NIP THEORIES AND SHELAH'S THEOREM

ELÍAS BARO

These are the –informal and expanded– notes of a mini-course given in the Universidad Autónoma de Madrid, 29-30 of May 2018 (6 hours). The mini-course was given at the end of a master course on Model Theory with applications to Algebra taught by Prof. Margarita Otero. The purpose of the mini-course is to introduce the students to modern pure model theoretic tools. Specifically, our purpose is the give the definition of stable theory, to give the definition of NIP theory and to give the statement and some hints of the proof of Shelah's Theorem [8] (which says that if we add to the language the externally definable sets of a model of a NIP theory, then the theory remains NIP).

We will work with a language L and an L-theory T complete and without finite models. We denote by M a model of T, and $\mathfrak{C}$ denotes an elementary extension of M which is κ-saturated and strongly κ-homogeneous for a "big" cardinal κ, say greater than $2^{2^{|M|}}$ (the model $\mathfrak{C}$ is what is known as the *monster model*, see [9, Thm.6.1.7]). We will use small letters $a, b, c, \ldots$ for tuples of $\mathfrak{C}$ and capital letters $A, B, C, \ldots$ for subsets of $\mathfrak{C}$ of cardinality less than κ. We will denote $\phi(x, y) \in L_A$ to say that $\phi(x, y)$ is a formula in the language L with parameters in a subset A. The notation $\models \phi(a)$ means $\mathfrak{C} \models \phi(a)$.

Along the document, we have introduced some paragraphs called Miscellany which are intended to give intuitions.

Finally, we would like to stress that there is *no originality* at all in the whole text. Everything has been extracted from the references listed at the end of the document. The Introduction is based on [4] and [5]; Section 2 is based on [1] and [7]. The proof Shelah's Theorem is presented along the rest of the sections and it follows the preprint [10] (which in turn is based on [2]).

1. Introduction

Recall the following statement from the master course:

Theorem 1.1 (Morley'65)**.** *If T is countable and T is κ_0-categorical for some $\kappa_0 > \aleph_0$ then T is κ-categorical for all $\kappa > \aleph_0$.*

Because of the methods and notions used in the proof, this theorem is commonly recognized as the starting point of modern Model Theory. Morley proposed several questions concerning the number $I_T(\kappa)$ of models of a theory T of cardinality κ up to isomorphism. These questions were analised by several model theorists, specially by S. Shelah in the 70' and 80'. One of the first notions he introduced to attack this problem is the following:

Definition 1.2. Let $\phi(x, y) \in L_{\mathfrak{C}}$ a formula. We say that ϕ has the *order property* (OP) in T if there are sequences $(a_i : i < \omega)$ and $(b_j : i < \omega)$ such that

$$\models \phi(a_i, b_j) \Longleftrightarrow i \leq j.$$

Otherwise, we say that $\phi(x, y)$ has the NOP. We say that the theory T is *unstable* if some formula without parameters has the OP in T. Otherwise, we say that it T is *stable*.

Henceforth, we will just write that a formula "has OP" instead of "has OP in T".

Date: 6 June, 2018.

Remark 1.3. 1) If $\phi(x, y, c)$ with $c \in \mathfrak{C}$ has the OP then $\phi(x, y, z)$ has the OP. For, if $(a_i : i < \omega)$ and $(b_j : i < \omega)$ witness the OP of $\phi(x, y, c)$, then $(a_i : i < \omega)$ and $(\widetilde{b}_j : j < \omega)$ witness the OP of $\phi(x, y, z)$, where $\widetilde{b}_j = (b_j, c)$.

2) By compactness, if $\phi(x, y) \in L_{\mathfrak{C}}$ has the NOP then there is $\ell < \omega$ such that for all $n > \ell$ there do not exist sequences $(a_i : i < n)$ and $(b_j : j < n)$ with

$$\models \phi(a_i, b_j) \iff i \leq j.$$

In other words, for each $n > \ell$, the sentence ψ_n^ℓ which says that there are not $x_0, \ldots, x_n, y_0, \ldots, y_n$ such that

$$\bigwedge_{i \leq j} \phi(x_i, y_j) \wedge \bigwedge_{i > j} \neg\phi(x_i, y_j)$$

is implied by T. In particular, note that by 1) and 2), we could give an equivalent definition of stability without mention the model $\mathfrak{C}$.

3) Again by compactness, if $(I, \leq)$ is a linear order and $\phi(x, y)$ has the OP, then there exists $(a_i : i \in I)$ and $(b_j : j \in I)$ such that

$$\models \phi(a_i, b_j) \iff i \leq j.$$

4) If $\phi(x, y) \in L_{\mathfrak{C}}$ has the OP, then the formula $\phi_{\text{opp}}(y, x) := \phi(x, y)$ has the OP. Indeed, let $\leq_*$ be the linear order in ω given by $i \leq_* j$ if and only if $i \geq j$. Then by 3) there are sequences $(a_i : i \in (\omega, \leq_*))$ and $(b_j : j \in (\omega, \leq_*))$ such that

$$\models \phi(a_i, b_j) \iff i \leq_* j \iff i \geq j$$

In particular, the sequences $(b_i : i < \omega)$ and $(a_j : j < \omega)$ satisfying

$$\models \phi_{\text{opp}}(b_i, a_j) \iff \models \phi(a_j, b_i) \iff i \leq j$$

as required.

The above notion provide a first *dividing line* towards the study of $I_T(-)$.

Theorem 1.4 (Shelah'71)**.** *If T is unstable then $I_T(\kappa) = 2^\kappa$ for all $\kappa > |T| + \aleph_0$.*

Example 1.5. 1) As an immediate consequence of the above theorem we have that the $\aleph_1$-categorical and complete theories ACF_0 and ACF_p for p prime, are stable. Let us show by hand that every formula $\phi(x, y)$ in the language of rings and with $\lg(x) = 1$ has the NOP in the theory of ACF. Indeed, by the quantifier elimination we can assume that $\phi(x, y)$ is of the form

$$\phi(x, y) : \bigvee_i \bigwedge_j \big(p_{ij}(x) = 0 \wedge q_i(x) \neq 0\big)$$

Let n_0 be a natural number bigger than the degree of the polynomials involved in the above expression. Then it follows that for every b, if $\phi(\mathfrak{C}, b)$ is finite then it has cardinality less than n_0, and if $\phi(\mathfrak{C}, b)$ is infinite then $\neg\phi(\mathfrak{C}, b)$ has cardinality less than n_0 (so definable sets of $\mathfrak{C}^1$ are finite or cofinite). Thus, if $(a_i : i < \omega)$ and $(b_j : j < \omega)$ satisfy that

$$\phi(a_i, b_j) \iff i \leq j$$

then both the sets $\phi(\mathfrak{C}, b_{n_0+1})$ and $\neg\phi(\mathfrak{C}, b_{n_0+1})$ have cardinality greater than n_0, a contradiction.

2) The o-minimal theory $\mathrm{Th}(\mathbb{R}, <, +, \cdot, -0, 1)$ is clearly unstable.

Because of the above theorem, the analysis of $I_T(-)$ focused in the stable case. Along the way, a lot of structural properties of the definable sets of models of stable theories were discovered. For example, recall that given $M \models T$ we say that $X \subseteq M^n$ is *externally definable* if for some formula $\phi(x, y)$ and parameter $a \in \mathfrak{C}$ we have

$$X = \phi(a, M) = \{m \in M : \mathfrak{C} \models \phi(a, m)\}.$$

Proposition 1.6. *Let T be stable and $M \models T$. Then every externally definable subset of M^n is definable in M.*

Proof. Consider the externally definable subset $\phi(a, M)$ of M^n, where $\phi(x, y) \in L$ and $a \in \mathfrak{C}$. We want to prove:

$$(\star) \qquad \text{there is } \delta(y) \in L_M \text{ such that } \delta(M) = \phi(a, M).$$

We first show that a suitable intersection of definable sets is contained in $\phi(a, M)$.

