RINGS OF DIFFERENTIABLE SEMIALGEBRAIC FUNCTIONS

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Abstract. In this work we analyze the main properties of the Zariski and maximal spectra of a ring $\mathcal{S}^r(M)$ of differentiable semialgebraic functions of class $\mathcal{C}^r$ on a semialgebraic set $M \subset \mathbb{R}^m$. Denote $\mathcal{S}^0(M)$ the ring of semialgebraic functions on M that admit a continuous extension to an open semialgebraic neighborhood of M in $\operatorname{Cl}(M)$, which is the real closure of $\mathcal{S}^r(M)$. Despite $\mathcal{S}^r(M)$ it is not real closed for $r \geq 1$, the Zariski and maximal spectra are homeomorphic to the corresponding ones of the real closed ring $\mathcal{S}^0(M)$. Moreover, we show that the quotients of $\mathcal{S}^r(M)$ by its prime ideals have real closed fields of fractions, so the ring $\mathcal{S}^r(M)$ is close to be real closed. The equality between the spectra of $\mathcal{S}^r(M)$ and $\mathcal{S}^0(M)$ guarantee that the properties of these rings that depend on such spectra coincide. For instance the ring $\mathcal{S}^r(M)$ is a Gelfand ring and its Krull dimension is equal to $\dim(M)$. If M is locally compact, the ring $\mathcal{S}^r(M)$ enjoys a Nullstellensatz result and Łojasiewicz inequality. We also show similar results for the ring $\mathcal{S}^{r*}(M)$ of differentiable bounded semialgebraic functions.

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1. Introduction

Recall that a *semialgebraic set* $M \subset \mathbb{R}^m$ is a set that can be described as a finite boolean combination of polynomial equalities and inequalities. A map $f : M \to N$ between semialgebraic sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ is *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$.

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However, in order to lighten notation we call *semialgebraic* the functions $f : M \to \mathbb{R}$ that are continuous and have semialgebraic graph.

Let $r \geq 1$ be a positive integer. An initial problem when dealing with differentiable semialgebraic functions of class $\mathcal{C}^r$ on a semialgebraic set $M \subset \mathbb{R}^m$ is to find an intrinsic definition of such type of functions. If $M \subset \mathbb{R}^m$ is an open subset, $f : M \to \mathbb{R}$ is a $\mathcal{S}^r$ *function* if it is a differentiable function of class $\mathcal{C}^r$ with semialgebraic graph. In [KP2, ATh, Th] the authors made a careful analysis of an intrinsic definition of $\mathcal{S}^r$ function in terms of jets of order r of (continuous) semialgebraic functions [ATh, Def.1.1], in order to achieve a semialgebraic Whitney's extension theorem when M is closed in $\mathbb{R}^m$, that is, *if $f : M \to \mathbb{R}$ is an $\mathcal{S}^r$ function on a closed semialgebraic subset of $\mathbb{R}^n$, there exists an $\mathcal{S}^r$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F|_M = f$*, see [ATh, Thm.1.2]. This approach follows the same kind of ideas developed when proving Whitney's extension theorem (see for instance [M, §1]) adapted to the semialgebraic case.

In case $r = 1$ the authors go beyond and prove, following the strategy developed in [F1, F2] adapted to the semialgebraic case, that *a semialgebraic function $f : M \to \mathbb{R}$ on a closed semialgebraic subset of $\mathbb{R}^n$ is $\mathcal{S}^1$ if and only if for each point $x \in M$ there exists a (non-necessarily semialgebraic) $\mathcal{C}^r$ extension $F_x : W^x \to \mathbb{R}$ to an open neighborhood $W^x \subset \mathbb{R}^n$ of the restriction* $f|_{M\cap W^x}$, see [ATh, Cor.7.14]. The authors hope that a suitable adaptation to the semialgebraic case of the very sophisticated techniques developed in [F1] will allow to prove the analogous result for each positive integer $r \geq 1$.

