mathbf{E}}) \leq D_2$. The largest coherence occurs when $\tilde{\mathbf{q}}^2 = \tilde{q}_0^2$, or when $\tilde{q}_3 = 0$, with $\mu_{\max}^2 = D_2$, while maximum visibility holds when $\tilde{q}_3 = 0$, with $v_{\max}^2 = D_2$. Therefore, D is the maximum degree of coherence and the maximum fringe visibility that can be achieved by any pair of fields obtained by energy-conserving linear combinations of the original pair's E_1, E_2 .

B. Four-Dimensional Case

Let us consider pairs of scalar fields \tilde{E}_1, \tilde{E}_2 obtained as linear combinations of four original fields E_j . They can always be considered as the two first components of the four-dimensional complex vectors $\tilde{\mathbf{E}}$, with $\tilde{\mathbf{E}} = \mathbf{U}\mathbf{E}$ obtained from the original pair's \mathbf{E} via energy-conserving linear unitary 4×4 matrices \mathbf{U} .

Since \mathbf{U} is unitary we have that $\mathbf{M}(\mathbf{E})$ and $\tilde{\mathbf{M}} = \mathbf{M}(\tilde{\mathbf{E}})$ lead to the same D . We can isolate the contribution of the two components \tilde{E}_1, \tilde{E}_2 in the form

$$\tilde{\mathbf{M}} = \begin{pmatrix} \mathbf{M}' & \mathbf{N} \\ \mathbf{N}^\dagger & \mathbf{K} \end{pmatrix}, \quad (14)$$

where

$$\mathbf{M}' = \begin{pmatrix} \langle \tilde{E}_1^* \tilde{E}_1 \rangle & \langle \tilde{E}_1^* \tilde{E}_2 \rangle \\ \langle \tilde{E}_2^* \tilde{E}_1 \rangle & \langle \tilde{E}_2^* \tilde{E}_2 \rangle \end{pmatrix}, \quad (15)$$

and \mathbf{N} and \mathbf{K} are 2×2 matrices.

We can express $\tilde{\mathbf{M}}$ in a suitable basis of 16 trace orthogonal matrices Λ_j ,^{23,24,26,27,30-34}

$$\tilde{\mathbf{M}} = q_0 \Lambda_0 + 2 \sum_{j=1}^{15} \tilde{q}_j \Lambda_j, \quad (16)$$

with $\text{tr}(\Lambda_j \Lambda_k) = (\delta_{j,k} + \delta_{j,0} \delta_{k,0})/2$. For our purposes we only need the explicit form of the first four matrices,

$$\Lambda_0 = \frac{1}{2} \mathbf{I}, \quad \Lambda_j = \frac{1}{2} \begin{bmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (17)$$

where \mathbf{I} is the 4×4 identity matrix, σ_j for $j=1,2,3$, are the 2×2 Pauli matrices, and $\mathbf{0}$ are 2×2 null matrices.

With these definitions the degree of coherence for the two scalar fields \tilde{E}_1, \tilde{E}_2 is expressed exactly as in Eq. (13),

$$\mu^2 = \frac{\tilde{q}_1^2 + \tilde{q}_2^2}{\tilde{q}_0^2 - \tilde{q}_3^2}, \quad (18)$$

where the four parameters $\tilde{q}_0, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ are defined as in Eq. (11) replacing \mathbf{E} by $\tilde{\mathbf{E}}$ [note that \tilde{q}_0 does not coincide with the q_0 in Eq. (16)]. Using the same reasoning employed in Eq. (13), and taking into account that $\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{q}_3^2 \leq \tilde{q}_0^2$, we get

$$\mu^2 \leq \frac{\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{q}_3^2}{\tilde{q}_0^2} \leq \frac{\tilde{\mathbf{q}}^2}{\tilde{q}_0^2}, \quad (19)$$

where $\tilde{\mathbf{q}}$ is the 15-dimensional real vector with components \tilde{q}_j with $j \neq 0$. The first inequality becomes equality when $\tilde{q}_3 = 0$ or when $\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{q}_3^2 = \tilde{q}_0^2$, while the second inequality becomes equality when $\tilde{q}_j = 0$ for $j > 3$.