Claim. There is $\ell \in \mathbb{N}$ such that for every $\psi(x) \in \mathrm{tp}(a|M)$

$$(*) \qquad \text{there are } a_0, \ldots, a_\ell \in \psi(M) \text{ with } \bigcap_{i=0}^{\ell} \phi(a_i, M) \subseteq \phi(a, M).$$

Proof of the Claim. Let ℓ be the number that witness the NOP of the formula $\phi(x, y)$ as in Remark 1.3.(2), and fix $\psi(x) \in \mathrm{tp}(a|M)$. Suppose $(*)$ is false. Since $\models \psi(a)$ and M is a model, it follows $M \models \exists x \psi(x)$. Pick $a_0 \in \psi(M)$. If $\phi(a_0, M) \subseteq \phi(a, M)$ then we are done. Otherwise, let $b_0 \in \phi(a_0, M)$ with $b_0 \notin \phi(a, M)$. Since $M \models \exists x(\psi(x) \wedge \neg\phi(x, b_0))$, there is $a_1 \in M$ with $a_1 \in \psi(M)$ and such that $\models \neg\phi(a_1, b_0)$. If $\phi(a_0, M) \wedge \phi(a_1, M) \subseteq \phi(a, M)$ then we are done. Otherwise, let $b_1 \in \phi(a_0, M) \cap \phi(a_1, M)$ such that $b_1 \notin \phi(a, M)$. Recursively, we find $a_0, \ldots, a_{\ell+1}$ and $b_0, \ldots, b_{\ell+1}$ such that $\models \psi(a_i)$ for all $i = 0, \ldots, \ell + 1$ and

$$\phi(a_i, b_j) \iff i \le j,$$

which is a contradiction with Remark 1.3(2).

Miscellany: We can imagine that the tuples $a_0, \ldots, a_\ell$ are "near" the tuple a (because they satisfy any fixed $\psi(x) \in \mathrm{tp}(a|M)$). So that in some sense, the above claim says that a intersection of "translations" $\phi(a_0, M), \ldots, \phi(a_\ell, M)$ of $\phi(x, M)$ by suitable elements $a_0, \ldots, a_\ell$ "near" a is contained in $\phi(a, M)$. Now, since each a_i is "near" a we can imagine that each $\phi(a_i, M)$ is "big" in $\phi(a, M)$. If we think in terms of algebraic geometry, the intersection of "big" sets is "big". So we have a family of "big" subsets of $\phi(a, M)$ which is parametrized by tuples in M which are as "near" to a as we want. Thus, the union of finitely many of these subsets should cover $\phi(a, M)$.

Define the formula $\Phi(x_0, \ldots, x_\ell, y) : \bigwedge_{i=0}^{\ell} \phi(x_i, y)$, so by the claim there are $a_0, \ldots, a_\ell$ such that

$$\Phi(a_0, \ldots, a_\ell, M) \subseteq \phi(a, M).$$

Assume that $(\star)$ is false. Then there is $b_0 \in \phi(a, M)$ such that $b_0 \notin \Phi(a_0, \ldots, a_\ell, M)$. Now, since $\phi(x, b_0) \in \mathrm{tp}(a|M)$, again by the claim there are $a'_0, \ldots, a'_\ell \in \phi(M, b_0)$ such that

$$\Phi(a'_0, \ldots, a'_\ell, M) \subseteq \phi(a, M).$$

Note that $b_0 \in \Phi(a'_0, \ldots, a'_\ell, M)$. Let us denote $c_0 := (a'_0, \ldots, a'_\ell)$, so that the above reads:

$$\models \Phi(c_0, b_0) \quad \& \quad \Phi(c_0, M) \subseteq \phi(a, M).$$

Since $(\star)$ is false, there is $b_1 \in \phi(a, M) \setminus \Phi(c_0, M)$. Again, there are $a''_0, \ldots, a''_\ell \in \phi(M, b_1)$ such that

$$\Phi(a''_0, \ldots, a''_\ell, M) \subseteq \phi(a, M).$$

Denote $c_1 := (a''_0, \ldots, a''_\ell)$, so that

$$\models \Phi(c_1, b_1) \quad \& \quad \models \neg\Phi(c_0, b_1).$$

Since $(\star)$ is false, there is $b_2 \in \phi(a, M) \setminus [\Phi(c_0, M) \cup \Phi(c_1, M)]$. Recursively, we construct sequences $(c_i : i < \omega)$ and $(b_j : i < \omega)$ such that

$$\models \Phi(c_i, b_j) \iff i \ge j$$

and so Φ_{opp} has the OP, a contradiction. □

The reciprocal of the above proposition is true and gives a characterization of stability:

Theorem 1.7. [5, Cor.2.10] *The following conditions are equivalent:*

(1) T *is stable.*
(2) *Every externally definable set of* $M \models T$ *is definable.*
(3) *For every* $M \models T$ *of cardinality* $\lambda > |T|$ *we have that* $|S_n(M)| = \lambda$.

As an easy consequence (exercise), to check that a theory is stable it suffices to show that the formulas of the form $\phi(x, y)$ with $\lg(x) = 1$ have the NOP, as we did with ACF.

Let us see another important example.

Example 1.8. Let K be a field. We say that a map $\delta : K \to K$ is a *derivation* if $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in K$. We say that a field K of characteristic 0 with a derivation d is *differentially closed* if for every $P \in K[x_0, \ldots, x_n] \setminus K[x_0, \ldots, x_{n-1}]$ and $g \in K[x_0, \ldots, x_{n-1}]$, $g \neq 0$, there is $a \in K$ such that $P(a, da, \ldots, , d^n a) = 0$ and $g(a, da, \ldots, , d^{n-1} a) \neq 0$. The theory DCF_0 of differentially closed fields in the language of fields with a symbol for the derivation, has QE and is stable.

Miscellany: Let k and K be algebraic closed fields of char $= 0$ with $k \subseteq K$. In Algebraic Geometry, we often handle an algebraic subset of K^n and we want to find an algebraic subset defined over k with certain nice properties. Of course, k is not a definable subset of K in the ring language (recall that every definable subset of K is finite or cofinite). However, we can find a derivation $\delta : K \to K$ such that $k = \{x \in K : \delta(x) = 0\}$. Moreover, we can even find a differentially closed extension $(\hat{K}, \hat{\delta})$ of (K, δ) such that still $k = \{x \in K : \hat{\delta}(x) = 0\}$ (see [6, §6.2] and [3, Ex.1.14]). Thus, our field k is *now* a definable object in the language of rings with the symbol $\hat{\delta}$, and this is a good point to achieve our purpose. Since we have added a new function symbol for the derivation, we have more definable sets. But we did not lose at all the "tame" behaviour of ACF_0 because DFC_0 is still stable. That is why studying extensions of well-known theories is a useful matter.

From the point of view of the study $I_T(-)$, the unstable theories are complicated. But we cannot say a priori that they are "wild". For, there are even stable theories with $I_T(\kappa) = 2^\kappa$, though we already mentioned that they have nice properties. Shelah (and others) introduced more dividing lines in order to handle unstable theories. The following web page gives a beautiful panoramic photo of *the map of the universe*:

www.forkinganddividing.com

One of these dividing lines is the *independence property*, that we will study in the next section (and which includes o-minimal theories).