In this work we adopt the definition of $\mathcal{S}^r$ functions proposed in [ATh] in terms of jets of order r of (continuous) semialgebraic functions and we denote $\mathcal{S}^r(M)$ the set of $\mathcal{S}^r$ functions on M, which is an $\mathbb{R}$-algebra with respect to the usual sum and product of functions (see Definition 2.1). We will denote by $\mathcal{S}(M)$ the set of continuous functions on M, and by $\mathcal{S}^0(M)$ the set of continuous functions on M that can be extended to an open semialgebraic neighborhood of M in $\operatorname{Cl}(M)$, or equivalently, to an open semialgebraic neighborhood of M in $\mathbb{R}^n$ (see Lemma 2.10). For $r \geq 0$ we denote by $\mathcal{S}^{r*}(M)$ and $\mathcal{S}^*(M)$ the corresponding bounded functions. We will use also the notation $\mathcal{S}^{r\diamond}(M)$ to refer indistinctly to both rings $\mathcal{S}^r(M)$ and $\mathcal{S}^{r*}(M)$, and similarly with $\mathcal{S}^\diamond(M)$.

In Section 2 we undertake a development of $\mathcal{S}^r$ functions for $r \geq 0$. One of our first observations (Lemma 2.12) is that even though the definition of $\mathcal{S}^r$ function on an *arbitrary* semialgebraic set M is intrinsic in nature, it enclose a Lipschitz condition and therefore they own a $\mathcal{S}^{r-1}$ extension to an open semialgebraic neighborhood of M. In particular, we have the following inclusion of rings:

$$\mathcal{S}^{r\diamond}(M) \to \mathcal{S}^{0\diamond}(M) \to \mathcal{S}^\diamond(M).$$

If M is locally compact then it is well-known that $\mathcal{S}^{0\diamond}(M) = \mathcal{S}^\diamond(M)$. In this paper we are also interested in *arbitrary* semialgebraic sets, and therefore it is natural to ask in which situations the equality $\mathcal{S}^{0\diamond}(M) = \mathcal{S}^\diamond(M)$ holds. To that aim, we introduce the following definitions. Let $M \subset \mathbb{R}^m$ be a semialgebraic set. We say that a semialgebraic set $M \subset \mathbb{R}^n$ is *problematic at* $x \in M$ if there exist a sequence of points $\{x_k\}_k \subset \operatorname{Cl}(M)$ converging to x such that each germ M_{x_k} is disconnected. Observe that if M is locally compact at $x \in M$ then there is an open ball B centered in x such that $M \cap B = \operatorname{Cl}(M) \cap B$ and therefore M is not problematic at x. In particular, locally compact semialgebraic sets are *non-problematic.*

On the other hand, as it is well-known, given the set of points of M of dimension k is a semialgebraic subset M. Denote $M_1, \dots, M_s$ the closures in M of those which are non empty and with $\dim(M_i) > \dim(M_{i+1})$. Each M_i is a pure dimensional semialgebraic set and $M = \bigcup_{i=1}^s M_i$. The semialgebraic sets M_i are univocally determined by M. Note also that M is locally compact at $x \in M$ if and only if each M_i is locally compact at x. Thus, M is locally compact if and only if each M_i is locally compact.

main2 **Theorem 1.1.** *Let $M \subset \mathbb{R}^m$ be a semialgebraic set. The following assertions are equivalent:*

(i) *M is either locally compact or the set $\mathfrak{F}$ of the indexes $i = 1, \dots, s$ such that M_i is non-locally compact is a singleton $\{i_0\}$, $\dim(M_{i_0}) = 2$ and M_{i_0} is non-problematic.*
(ii) $\mathcal{S}^{0\diamond}(M) = \mathcal{S}^\diamond(M)$.

(iii) *The map* $\varphi : \operatorname{Spec}^\diamond(M) \to \operatorname{Spec}^{0\diamond}(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{0\diamond}(M)$ *is a homeomorphism.*

Another key result from Section 2 is that if $M \subset \mathbb{R}^m$ is a semialgebraic set, then

$$\mathcal{S}^r(M) \cong \varinjlim(\mathcal{S}^r(E), \mathtt{j}) \tag{1.1}$$ dlim

where $E \subset \mathbb{R}^m$ is a locally compact semialgebraic set and $\mathtt{j} := (\mathtt{j}_1, \ldots, \mathtt{j}_m) : M \to E$ is an $\mathcal{S}^r$ embedding such that $E \subset \operatorname{Cl}(\mathtt{j}(M))$ (and a similar result holds in the bounded case). We will show also how some of the properties of $\mathcal{S}^r$ functions on locally compact semialgebraic sets transfer through the direct limit. This transfer method has limitations: for instance, in [FG5] it is shown that Łojasiewicz's inequality [BCR, Cor.2.6.7] hold in the $\mathbb{R}$-algebra $\mathcal{S}(M)$ if and only if M is locally compact.

