Duality for logarithmic interpolation spaces when $0 < q < 1$ and applications

Blanca F. Besoy[a,1], Fernando Cobos[a,1,*]

[a] *Departamento de Análisis Matemático y Matemática Aplicada , Facultad de Matemáticas, Universidad Complutense de Madrid. Plaza de Ciencias 3, 28040 Madrid, Spain.*

Abstract

We work with spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ which are logarithmic perturbations of the real interpolation spaces. We determine the dual of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ when $0 < q < 1$. As we show, if $\theta = 0$ or 1 then the dual space depends on the relationship between q and $\mathbb{A}$. Furthermore we apply the abstract results to compute the dual space of Besov spaces of logarithmic smoothness and the dual space of spaces of compact operators in a Hilbert space which are close to the Macaev ideals.

Keywords: Logarithmic interpolation methods, duality, Besov spaces, Macaev operator spaces.
2010 MSC: Primary 46B70, 46E35, 47B10.

Dedicated to Professor Bohumir Opic on the occasion of his 70th birthday.

1. Introduction

The description of the dual space is an important and useful item on the list of properties of any interpolation method. For the case of the real method $(A_0, A_1)_{\theta,q}$ where $0 < \theta < 1$ and $1 \le q \le \infty$, this problem was considered by Lions and Peetre [32] in their foundational paper of the real method (see also the paper by Lions [31]). Later Peetre [39] studied the case when $0 < q < 1$. Then $(A_0, A_1)_{\theta,q}$ is no longer a Banach space but a quasi-Banach space. He proved that $((A_0, A_1)_{\theta,q})' = (A_0', A_1')_{\theta,\infty}$.

*Corresponding author.
Email addresses: blanca.f.besoy@ucm.es (Blanca F. Besoy), cobos@mat.ucm.es (Fernando Cobos)
[1] The authors have been supported in part by MTM2017-84058-P (AEI/FEDER, UE). B.F. Besoy has also been supported by FPU grant FPU16/02420 of the Spanish Ministerio de Educación, Cultura y Deporte.

Some applications of the ideas of real interpolation have required to consider logarithmic perturbations of the real method $(A_0, A_1)_{\theta,q,\mathbb{A}}$, quasi-normed by

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty [t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a)]^q \frac{dt}{t} \right)^{1/q}$$

where $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell^{\mathbb{A}}(t) = (1 - \log t)^{\alpha_0}$ if $0 < t \le 1$, $\ell^{\mathbb{A}}(t) = (1 + \log t)^{\alpha_\infty}$ if $1 < t < \infty$ and $K(t,a)$ is the Peetre's K-functional. See the papers by Evans and Opic [23], Evans, Opic and Pick [24], Edmunds and Opic [21], Cobos and Segurado [17] and the references given there. Now θ can also take the limit values 0 and 1, producing spaces very close to A_1 if $\theta = 1$ and very close to A_0 if $\theta = 0$. Consider, for example, the couple of Lebesgue spaces (L_{p_0}, L_{p_1}). Applying to this couple the real method we can only obtain Lebesgue and Lorentz spaces, but using the method $(\cdot,\cdot)_{\theta,q,\mathbb{A}}$ we can also get Lorentz-Zygmund spaces.

If $0 < \theta < 1, \mathbb{A} \in \mathbb{R}^2$ and $1 \le q \le \infty$ (respectively, $0 < q < 1$) the dual of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ follows from the general results of Persson [40] (respectively, Cobos [7]). The dual spaces of $(A_0, A_1)_{1,q,\mathbb{A}}$ and $(A_0, A_1)_{0,q,\mathbb{A}}$ have been described by Cobos and Segurado [17] when $1 \le q \le \infty$. In this paper we continue those investigations by studying the case $0 < q < 1$ and determining the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ and $(A_0, A_1)_{0,q,\mathbb{A}}$ in terms of the K-functional.

The main feature of our results is that the parameter q always appears in the exponent of the logarithms of the resulting spaces. This is a remarkable difference with respect to the results of [17]. On the other hand, it makes sense to point out a connection between these abstract results, given by Theorem 4.3 below, and those on associate spaces of generalized Lorentz-Zygmund spaces given in [38, Theorem 6.6 (ii)]. See Remark 4.5 below.

The duality results are given in Section 4. They require the description of the logarithmic spaces in terms of the J-functional that we establish in the previous Section 3. These results of equivalence are of independent interest and complement those proved by Cobos and Segurado [17, Section 3].

In the last two sections of the paper we show applications of the duality results to function spaces and to operator spaces. In Section 5, among other things, we study the dual of the Besov spaces $\mathbf{B}^{0,b}_{p,q}$ defined by using the modulus of smoothness associated to L_p $(1 < p < \infty)$, having classical smoothness 0 and logarithmic smoothness with exponent b. These spaces of smoothness close to zero are attracting considerable attention lately (see, for example, [5, 9, 12, 11]). With the help of logarithmic Lispchitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ (see [27, 28]), Cobos and Domíngez [9] have described the dual of $\mathbf{B}^{0,b}_{p,q}$ if $1 \le q < \infty$. We consider here the case $0 < q < 1$. On the contrary to the case $1 \le q < \infty$, if $0 < q < 1$ we show that the Lipschitz space involved in the description of the dual has logarithmic smoothness that changes with q. In particular, when $b = 0$ we show that the dual of $\mathbf{B}^{0}_{p,q}$ is isomorphic to

$\mathrm{Lip}^{(1,-1/q)}_{p',\infty}$ for $0 < q < 1$.

In last Section 6 we consider two scales of spaces of compact operators on a Hilbert space related to the Macaev ideals (see [33, 25]) and we describe the duality relationships between them.

2. Logarithmic interpolation methods

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is to say two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. For $t > 0$, the *Peetre's K- and J-functionals* are defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j, j = 0, 1\}$$

where $a \in A_0 + A_1$ and

$$J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\,\|a\|_{A_1}\},\ a \in A_0 \cap A_1.$$

Observe that $K(1, \cdot)$ is the norm of $A_0 + A_1 = \Sigma(\bar{A})$ and $J(1, \cdot)$ the norm of $A_0 \cap A_1 = \Delta(\bar{A})$.

Let $\ell(t) = 1 + |\log t|$, $\ell\ell(t) = 1 + \log(1 + |\log t|)$ and for $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ put

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \le 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty, \end{cases}$$

and define $\ell\ell^{\mathbb{A}}(t)$ similarly. We also put $-\mathbb{A} = (-\alpha_0, -\alpha_\infty)$.

Given $0 \le \theta \le 1$, $0 < q \le \infty$ and $\mathbb{A} \in \mathbb{R}^2$, the *logarithmic interpolation space* $(A_0, A_1)_{\theta,q,\mathbb{A}}$ consists of all $a \in A_0 + A_1$ such that

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty [t^{-\theta} \ell^{\mathbb{A}}(t) K(t, a)]^q \frac{dt}{t} \right)^{1/q} < \infty$$

(as usual the integral should be replaced by the supremum when $q = \infty$). The space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ turns out to be a Banach space provided that $1 \le q \le \infty$ and a quasi-Banach space if $0 < q < 1$ (see [23, 24, 17]).

For $0 < \theta < 1$ and $\mathbb{A} = (0, 0)$, the space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ coincides with the real interpolation space $(A_0, A_1)_{\theta,q}$ realized as a K-space (see [3, 42, 2, 4]). If $0 < \theta < 1$ and $\mathbb{A} \ne (0, 0)$, then $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is a special case of the real method with a function parameter (see [26, 40]). In these two cases, the properties of the spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ are well-known. Concerning the remaining cases, namely $\theta = 0$ and $\theta = 1$, it is enough to study just one of the two because $K(t, a; A_0, A_1) = tK(t^{-1}, a; A_1, A_0)$ and therefore

$$(A_0, A_1)_{0,q,(\alpha_0,\alpha_\infty)} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)}. \tag{2.1}$$

Subsequently we focuss on the case $\theta = 1$.

In order to avoid that $(A_0, A_1)_{1,q,\mathbb{A}} = \{0\}$ we should assume

$$\begin{cases} \alpha_0 + 1/q < 0 & \text{if } 0 < q < \infty, \\ \alpha_0 \le 0 & \text{if } q = \infty, \end{cases} \tag{2.2}$$

(see [24, Theorem 2.2]). If $\alpha_0 = -1/q$ and $0 < q < \infty$, we still get a non-trivial space if we replace $\ell^{(-1/q,\alpha_\infty)}(t)$ by $\ell^{(-1/q,\alpha_\infty)}(t)\ell\ell^{(\beta_0,\beta_\infty)}(t)$ with $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ and $\beta_0 + 1/q < 0$. This leads to the space $(A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}}$ formed by all those $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A},\mathbb{B}}} = \left(\int_0^\infty [t^{-1}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)K(t,a)]^q \frac{dt}{t} \right)^{1/q}.$$

Under the above mentioned assumptions, i.e.

$$\begin{cases} \alpha_0 + 1/q < 0, \text{ or } \alpha_0 = -1/q \text{ and } \beta_0 + 1/q < 0 & \text{if } q < \infty, \\ \alpha_0 < 0, \text{ or } \alpha_0 = 0 \text{ and } \beta_0 \le 0 & \text{if } q = \infty, \end{cases} \tag{2.3}$$

it turns out that $\left(\int_0^\infty [\min(1,t)t^{-1}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)]^q \frac{dt}{t}\right)^{1/q} < \infty$. Hence, using that

$$\min(1,t)K(1,a) \le K(t,a) \text{ for } a \in A_0 + A_1$$

and

$$K(t,a) \le \min(1,t)J(1,a) \text{ for } a \in A_0 \cap A_1,$$

we derive that

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}} \hookrightarrow A_0 + A_1$$

where $\hookrightarrow$ means continuous embedding.

It is easy to check that the quasi-norm of $(A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}}$ is equivalent to

$$\|a\|_{(A_0,A_1)^{\diamond}_{1,q,\mathbb{A},\mathbb{B}}} = \left(\sum_{m=-\infty}^{\infty} [2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)K(2^m,a)]^q \right)^{1/q}$$

(the ℓ_q-quasi-norm should be replaced by the ℓ_∞-quasi-norm if $q = \infty$).

Subsequently, we follow the usual notation concerning the symbols $\lesssim$ and $\sim$: If X and Y are quantities depending on certain parameters, we put $X \lesssim Y$ if $X \le cY$ with a constant c independent of the parameters. We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. A similar notation is used for quasi-norms.