2. Formulas with IP

Let us give the definition of the independence property:

Definition 2.1. Let $\phi(x, y) \in L_{\mathfrak{C}}$. We say that $\phi(x, y)$ has the *the independence property* (IP) if there are $(a_i : i < \omega)$, $(b_I : I \subseteq \omega)$ such that

$$\models \phi(a_i, b_I) \iff i \in I.$$

If ϕ does not have IP we say it has NIP. We say that T has IP if some formula without parameters has IP, and otherwise it is said that T has NIP.

Remark 2.2. 1) If $\phi(x, y, z) \in L$ and $\phi(x, y, a)$ has IP, then $\phi(x; y, z)$ has IP.

2) If $\phi(x,y) \in L_{\mathfrak{C}}$ is a formula with NOP, by compactness there is $\ell < \omega$ such that for every $n > \ell$ there do not exist a sequence $(a_i : i < n)$ such that for all $I \subseteq n$,

$$\{\phi(a_i,y) : i \in I\} \cup \{\neg\phi(a_i,y) : i \in n \setminus I\}$$

is consistent.

3) By compactness, if $\phi(x,y) \in L_{\mathfrak{C}}$ has IP then for every set X there are $(a_i : i \in X)$, $(b_I : I \subseteq X)$ such that

$$\models \phi(a_i, b_I) \iff i \in I$$

4) If $\phi(x,y) \in L_{\mathfrak{C}}$ has IP, then the formula $\phi_{\mathrm{opp}}(y,x) := \phi(x,y)$ has IP. Indeed, let us show that ϕ_{opp} satisfies (2) for a fixed $n < \omega$. Since ϕ has IP and by (3), there are $(a_X : X \in \mathcal{P}(n)\}$, $(b_I : I \subseteq \mathcal{P}(n))$ such that

$$\models \phi(a_X, b_I) \iff X \in I.$$

Given $i < n$, define $U_i := \{X \subseteq n : i \in X\}$ and $c_i := b_{U_i}$. Then, for the sequence $(c_i : i < n)$ we have that

$$\models \phi_{\mathrm{opp}}(c_i, a_X) \iff \models \phi(a_X, c_i) \iff X \in U_i \iff i \in X$$

and so for every $X \subseteq n$ the set

$$\{\phi_{\mathrm{opp}}(c_i, x) : i \in X\} \cup \{\neg\phi_{\mathrm{opp}}(c_i, x) : i \in n \setminus X\}$$

is consistent, as required.

As in the stable setting, we have the following:

Proposition 2.3. *If T es IP then there is $\phi(x,y) \in L$ which has IP and such that $lg(x) = 1$.*

Of course, this result is very useful to check that a given theory has NIP. To prove it, we must characterize IP in another way. We will do it in Section 3, but we point out that it is not necessary to prove Shelah's Theorem (so the reader just interested in the latter can skip Section 3). Instead, we just will need to check that a certain property implies IP. We recall the following model theoretic basic concept:

Definition 2.4. Let I be a linear order and M an L-structure. A sequence $(a_i : i < \omega)$ of elements of M is called an *indiscernible sequence* if for all L-formulas $\phi(x_1, \ldots, x_n)$ and indexes $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ from I we have that

$$M \models \phi(a_{i_1}, \ldots, a_{i_n}) \iff M \models \phi(a_{j_1}, \ldots, a_{j_n}).$$

Proposition 2.5. *Let $\phi(x,y)$ be an L-formula such that there are an indiscernible sequence $(a_i : i < \omega)$ and a tuple b with*

$$\models \phi(a_i, b) \iff i \text{ is even.}$$

Then ϕ has IP.

Proof. Let us see that for all $n < \omega$ and $I \subseteq n$, the set of formulas

$$\{\phi(a_i,y) : i \in I\} \cup \{\neg\phi(a_i,y) : i \in n \setminus I\}$$

is consistent. Indeed, denote $I = \{m_0 < \cdots < m_\ell\}$ and

$$n \setminus I = \{k_0 < \cdots < k_s\}.$$

Define $m'_i := 2m_i$ and $k'_i := 2k_i + 1$. Is is easy to see that

$$\begin{cases} m_i < m_j \iff m'_i < m'_j \\ k_i < k_j \iff k'_i < k'_j \\ m_i < k_j \iff m'_i < k'_j \end{cases}$$

Let $I' := \{m'_0 < \cdots < m'_\ell\}$ and $J' := \{k'_0 < \cdots < k'_s\}$. Then by hypothesis

$$\{\phi(a_i,y) : i \in I'\} \cup \{\neg\phi(a_i,y) : i \in J'\}$$

is consistent because it is realized by b. Thus, by indiscernibility

$$\{\phi(a_i, y) : i \in I\} \cup \{\neg\phi(a_i, y) : i \in n \setminus I\}$$

is consistent. □

Miscellany: For example, suppose that $n = 5$ and let $I = \{0, 1, 4\}$, so that $n \setminus I = \{2, 3\}$. We want to code the following configuration:

$$\begin{array}{ccccc} \checkmark & \checkmark & \times & \times & \checkmark \\ \bullet_0 & \bullet_1 & \bullet_2 & \bullet_3 & \bullet_4 \end{array}$$

In this case we would obtain $I' = \{0, 2, 8\}$ and $J' = \{5, 7\}$, and we see that the sequence $a_0, a_2, a_8, a_5, a_7, a_8$ codes the above configuration:

$$\begin{array}{ccccccccccc} \checkmark & \times & \checkmark & \times & \checkmark & \times & \checkmark & \times & \checkmark & \times & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdots \end{array}$$

The reciprocal of the above proposition is true (and again is essentially the content of Section 3, see Remark 3.5).

We finish this section with some examples of NIP theories, we take as granted the results in Section 3.

Example 2.6. If T is stable. Recall that T is stable if and only if no formula has the order property in T. Recall that $\phi(x, y)$ has the order property if and only if there are sequences $(a_i : i < \omega)$ and $(b_i : i < \omega)$ such that

$$\models \psi(a_i, b_i) \iff i \leq j.$$

If T has IP then there exist $\phi(x, y)$ and sequences $(a_i : i < \omega)$ and $(b_I : I \subseteq \omega)$ with

$$\models \psi(a_i, b_I) \iff i \in I.$$

Define $\tilde{b}_j := b_{j+1}$ for each $j < \omega$. Then for the sequences $(a_i : i < \omega)$ and $(\tilde{b}_j : j < \omega)$ we have

$$\models \psi(a_i, \tilde{b}_j) \iff \models \psi(a_i, b_{j+1}) \iff i < j + 1 \iff i \leq j$$

and therefore $\psi(x, y)$ has the order property.

Example 2.7. It T is o-minimal then T is NIP. Indeed, by Proposition 2.3, if T has IP then there exists a formula $\psi(x, y)$ with IP and where x is a single variable. Then there is some indiscernible sequence of elements $(a_i : i < \omega)$ and some tuple b such that

$$\models \psi(a_i, b) \iff i \text{ is even}$$

By o-minimality $\psi(x, b)$ defines a finite union of interval and points. Hence, for some boolean combination $\phi(x, z)$ of formulas $x < z_k$ and some tuple c, the formula $\psi(x, b)$ is equivalent to $\phi(x, c)$. Since

$$\models \phi(a_i, c) \iff i \text{ is even}$$

we have that $\phi(x, z)$ is IP too. In fact, by Corollary 3.6 we can assume that $\phi(x, z)$ is the formula $x < z$. Therefore there exists an indiscernible sequence of elements $(d_i : i < \omega)$ and some element c such that

$$\models d_i < c \iff i \text{ is even}.$$

In particular we have that $d_0 < c$ and $c \leq d_1$. By indiscernibility and since $d_0 < d_1$, we also have that $d_1 < d_2$, so that $c \leq d_1 < d_2$. However $d_2 < c$ since 2 is even, a contradiction.