The desired relation between D and μ emerges once it is noted that $\text{tr} \tilde{\mathbf{M}} = 2q_0$, $\text{tr}(\tilde{\mathbf{M}}^2) = q_0^2 + 2\tilde{\mathbf{q}}^2$, so that

$$D = \frac{4}{3} \left[\frac{\text{tr}(\tilde{\mathbf{M}}^2)}{(\text{tr} \tilde{\mathbf{M}})^2} - \frac{1}{4} \right] = \frac{2\tilde{\mathbf{q}}^2}{3q_0^2}. \quad (20)$$

Combining Eqs. (19) and (20) we get the desired result,

$$\mu^2 \leq \frac{3q_0^2}{2\tilde{q}_0^2} D, \tag{21}$$

which shows that μ is bounded by D , the factor q_0/\tilde{q}_0 arising because the intensity normalizations for μ and D are naturally different. Equivalently for the visibility we have

$$v^2 = \frac{\tilde{q}_1^2 + \tilde{q}_2^2}{\tilde{q}_0^2} \leq \frac{3q_0^2}{2\tilde{q}_0^2} D. \tag{22}$$

This result can also be extended to the incoherent superposition of two fields or two interference patterns. The scalar degree of coherence μ for the fields (E_1, E_2) , with $E_1 = E_{1,a} + E_{1,b}$, $E_2 = E_{2,a} + E_{2,b}$ being the mutually incoherent fields $E_{j,a}$, $E_{k,b}$, $\langle E_{j,a} E_{k,b} \rangle = 0$ for $j, k = 1, 2$, is less than or equal to the maximum of the degrees of coherence μ_a, μ_b for the pairs $(E_{1,a}, E_{2,a})$ and $(E_{1,b}, E_{2,b})$, i.e., $\mu \leq \max(\mu_a, \mu_b)$. Similarly, the visibility v of the incoherent superposition of two interferometric distributions with visibilities v_a, v_b satisfies that $v \leq \max(v_a, v_b)$. Therefore, the bounds [Eqs. (21) and (22)] derived above also extend to incoherent superpositions of the four original fields.

4. DISCUSSION

As we mentioned in Section 1 there are two previously proposed conflicting definitions of coherence for pairs of vectorial electromagnetic fields. These definitions are the one in Eq. (7)^{4,5,7} and^{1-3,6}

$$\tilde{\mu}'^2(\mathbf{r}_1, \mathbf{r}_2) = \frac{[\text{tr } \Gamma(\mathbf{r}_1, \mathbf{r}_2)]^2}{\text{tr } \Gamma(\mathbf{r}_1, \mathbf{r}_1) \text{tr } \Gamma(\mathbf{r}_2, \mathbf{r}_2)}. \tag{23}$$

Equation (7) was introduced directly as a generalization of the scalar case. It appears very naturally within the formalism presented in this work as a contribution to D . It differs from D in that $\tilde{\mu}$ does not include the coherence between polarization components at the same point. Moreover, no definite relation between $\tilde{\mu}$ and the visibility of interference fringes has yet been elucidated.

Equation (23) is motivated by the visibility of particular two-beam interferometric arrangements when polarization plays no role, being discarded in the observation of the interference. The main flaw of Eq. (23) is that it actually ignores polarization, when the interest in generalizing coherence to the vectorial case relies on involving polarization, not in ignoring it. In this regard D accounts for visibility of interference, fully taking into account polarization.

Recently, a more detailed approach along this same line has been presented in Ref. 35, defining a degree of coherence $\tilde{\mu}''$ in terms of the maximum singular value of the matrix

$$\Gamma^{-1/2}(\mathbf{r}_2, \mathbf{r}_2) \Gamma(\mathbf{r}_1, \mathbf{r}_2) \Gamma^{-1/2}(\mathbf{r}_1, \mathbf{r}_1), \tag{24}$$

which is a normalized version of $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ in terms of equal intensity beams, so that $\tilde{\mu}''$ becomes the maximum degree of coherence achievable between arbitrary field components from different points. This provides an improvement of $\tilde{\mu}$ by including a direct connection with visibility of interference fringes. In comparison with D we have that this approach does not include the coherence

between field components at the same point. This difference is revealed, for example, when $\Gamma(\mathbf{r}_1, \mathbf{r}_2) \rightarrow \mathbf{0}$ with nonvanishing degrees of polarization $P(\mathbf{r}_1), P(\mathbf{r}_2) \neq 0$. In such a case $\tilde{\mu}'' \rightarrow 0$ while $D \neq 0$ because of the coherence required to sustain partial polarization.