In Section 3 we analyze the Zariski spectra of the ring $\mathcal{S}^{r\diamond}(M)$. Given a commutative ring A, we denote $\operatorname{Spec}(A)$ its Zariski spectrum endowed with the Zariski topology, which has as a subbasis of open sets the family of *basic open sets* given $\mathcal{D}(a) := \{\mathfrak{p} \in \operatorname{Spec}(A) : \ a \notin \mathfrak{p}\}$ for $a \in A$. The *constructible* subsets of $\operatorname{Spec}(A)$ are the Boolean combination of the latter basic open sets. To lighten notations, for each $r \geq 0$ we write

$$\operatorname{Spec}^{r\diamond}(M) := \operatorname{Spec}(\mathcal{S}^{r\diamond}(M)).$$

and $\operatorname{Spec}^\diamond(M) := \operatorname{Spec}(\mathcal{S}^\diamond(M))$. The main result of this section is that the Zariski spectra of $\mathcal{S}^{r\diamond}(M)$ and $\mathcal{S}^{0\diamond}(M)$ are homeomorphic.

main1 **Theorem 1.2.** *Let* $M \subset \mathbb{R}^m$ *be a semialgebraic set. Then the map*

$$\varphi : \operatorname{Spec}^{0\diamond}(M) \to \operatorname{Spec}^{r\diamond}(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{r\diamond}(M)$$

is a homeomorphism, whose inverse map is

$$\psi : \operatorname{Spec}^{r\diamond}(M) \to \operatorname{Spec}^{0\diamond}(M),\ \mathfrak{q} \mapsto \sqrt{\mathfrak{q}\mathcal{S}^0(M)}.$$

The previous result implies in particular that for each $r \geq 0$, the rings $\mathcal{S}^r(M)$ and $\mathcal{S}^{r*}(M)$ are Gelfand rings and their Krull dimensions coincide with the dimension of M. Denote $\beta_s^{r\diamond}M$ the maximal spectra of $\mathcal{S}^{r\diamond}(M)$. As homeomorphism between topological spaces map closed points to closed points, the restriction

$$\psi|_{\beta^{0\diamond}M} : \beta^{0\diamond}M \to \beta^{r\diamond}M, \mathfrak{m} \mapsto \mathfrak{m} \cap \mathcal{S}^{r\diamond}(M)$$

is also a homeomorphism.

The proof of Theorem 1.2 relies on the fact that each ring $\mathcal{S}^r(M)$ enjoys a Łojasiewicz's Nullstellesatz if M is locally compact. Denote $Z(f)$ the zero-set of a function $f \in \mathcal{S}^r(M)$. We have the following.

null2 **Theorem 1.3** (Łojasiewicz's Nullstellensatz)**.** *Let* $M \subset \mathbb{R}^m$ *be a locally compact semialgebraic set and let* $f, g \in \mathcal{S}^r(M)$ *with* $Z(f) \subset Z(g)$*. Then there exist an integer* $\ell > 0$ *and* $h \in \mathcal{S}^r(M)$ *such that* $g^\ell = hf$ *and* $Z(g) = Z(h)$*.*

In Section 4 we analyze next the fields of fractions of the quotients $\mathcal{S}^{r\diamond}(M)/\mathfrak{q}$ of the a ring $\mathcal{S}^{r\diamond}(M)$ of $\mathcal{S}^{r\diamond}$ functions on a semialgebraic set M modulo a prime ideal $\mathfrak{q}$ and we prove the following.

main3 **Theorem 1.4.** *Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{S}^{0\diamond}(M)$ *and consider the natural inclusion*

$$\mathtt{j} : \mathcal{S}^{r\diamond}(M)/(\mathfrak{p} \cap \mathcal{S}^{r\diamond}(M)) \to \mathcal{S}^{0\diamond}(M)/\mathfrak{p}.$$