We enumerate some examples of NIP theories, see Appendix A in [7] for details.

Example 2.8. (1) The theory of an abelian ordered groups.
(2) The theory of the p-adic numbers.
(3) The theory of an algebraic closed valued field.
(4) The theory of a Henselian field of residue field of char $= 0$ with a NIP theory.

3. Alternation number

Our objective in this section is to prove Proposition 2.3. For that aim, we will use indiscernible sequences introduced in the preceding section. The following lemma is a basic model theoretic tool which ensures the existence of indiscernibles (see [9, Lemma 5.1.3]).

Definition 3.1. Let I be a linear order, let M be an L-structure and let $(a_i : i \in I)$ be a sequence of elements of M. The *Ehrenfeucht-Mostowski type* of the sequence is the set of formulas $\phi(x_1, \dots, x_n)$ such that $M \models \phi(a_{i_1}, \dots, a_{i_n})$ for all $i_1 < \cdots < i_n$.

Lemma 3.2 (Standard Lemma). *Let I be a linear order and M an L-structure. Given a sequence $(a_i : i \in I)$ of elements of M, there is an* indiscernible sequence *satisfying the EM-type of $(a_i : i \in I)$.*

Now, we can characterize IP in another way as follows:

Definition 3.3. Let $\phi(x, y) \in L$. We write $\mathrm{alt}(\phi) < \infty$ if for each indiscernible sequence $(a_i : i \in I)$ and each tuple b there is $n < \omega$ such that either $\models \phi(a_i, b)$ for all $i > n$, or $\models \neg\phi(a_i, b)$ for all $i > n$. Otherwise, we write $\mathrm{alt}(\phi) = \infty$.

Proposition 3.4. *The formula $\phi(x, y)$ has IP if and only if $\mathrm{alt}(\phi) = \infty$.*

Proof. Left to right: Let $(a_i : i \in \omega)$ and $(b_I : I \subseteq \omega)$ witness the IP of $\phi(x, y)$. We first show that the IP of $\phi(x, y)$ is also witnessed by an indiscernible sequence $(a'_i : i \in \omega)$ and a sequence $(b'_I : I \subseteq \omega)$.

For any $n < \omega$ and any $I \subseteq n$ consider the formula

$$\psi_I(x_0, \dots, x_n) : \exists y (\bigwedge_{i \in I} \phi(x_i, y) \wedge \bigwedge_{i \in n \setminus I} \neg\phi(x_i, y)).$$

Note that for any $j_0 < \cdots < j_n$ we have that $\models \psi_I(a_{j_0}, \dots, a_{j_n})$. Indeed, if we define $J := \{j_k : k \in I\}$ then $(a_{j_0}, \dots, a_{j_n}, b_J)$ satisfies $\bigwedge_{i \in I} \phi(x_i, y) \wedge \bigwedge_{i \in n \setminus I} \neg\phi(x_i, y)$. This proves that $\psi_J(x_0, \dots, x_n)$ belongs to the EM-type of the sequence $(a_i : i < \omega)$ and so there exists by Lemma 3.2 an indiscernible sequence $(a'_i : i < \omega)$ satisfying all the formulas $\psi_I(x_1, \dots, x_n)$. By compactness there exists $(b'_I : i \in I)$ such that

$$\models \phi(a'_i, b'_I) \iff i \in I,$$

as required.

Finally, denote the even numbers by $I_0 \subseteq \omega$ and pick $c := b'_{I_0}$. Since

$$\models \phi(a'_i, c) \iff i \text{ is even}$$

it follows that $\mathrm{alt}(\phi) = \infty$.

Right to left: let $(a_i : i \in \omega)$ be an indiscernible sequence and a tuple b such that for every $n < \omega$ there are $i, j > n$ with $\models \phi(a_i, b)$ and $\models \neg\phi(a_j, b)$. By Remark 2.2(2) it suffices to prove that for any $n < \omega$ and for all $I \subseteq n$, the set of formulas

$$\{\phi(a_i, y) : i \in I\} \cup \{\neg\phi(a_i, y) : i \in n \setminus I\}$$

is consistent. For that aim, it is enough to argue as in the proof of Proposition 2.5. Indeed, denote $I = \{m_0 < \cdots < m_\ell\}$ and $n \setminus I = \{k_0 < \cdots < k_s\}$. Since $\mathrm{alt}(\phi) = \infty$, we can find indexes $m'_0 < \cdots < m'_\ell$ and indexes $k'_0 < \cdots < k'_s$ such that

$$\models \phi(a_{m'_i}, b) \qquad \& \qquad \models \neg\phi(a_{k'_i}, b) \qquad \& \qquad m_i < k_j \iff m'_i < k'_j.$$

Denote $I' := \{m'_0 < \cdots < m'_\ell\}$ and $J' := \{k'_0 < \cdots < k'_s\}$. Then

$$\{\phi(a_i, y) : i \in I'\} \cup \{\neg\phi(a_i, y) : i \in J'\}$$

is consistent. By indiscernibility $\{\phi(a_i, y) : i \in I\} \cup \{\neg\phi(a_i, y) : i \in n \setminus I\}$ is consistent, as required. □

Remark 3.5. Note that in the proof of the left to right implication we have proved the reciprocal implication of Proposition 2.5.

Corollary 3.6. *Any Boolean combination of formulas with the NIP has the NIP.*

Proof. For negations of formulas it is clear. Pick formulas $\phi_1(x, y)$ and $\phi_2(x, y)$ with the NIP and let us show that $\phi(x, y) := \phi_1(x, y) \wedge \phi_2(x, y)$ has the NIP. Let $(a_i : i < \omega)$ be an indiscernible sequence and let b a tuple. Then by Proposition 3.4 there is $n_1 < \omega$ such that

$$\models \phi_1(a_i, b) \text{ for all } i > n_1 \qquad \text{or} \qquad \models \neg\phi_1(a_i, b) \text{ for all } i > n_1.$$

Again by Proposition 3.4 there is $n_2 < \omega$ such that

$$\models \phi_2(a_i, b) \text{ for all } i > n_2 \qquad \text{or} \qquad \models \neg\phi_2(a_i, b) \text{ for all } i > n_2$$

Then for $n := \max\{n_1, n_2\}$ we have that

$$\models \phi(a_i, b) \text{ for all } i > n \qquad \text{or} \qquad \models \neg\phi(a_i, b) \text{ for all } i > n$$

as required. □

We already have all the ingredients to prove Proposition 2.3.

Proof of Proposition 2.3. Let us assume that all formulas $\phi(x, y)$ with $\lg(y) = 1$ have the NIP.

Claim. Let $(a_i : i < |T|^+)$ be an indiscernible sequence and let b be an element. Then there is some $\alpha < |T|^+$ such that the sequence $(a_i : \alpha < i < |T|^+)$ is indiscernible over b.