When $A_1 \hookrightarrow A_0$ with the embedding having norm less than or equal to 1, then $K(t,a) = \|a\|_{A_0}$ for $t \ge 1$. If $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and q satisfy (2.2), we have

$$\begin{aligned}\left(\int_1^\infty [t^{-1}\ell^{\alpha_\infty}(t)K(t,a)]^q\frac{dt}{t}\right)^{1/q} &= \|a\|_{A_0}\left(\int_1^\infty [t^{-1}\ell^{\alpha_\infty}(t)]^q\frac{dt}{t}\right)^{1/q}\\ &\le c\,\|a\|_{A_0}\left(\int_0^1 \ell^{\alpha_0 q}(t)\frac{dt}{t}\right)^{1/q}\\ &\le c\left(\int_0^1 [t^{-1}(1-\log t)^{\alpha_0}K(t,a)]^q\frac{dt}{t}\right)^{1/q}\end{aligned}$$

where we have used in the last inequality that $t^{-1}K(t,a)$ is a non-increasing function with $K(1,a)=\|a\|_{A_0}$. Whence, we obtain

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} \sim \left(\int_0^1 \left[\frac{K(t,a)}{t(1-\log t)^{-\alpha_0}}\right]^q\frac{dt}{t}\right)^{1/q}. \qquad (2.4)$$

Equivalence (2.4) shows the connection between logarithmic interpolation spaces and the so-called *limiting real interpolation spaces*, studied in [14, 8, 9, 12] among other papers.

If $A_0 \hookrightarrow A_1$, with the embedding having norm less than or equal to 1, then $K(t,a)=t\,\|a\|_{A_1}$ for $0<t\le 1$. In this case if $\mathbb{A}=(\alpha_0,\alpha_\infty)$ and q satisfy (2.2), then

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} \sim \left(\int_1^\infty \left[\frac{K(t,a)}{t(1+\log t)^{-\alpha_\infty}}\right]^q\frac{dt}{t}\right)^{1/q} \qquad (2.5)$$

because

$$\begin{aligned}\left(\int_0^1 [t^{-1}\ell^{\alpha_0}(t)K(t,a)]^q\frac{dt}{t}\right)^{1/q} &= \|a\|_{A_1}\left(\int_0^1 \ell^{\alpha_0 q}(t)\frac{dt}{t}\right)^{1/q}\\ &\le c\,\|a\|_{A_1}\left(\int_1^\infty [t^{-1}\ell^{\alpha_\infty}(t)]^q\frac{dt}{t}\right)^{1/q}\\ &\le c\left(\int_1^\infty [t^{-1}\ell^{\alpha_\infty}(t)K(t,a)]^q\frac{dt}{t}\right)^{1/q}.\end{aligned}$$

To establish some important points of the theory of the spaces $(A_0,A_1)_{1,q,\mathbb{A}}$ the following J-spaces are useful. Assume that

$$\begin{cases}\alpha_\infty>0, \text{ or } \alpha_\infty=0 \text{ and } \beta_\infty\ge 0 & \text{if } 0<q\le 1,\\ \alpha_\infty-1/q'>0, \text{ or } \alpha_\infty=1/q' \text{ and } \beta_\infty-1/q'>0 & \text{if } 1<q\le\infty,\end{cases} \qquad (2.6)$$

where $1/q+1/q'=1$ if $1\le q\le\infty$. The space $(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ is formed by all those $a\in A_0+A_1$ for which there exists $(u_m)_{m\in\mathbb{Z}}\subseteq A_0\cap A_1$ such that

$$a=\sum_{m=-\infty}^{\infty} u_m \text{ (convergence in } A_0+A_1) \qquad (2.7)$$

and

$$\left(\sum_{m=-\infty}^{\infty}[2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m,u_m)]^q\right)^{1/q}<\infty \tag{2.8}$$

(again, the ℓ_q-quasi-norm should be replaced by the ℓ_∞-quasi-norm if $q=\infty$). We set

$$\|a\|_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}}=\inf\left\{\left(\sum_{m=-\infty}^{\infty}[2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m,u_m)]^q\right)^{1/q}\right\}$$

where the infimum is taken over all representations (u_m) satisfying (2.7) and (2.8). If $\mathbb{B}=(0,0)$, we simply write $(A_0,A_1)^J_{1,q,\mathbb{A}}$.

Condition (2.6) yields that

$$\begin{cases}\sup_{m\in\mathbb{Z}}\left\{\min(1,2^m)\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)\right\}<\infty & \text{if } 0<q\le 1,\\ \left(\sum_{m=-\infty}^{\infty}[\min(1,2^m)\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q'}\right)^{1/q'}<\infty & \text{if } 1<q\le\infty.\end{cases} \tag{2.9}$$

Hence, if $a=\sum_{m=-\infty}^{\infty}u_m$ is a representation of a satisfying (2.7), (2.8) and $1\le q\le\infty$, it follows from the inequality

$$\|u_m\|_{A_0+A_1}\le\min(1,2^{-m})J(2^m,u_m)$$

by using Hölder's inequality and (2.9) that the series $\sum_{m=-\infty}^{\infty}u_m$ is absolutely convergent in A_0+A_1. If $0<q<1$, using Jensen inequality and (2.9) we obtain

$$\begin{aligned}\sum_{m=-\infty}^{\infty}\|u_m\|_{A_0+A_1} &\le \sum_{m=-\infty}^{\infty}\min(1,2^{-m})J(2^m,u_m)\\ &\le \left(\sum_{m=-\infty}^{\infty}\left(\min(1,2^{-m})J(2^m,u_m)\right)^q\right)^{1/q}\\ &\le \sup_{m\in\mathbb{Z}}\left\{\min(1,2^m)\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)\right\}\\ &\quad\times\left(\sum_{m=-\infty}^{\infty}[2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m,u_m)]^q\right)^{1/q}<\infty\,.\end{aligned}$$

These arguments also show that $(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}\hookrightarrow A_0+A_1$. On the other hand, embedding $A_0\cap A_1\hookrightarrow(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ holds because for any $a\in A_0\cap A_1$ we can represent a as $a=\sum_{m=-\infty}^{\infty}\delta^0_m a$, where δ^i_m is the Kronecker delta. Hence,

$$\|a\|_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}}\le J(1,a)=\|a\|_{A_0\cap A_1}.$$

If $1 \le q \le \infty$ there is a continuous representation for the J-spaces. It can be established proceeding as in the case of the real method (see [3, Section 3.2]). Hölder's inequality is used in the arguments. The outcome is that $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ consists of all those $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t)\frac{dt}{t} \text{ (convergence in } A_0 + A_1)$$

and

$$\left(\int_0^\infty [t^{-1}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Moreover

$$\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}} = \inf\left\{\left(\int_0^\infty [t^{-1}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} : a = \int_0^\infty u(t)\frac{dt}{t}\right\}$$

is an equivalent quasi-norm in $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$.

Let $\Sigma(\bar{A})^\circ$ be the closure of $\Delta(\bar{A})$ in $\Sigma(\bar{A})$. We have that

$$a \in \Sigma(\bar{A})^\circ \text{ if and only if } \min(1, 1/t)K(t,a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty. \tag{2.10}$$

Indeed, it follows from *The fundamental lemma* (see [3, Lemma 3.3.2]) that if $\min(1, 1/t)K(t,a) \to 0$ as $t \to 0$ and as $t \to \infty$, then $a \in \Sigma(\bar{A})^\circ$. Conversely, given any $a \in \Sigma(\bar{A})^\circ$ and any $\varepsilon > 0$, we can choose $b \in \Delta(\bar{A})$ such that $K(1, a-b) \le \varepsilon/2$. Choose $0 < L < 1$ such that for any $0 < t < L$ we have $t\|b\|_{A_1} \le \varepsilon/2$. Then, if $0 < t < L$, we obtain

$$K(t,a) \le K(t, a-b) + K(t,b) \le K(1, a-b) + t\|b\|_{A_1} \le \varepsilon.$$

This establishes that if $a \in \Sigma(\bar{A})^\circ$ then $K(t,a) \to 0$ as $t \to 0$. A similar argument shows that $t^{-1}K(t,a) \to 0$ as $t \to \infty$.

By the construction of the J-space, we have that $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}} \subseteq \Sigma(\bar{A})^\circ$, but this is not always the case for the K-space as we show next by means of an example.

Example 2.1. *Consider the Banach couple of sequence spaces $\bar{A} = (\ell_1, \ell_\infty)$, let $\xi = (1, 1, 1, ...)$, $0 < q \le \infty$ and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that*

$$\begin{cases} \alpha_0 + 1/q < 0 \text{ and } \alpha_\infty + 1/q < 0 & \text{if } 0 < q < \infty, \\ \alpha_0 \le 0 \text{ and } \alpha_\infty \le 0 & \text{if } q = \infty. \end{cases} \tag{2.11}$$

According to [42, Theorem 1.18.3] we have that $K(2^m, \xi) = 2^m$ for $m \in \mathbb{Z}$. So, ξ belongs to $(\ell_1, \ell_\infty)_{1,q,\mathbb{A}}$ because

$$\|\xi\|_{(\ell_1,\ell_\infty)^\diamond_{1,q,\mathbb{A}}} = \left(\sum_{m=-\infty}^{0} \ell^{\alpha_0 q}(2^m) + \sum_{m=1}^{\infty} \ell^{\alpha_\infty q}(2^m)\right)^{1/q} < \infty.$$

However ,

$$\lim_{t\to\infty} \min(1,1/t)K(t,\xi) = \lim_{t\to\infty} K(t,\xi)/t = 1.$$

Outside the rank pointed out for α_∞ in (2.11), the K-space is always contained in $\Sigma(\bar{A})^\circ$ as we show next.

Lemma 2.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (2.3). Assume in addition that*

$$\begin{cases} \alpha_\infty + 1/q > 0, \text{ or } \alpha_\infty = -1/q \text{ and } \beta_\infty + 1/q \ge 0 & \text{if } 0 < q < \infty, \\ \alpha_\infty > 0, \text{ or } \alpha_\infty = 0 \text{ and } \beta_\infty > 0 & \text{if } q = \infty. \end{cases} \tag{2.12}$$

Then $(A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}} \subseteq \Sigma(\bar{A})^\circ$.