Then the inclusion $\mathtt{j}$ *induces an isomorphism between the field of fractions* $\kappa(\mathfrak{p} \cap \mathcal{S}^{r\diamond}(M))$ *and* $\kappa(\mathfrak{p})$ *of the integral domains* $\mathcal{S}^{r\diamond}(M)/(\mathfrak{p} \cap \mathcal{S}^{r\diamond}(M))$ *and* $\mathcal{S}^{0\diamond}(M)/\mathfrak{p}$*. In particular, the field* $\kappa(\mathfrak{p} \cap \mathcal{S}^{r\diamond}(M))$ *is real closed.*

In the final Section 5 of this paper we contextualise the ring of $\mathcal{S}^{r\diamond}$ functions within the theory of *real closed rings* (Definition 5.1), introduced by Schwartz in the 80's of the last century [S1]. As it is well-known, the rings $\mathcal{S}^\diamond(M)$ are particular cases of the such real closed rings. The theory of real closed rings has been deeply developed till now in a fruitful attempt to establish new foundations for semi-algebraic geometry with relevant interconnections to model theory, see [CD1, CD2], [S1, S2, S3, S4], [PS, SM, ST] and [T1, T2, T3]. In addition, this theory, which generalizes the classical techniques concerning the semi-algebraic spaces of Delfs-Knebusch (see [DK2]), provides a powerful machinery to approach problems concerning certain rings of real-valued functions and contributes to achieve a better understanding of the algebraic and topological properties of such rings. We highlight some of them:

- Rings of real-valued continuous functions on Tychonoff spaces.
- Rings of semi-algebraic functions on semi-algebraic sets of an arbitrary real closed field.
- Rings of definable continuous functions on definable sets in o-minimal expansions of fields.

Every commutative unital ring A has a so-called real closure $\operatorname{rcl}(A)$ and this is unique up to a unique ring homomorphism over A. This means that $\operatorname{rcl}(A)$ is a real closed ring and there is a (not necessarily injective) ring homomorphism $\mathtt{i} : A \to \operatorname{rcl}(A)$ such that for every ring homomorphism $f : A \to B$ to some other real closed ring B there exists a unique ring homomorphism $\overline{f} : \operatorname{rcl}(\mathrm{A}) \to B$ satisfying $f = \overline{f} \circ \mathtt{i}$. Moreover, the real spectra of $\operatorname{rcl}(A)$ and A are homeomorphic. For instance, the real closure of the polynomial ring $\mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$ is the ring of continuous semialgebraic functions $\mathcal{S}^0(\mathbb{R}^n)$. More generally if $Z \subset \mathbb{R}^n$ is an algebraic set, the real closed ring $\mathcal{S}^0(Z)$ is the real closure of the ring of polynomial functions on Z.

In our setting, we show that the real closure of the ring $\mathcal{S}^{r\diamond}(M)$ is $\mathcal{S}^{0\diamond}(M)$, see Proposition 5.6. In case that M is locally compact, the latter can be deduced almost directly from the results by Schwartz and Tressl, so even from the very beginning we know that the real spectra of $\mathcal{S}^{r\diamond}(M)$ and $\mathcal{S}^{0\diamond}(M)$ are homeomorphic (and the residue fields of the latter are the real closures of the former). We would like to stress that in Theorem 1.2 and 1.4 we are giving more (explicit) information about the Zariski spectra and the residue fields of $\mathcal{S}^{r\diamond}(M)$, an information that suggests that the ring $\mathcal{S}^{r\diamond}(M)$ is closed to be a real closed ring. Furthermore, we would like to point out also that the only proof we know of Proposition 5.6 for an arbitrary M is the one we provide here, that uses in a crucial way Theorems 1.2 and 1.4.