Proof. Otherwise, for every $\alpha < |T|^+$ there is a formula $\delta_\alpha(x_1, \dots, x_{k(\alpha)}, y)$ where $\lg(x_i) = \lg(x)$, and indexes $\alpha < i_1 < \dots < i_{k(\alpha)}$ and $\alpha < j_1 < \dots < j_{k(\alpha)}$ such that

$$\models \delta_\alpha(a_{i_1}, \dots, a_{i_{k(\alpha)}}, b) \qquad \& \qquad \models \neg\delta_\alpha(a_{j_1}, \dots, a_{j_{k(\alpha)}}, b).$$

Denote $c^1_\alpha := (a_{i_1}, \dots, a_{i_{k(\alpha)}})$ and $c^0_\alpha := (a_{j_1}, \dots, a_{j_{k(\alpha)}})$ and $\mathrm{Bd}(\alpha) := \max\{i_{k(\alpha)}, j_{k(\alpha)}\}$.

By cardinality, there is a formula $\delta(x_1, \dots, x_k, y)$ such that $J := \{\alpha \mid \delta_\alpha = \delta\}$ is cofinal in $|T|^+$. Indeed, consider the function

$$f : |T|^+ \to L : \alpha \mapsto \delta_\alpha.$$

Suppose that $f^{-1}(\delta)$ is not cofinal in $|T|^+$ for every $\delta \in \mathrm{Im}(f)$. That is, for every $\delta \in \mathrm{Im}(f)$ there is $\alpha_\delta < |T|^+$ such that for all $\beta \in f^{-1}(\delta)$ we have $\beta < \alpha_\delta$. We get a function

$$g : \mathrm{Im}(f) \to |T|^+$$

which is cofinal because $\alpha < g(f(\alpha))$ for all $\alpha < |T|^+$, and so $|L| \geq \mathrm{cf}(|T|^+) = |T|^+$, a contradiction.

Let α_0 be the minimum element of J and define $c_0 := c^0_{\alpha_0}$. Let $\alpha_1 \in J$ with $\alpha_1 > \mathrm{Bd}(\alpha_0)$ and define $d_1 := c^1_{\alpha_1}$. Inductively, we construct a sequence $(d_i : i < \omega)$ such that

$$\models \delta(d_i, b) \iff i \text{ is even.}$$

Moreover, the sequence $(d_i : i < \omega)$ is indiscernible. For, given $j_1 < j_2$ we have that the k indexes of the instances of elements of $(a_i : i < \omega)$ in d_{j_2} are greater than α_{j_2}. On the other hand, the corresponding k indexes of the instances of elements of $(a_i : i < \omega)$ in d_{j_1} are less than $\mathrm{Bd}(\alpha_1)$. Since $\alpha_2 > \mathrm{Bd}(\alpha_1)$, we deduce the k indexes corresponding to d_{j_2} are greater than the k indexes corresponding to d_{j_1}. Hence, the indiscernibility of $(a_i : i < \omega)$ implies that $(d_i : i < \omega)$ is indiscernible. All in all, we deduce that δ has the IP, a contradiction with our hypothesis. □

Now, let $\phi(x,y) \in L$ with $\ell := \lg(y) > 1$. Suppose that ϕ has IP, so by Proposition 3.4 there exist an indiscernible sequence $(a_i : i < \omega)$ and a tuple $b = (b_1, \dots, b_\ell)$ such that

$$(*) \qquad \text{for all } n < \omega \text{ there are } i, j > n \text{ with } \models \phi(a_i, b) \text{ and } \models \neg\phi(a_j, b).$$

Consider the partial type p in the variables $(x_i : i < |T|^+)$ and y which says that $(x_i : i < |T|^+)$ is an indiscernible sequence and for all $i < |T|^+$ we have that

$$\phi(x_i, y) \longleftrightarrow \neg\phi(x_{i+1}, y).$$

Given any finite subset of p, by $(*)$ we can choose finitely many tuples in $(a_i : i < \omega)$ and b satisfying all the formulas in the subset. Hence, there exists a realization of the type, that we will denote again by $(a_i : i < |T|^+)$ and b.

By the claim, there is $\alpha_1 < |T|^+$ such that $(a_i : \alpha_1 < i < \aleph_1)$ is indiscernible over b_1. In other words, the sequence $(a_i b_1 : \alpha_1 < i < |T|^+)$ is indiscernible. Again by the claim there is $\alpha_2 < |T|^+$ such that $(a_i b_1 : \alpha_2 < i < |T|^+)$ is indiscernible over b_2. Inductively, we find $\alpha < |T|^+$ such that $(a_i b_1 \cdots b_{\ell-1} : \alpha < i < \aleph_1)$ is indiscernible over b_ℓ.

On the other hand, by hypothesis the formula $\phi(x, y_1, \dots, y_{\ell-1}; y_\ell)$ has NIP. Define inductively $a'_i := a_{\alpha+i+1}$ for each $i < \omega$ and note that $(a'_i b_1, \dots, b_{\ell-1} : i < \omega)$ is an indiscernible sequence. Thus, by Proposition 3.4 we can assume that there is $N < \omega$ such that

$$\models \phi(a'_i, b_1, \dots, b_{\ell-1}, b_\ell) \text{ for all } i > N.$$

This is a contradiction since by definition of the type p we have that

$$\models \phi(a'_i, b) \Longleftrightarrow \models \neg\phi(a'_{i+1}, b)$$

for all $i < \omega$. □

4. Externally definable sets

Recall that a subset $X \subseteq M^n$ is *externally definable* if for some formula $\phi(x,y)$ and parameter $c \in \mathfrak{C}$ we have $X = \phi(x, c)$. Now, we want to study externally definable sets in NIP theories. Of course, we know that they will not be definable in general, otherwise the theory would be stable. However, we will see that we do not lose the NIP if include them in our language (and this is an example of why we say that NIP theories do not have a 'wild' behaviour). Specifically:

Definition 4.1. Let L^{sh} be the language L together with a relation symbol R_X for each externally definable set X of M, and let M^{sh} the obvious L^{sh}-structure.

Example 4.2. Let T be the theory of real closed field in the language of ordered fields. Then both $\mathbb{R}$ and the real algebraic numbers $\mathbb{R}_{alg}$ are models of T. Take the formula $\psi(x,y) : -y < x < y$. Then the set

$$\psi(\mathbb{R}_{alg}, \pi) = \{a \in \mathbb{R}_{alg} : -\pi < a < \pi\}$$

is an externally definable subset of M. Note that there are not $b_1, b_2 \in \mathbb{R}_{alg}$ such that $\psi(\mathbb{R}_{alg}, \pi)$ equals the interval (b_1, b_2) in $\psi(\mathbb{R}_{alg}, \pi)$. However, note that $\psi(\mathbb{R}_{alg}, \pi)$ is convex.

Our aim is to prove:

Theorem 4.3 (Shelah's Theorem)**.** *If T has NIP then the theory $Th(\mathcal{M}^{sh})$ has elimination of quantifiers.*

We will prove Shelah's theorem below, let us see first some of its consequences.

Corollary 4.4. *If T has NIP then $Th(\mathcal{M}^{sh})$ has NIP.*

Proof. First note that for every $\psi(x,y) \in L$ and $m \in M$ we have that the definable subset $\psi(M,m) \subseteq M^n$ is (externally) definable and therefore $R_{\psi(M,m)} = \psi(M,m)$. That is, the formulas in L with parameters in M are equivalent in $Th(M^{sh})$ to quantifier-free formulas in L^{sh}. Therefore, we can assume that a quantifier-free formula in L^{sh} is a Boolean combination of formulas of the form R_X for X an externally definable set. Since a Boolean combination of externally definable sets is a externally definable set, we can assume that all quantifier-free in formulas in L^{sh} are of the form R_X for X an externally definable set.