Proof. Let $a \in (A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}}$. Then

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A},\mathbb{B}}} = \left(\int_0^\infty [t^{-1}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)K(t,a)]^q \frac{dt}{t} \right)^{1/q} < \infty$$

but

$$\int_0^1 [t^{-1}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)]^q \frac{dt}{t} = \infty$$

and, by (2.12),

$$\int_1^\infty [\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)]^q \frac{dt}{t} = \infty.$$

Therefore

$$K(t,a) \to 0 \text{ as } t \to 0 \text{ and } K(t,a)/t \to 0 \text{ as } t \to \infty.$$

According to (2.10), we conclude that $a \in \Sigma(\bar{A})^\circ$. □

3. Equivalence theorems between K- and J-spaces

We start with an auxiliary result. If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we put $\mathbb{A}\lambda = (\alpha_0\lambda, \alpha_\infty\lambda)$ and $\mathbb{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$.

Lemma 3.1. *Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le 1$ such that $\alpha_0 + 1/q < 0$. Put*

$$V(2^k) = \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{m-k})2^{-m}\ell^{\mathbb{A}}(2^m)]^q \right)^{1/q}, \ k \in \mathbb{Z}.$$

Then we have:

(a) If $\alpha_\infty + 1/q > 0$, $V(2^k) \sim 2^{-k}\ell^{\mathbb{A}+1/q}(2^k)$.

(b) If $\alpha_\infty + 1/q < 0$, $V(2^k) \sim 2^{-k}\ell^{(\alpha_0+1/q,0)}(2^k)$.

(c) If $\alpha_\infty + 1/q = 0$, $V(2^k) \sim \begin{cases} 2^{-k}\ell^{\alpha_0+1/q}(2^k) & \text{if } k \le 0, \\ 2^{-k}\ell\ell^{1/q}(2^k) & \text{if } k > 0. \end{cases}$

PROOF. We have

$$\begin{aligned} V(2^k)^q &= 2^{-kq}\sum_{m=-\infty}^{k} \ell^{\mathbb{A}q}(2^m) + \sum_{m=k+1}^{\infty} 2^{-mq}\ell^{\mathbb{A}q}(2^m) \\ &\sim 2^{-kq}\int_0^{2^k} \ell^{\mathbb{A}q}(t)\frac{dt}{t} + 2^{-kq}\ell^{\mathbb{A}q}(2^k). \end{aligned}$$

To estimate the integral $I = \int_0^{2^k} \ell^{\mathbb{A}q}(t)\frac{dt}{t}$ we distinguish several cases. If $k \in \mathbb{Z}$ and $k \le 0$, since $\alpha_0 + 1/q < 0$, we get

$$I = \int_0^{2^k} (1 - \log t)^{\alpha_0 q}\frac{dt}{t} \sim (1 - \log 2^k)^{\alpha_0 q+1} = \ell^{\mathbb{A}q+1}(2^k).$$

This yields that

$$V(2^k)^q \sim 2^{-kq}\ell^{\mathbb{A}q+1}(2^k) \text{ for } k \le 0,\ k \in \mathbb{Z}.$$

Assume now that $k > 0$, $k \in \mathbb{Z}$, then

$$\begin{aligned} I &= \int_0^1 (1 - \log t)^{\alpha_0 q}\frac{dt}{t} + \int_1^{2^k} (1 + \log t)^{\alpha_\infty q}\frac{dt}{t} \\ &\sim \int_1^{2^k} (1 + \log t)^{\alpha_\infty q}\frac{dt}{t}. \end{aligned}$$

In the case (a) we obtain $I \sim \ell^{\alpha_\infty q+1}(2^k)$, which implies that

$$V(2^k)^q \sim 2^{-kq}\ell^{\mathbb{A}q+1}(2^k), \text{ for } k > 0,\ k \in \mathbb{Z}.$$

In the case (b) we get $I \sim 1$ and so

$$V(2^k)^q \sim 2^{-kq} \text{ for } k > 0,\ k \in \mathbb{Z}.$$

Finally, in the case (c),

$$I \sim \int_1^{2^k} (1 + \log t)^{-1}\frac{dt}{t} \sim \ell\ell(2^k),$$

which yields that

$$V(2^k)^q \sim 2^{-kq}\ell\ell(2^k) \text{ for } k > 0, k \in \mathbb{Z}.$$

The proof is finished. □

Given a Banach couple $\bar{A} = (A_0, A_1)$, the *Gagliardo completion* $A_j^{\sim}$ of A_j is formed of all those $a \in A_0 + A_1$ for which there exists a bounded sequence (a_n) in A_j which converges to a in $A_0 + A_1$. The norm in $A_j^{\sim}$ is given by

$$\|a\|_{A_j^{\sim}} = \inf_{(a_n)} \left(\sup_{n \in \mathbb{N}} (\|a_n\|_{A_j}) \right)$$

(see [3, 2, 4]). Clearly $A_j \hookrightarrow A_j^{\sim} \hookrightarrow A_0 + A_1$. By [2, Theorem 5.1.4] we have

$$\|a\|_{A_0^{\sim}} = \lim_{t \to \infty} K(t, a) \text{ and } \|a\|_{A_1^{\sim}} = \lim_{t \to 0} \frac{K(t, a)}{t}.$$

We put $\overline{A^{\sim}} = (A_0^{\sim}, A_1^{\sim})$. According to [2, Theorem 5.1.5]

$$K(t, a; A_0, A_1) = K(t, a; A_0^{\sim}, A_1^{\sim}), \ t > 0, \ a \in A_0 + A_1. \tag{3.1}$$

The Banach couple $\bar{A}$ is called *mutually closed* if $A_j = A_j^{\sim}$ for $j = 0, 1$.

We say that $\bar{A}$ is *regular* if $A_0 \cap A_1$ is dense in A_j for $j = 0, 1$.

Theorem 3.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq 1$ such that $\alpha_0 + 1/q < 0$. Then we have with equivalence of quasi-norms*

(i) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q}$ *if* $\alpha_\infty + 1/q > 0$,

(ii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q,(0,1/q)}$ *if* $\alpha_\infty + 1/q = 0$.

If in addition $\bar{A}$ is regular, then

(iii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})^J_{1,q,(\alpha_0+1/q,0)}$ *if* $\alpha_\infty + 1/q < 0$.

Proof. It is enough to work with the couple $(A_0^{\sim}, A_1^{\sim})$ because, by (3.1), we know that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{1,q,\mathbb{A}}$. Let a be any element of the J-spaces of the statement and let $a = \sum_{k=-\infty}^{\infty} u_k$ be a discrete J-representation of a. Since $0 < q \leq 1$, we have

$$\begin{aligned} K(2^m, a)^q &\leq \left(\sum_{k=-\infty}^{\infty} K(2^m, u_k) \right)^q \leq \sum_{k=-\infty}^{\infty} K(2^m, u_k)^q \\ &\leq \sum_{k=-\infty}^{\infty} [\min(1, 2^{m-k}) J(2^k, u_k)]^q. \end{aligned}$$

Hence

$$\begin{aligned} \|a\|_{(A_0,A_1)^{\diamond}_{1,q,\mathbb{A}}} &= \left(\sum_{m=-\infty}^{\infty} [2^{-m} \ell^{\mathbb{A}}(2^m) K(2^m, a)]^q \right)^{1/q} \\ &\leq \left(\sum_{k=-\infty}^{\infty} J(2^k, u_k)^q \sum_{m=-\infty}^{\infty} [\min(1, 2^{m-k}) 2^{-m} \ell^{\mathbb{A}}(2^m)]^q \right)^{1/q}. \end{aligned}$$

The interior sum can be estimated by using Lemma 3.1, with the outcome that in any of the cases (i), (ii) and (iii), the J-space of the statement is continuously embedded in $(A_0^{\sim}, A_1^{\sim})_{1,q,\mathbb{A}}$.

Suppose now that $a \in (A_0^{\sim}, A_1^{\sim})_{1,q,\mathbb{A}}$. In the cases (i) and (ii), it follows from Lemma 2.2 that $a \in \Sigma(\overline{A^{\sim}})^{\circ}$. In the case (iii), the additional assumption that $\bar{A}$ is regular and [4, Corollary 2.2.23] imply that $a \in \Sigma(\overline{A^{\sim}})^{\circ}$ as well. According to [37, Theorem 3.2] there exists a representation $a = \sum_{k=-\infty}^{\infty} u_k$ (convergence in $A_0^{\sim} + A_1^{\sim}$) with $(u_k) \subseteq A_0^{\sim} \cap A_1^{\sim}$ and

$$\left(\sum_{k=-\infty}^{\infty} [\min(1, 2^{m-k}) J(2^k, u_k)]^q \right)^{1/q} \leq cK(2^m, a), \ m \in \mathbb{Z}, \tag{3.2}$$

where c is a constant depending on q only. By Lemma 3.1 the weight in front of $J(2^k, u_k)$ in the discrete J-norm $\|a\|_J^{\diamondsuit}$ is equivalent to $V(2^k)$ in any of the cases (i), (ii) and (iii). Consequently,

$$\begin{aligned} \|a\|_J^{\diamondsuit} &\lesssim \left(\sum_{k=-\infty}^{\infty} (V(2^k) J(2^k, u_k))^q \right)^{1/q} \\ &= \left(\sum_{m=-\infty}^{\infty} 2^{-mq} \ell^{\mathbb{A}q}(2^m) \sum_{k=-\infty}^{\infty} \min(1, 2^{m-k})^q J(2^k, u_k)^q \right)^{1/q} \\ &\lesssim \left(\sum_{m=-\infty}^{\infty} [2^{-m} \ell^{\mathbb{A}}(2^m) K(2^m, a)]^q \right)^{1/q} = \|a\|_{(A_0,A_1)^{\diamondsuit}_{1,q,\mathbb{A}}} \end{aligned}$$

where the last inequality follows from (3.2). This completes the proof. □

Remark 3.3. Let $\bar{B} = (B_0, B_1)$ be a Banach couple. Under the assumptions on $\mathbb{A}$ and q in the case *(iii)* of Theorem 3.2, it follows from Example 2.1 and the previous discussion to Example 2.1 that $(B_0, B_1)_{1,q,\mathbb{A}}$ may not have a description in terms of the J-functional. However, if $\bar{B}$ is regular then the J-description does exist as we have just shown in Theorem 3.2/(iii).

Next we show other two equivalence results between K- and J-spaces that will be useful later. They refers to the Banach case where $1 \leq q \leq \infty$. The first one extends [17, Theorem 5.7] allowing q to take the value 1 and β_0 be any number less than $-1/q$.