The degree of coherence D embraces the merits of previous approaches while avoiding their flaws. On the one hand D is sensitive to polarization, extending the idea in Refs. 4, 5, 7, and 35 to include phase relations between all field components on the same footing. On the other hand, D goes beyond the main idea of Refs. 1-3, 6, and 35 since it establishes a bound to the visibility of arbitrary two-beam interferometric arrangements, including those sensitive to polarization. Next we show that D provides satisfactory answers to the two counterexamples presented in Refs. 4 and 6.

A. Fully Coherent Fields with Orthogonal Polarizations

Let us consider first the example presented in Ref. 4 where the only components different from zero are

$$E_x(\mathbf{r}_1, t) = C e^{-i\omega t}, \quad E_y(\mathbf{r}_2, t) = C e^{-i\omega t}, \tag{25}$$

where C is a constant. In this case

$$\mathbf{M} = C^2 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

so that D is maximum $D=1$. This agrees with $\tilde{\mu} = \tilde{\mu}'' = 1$.

On the other hand, Eq. (23) predicts the fully opposite result, $\tilde{\mu}' = 0$, which is unsuitable since certainly the fields in Eq. (25) are coherent and can lead to interference fringes with unit visibility simply using a phase plate.

B. Interference with Unpolarized Light

The example presented in Ref. 6 deals with the interference produced by the unpolarized light

$$\mathbf{M} = C^2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \tag{27}$$

where C is a constant, leading to $D = \frac{1}{3}$.

For this example we have $\tilde{\mu} = 1/\sqrt{2}$ while $\tilde{\mu}' = \tilde{\mu}'' = 1$, which agrees with the maximum visibility of interference fringes obtained for this example. In this case the flaw of $\tilde{\mu}$ in Eq. (7) is that it seems to suggest that these fields are not able to reach maximum interferometric visibility.

The question addressed by this example is whether for values of $\tilde{\mu}$, and D less than unity, it is possible to infer the maximum visibility achievable by any two-beam interference fringes produced by arbitrary superpositions of the four original fields. One of the advantages of the approach presented in this work is that it always allows us to answer this question via Eqs. (21) and (22), while no equivalent relation is known for the other approaches.

Although D does not reach its maximum value (there are field components mutually incoherent since the fields are unpolarized) nevertheless the bound [Eq. (22)] allows

the existence of fringes with maximum visibility. In this case we have the incoherent superposition of two identical interferometric distributions produced by each polarization component separately with maximum visibilities $v_a = v_b = 1$, leading to a total visibility $v = 1$. The interferometric distributions produced by each polarization component contain half of the total intensity so that $q_0/\tilde{q}_0 = 2$ and the upper bound [Eq. (22)] leads to $v \leq \sqrt{2}$, which is fully compatible with the maximum visibility of interference fringes $v = 1$.

We stress that, in contrast with the scalar case, in the vectorial case it is by no means obvious whether some particular fields with a given value of the degree of coherence can or cannot produce fringes with unit visibility. It turns out that for this example this is the case, even after disregarding polarization. However, for other particular examples with other values of D , q_0 , and \tilde{q}_0 the answer can be negative.

5. GLOBAL AND LOCAL COHERENCE

In this section we show that there is a definite relation between a global degree of coherence for vectorial waves μ_g recently introduced and the local definition $D(\mathbf{r}_1, \mathbf{r}_2)$ presented in this work.

The global degree of coherence for scalar waves^{36–40} has been translated to the vectorial case in Ref. 22 in terms of the spatial weighted average of the square of the local degree of coherence for vectorial fields $\tilde{\mu}(\mathbf{r}_1, \mathbf{r}_2)$ in Eq. (7),

$$\mu_g^2 = \frac{\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \text{tr} \Gamma(\mathbf{r}_1, \mathbf{r}_1) \text{tr} \Gamma(\mathbf{r}_2, \mathbf{r}_2) \tilde{\mu}^2(\mathbf{r}_1, \mathbf{r}_2)}{\left[\int d^2\mathbf{r} \text{tr} \Gamma(\mathbf{r}, \mathbf{r}) \right]^2}. \quad (28)$$

We apply this approach to the paradigmatic case of the Young interferometer. This is particularly interesting since for small enough apertures the global degree of coherence after the apertures should define a local degree of coherence for the input field particularized to the points of the apertures.