Now, take $\phi(x,y) \in L^{sh}$ with IP. By Shelah's theorem the formula $\phi(x,y)$ is equivalent to some $R_X(x,y)$ where $R_X \in L^{sh}$ for some externally definable set $X = \psi(x,y,c)$. Therefore, for any $n < \omega$ and $I \subseteq \{0,\dots,n\}$ we have that

$$M^{sh} \models \exists x_0 \dots x_n y\big(\{R_X(x_i,y) : i < \omega) : i \in I\} \cup \{\neg R_X(x_i,y) : i < \omega) : i \notin I\}\big)$$

is consistent. This means that $\phi(x,y,c)$ has IP, a contradiction. □

Corollary 4.5. *If T is o-minimal then $Th(M^{sh})$ is weakly o-minimal, i.e., every subset of M definable in M^{sh} is a finite union of convex subsets.*

Proof. Take $\phi(x,y) \in L^{sh}$ where x is a single variable. Then by Shelah's theorem the formula $\phi(x,y)$ is equivalent to some $R_X(x,y)$ where $R_X \in L^{sh}$ for some externally definable set $X = \psi(x,y,c)$. By o-minimality for all $b \in \mathfrak{C}$ the set $\psi(x,b,c)$ is a finite union of intervals (we can consider a point as a closed interval). We know that there exists $N \in \mathbb{N}$ such that for all $b \in \mathfrak{C}$ the set $\psi(x,b,c)$ is the union of at most N intervals. Moreover, there are definable functions $f_1,\dots,f_N$ such that $f_i(b)$ belongs to an interval of $\psi(x,b,c)$ and each one of these intervals intersects with $\{f_1(b),\dots,f_N(b)\}$ (this is true because of the so called cell decomposition of o-minimal structures). In particular in $Th(M^{sh})$ it is true that for every y we have that $R_X(-,y)$ is empty or

$$\forall z \in R_X(-,y) \bigvee_i (\text{the points between } f_i(y) \text{ and } z \text{ are in } R_X(-,y))$$

This implies that if $\mathcal{N} \models Th(M^{sh})$, for any $d \in N$ we have that $\phi(x,d)$ is a finite union of (at most N) convex sets. Hence $Th(M^{sh})$ is weakly o-minimal. □

Caution 4.6. Take $\phi(x,y,c)$ a formula with $c \in \mathfrak{C}$ and consider the externally definable set $X = \phi(M,c)$. If we denote $\pi : M \times M \to M : (x,y) \mapsto y$, then to prove Shelah's Theorem it suffices to show that $\pi(X)$ is externally definable. We have to be aware that the formula $\varphi(y,c) : \exists x \phi(x,y,c)$ satisfy $\pi(X) \subseteq \varphi(M,c)$ but the equality is *not* necessarily true. For example, consider $\mathbb{R}$ and $\mathbb{R}_{alg}$ as in Example 4.2 and let $\phi(x,y,z) : y = xz$ and $c \in \mathbb{R} \setminus \mathbb{R}_{alg}$. Clearly, we have $\phi(\mathbb{R}_{alg},c) = \{(0,0)\}$. But for the formula $\varphi(y,c) : \exists x \phi(x,y,c)$ we have $\varphi(\mathbb{R}_{alg},c) = \mathbb{R}_{alg}$

Clearly, to use the line $y = xc$ in order to describe the (externally) definable set $\{(0,0)\}$ is a bad idea. Somehow, we would like to have a formula defining our externally definable set such that its behaviour in M is not so different than in $\mathfrak{C}$. The following notion tries to capture the later:

Definition 4.7. [2] Let $X \subseteq M^n$ be externally definable. We say that the formula $\theta(x) \in L_{\mathfrak{C}}$ is a *honest definition* of X if for every $\psi(x) \in L_M$ we have that

$$\theta(M) \subseteq \psi(M) \Longrightarrow \theta(\mathfrak{C}) \subseteq \psi(\mathfrak{C}).$$

(Note that the implication from right to left is obvious.)

Our aim in the next section is to prove the following:

Theorem 4.8. [2] *In a NIP theory, every externally definable set on $M \models T$ has a honest definition.*

This proposition implies Shelah's Theorem.

Proof of Shelah's Theorem. Let $X \subseteq M^{1+k}$ be externally definable and $\pi : M \times M^k \to M^k : (x, y) \to y$ be the projection. It suffices to show that $\pi(X) \subseteq M^k$ is externally definable. By Proposition 4.8 there exists a honest definition $\theta(x, y) \in L_{\mathfrak{C}}$ of X, we want to prove that for the formula $\varphi(y) : \exists x\theta(x, y)$ we have $\pi(X) = \varphi(M)$. Fix $m \in M^k$ and consider the formula $\psi(x, y) : y \neq m$. Then,

$$\begin{aligned} m \in \pi(X) &\iff \theta(n, m) \text{ for some } n \in M \\ &\iff \theta(M) \nsubseteq \psi(M) \\ &\iff \theta(\mathfrak{C}) \nsubseteq \psi(\mathfrak{C}) \\ &\iff \theta(d, m) \text{ for some } d \in \mathfrak{C} \\ &\iff \mathfrak{C} \models \varphi(m) \end{aligned}$$

□

5. Coheirs

To prove the existence of honest definitions we need first to analyse an abstract notion of independence. We write $a \equiv_B a'$ if $\text{tp}(a|B) = \text{tp}(a'|B)$.

Definition 5.1. Let A be a set with $M \subseteq A$. We say that $p(x) \in S(A)$ is *coheir* over M if for every $\phi(x, a) \in p(x)$ there is $m \in M$ such that $\models \phi(m, a)$. For any sets A and C we write

$$A \underset{M}{\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.9ex\hbox{$|$}}} C$$

if for every tuple $a \in A$ the type $tp(a|MC)$ is a coheir over M.

Miscellany: Let us see what is the $\smile\!\!\!\!|$ relation in the case of an algebraic closed field K. Pick two elements a and c in an extension of K. If $a \in K$ then clearly $a \smile\!\!\!\!|_K c$. If $c \in K$ we also have that tp$(a|Kc)$ is a coheir over K and so $a \smile\!\!\!\!|_K c$. For, if $\phi(x) \in \text{tp}(a|Kc) = \text{tp}(a|K)$ then $K \models \exists x\phi(x)$ and therefore there is $m \in K$ such that $K \models \phi(m)$. So assume that a and c are not in K. Let us show that

$$a \underset{K}{\smile\!\!\!\!|} c \iff a \text{ is not algebraic over } Kc.$$

Suppose that $a \smile\!\!\!\!|_K c$ and there is a polynomial $P(x)$ with coefficient in Kc such that $P(a) = 0$. Since tp$(a|Kc)$ is a coheir over K and the formula $P(x) = 0$ belongs to tp$(a|Kc)$, there is $m \in K$ such that $P(m) = 0$. Then c is algebraic over K, so $c \in K$, a contradiction.

Suppose now that a is not algebraic over Kc. Take $\phi(x, y) \in L$ such that $\phi(x, c) \in \text{tp}(a|Kc)$. By quantifier elimination of ACF, if $\phi(x, c)$ is finite then a is algebraic over Kc, a contradiction. So $\phi(x, c)$ is not finite and so $\neg\phi(x, c)$ is finite. In particular, there is $m \in K$ such that $K \models \phi(m, c)$.

We see that in ACF the $\smile\!\!\!\!|$ relation is algebraic independence. In general, in stable theories the relation $\smile\!\!\!\!|$ has good properties (for example, symmetry) which allow us to say that it is an independence relation (see [9, §8.5]). In other contexts, this notion do not have such properties and therefore it only give us a rudimentary notion of independence.