Theorem 3.4. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $1 \leq q \leq \infty$, $1/q + 1/q' = 1$ and $\alpha_\infty, \beta_0 \in \mathbb{R}$ such that*

$$\begin{cases} \alpha_\infty + 1/q > 0 \text{ and } \beta_0 + 1/q < 0 & \text{if } 1 \leq q < \infty, \\ \alpha_\infty > 0 \text{ and } \beta_0 < 0 & \text{if } q = \infty. \end{cases} \tag{3.3}$$

Then we have with equivalent norms

$$(A_0, A_1)_{1,q,(-1/q,\alpha_\infty),(\beta_0,0)} = (A_0, A_1)^J_{1,q,(1/q',\alpha_\infty+1),(\beta_0+1,0)} \,.$$

Proof. Let us call for short $\bar{A}_K$ to the K-space and $\bar{A}_J$ to the J-space. Assume that $1 \le q < \infty$. The case $q = \infty$ can be treated analogously. Take any $a \in \bar{A}_J$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a J-representation of a such that

$$\left(\int_0^1 \left[\frac{\ell^{1/q'}(t)\ell\ell^{\beta_0+1}(t)J(t,u(t))}{t}\right]^q \frac{dt}{t} + \int_1^\infty \left[\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t}\right]^q \frac{dt}{t}\right)^{1/q} \le 2\,\|a\|_{\bar{A}_J}.$$

Using that

$$\begin{aligned}
\frac{1}{t}K(t,a) &\le \frac{1}{t}\int_0^\infty K(t,u(s))\frac{ds}{s} \\
&\le \frac{1}{t}\int_0^\infty \min(1,t/s)J(s,u(s))\frac{ds}{s} \\
&\le \frac{1}{t}\int_0^t J(s,u(s))\frac{ds}{s} + \int_t^\infty \frac{1}{s}J(s,u(s))\frac{ds}{s},
\end{aligned}$$

we obtain

$$\begin{aligned}
\|a\|_{\bar{A}_K} \le\ & \left(\int_0^1 \left[\frac{(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0}}{t}\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
&+ \left(\int_0^1 \left[(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
&+ \left(\int_0^1 \left[(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0}\int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
&+ \left(\int_1^\infty \left[\frac{(1+\log t)^{\alpha_\infty}}{t}\int_0^1 J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
&+ \left(\int_1^\infty \left[\frac{(1+\log t)^{\alpha_\infty}}{t}\int_1^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
&+ \left(\int_1^\infty \left[(1+\log t)^{\alpha_\infty}\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\
=\ & I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Next we show that $I_j \lesssim \|a\|_{\bar{A}_J}$ for $j = 1, ..., 6$. We start with I_1 in the special case $q = 1$. Taking any $0 < \varepsilon < 1$ and using Fubini's theorem, we get

$$\begin{aligned}
I_1 &= \int_0^1 J(s,u(s)) \int_s^1 \frac{(1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0}}{t}\frac{dt}{t}\frac{ds}{s} \\
&\lesssim \int_0^1 J(s,u(s)) \frac{(1-\log s)^{-1}(1+\log(1-\log s))^{\beta_0}}{s^{\varepsilon}} \int_s^1 t^{\varepsilon-1}\frac{dt}{t}\frac{ds}{s} \\
&\lesssim \int_0^1 J(s,u(s)) \frac{(1-\log s)^{-1}(1+\log(1-\log s))^{\beta_0}}{s}\frac{ds}{s} \\
&\leq \int_0^1 J(s,u(s)) \frac{(1+\log(1-\log s))^{\beta_0+1}}{s}\frac{ds}{s} \\
&\lesssim \|a\|_{\bar{A}_J}.
\end{aligned}$$

Consider now the case $1<q<\infty$. By Hölder's inequality, for the interior integral in I_1 we get

$$\begin{aligned}
\int_0^t J(s,u(s))\frac{ds}{s} &\leq \left(\int_0^t \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad \times \left(\int_0^t \left[s\ell^{-1/q'}(s)\ell\ell^{-(\beta_0+1)}(s)\right]^{q'} \frac{ds}{s}\right)^{1/q'} \\
&\lesssim t\ell^{-1/q'}(t)\ell\ell^{-\beta_0-1}(t)\left(\int_0^t \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q}.
\end{aligned}$$

Whence, by Fubini's theorem,

$$\begin{aligned}
I_1 &\lesssim \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \int_s^1 \left[\ell^{-1}(t)\ell\ell^{-1}(t)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\
&\lesssim \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\lesssim \|a\|_{\bar{A}_J}.
\end{aligned}$$

To estimate I_2 when $q=1$, since $\beta_0+1<0$, using Fubini's theorem, we get

$$\begin{aligned}
I_2 &= \int_0^1 \frac{J(s,u(s))}{s}\int_0^s (1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0}\frac{dt}{t}\frac{ds}{s} \\
&= \int_0^1 \frac{J(s,u(s))}{s}(1+\log(1-\log s))^{\beta_0+1}\frac{ds}{s} \\
&\lesssim \|a\|_{\bar{A}_J}.
\end{aligned}$$

If $1 < q < \infty$, taking $\varepsilon > 0$ such that $\beta_0 + 1 + 1/q < \varepsilon < 1$, we obtain

$$\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \le \left(\int_t^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\right]^q \frac{ds}{s}\right)^{1/q} \times \left(\int_t^1 \ell^{-1}(s)\ell\ell^{-\varepsilon q'+q'/q}(s)\frac{ds}{s}\right)^{1/q'}$$
$$\lesssim \ell\ell^{-\varepsilon+1}(t)\left(\int_t^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\right]^q \frac{ds}{s}\right)^{1/q}.$$

Hence, by Fubini's theorem,

$$\begin{aligned} I_2 &\lesssim \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\right]^q \int_0^s \ell^{-1}(t)\ell\ell^{(\beta_0-\varepsilon+1)q}(t)\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\lesssim \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\ &\lesssim \|a\|_{\bar{A}_J}. \end{aligned}$$

As for I_3, since $\beta_0 + 1/q < 0$ and $\alpha_\infty + 1/q > 0$, for any $1 \le q < \infty$, we derive

$$\begin{aligned} I_3 &= \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\left(\int_0^1 (1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0 q}\frac{dt}{t}\right)^{1/q} \\ &\lesssim \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} \\ &\lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s}\ell^{\alpha_\infty+1}(s)\right]^q \frac{ds}{s}\right)^{1/q}\left(\int_1^\infty \ell^{-(\alpha_\infty+1)q'}(s)\frac{ds}{s}\right)^{1/q'} \\ &\lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s}\ell^{\alpha_\infty+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\ &\lesssim \|a\|_{\bar{A}_J}. \end{aligned}$$

Consider now I_4 for any $1 \le q < \infty$. We obtain

$$\begin{aligned} I_4 &\lesssim \int_0^1 J(s,u(s))\frac{ds}{s} \\ &\le \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\ &\quad\times \left(\int_0^1 \left[s\ell^{-1/q'}(s)\ell\ell^{-(\beta_0+1)}(s)\right]^{q'} \frac{ds}{s}\right)^{1/q'} \\ &\lesssim \left(\int_0^1 \left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\right]^q \frac{ds}{s}\right)^{1/q} \\ &\lesssim \|a\|_{\bar{A}_J}. \end{aligned}$$

The integral I_5 (respectively, I_6) coincides with J_4 (respectively, I_4) in the proof of [17, Theorem 3.5]. Therefore,

$$I_5 \lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s}(1+\log s)^{\alpha_\infty}\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \|a\|_{\bar{A}_J}$$

and

$$I_6 \lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s}(1+\log s)^{\alpha_\infty+1}\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \|a\|_{\bar{A}_J}.$$

This shows the embedding $\bar{A}_J \hookrightarrow \bar{A}_K$.

Next we proceed with the converse embedding. Take any $a \in \bar{A}_K$. By Lemma 2.2 and (2.10), we have that

$$\min(1,1/t)K(t,a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty. \tag{3.4}$$

For $\nu \in \mathbb{Z}$, we put

$$\gamma_\nu = \begin{cases} 2^{-2^{2^{-\nu-1}}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

We can decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$, $j = 0,1$, such that

$$\|a_{0,\nu}\|_{A_0} + \gamma_{\nu-1}\|a_{1,\nu}\|_{A_1} \le 2K(\gamma_{\nu-1}, a), \ \nu \in \mathbb{Z}. \tag{3.5}$$

Write

$$u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1.$$

According to (3.5) we have

$$\begin{aligned} \left\| a - \sum_{\nu=N}^{M} u_\nu \right\|_{A_0+A_1} &\le \|a_{0,N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} \\ &\lesssim K(\gamma_{N-2}, a) + \frac{K(\gamma_{M-1}, a)}{\gamma_{M-1}} \to 0 \text{ as } M \to \infty \text{ and } N \to -\infty \end{aligned}$$

by (3.4). So $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$.

Let $D_\nu = (\gamma_{\nu-1}, \gamma_\nu]$, $\nu \in \mathbb{Z}$. We have

$$\int_{D_\nu} \frac{dt}{t} = \begin{cases} \log 2 & \text{if } \nu = 1, \\ 2^{\nu-2}\log 2 & \text{if } \nu > 1. \end{cases}$$

For $\nu \le 0$ put

$$\delta_\nu = \int_{D_\nu} (1-\log t)^{-1}(1+\log(1-\log t))^{-1}\frac{dt}{t} \sim 1.$$

Consider the function

$$w(t) = \begin{cases} \frac{u_\nu}{\delta_\nu \ell(t)\ell\ell(t)} & \text{if } t \in D_\nu \text{ and } \nu \le 0, \\ \frac{u_1}{\log 2} & \text{if } t \in D_1, \\ \frac{u_\nu}{2^{\nu-2}\log 2} & \text{if } t \in D_\nu \text{ and } \nu > 1. \end{cases}$$

Then

$$\int_0^\infty w(t)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} u_\nu = a \text{ in } A_0 + A_1.$$

If $\nu \le 0$ and $t \in D_\nu$, using (3.5), we get

$$\begin{aligned} \frac{J(t, w(t))}{t} &\sim \ell^{-1}(t)\ell\ell^{-1}(t)\frac{J(t, u_\nu)}{t} \le \ell^{-1}(t)\ell\ell^{-1}(t)\frac{J(\gamma_{\nu-1}, u_\nu)}{\gamma_{\nu-1}} \\ &\lesssim \ell^{-1}(t)\ell\ell^{-1}(t)\frac{K(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}. \end{aligned}$$