The two apertures of the Young interferometer are located at coordinates $\mathbf{r}_\pm = \pm \mathbf{a}$ in the plane $z=0$. They are described with real field-amplitude transmission coefficients t identical for all field components. Since the apertures do not overlap, we have $\int d^2\mathbf{r} t(\mathbf{r}-\mathbf{a})t(\mathbf{r}+\mathbf{a})=0$.

The form of the aperture is not essential since throughout we will consider the limit of vanishing width. In such a limit the input field \mathbf{E}_i at $z=0$ is constant in each aperture, so that the field at $z=0$ just after the aperture \mathbf{E}_t is

$$\mathbf{E}_t(\mathbf{r}) \simeq t(\mathbf{r}-\mathbf{a})\mathbf{E}_i(\mathbf{a}) + t(\mathbf{r}+\mathbf{a})\mathbf{E}_i(-\mathbf{a}), \quad (29)$$

and the cross-spectral density matrix immediately after the aperture Γ_t is related to the input cross-spectral density matrix Γ_i in the form

$$\Gamma_t(\mathbf{r}_1, \mathbf{r}_2) \simeq \sum_{j,l=\pm} t(\mathbf{r}_1-\mathbf{a}_j)t(\mathbf{r}_2-\mathbf{a}_l)\Gamma_i(\mathbf{a}_j, \mathbf{a}_l), \quad (30)$$

with $\mathbf{a}_\pm = \pm \mathbf{a}$.

When we apply Eq. (28) to the field after the aperture \mathbf{E}_t we get

$$\mu_g^2 = \frac{\text{tr}(\mathbf{M}_i^2)}{(\text{tr} \mathbf{M}_i)^2}, \quad (31)$$

where \mathbf{M}_i is the Hermitian positive definite 4×4 matrix in Eq. (2) particularized to the input beam illuminating the aperture \mathbf{E}_i .

From Eq. (31) we get $1 \geq \mu_g^2 \geq \frac{1}{4}$. The maximum $\mu_g^2 = 1$ is obtained when \mathbf{M}_i has only one nonvanishing eigenvalue, while the minimum $\mu_g^2 = \frac{1}{4}$ is obtained when \mathbf{M}_i is proportional to the 4×4 identity. The range of variation for μ_g suggests a further renormalization so that it runs between 0 and 1. When this is done we get precisely the very same D in Eq. (4), i.e.,

$$D = \frac{4}{3} \left(\mu_g^2 - \frac{1}{4} \right). \quad (32)$$

Therefore, the properly normalized global degree of coherence for the field after the small apertures coincides exactly with the local degree of coherence D for the input fields at the position of the apertures.

There is no so direct relationship with other local measures of coherence for vectorial waves. In particular, for Eq. (7) we have from Eqs. (5) and (31) that

$$\mu_g^2 = \frac{\mathcal{I}_+^2(1+P_+^2) + \mathcal{I}_-^2(1+P_-^2) + 4\mathcal{I}_+\mathcal{I}_-\tilde{\mu}^2}{2(\mathcal{I}_+ + \mathcal{I}_-)^2}, \quad (33)$$

where P_\pm and \mathcal{I}_\pm are the degrees of polarization and intensities, respectively, at the apertures, and $\tilde{\mu}$ is the local degree of coherence [Eq. (7)] for the input fields at the apertures. This means that μ_g depends not only on $\tilde{\mu}$ but also on the intensity and polarization at the apertures. On the other hand, no general relation can be found between μ_g and $\tilde{\mu}'$ or $\tilde{\mu}''$.

6. CONCLUSIONS

In this work we have developed an alternative approach to the assessment of coherence for vectorial fields. The proposal is to measure the amount of coherence as the distance between the cross-spectral density matrix and the identity matrix associated with fully incoherent and fully unpolarized light. This approach satisfies three properties not satisfied by any one of the previous approaches to this issue, namely:

1. This approach is valid for the assessment of the coherence of an arbitrary number of fields, treating all field components on the same footing, i.e., encompassing correlations between fields at different points as well as correlations between polarization components at the same point. This appears to be appropriate since both contributions can manifest in practice (alone or in combination) depending on the particular practical arrangement considered.

2. This measure of coherence has a very definite and practical relation with the standard degree of coherence and visibility for pairs of scalar fields. This is because it establishes an upper bound for the degree of coherence and for the visibility of interference fringes for any pair of scalar waves obtained by linear combination of the original fields.

3. This local measure of coherence can be derived from a previously introduced global measure of coherence by applying it to a Young interferometer with apertures of small enough area.

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