Lemma 5.2. *If $a \smile\!\!\!\!|_M Bc$ and $B \smile\!\!\!\!|_M c$ then $aB \smile\!\!\!\!|_M c$.*

Proof. Let $b \in B$ and $\phi(x, y, c) \in tp(ab|Mc)$. In particular $\phi(x, b, c) \in tp(a|Mbc)$, so there exists $m \in M$ such that $\models \phi(m, b, c)$. Therefore $\phi(m, y, c) \in tp(b|Mc)$ and therefore there is $n \in M$ such that $\models \phi(m, n, c)$, as required. □

Lemma 5.3. *Let A and c be such that $M \subseteq A$ and $A \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$. Then for every $b \in \mathfrak{C}$ there exists $b' \in \mathfrak{C}$ such that $b' \equiv_A b$ and $b'A \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$.*

Proof. Denote $p := \text{tp}(b|A)$ and consider the set

$$\Sigma(x) := \{\psi(x, a, c) : a \in A, \psi(x, y, z) \in L_M \text{ such that } \models \psi(m_1, m_2, c) \text{ for all } m_1, m_2 \in M\}.$$

Suppose that $p \cup \Sigma$ is coherent and take $b' \models \Sigma$. Indeed, we clearly have that $b' \equiv_A b$ and let us check $b'A \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$. Pick $a \in A$ and suppose that $\text{tp}(b'a|Mc)$ is not coheir over M. Then there exists $\psi(x, y, z) \in L_M$ with $\models \psi(b', a, c)$ and such that for all $m_1, m_2 \in M$

$$\models \neg\psi(m_1, m_2, c).$$

Then $\neg\psi(x, a, c) \in \Sigma$ and so $\models \neg\psi(b', a, c)$, a contradiction.

Let us show that $p \cup \Sigma$ is coherent. Take $\phi(x, a) \in p$ and $\psi(x, a, c) \in \Sigma$. Suppose that $\phi(x, a) \wedge \psi(x, a, c)$ is not coherent, so

$$\models \exists x\phi(x, a) \wedge \forall x(\phi(x, a) \rightarrow \neg\psi(x, a, c)).$$

Since $A \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$ there exists $m_2 \in M$ such that

$$\models \exists x\phi(x, m_2) \wedge \forall x(\phi(x, m_2) \rightarrow \neg\psi(x, m_2, c)).$$

Since M is a model, there is $m_1 \in M$ such that $\phi(m_1, m_2)$. In particular we have $\neg\psi(m_1, m_2, c)$, a contradiction with $\psi(x, a, c) \in \Sigma$. □

Lemma 5.4. *Coh(M) $:= \{q \in S(\mathfrak{C}) : q$ is a coheir over $M\}$ is closed and its cardinality is less or equal than $2^{2^{|M|}}$.*

Proof. $\text{Coh}(M)$ is closed because $\text{Coh}(M) = \bigcap_{\psi \in \mathcal{F}}[\psi]$ where

$$\mathcal{F} := \{\psi(x, b) : \psi(x, y) \in L \text{ and } b \in \mathfrak{C} \text{ such that } \models \psi(m, b) \text{ for all } m \in \mathfrak{C}\}.$$

On the other hand, the map

$$\begin{array}{rcl} \rho : \text{Coh}(M) & \rightarrow & \mathcal{P}(\mathcal{P}(M)) \\ p & \mapsto & \{\phi(M) : \phi \in p\} \end{array}$$

is injective and therefore $\text{Coh}(M)$ is bounded. Indeed, let $p, q \in \text{Coh}(M)$ be such that $\rho(p) = \rho(q)$. Take $\psi \in p$ possibly with parameters and suppose that $\psi \notin q$, so that $\neg\psi \in q$. In particular, $\psi(M) \in \rho(p)$ and $\neg\psi(M) \in \rho(q)$. Since $\neg\psi(M) \in \rho(q) = \rho(p)$ there is $\phi \in p$ such that $\neg\psi(M) = \phi(M)$. In particular, $\psi \wedge \phi \in p$ and therefore

$$(\psi \wedge \phi)(M) = \psi(M) \cap \phi(M) = \psi(M) \cap \neg\psi(M) = \emptyset$$

which is a contradiction because p is a coheir over M and therefore $(\psi \wedge \phi)(M) \neq \emptyset$. □

6. The existence of honest definitions

Proposition 6.1. *Let $\phi(x, y)$ a formula with NIP and $c \notin M$. Then there exists A and $\theta(x) \in L_A$ such that*

a) $A \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$,
b) $\theta(M) = \phi(M, c)$,
c) *If* $bA \mathop{\smile\!\!\!\!\!\!\!\perp}_M c$ *then* $\mathfrak{C} \models \theta(b) \rightarrow \phi(b, c)$.

Proof. The set

$$\mathcal{Q} := \{q \in S(\mathfrak{C}) : q \text{ is a coheir over } M \ \&\ \phi(x,c) \in q\} = \mathrm{Coh}(M) \cap [\phi(x,c)],$$

which is closed and its cardinality is less or equal than $2^{2^{|M|}}$ by Lemma 5.4. So let us consider a enumeration $\mathcal{Q} = \{q_\alpha : \alpha < \lambda\}$ of $\mathcal{Q}$. We will construct an ascending chain of sets A_α and formulas $\theta_\alpha(x) \in q_\alpha|_{A_\alpha}$ such that

i) $A_\alpha \mathop{\smile\!\!\!\!\!\mid}_M c$,
ii) for each b with $bA_\alpha \mathop{\smile\!\!\!\!\!\mid}_M c$ we have that $\models \theta_\alpha(b) \to \phi(b,c)$.

We describe step α (in step 0 we have to apply the following argument taking as A' a realization of $q_0|_{Mc}$). Take $A' := \bigcup_{\beta<\alpha} A_\beta$ and note that $A' \mathop{\smile\!\!\!\!\!\mid}_M c$ by i). We claim:

Claim. If there exist sequences $(a_i : i < \omega)$ and $(b_i : i < \omega)$ such that for all $i < \omega$ we have

$$a_i \models q_\alpha|_{a_0 b_0 \cdots a_{i-1} b_{i-1} M A' c} \qquad \& \qquad b_i \models q_\alpha|_{a_0 b_0 \cdots a_i M A'} \cup \{\neg\phi(x,c)\}$$

and $a_0 b_0 \cdots a_i b_i A' \mathop{\smile\!\!\!\!\!\mid}_M c$, then ϕ has IP.

Proof. Denote $d_{2i} := a_i$ and $d_{2i+1} := b_i$ for each $i < \omega$. Since

$$\models \phi(d_i, c) \Longleftrightarrow i \text{ is even},$$

by Lemma 2.5 it suffices to show that the sequence $(d_i : i < \omega)$ is indiscernible. First note that by construction, for any $i \leq j$ we have that

$$(*) \qquad d_i \equiv_{M d_1 \cdots d_{i-1}} d_j \qquad \& \qquad tp(d_j | M d_1 \cdots d_i) \text{ is a coheir over M.}$$

Let us show that for every $n \geq 1$ and indexes $i_1 < \cdots < i_n$ we have that

$$d_1 \cdots d_n \equiv_M d_{i_1} \cdots d_{i_n}.$$

We prove it by induction, the initial step $n = 1$ follows by $(*)$. So let $n > 1$ and suppose there exists $\psi(x_1, \ldots, x_n) \in L_M$ such that $\models \psi(d_1, \ldots, d_n)$ and $\models \neg\psi(d_{i_1}, \ldots, i_n)$. Since $n \leq i_n$, by $(*)$ we deduce

$$tp(d_n | M d_1 \cdots d_{n-1}) = tp(d_{i_n} | M d_1 \cdots d_{n-1})$$

and therefore $\models \psi(d_1, \ldots, d_{n-1}, d_{i_n})$. In particular,

$$\phi(d_1, \ldots, d_{n-1}, x) \wedge \neg\phi(d_{i_1}, \ldots, d_{i_{n-1}}, x) \in tp(d_{i_n} | M d_1 \cdots d_{i_{n-1}}).$$