Whence,

$$\begin{aligned} &\left(\int_{D_\nu}\left[\ell^{1/q'}(t)\ell\ell^{\beta_0+1}(t)\frac{J(t, w(t))}{t}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\qquad\lesssim \frac{K(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}\left(\int_{D_\nu}\left[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\ &\qquad\sim \frac{K(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}} 2^{|\nu|(\beta_0+1/q)} \\ &\qquad\sim \frac{K(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}\left(\int_{D_{\nu-2}}\left[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\ &\qquad\le \left(\int_{D_{\nu-2}}\left[\frac{K(t, a)}{t}\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} \end{aligned}$$

because $t^{-1}K(t, a)$ is a non-increasing function. If $\nu > 2$, proceeding as in [17, Theorem 3.5] we obtain

$$\left(\int_{D_\nu}\left[\ell^{\alpha_\infty+1}(t)\frac{J(t, w(t))}{t}\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_{D_{\nu-2}}\left[\frac{K(t, a)}{t}\ell^{\alpha_\infty}(t)\right]^q \frac{dt}{t}\right)^{1/q}.$$

For $\nu = 1, 2$ we derive

$$\left(\int_{D_\nu}\left[\ell^{\alpha_\infty+1}(t)\frac{J(t, w(t))}{t}\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_{D_{\nu-2}}\left[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\frac{K(t, a)}{t}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Consequently,

$$\begin{aligned}
\|a\|_{\bar{A}_J} &\leq \left(\sum_{\nu=-\infty}^{0} \int_{D_\nu} \left[\ell^{1/q'}(t) \ell\ell^{\beta_0+1}(t) \frac{J(t,w(t))}{t} \right]^q \frac{dt}{t} \right. \\
&\quad \left. + \sum_{\nu=1}^{\infty} \int_{D_\nu} \left[\ell^{\alpha_\infty+1}(t) \frac{J(t,w(t))}{t} \right]^q \frac{dt}{t} \right)^{1/q} \\
&\lesssim \left(\int_0^1 \left[\frac{K(t,a)}{t} \ell^{-1/q}(t) \ell\ell^{\beta_0}(t) \right]^q \frac{dt}{t} + \int_1^\infty \left[\frac{K(t,a)}{t} \ell^{\alpha_\infty}(t) \right]^q \frac{dt}{t} \right)^{1/q} \\
&= \|a\|_{\bar{A}_K} .
\end{aligned}$$

This completes the proof. □

The following result can be established by using similar arguments to those of Theorem 3.4.

Theorem 3.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $1 \leq q \leq \infty$, $1/q + 1/q' = 1$ and $\alpha_0, \beta_\infty \in \mathbb{R}$ such that*

$$\begin{cases} \alpha_0 + 1/q < 0 \text{ and } \beta_\infty + 1/q > 0 & \text{if } 1 \leq q < \infty, \\ \alpha_0 < 0 \text{ and } \beta_\infty > 0 & \text{if } q = \infty. \end{cases}$$

Then

$$(A_0, A_1)_{1,q,(\alpha_0,-1/q),(0,\beta_\infty)} = (A_0, A_1)^J_{1,q,(\alpha_0+1,1/q'),(0,\beta_\infty+1)} .$$

Remark 3.6. Note the difference between the equivalence results in the quasi-Banach case and in the Banach case: If $1 \leq q \leq \infty$ the exponents of the logarithms in the J-space in the equivalence theorems are obtained either by adding an unit to the exponents of the K-space (see Theorems 3.4 and 3.5 and [17, Theorems 3.5 and 5.7]) or by adding an unit to the exponents and multiplying by an iterated logarithm if $t > 1$ (see [17, Theorem 3.6]). However, if $0 < q < 1$ the correction may have three distinct shapes and each one of them involves the parameter q. In this case, the exponents of the logarithms in the J-space are obtained either adding the term $1/q$ to the exponents of the K-space or adding $1/q$ to α_0 and replacing α_∞ by 0 or adding $1/q$ to the exponents and, moreover, multiplying by an iterated logarithm to the power $1/q$ if $t > 1$ (see Theorem 3.2).

4. Duality

Let $(E, \|\cdot\|)$ be a quasi-Banach space. Consider in E the semi-norm

$$\|x\|^{\#} = \inf\{ \sum_{k=1}^{n} \|x_k\|_E : x = \sum_{k=1}^{n} x_k \}$$

and let $N = \{x \in E : \|x\|^{\#} = 0\}$. We designate as $E^{\#}$ to the completion of the quotient space E/N with the quotient norm induced by $\|x\|^{\#}$.

As it is shown in [39, p.125], the dual space $(E^{\#})'$ of $E^{\#}$ coincides with the dual of E:

$$(E^{\#})' = E'. \tag{4.1}$$

Lemma 4.1. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty), \mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ and $0 < q < 1$ such that $\alpha_\infty > 0$, or $\alpha_\infty = 0$ and $\beta_\infty \geq 0$. Then*

$$\left((A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}\right)^{\#} = (A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}}.$$

PROOF. Since $\ell_q \hookrightarrow \ell_1$, the discrete representation of the J-spaces shows that $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}} \hookrightarrow (A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}}$. Hence, if $a \in (A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ and $a = \sum_{k=1}^n a_k$, using that $(A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}}$ is a norm space, we obtain

$$\|a\|_{(A_0,A_1)^{J,\diamond}_{1,1,\mathbb{A},\mathbb{B}}} \leq \sum_{k=1}^n \|a_k\|_{(A_0,A_1)^{J,\diamond}_{1,1,\mathbb{A},\mathbb{B}}} \leq \sum_{k=1}^n \|a_k\|_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}}.$$

Taking the infimum over all finite decompositions $a = \sum_{k=1}^n a_k$ of a in $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$, it follows that

$$\|a\|_{(A_0,A_1)^{J,\diamond}_{1,1,\mathbb{A},\mathbb{B}}} \leq \|a\|^{\#}_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}}.$$

In particular, we get that $\|\cdot\|^{\#}_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}}$ is not only a semi-norm but a norm and that

$$\left((A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}\right)^{\#} \hookrightarrow (A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}}. \tag{4.2}$$

In order to establish the converse embedding, take any $u \in A_0 \cap A_1$ and let δ^k_m be the Kronecker delta. For any $k \in \mathbb{Z}$, using the J-representation $u = \sum_{m=-\infty}^{\infty} \delta^k_m u$, we obtain

$$\|u\|_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}} \leq 2^{-k}\ell^{\mathbb{A}}(2^k)\ell\ell^{\mathbb{B}}(2^k)J(2^k, u).$$

Let now $a \in (A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}}$ and take any J-representation $a = \sum_{m=-\infty}^{\infty} u_m$ of a. The previous estimate yields that

$$\begin{aligned}\sum_{m=N}^{M} \|u_m\|^{\#}_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}} &\leq \sum_{m=N}^{M} \|u_m\|_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}} \\ &\leq \sum_{m=N}^{M} 2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m, u_m).\end{aligned}$$

It follows that the series $\sum_{m=-\infty}^{\infty} u_m$ is convergent in $\left((A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}\right)^{\#}$. The sum of the series should be also a because of the embedding (4.2). Therefore,

$$\begin{aligned}
\|a\|^{\#}_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}} &\leq \sum_{m=-\infty}^{\infty} \|u_m\|^{\#}_{(A_0,A_1)^{J,\diamond}_{1,q,\mathbb{A},\mathbb{B}}} \\
&\leq \sum_{m=-\infty}^{\infty} 2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m) J(2^m, u_m).
\end{aligned}$$

This yields the embedding

$$(A_0, A_1)^J_{1,1,\mathbb{A},\mathbb{B}} \hookrightarrow \left((A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}\right)^{\#}$$

and completes the proof. □

Next we show the corresponding result for K-spaces.

Theorem 4.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $0 < q < 1$ such that $\alpha_0 + 1/q < 0$. Then we have:*

(i) if $\alpha_\infty + 1/q > 0$, $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,\mathbb{A}+1/q-1}$,

(ii) if $\alpha_\infty + 1/q = 0$, $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)}$,

(iii) if $\alpha_\infty + 1/q < 0$ and, in addition, $\bar{A}$ is regular and $\delta < -1$, then $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)}$.

Proof. For the case (i), according to Theorem 3.2, Lemma 4.1 and (3.1), we derive

$$\begin{aligned}
(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \left((A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}\right)^{\#} \\
&= (A_0^\sim, A_1^\sim)^J_{1,1,\mathbb{A}+1/q} \\
&= (A_0^\sim, A_1^\sim)_{1,1,\mathbb{A}+1/q-1} = (A_0, A_1)_{1,1,\mathbb{A}+1/q-1}.
\end{aligned}$$

For the case (ii), we proceed similarly but using also Theorem 3.5. We obtain

$$\begin{aligned}
(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \left((A_0^\sim, A_1^\sim)^J_{1,q,(\alpha_0+1/q,0),(0,1/q)}\right)^{\#} \\
&= (A_0^\sim, A_1^\sim)^J_{1,1,(\alpha_0+1/q,0),(0,1/q)} \\
&= (A_0, A_1)_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)}.
\end{aligned}$$

Finally, in the case (iii), we have, by Theorem 3.2 and Lemma 4.1,

$$\begin{aligned}(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \Big((A_0^{\sim}, A_1^{\sim})^J_{1,q,(\alpha_0+1/q,0)} \Big)^{\#} \\ &= (A_0^{\sim}, A_1^{\sim})^J_{1,1,(\alpha_0+1/q,0)} \\ &= (A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)} \,.\end{aligned}$$

□

In what follows we assume that $\bar{A}$ is regular. Then the dual space A_j' of A_j is continuously embedded in $(A_0 \cap A_1)'$ and so (A_0', A_1') is a Banach couple too.

Note that the several instances shown in Theorem 4.2 do not appear when $0 < \theta < 1$. Indeed, it follows from [7, Theorem 3.1] (see also [34, Corollary 2]) that for $0 < q < 1$ and $(\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ we have

$$(A_0, A_1)^{\#}_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0, A_1)_{\theta,1,(\alpha_0,\alpha_\infty)} \,. \tag{4.3}$$

Equalities (4.1), (4.3) and [19, Theorem 3.1] (or [40, Theorem 2.4]) yield

$$\Big((A_0, A_1)_{\theta,q,(\alpha_0,\alpha_\infty)} \Big)' = \big(A_0', A_1' \big)_{\theta,\infty,(-\alpha_\infty,-\alpha_0)} \,. \tag{4.4}$$

Next we proceed to determine the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ for $0 < q < 1$. Our arguments are based on the previous results of this paper and the duality theorems for the case $q = 1$ established by Cobos and Segurado [17].

Theorem 4.3. *Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $0 < q < 1$ such that $\alpha_0 + 1/q < 0$. Then for the dual space W of $(A_0, A_1)_{1,q,\mathbb{A}}$ we have:*

(i) if $\alpha_\infty + 1/q > 0$, then $W = (A_0', A_1')_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}$;

(ii) if $\alpha_\infty + 1/q = 0$, then $W = (A_0', A_1')_{1,\infty,(0,-\alpha_0-1/q),(-1/q,0)}$;

(iii) if $\alpha_\infty + 1/q < 0$, then $W = A_1' \cap (A_0', A_1')_{1,\infty,(-1,-\alpha_0-1/q)}$.

Proof. In the case (i), according to (4.1) and Theorem 4.2, we obtain

$$W = \Big((A_0, A_1)^{\#}_{1,q,\mathbb{A}} \Big)' = \Big((A_0, A_1)_{1,1,\mathbb{A}+1/q-1} \Big)' .$$

The last dual space has been determined in [17, Theorem 5.6] with the result that $W = (A_0', A_1')_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}$.

For the case (ii), applying Theorem 4.2 and [19, Theorem 3.1], we get

$$\begin{aligned}W &= \Big((A_0, A_1)_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)} \Big)' \\ &= \big(A_0', A_1' \big)^J_{1,\infty,(1,-\alpha_0-1/q+1),(-1/q+1,0)} \,.\end{aligned}$$

We can describe the J-space in terms of the K-functional by using Theorem 3.4. We derive that

$$W = \left(A_0', A_1'\right)_{1,\infty,(0,-\alpha_0-1/q),(-1/q,0)} .$$

In the last case (iii), take any $\delta < -1$. Using Theorem 4.2 we obtain

$$W = \Big((A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)}\Big)'$$

and, by [17, Theorem 5.10], we get

$$W = A_1' \cap \left(A_0', A_1'\right)_{1,\infty,(-1,-\alpha_0-1/q)} .$$

□

Remark 4.4. The corresponding duality results for $(A_0, A_1)_{0,q,\mathbb{A}}$ can be derived from Theorem 4.3 by using (2.1).

Remark 4.5. Let $L_{(p,q;\mathbb{A},\mathbb{B})}$ be the *generalized Lorentz-Zygmund (GLZ) space* as defined in [38, Definition 3.3]. We put $L_{(p,q;\mathbb{A})} = L_{(p,q;\mathbb{A},\mathbb{B})}$ if $\mathbb{B} = (0,0)$. We write X^* for the associate space (see [38, page 396]) of the GLZ space X. Then, by [38, Theorems 6.6 (ii) and 3.8], if $0 < q < 1$ and $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$, we have

$$(L_{(\infty,q;\mathbb{A})})^* = L_{(1,\infty;-\mathbb{A}-1/q)} .$$

Moreover (cf., e.g., [24, Lemma 8.3] and [38, Theorem 3.8]),

$$L_{(\infty,q;\mathbb{A})} = (L_1, L_\infty)_{1,q,\mathbb{A}} ,$$

and

$$\begin{aligned} L_{(1,\infty;-\mathbb{A}-1/q)} &= (L_1, L_\infty)_{0,\infty,-\mathbb{A}-1/q} = (L_\infty, L_1)_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)} \\ &= (L_1^*, L_\infty^*)_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)} . \end{aligned}$$

Consequently,

$$\Big((L_1, L_\infty)_{1,q,\mathbb{A}}\Big)^* = (L_1^*, L_\infty^*)_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)} ,$$

which corresponds to the result on dual spaces in Theorem 4.3 (i) provided that $A_0 = L_1$ and $A_1 = L_\infty$ if we disregard the fact that (L_1, L_∞) is not a regular couple.

Similarly, such a correspondence can be verified in the remaining cases of Theorem 4.3.

5. Applications to Function Spaces

Besov spaces with logarithmic smoothness are attracting considerable attention in recent years (see, for example, the papers by Caetano, Gogatishvili and Opic [5], Cobos and Domínguez [8, 9, 10], Cobos, Domínguez and Triebel [12] or Cobos, Domínguez and Kühn [11]). These spaces are very near to the Lebesgue space L_p but they have additional properties than L_p due to their smoothness and their structure of Besov spaces (see, for example, [8, Theorem 5.1]). There are two natural ways to introduce them that we review next.

Let f be a (complex-valued) function on $\mathbb{R}^d$, let $h \in \mathbb{R}^d$ and $1 < p < \infty$. If f belongs to L_p, the *modulus of smoothness* of f is defined by

$$\omega(f,t)_p = \sup\{\|f(\cdot + h) - f(\cdot)\|_{L_p} : |h| \leq t\}.$$

For $0 < q \leq \infty$ and $-\infty < b < \infty$, the Besov space $\mathbf{B}^{0,b}_{p,q}$ is formed by all $f \in L_p$ having a finite quasi-norm

$$\|f\|_{\mathbf{B}^{0,b}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 [(1 - \log t)^b \omega(f,t)_p]^q \frac{dt}{t} \right)^{1/q}$$

(see [20, 9, 12]). The case of interest is $b \geq -1/q$ because if $b < -1/q$ then $\int_0^1 (1 - \log t)^{bq} dt/t < \infty$ and so $\mathbf{B}^{0,b}_{p,q} = L_p$ with equivalence of quasi-norms. Also $\mathbf{B}^{0,0}_{p,\infty} = L_p$.

Note that $\mathbf{B}^{0,b}_{p,q}$ has zero classical smoothness and logarithmic smoothness with exponent b. We put $\mathbf{B}^0_{p,q} = \mathbf{B}^{0,b}_{p,q} = L_p$ when $b = 0$.

Similar spaces may be introduced by following the Fourier analytic approach as we recall now. Let $\mathcal{S}$ and $\mathcal{S}'$ be the Schwartz space of all (complex-valued) rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. We write $\mathcal{F}$ for the Fourier transform on $\mathcal{S}'$ and $\mathcal{F}^{-1}$ for the inverse Fourier transform. Take $\varphi_0 \in \mathcal{S}$ such that

$$\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^d : |x| \leq 2\} \text{ and } \varphi_0(x) = 1 \text{ if } |x| \leq 1.$$

For $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$ let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. The sequence $(\varphi_j)_{j=0}^\infty$ is a smooth dyadic resolution of unity.

For $1 < p < \infty$, $0 < q \leq \infty$ and $b \in \mathbb{R}$, the Besov space $B^{0,b}_{p,q}$ consists of all $f \in \mathcal{S}'$ having a finite quasi-norm

$$\|f\|_{B^{0,b}_{p,q}} = \left(\sum_{j=0}^\infty [(1+j)^b \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\right\|_{L_p}]^q \right)^{1/q}$$

(see [36, 13, 30]). Again if $b = 0$, we simply write $B^0_{p,q}$.

Spaces $\mathbf{B}^{0,b}_{p,q}$ and $B^{0,b}_{p,q}$ are different but they are closely related as it is shown in [9, Theorem 3.3] and [10, Theorem 3.2]. Duality for these spaces has been studied in [9, Section 4] for the Banach case $1 \le q < \infty$. Next we consider the quasi-Banach case $0 < q < 1$. We start with the spaces $B^{0,b}_{p,q}$.

Peetre [39, Example 3.2] showed that if $b = 0$ then $\left(B^0_{p,q}\right)' = B^0_{p',\infty}$ if $1 < p < \infty$, $1/p + 1/p' = 1$ and $0 < q < 1$. The corresponding formula for $b \neq 0$ is as follows.

Theorem 5.1. *Let $1 < p < \infty$, $1/p + 1/p' = 1$, $0 < q < 1$ and $b \in \mathbb{R}$. Then*

$$\left(B^{0,b}_{p,q}\right)' = B^{0,-b}_{p',\infty}.$$

PROOF. Let H^s_p be the fractional Sobolev space. We recall that $(H^s_p)' = H^{-s}_{p'}$ (see [42, Theorem 2.6.1]). It follows from [13, Theorem 5.3 and Remark 5.4] that $B^{0,b}_{p,q} = \left(H^{-1}_p, H^1_p\right)_{1/2,q,(b,b)}$. Consequently, using (4.4) and again [13, Theorem 5.3], we obtain

$$\begin{aligned} \left(B^{0,b}_{p,q}\right)' &= \left(\left(H^{-1}_p, H^1_p\right)_{1/2,q,(b,b)}\right)' \\ &= \left(H^1_{p'}, H^{-1}_{p'}\right)_{1/2,\infty,(-b,-b)} = B^{0,-b}_{p',\infty}. \end{aligned}$$

□

Next we determine the dual of spaces $\mathbf{B}^{0,b}_{p,q}$ with the help of logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}$ studied by Haroske in [27, 28] (see also [22, p.149]) and the references given there.