Again by $(*)$ there is $m \in M$ such that

$$\models \phi(d_1, \ldots, d_{n-1}, m) \wedge \neg\phi(d_{i_1}, \ldots, d_{i_{n-1}}, m)$$

and therefore $d_1 \cdots d_{n-1} \not\equiv_M d_{i_1} \cdots d_{i_{n-1}}$, a contradiction with our induction hypothesis. □

Hence, there are $a_0, \ldots, a_N$ and $b_0, \ldots, b_{N-1}$ such that for all $i < N$

$$a_i \models q_\alpha|_{a_0 b_0 \cdots a_{i-1} b_{i-1} M A' c} \qquad \& \qquad b_i \models q_\alpha|_{a_0 b_0 \cdots a_i M A'} \cup \{\neg\phi(x,c)\}$$

and $a_0 b_0 \cdots b_{N-1} a_N A' \mathop{\smile\!\!\!\!\!\mid}_M c$ and with

$$\{b : a_0 b_0 \cdots a_N b A' \mathop{\smile\!\!\!\!\!\mid}_M c \ \&\ b \models q_\alpha|_{a_0 b_0 \cdots a_N M A'} \cup \{\neg\phi(x,c)\}\} = \emptyset.$$

Define $A_\alpha := A' \cup \{a_0, \ldots, a_N, b_0, \ldots, b_{N-1}\}$ which clearly satisfies $A_\alpha \mathop{\smile\!\!\!\!\!\mid}_M c$ and note that

$$\{b : bA_\alpha \mathop{\smile\!\!\!\!\!\mid}_M c\} = \{b : b \models \Sigma(x)\}$$

where $\Sigma(x)$ is

$$\{\psi(x,a,c) : \psi(x,y,z) \in L_M \text{ and } a \in A_\alpha \text{ such that } \psi(x,a,c) \text{ is not satisfiable in } M\}.$$

Thus,

$$\Sigma(x) \cup q_\alpha|_{A_\alpha M} \vdash \phi(x,c)$$

and therefore by compactness there exists $\theta_\alpha \in q_\alpha|_{A_\alpha M}$ such that for all b with $bA_\alpha \mathop{\smile\!\!\!\!\!\!|}_M c$ we have that

$$\models \theta_\alpha(b) \to \phi(b,c),$$

as required.

Finally, define $A := \bigcup_{\alpha<\lambda} A_\alpha$. Since $\mathcal{Q} \subseteq \bigcup_{\alpha<\lambda}[\theta_\alpha]$, by compactness there exist $\alpha_1,\dots,\alpha_\ell < \lambda$ such that $\mathcal{Q} \subseteq [\theta_{\alpha_1}] \cup \cdots \cup [\theta_{\alpha_\ell}] = [\theta_{\alpha_1} \vee \cdots \vee \theta_{\alpha_\ell}]$. Let us show that the set of parameters A and the formula

$$\theta := \theta_{\alpha_1} \vee \cdots \vee \theta_{\alpha_\ell} \in L_A$$

satisfies the properties of the statement. Property (a) is obvious because $A_\alpha \mathop{\smile\!\!\!\!\!\!|}_M c$ for each $\alpha < \lambda$. For (c), take b with $bA \mathop{\smile\!\!\!\!\!\!|}_M c$ such that $\models \theta(b)$, so $\models \theta_{\alpha_j}(b)$ for some $j = 1,\dots,\ell$. Since $bA_{\alpha_j} \mathop{\smile\!\!\!\!\!\!|}_M c$, by (ii) it follows $\models \phi(b,c)$, as required. Let us check (b). Indeed, for every $m \in M$ we have that $mA \mathop{\smile\!\!\!\!\!\!|}_M c$ and therefore by (c) we get that $\models \theta(m) \to \phi(m,c)$. That is, $\theta(M) \subseteq \phi(M,c)$. On the other hand, if $m \in M$ is such that $\models \phi(m,c)$ then

$$tp(m|\mathfrak{C}) \in \mathcal{Q} \subseteq [\theta]$$

and therefore $\theta \in tp(m|\mathfrak{C})$, i.e. $\models \theta(m)$, as required. □

Proof of Theorem 4.8. Let $X \subseteq M^n$ be an externally definable subset. We can assume that X is not definable in M, so there is a formula $\phi(x,y) \in L$ and $c \in \mathfrak{C} \setminus M$ such that $X = \phi(M,c)$. Let us check that the formula $\theta(x) \in L_A$ given by Proposition 6.1 is a honest definition of X. Let $\psi(x) \in L_M$ be a formula such that $\theta(M) \subseteq \psi(M)$. We have to show that

$$\models \theta(b) \to \psi(b).$$

for every $b \in \mathfrak{C}$. The formula $\theta(x) \to \psi(x)$ has parameters in A and therefore by Lemma 5.3 it is enough to prove that for every $b \in \mathfrak{C}$ with $bA \mathop{\smile\!\!\!\!\!\!|}_M c$ we have that $\mathfrak{C} \models \theta(b) \to \psi(b)$. Moreover, by c) of Proposition 6.1 it suffices to prove that

$$\models \phi(b,c) \to \psi(b)$$

for every $b \in \mathfrak{C}$ with $bA \mathop{\smile\!\!\!\!\!\!|}_M c$. Since $tp(b|Mc)$ is a coheir over M and $\phi(x,c) \to \psi(x)$ is a formula in L_{Mc}, if $\mathfrak{C} \not\models \phi(b,c) \to \psi(b)$ then

$$\neg(\phi(x,c) \to \psi(x)) \in \mathrm{tp}(b|Mc)$$

and so there is $m \in M$ such that $\models \phi(m,c) \wedge \neg\psi(m)$, a contradiction with $\phi(M,c) = \theta(M) \subseteq \psi(M)$. □

References

[1] E. Casanovas, NIP formulas and theories. http://www.ub.edu/modeltheory/casanovas/e.html.
[2] A. Chernikov and P. Simon. Externally definable sets and dependent pairs. *Israel J. Math.*, 194(1):409–425, 2013.
[3] A.R. Magid. *Lectures on differential Galois theory.* In *University Lecture Series* (7). American Mathematical Society, Providence, RI, 1994.
[4] A. Pillay. *An introduction to stability theory.* In *Oxford Logic Guides* (8). Oxford University Press, 1983.
[5] A. Pillay. Stability theory. Lecture notes. https://www3.nd.edu/ apillay/.
[6] B. Poizat. *A course in model theory.* In *Universitext.* Springer-Verlag, New York, 2000.
[7] P. Simon. *A guide to NIP theories.* In *Lecture Notes in Logic.* Association for Symbolic Logic, Cambridge Scientific Publishers, 2015.
[8] S. Shelah. Strongly dependent theories, *Israel J. Math.*, 204 (1): 1–83, 2014.
[9] K.Tent and M.Ziegler. *A course in Model Theory.* Lecture Notes in Logic, Cambridge University Press, 2012.
[10] M. Ziegler. Chernikov and Simons proof of Shelahs theorem, preprint, 2010.

Departamento de Álgebra, Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain. E-mail address: ebaro@ucm.es