For $1 < p < \infty$ and $\alpha \ge 0$, the space $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}$ consists of all functions $f \in L_p$ having a finite norm

$$\|f\|_{\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}} = \|f\|_{L_p} + \sup_{0<t<1} \left(\frac{\omega(f,t)_p}{t(1-\log t)^\alpha}\right).$$

For $s \in \mathbb{R}$, we denote by I_s the lift operator defined by

$$I_s f = \mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F} f.$$

Theorem 5.2. *Let $1 < p < \infty$, $1/p + 1/p' = 1$, $0 < q < 1$ and $b + 1/q > 0$. The space $\left(\boldsymbol{B}^{0,b}_{p,q}\right)'$ is formed by all $f \in H^{-1}_{p'}$ such that $I_{-1} f \in Lip^{(1,-b-1/q)}_{p',\infty}$. Moreover*

$$\|f\|_{\left(\boldsymbol{B}^{0,b}_{p,q}\right)'} \sim \|I_{-1} f\|_{Lip^{(1,-b-1/q)}_{p',\infty}}.$$

PROOF. Consider the Banach couple formed by the Sobolev space W^1_p and L_p. By [2, Theorem 5.4.12], we get

$$K(t, f; W^1_p, L_p) = tK(t^{-1}, f; L_p, W^1_p) \sim \min(1,t)\,\|f\|_{L_p} + t\omega(f, t^{-1})_p.$$

Take any τ with $\tau + 1/q < 0$. Using (2.5), we obtain that

$$\mathbf{B}^{0,b}_{p,q} = \left(W^1_p, L_p\right)_{1,q,(\tau,b)} .$$

Applying Theorem 4.3 and having in mind that $(H^s_p)' = H^{-s}_{p'}$, we derive that

$$\left(\mathbf{B}^{0,b}_{p,q}\right)' = \left(H^{-1}_{p'}, L_{p'}\right)_{1,\infty,(-b-1/q,-\tau-1/q)} .$$

On the other hand, since $I_{-1} : H^s_{p'} \to H^{s+1}_{p'}$ is bijective and bounded, we have

$$K(t, f; H^{-1}_{p'}, L_{p'}) \sim K(t, I_{-1}f; L_{p'}, W^1_{p'}) \sim \min(1, t) \left\|I_{-1}f\right\|_{L_{p'}} + \omega(I_{-1}f, t)_{p'}.$$

Therefore, using (2.4), we conclude

$$\begin{aligned} \|f\|_{\left(\mathbf{B}^{0,b}_{p,q}\right)'} &\sim \sup_{0<t<1} \left(\frac{K(t, f; H^{-1}_{p'}, L_{p'})}{t(1-\log t)^{b+1/q}} \right) \\ &\sim \|I_{-1}f\|_{L_{p'}} + \sup_{0<t<1} \left(\frac{\omega(I_{-1}f, t)_{p'}}{t(1-\log t)^{b+1/q}} \right) \\ &= \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1/q)}_{p',\infty}} . \end{aligned}$$

□

For the special case $b = 0$ we obtain the following.

Corollary 5.3. *Let* $1 < p < \infty$, $1/p + 1/p' = 1$ *and* $0 < q < 1$. *The space* $\left(\boldsymbol{B}^0_{p,q}\right)'$ *is formed by all* $f \in H^{-1}_{p'}$ *such that* $I_{-1}f \in Lip^{(1,-1/q)}_{p',\infty}$. *Moreover*

$$\|f\|_{\left(\boldsymbol{B}^0_{p,q}\right)'} \sim \|I_{-1}f\|_{Lip^{(1,-1/q)}_{p',\infty}} .$$

Remark 5.4. As follows from Theorem 5.1, the dual of $B^{0,b}_{p,q}$ does not depend on q for $0 < q < 1$. However this is not the case for the space $\mathbf{B}^{0,b}_{p,q}$. Moreover, Theorem 5.2 point out a remarkable difference between the dual of $\mathbf{B}^{0,b}_{p,q}$ in the quasi-Banach case $0 < q < 1$ and the dual in the Banach case $1 \le q < \infty$ described in [9, Theorem 4.3]: $\|f\|_{\left(\mathbf{B}^{0,b}_{p,q}\right)'} \sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}}$.

As one can see, the role of q when $0 < q < 1$ is in the exponent, $-b-1/q$, of the logarithm in the associated Lipschitz space, while in the Banach case that exponent has the constant value $-b-1$ and q has a role in the second index, q', of the Lipschitz space.

6. Applications to Operator Spaces

Let H be a Hilbert space and let $\mathcal{L}(H)$ be the Banach space of all bounded linear operators acting from H into H. For $T \in \mathcal{L}(H)$, *the singular numbers* of T are defined by

$$s_n(T) = \inf\{\|T - R\|_{\mathcal{L}(H)} : R \in \mathcal{L}(H) \text{ with rank } R < n\}, \ n \in \mathbb{N}.$$

Let S_∞ be the subspace of $\mathcal{L}(H)$ formed by all compact operators and for $1 \le p < \infty$ let S_p be *the Schatten p-class*, formed by all those $T \in \mathcal{L}(H)$ having a finite norm

$$\|T\|_{S_p} = \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{1/p}$$

(see [25, 29]). The so-called *Macaev ideals* $S_\Pi, S_{\infty,1}$ are defined as the collection of all $T \in \mathcal{L}(H)$ which have a finite norm

$$\begin{aligned}\|T\|_{S_\Pi} &= \sup\Big\{(1+\log n)^{-1}\sum_{k=1}^{n} s_k(T) : n \in \mathbb{N}\Big\},\\ \|T\|_{S_{\infty,1}} &= \sum_{n=1}^{\infty} s_n(T) n^{-1},\end{aligned}$$

respectively (see [33, 25, 1, 18]). We have the following continuous embeddings

$$S_1 \hookrightarrow S_\Pi^\circ \hookrightarrow S_\Pi \hookrightarrow S_q \hookrightarrow S_{\infty,1} \hookrightarrow S_\infty \hookrightarrow \mathcal{L}(H).$$

Here $1 < q < \infty$ and S° stands for the closure in the space S of the set of all finite rank operators in H. Furthermore, the following duality formulae hold

$$(S_1)' = \mathcal{L}(H) \text{ and } (S_p)' = S_{p'} \text{ for } 1 < p \le \infty, 1/p + 1/p' = 1, \quad (6.1)$$
$$(S_\Pi^\circ)' = S_{\infty,1} \text{ and } (S_{\infty,1})' = S_\Pi, \quad (6.2)$$

(see [25, Theorems III.12.3 and III.15.2]).

The space $S_{\infty,1}$ is a member of the scale

$$S_{\infty,q,b} = \left\{T \in \mathcal{L}(H) : \|T\|_{S_{\infty,q,b}} = \left(\sum_{n=1}^{\infty}[(1+\log n)^b s_n(T)]^q n^{-1}\right)^{1/q} < \infty\right\}$$

where $0 < q \le \infty$, $b + 1/q \ge 0$ if $q < \infty$ and $b > 0$ if $q = \infty$ because otherwise the space is equal to $\mathcal{L}(H)$. Theses spaces make sense even for operators between Banach spaces provided that we replace the singular numbers by the approximation numbers. They have been studied in [6, 15] among other papers.

Associated to S_Π we may consider the scale

$$S_{\Pi,q,b} = \left\{ T \in \mathcal{L}(H) : \|T\|_{S_{\Pi,q,b}} = \left(\sum_{n=1}^{\infty} [(1+\log n)^b \sum_{k=1}^{n} s_k(T)]^q n^{-1} \right)^{1/q} < \infty \right\}$$

where $b + 1/q < 0$ if $q < \infty$ and $b \leq 0$ if $q = \infty$ because otherwise the space is just $\{0\}$.

All these spaces can be obtained by interpolation between S_1 and S_∞ by using logarithmic interpolation methods as we show next.

Lemma 6.1. *Let $0 < q \leq \infty$ and $b, \eta \in \mathbb{R}$.*

(i) If $b + 1/q < 0 \leq \eta + 1/q$ and $q < \infty$, or $b \leq 0 < \eta$ and $q = \infty$, then we have, with equivalent quasi-norms,

$$(S_\infty, S_1)_{1,q,(b,\eta)} = S_{\Pi,q,b}.$$

(ii) If $\eta + 1/q < 0 \leq b + 1/q$ and $q < \infty$, or $\eta \leq 0 < b$ and $q = \infty$, then we have, with equivalent quasi-norms,

$$(S_1, S_\infty)_{1,q,(\eta,b)} = S_{\infty,q,b}.$$

Proof. Since $S_1 \hookrightarrow S_\infty$ and

$$K(n^{-1}, T; S_\infty, S_1) = n^{-1} \sum_{k=1}^{n} s_k(T), \ n \in \mathbb{N} \tag{6.3}$$

(see [41, 35, 42]), using (2.4) we obtain

$$\begin{aligned}
\|T\|_{(S_\infty,S_1)_{1,q,(b,\eta)}} &\sim \left(\int_0^1 [t^{-1}(1-\log t)^b K(t,T)]^q \frac{dt}{t} \right)^{1/q} \\
&\sim \left(\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} [t^{-1}(1-\log t)^b K(t,T)]^q \frac{dt}{t} \right)^{1/q} \\
&\sim \left(\sum_{n=1}^{\infty} [(1+\log n)^b \sum_{k=1}^{n} s_k(T)]^q n^{-1} \right)^{1/q} \\
&\sim \|T\|_{S_{\Pi,q,b}}.
\end{aligned}$$

To prove (ii), we use (2.5) obtaining that

$$\begin{aligned}
\|T\|_{(S_1,S_\infty)_{1,q,(\eta,b)}} &\sim \left(\int_1^\infty [t^{-1}(1+\log t)^b K(t,T)]^q \frac{dt}{t}\right)^{1/q}\\
&\sim \left(\sum_{n=1}^\infty \int_n^{n+1} [t^{-1}(1+\log t)^b K(t,T)]^q \frac{dt}{t}\right)^{1/q}\\
&\sim \left(\sum_{n=1}^\infty [(1+\log n)^b \frac{1}{n}\sum_{k=1}^n s_k(T)]^q n^{-1}\right)^{1/q}\\
&\sim \left(\sum_{n=1}^\infty [(1+\log n)^b s_n(T)]^q n^{-1}\right)^{1/q}\\
&= \|T\|_{S_{\infty,q,b}}\ ,
\end{aligned}$$

where we have used the generalized Hardy's inequality established in [16, Theorem 1.2] for the last equivalence. □

Remark 6.2. Under the assumptions of Lemma 6.1, according to Lemma 2.2, we have

$$(\mathcal{L}(H), S_1)_{1,q,(b,\eta)} \subseteq S_\infty,\ (S_1, \mathcal{L}(H))_{1,q,(\eta,b)} \subseteq S_\infty.$$

Since $S_1 \hookrightarrow S_\infty \hookrightarrow \mathcal{L}(H)$, it is easy to check that

$$K(t,T;S_1,\mathcal{L}(H)) = K(t,T;S_1,S_\infty),\ T \in S_\infty.$$

Therefore, under the assumptions on b, q, η as in Lemma 6.1, we also have

$$(\mathcal{L}(H), S_1)_{1,q,(b,\eta)} = S_{\Pi,q,b},\ (S_1, \mathcal{L}(H))_{1,q,(\eta,b)} = S_{\infty,q,b}.$$

We close the paper by showing the duality relationships between these two scales of spaces.

Theorem 6.3. *Let $0 < q \le \infty$ and $b \in \mathbb{R}$ with $b + 1/q < 0$. Then we have*

(i) $(S_{\Pi,q,b})' = S_{\infty,\infty,-b-1/q}$ *if* $0 < q < 1$,

(ii) $(S_{\Pi,q,b})' = S_{\infty,q',-b-1}$ *if* $1 \le q < \infty$, $1/q + 1/q' = 1$,

(iii) $\left(S^\circ_{\Pi,\infty,b}\right)' = S_{\infty,1,-b-1}$.

Proof. By Lemma 6.1, we know that $S_{\Pi,q,b} = (S_\infty, S_1)_{1,q,(b,0)}$. Hence, if $0 < q < 1$, according to (6.1), Theorem 4.3/(i) and Remark 6.2, we obtain

$$(S_{\Pi,q,b})' = (S_1, \mathcal{L}(H))_{1,\infty,(-1/q,-b-1/q)} = S_{\infty,\infty,-b-1/q}.$$

If $1 \le q < \infty$, we proceed similarly but using now [17, Theorem 5.6]. Finally, if $q = \infty$, we get by [17, Theorem 5.9]

$$\begin{aligned} \left(S^{\circ}_{\Pi,\infty,b}\right)' &= \Big((S_\infty, S_1)^{\circ}_{1,\infty,(b,1)}\Big)' \\ &= (S_1, \mathcal{L}(H))_{1,1,(-2,-b-1)} = S_{\infty,1,-b-1}. \end{aligned}$$

□

Theorem 6.4. *Let $0 < q \le \infty$ and $0 < b + 1/q$. Then we have*

(i) $(S_{\infty,q,b})' = S_{\Pi,\infty,-b-1/q}$ *if* $0 < q < 1$,

(ii) $(S_{\infty,q,b})' = S_{\Pi,q',-b-1}$ *if* $1 \le q < \infty$, $1/q + 1/q' = 1$,

(iii) $(S^{\circ}_{\infty,\infty,b})' = S_{\Pi,1,-b-1}$.

Proof. For $0 < q < 1$, according to Lemma 6.1, Theorem 4.3/(i) and Remark 6.2, we derive

$$\begin{aligned} (S_{\infty,q,b})' &= \big((S_1, S_\infty)_{1,q,(-2/q,b)}\big)' \\ &= \big(\mathcal{L}(H), S_1\big)_{1,\infty,(-b-1/q,1/q)} = S_{\Pi,\infty,-b-1/q}. \end{aligned}$$

The case $1 \le q < \infty$ is similar but using [17, Theorem 5.6]. If $q = \infty$, by Lemma 6.1 and [17, Theorem 5.9], we obtain

$$\begin{aligned} (S^{\circ}_{\infty,\infty,b})' &= \Big((S_1, S_\infty)^{\circ}_{1,\infty,(-1,b)}\Big)' \\ &= \big(\mathcal{L}(H), S_1\big)_{1,1,(-b-1,0)} = S_{\Pi,1,-b-1}. \end{aligned}$$

□

Since $S_\Pi = S_{\Pi,\infty,-1}$ and $S_{\infty,1} = S_{\infty,1,0}$, formulae (6.2) follows from Theorems 6.3 and 6.4.

Remark 6.5. Looking at the statements of Theorems 6.3 and 6.4 one can see that the exponent of the logarithm depends on q when $0 < q < 1$, while this is not the case for $1 \le q \le \infty$.

The dual of the space $S_{\infty,q,-1/q}$ does not belong to the scale of spaces $S_{\Pi,p,b}$. We compute it in the case $0 < q < 1$.

Theorem 6.6. *Let $0 < q < 1$. Then $(S_{\infty,q,-1/q})'$ coincides with the collection of all $T \in \mathcal{L}(H)$ which have a finite norm*

$$\|T\| = \sup\Big\{(1 + \log(1 + \log n))^{-1/q} \sum_{k=1}^{n} s_k(T) : k \in \mathbb{N}\Big\}.$$

Proof. Since $S_{\infty,q,-1/q} = (S_1, S_\infty)_{1,q,(-2/q,-1/q)}$, it follows from Theorem 4.3/(ii) that

$$(S_{\infty,q,-1/q})' = (\mathcal{L}(H), S_1)_{1,\infty,(0,1/q),(-1/q,0)}.$$

In order to identify this interpolation space, we observe that $K(t, T; \mathcal{L}(H), S_1) = \|T\|_{\mathcal{L}(H)}$ for $t \geq 1$. Hence,

$$\sup\left\{t^{-1}K(t,T)\ell^{1/q}(t) : t \geq 1\right\} \sim \|T\|_{\mathcal{L}(H)}.$$

Consequently, using (6.3), we derive

$$\begin{aligned}
&\|T\|_{(\mathcal{L}(H),S_1)_{1,\infty,(0,1/q),(-1/q,0)}} \\
&\quad \sim \sup\{t^{-1}K(t,T;\mathcal{L}(H),S_1)\ell\ell^{-1/q}(t) : 0 < t < 1\} \\
&\quad = \sup\{K(s,T;S_1,\mathcal{L}(H))\ell\ell^{-1/q}(s) : 1 < s < \infty\} \\
&\quad \sim \sup\left\{(1+\log(1+\log n))^{-1/q}\sum_{k=1}^{n} s_k(T) : n \in \mathbb{N}\right\}.
\end{aligned}$$

□

Acknowledgements

The authors would also like to thank the referee for his/her useful comments and for drawing their attention to Remark 4.5.

References

[1] J. Arazy, S.D. Fisher and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. 110 (1988) 989–1053.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.

[3] J. Bergh and J. Löfström, *Interpolation Spaces. An introduction*, Springer, Berlin, 1976.

[4] Y. Brudnyĭ and N. Krugljak, *Interpolation Functors and Interpolation Spaces*, Vol. 1, North-Holland, Amsterdam, 1991.

[5] A.M. Caetano, A. Gogatishvili and B. Opic, *Sharp embeddings of Besov spaces involving only logarithmic smoothness*, J. Approx. Theory 152 (2008) 188–214.

[6] F. Cobos, *On the Lorentz-Marcinkiewicz operator ideal*, Math. Nachr. 126 (1986) 281–300.

[7] F. Cobos, *Duality and Lorentz-Marcinkiewicz operator spaces*, Math. Scand. 63 (1988) 261–267.

[8] F. Cobos and O. Domínguez, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory 189 (2015) 43–66.

[9] F. Cobos and O. Domínguez, *On Besov spaces of logarithmic smoothness and Lipschitz spaces*, J. Math. Anal. Appl. 425 (2015) 71–84.

[10] F. Cobos and O. Domínguez, *On the relationship between two kinds of Besov spaces with smoothness near zero and some other applications of limiting interpolation*, J. Fourier Anal. Appl. 22 (2016) 1174-1191.

[11] F. Cobos, O. Domínguez and T. Kühn, *Approximation and entropy numbers of embeddings between approximation spaces*, Constr. Approx. 47 (2018) 453-486.

[12] F. Cobos, O. Domínguez and H. Triebel, *Characterizations of logarithmic Besov spaces in terms of differences, Fourier-analytical decompositions, wavelets and semi-groups*, J. Funct. Anal. 270 (2016) 4386-4425.

[13] F. Cobos and D.L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, in: *Function Spaces and Applications*. Lect. Notes in Math., vol. 1302, Springer, Berlin, 1988, pp. 158–170.

[14] F. Cobos, L.M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. 256 (2009) 2321-2366.

[15] F. Cobos and I. Resina, *Representation theorems for some operator ideals*, J. Lond. Math. Soc. 39 (1989) 324-334.

[16] F. Cobos and I. Resina, *On Hardy's inequality and eigenvalue distributions*, Port. Math. 50 (1993) 151-155.

[17] F. Cobos and A. Segurado, *Description of logarithmic interpolation spaces by means of the J-functional and applications*, J. Funct. Anal. 268 (2015) 2906–2945.

[18] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[19] M. Cwikel and J. Peetre, *Abstract K and J spaces*, J. Math. Pures Appl. 60 (1981) 1–50.

[20] R.A. DeVore, S.D. Riemenschneider and R.C. Sharpley, *Weak interpolation in Banach spaces*, J. Funct. Anal. 33 (1979) 58–94.

[21] D. E. Edmunds and B. Opic, *Limiting variants of Krasnosel'skiĭ's compact interpolation theorem*, J. Funct. Anal. 266 (2014) 3265–3285.

[22] D.E. Edmunds and W.D. Evans, *Hardy Operators. Function Spaces and Embeddings*, Springer, Berlin, 2004.

[23] W.D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. 52 (2000) 920–960.

[24] W.D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. 7 (2002) 187–269.

[25] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1969.

[26] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. 42 (1978) 289–305.

[27] D.D. Haroske, *On more general Lipschitz spaces*, Z. Anal. Anwend. 19 (2000) 781–799.

[28] D.D. Haroske, *Envelopes and Sharp Embeddings of Function Spaces*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 437, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[29] H. König, *Eigenvalue Distribution of Compact Operators*, Birkhäuser, Basel, 1986.

[30] H.-G. Leopold, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, in: *Function Spaces*. Lect. Notes in Pure and Appl. Math., vol. 213, Marcel Dekker, New York, 2000, pp. 323–336.

[31] J. L. Lions, *Sur les espaces d'interpolation; dualité*, Math. Scand. 9 (1961) 147–177.

[32] J.-L. Lions and J. Peetre, *Sur une classe d´espaces d´interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964) 5–68.

[33] V.I. Macaev, *A class of completely continuous operators*, Soviet Math. Dokl. 2 (1961) 972–975.

[34] M. Mastyło, *Banach envelopes of some interpolation quasi-Banach spaces*, in: *Function Spaces and Applications*. Lect. Notes in Math., vol. 1302, Springer, Berlin, 1988, pp. 321–329.

[35] C. Merucci, *Interpolation dans* $C^{\omega}(H)$, C. R. Acad. Sc. Paris 274 (1972) 1163–1166.

[36] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, in: *Interpolation Spaces and allied Topics in Analysis*, Lect. Notes in Math., vol. 1070, Springer, Berlin, 1984, pp. 183–201.

[37] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. 132 (1982) 291–330.

[38] B. Opic and L. Pick, *On generalized Lorentz-Zygmund spaces*, Math. Inequal. Appl. 2 (1999) 391–467.

[39] J. Peetre, *Remark on the dual of an interpolation space*, Math. Scand. 34 (1974) 124-128.

[40] L.-E. Persson, *Interpolation with a parameter function*, Math. Scand. 59 (1986) 199–222.

[41] H. Triebel, *Über die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev-Besov-Räumen*, Invent. Math. 4 (1967) 275–279.

[42] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.