Finally, we consider the ring $\mathcal{S}^{\infty\diamond}(M) := \bigcap_{r\geq 0} \mathcal{S}^{r\diamond}(M)$, which we can also describe as the inverse limit of the family of rings $\{\mathcal{S}^r(M)\}_{r\geq 0}$ with respect to the inclusion relation. It is natural to compare the ring $\mathcal{S}^\infty(M)$ with its subring of Nash functions $\mathcal{N}(M)$, i.e., the collection of functions $f : M \to \mathbb{R}$ that admit a Nash extension to an open semialgebraic extension to an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of M. Recall that a real function g on an open semialgebraic set $U \subset \mathbb{R}^m$ is *Nash* if it is semialgebraic and smooth on U. This property is equivalent to be an analytic function and algebraic over the polynomials, that is, there exists a polynomial $P \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_m, \mathtt{y}]$ such that $P(x, f(x)) = 0$ for each $x \in U$. We show that the rings $\mathcal{S}(M)$ and $\mathcal{N}(M)$ are different in general (Example 5.8), and they coincide if M is a Nash coherent set (Corollary 5.10). Recall that a semialgebraic set $M \subset \mathbb{R}^m$ is a *Nash set* if it is locally compact and there exists a Nash function f on the open semialgebraic set $U := \mathbb{R}^m \setminus (\operatorname{Cl}(M) \setminus M)$ such that $M = \mathcal{Z}(f)$, see [FG2]. The coherent condition refers to Serre's notion, see [BFR, §2.B].

On the other hand, it is not true that the real closure of an inverse limit is the inverse limit of the real closures, and therefore we study in which situations the real closure of $\mathcal{S}^{\infty\diamond}(M)$ is the ring $\mathcal{S}^{0\diamond}(M)$.

nash **Theorem 1.5.** *Let $M \subset \mathbb{R}^m$ be a semialgebraic set. Consider the following assertions:*

(i) *There exists an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of M such that $\operatorname{Cl}(M) \cap U$ is a coherent Nash set.*
(ii) *The inclusion $\mathtt{j} : \mathcal{S}^{\infty\diamond}(M) \hookrightarrow \mathcal{S}^{0\diamond}(M)$ provides the real closure of $\mathcal{S}^{\infty\diamond}(M)$.*
(iii) *For each $x \in M$ the germ $\operatorname{Cl}(M)_x$ is a Nash germ.*

Then (i) implies (ii), and (ii) implies (iii).

An easy consequence of these results is that given an open semialgebraic subset U of $\mathbb{R}^n$, the real closure of $\mathcal{N}(U)$ is $\mathcal{S}(U)$ if and only if U is dense in $\mathbb{R}^n$.

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2. Rings of $\mathcal{S}^r$ functions

If $U \subset \mathbb{R}^n$ is an open semialgebraic set and $r \geq 0$ is an integer, we say that a semialgebraic function on U is an $\mathcal{S}^r$ *function* if it is a function of class $\mathcal{C}^r$ in the classical sense. Our purpose next is to extend the previous definition for functions on an arbitrary semialgebraic set. In the literature the $\mathcal{S}^r$ functions are considered over closed semialgebraic subsets and are introduced as the restriction of an $\mathcal{S}^r$ function defined on an open semialgebraic neighborhood of the original set. The existence of a (continuous) semialgebraic extensions is never an obstacle: *any locally compact semialgebraic set $M \subset \mathbb{R}^m$ is the semialgebraic retract of an small open neighborhood $V \subset \mathbb{R}^m$ of M*, see [DK1, Thm 1]. In the general case of a non locally compact subset even the existence of a semialgebraic extension map is a delicate point [Fe2]. As we see below, if $f : M \to \mathbb{R}$ is a semialgebraic function, there exists a semialgebraic embedding $\mathtt{j} : M \hookrightarrow \mathbb{R}^p$ and a semialgebraic function $F : W \to \mathbb{R}$ on an open semialgebraic neighborhood $W \subset \mathbb{R}^p$ of $\mathtt{j}(M)$ such that $F \circ \mathtt{j} = f$.

2.A. Differentiability for semialgebraic functions. Our purpose is to work in the differentiable case as in the continuous one, where we always deal with the classical intrinsic notion of continuity that avoids the intricate extension aspect. This is a classical problem that goes back to Whitney's extension theorem. Inspired on the latter and based on jets of semialgebraic functions, we recall and develop a notion of differentiable semialgebraic function of intrinsic nature proposed in [ATh] motivated by the proof of Whitney's extension theorem collected in [M, §1]. We denote $\mathbb{N} := \{0, 1, 2, \ldots\}$ the set of natural numbers including 0.

def:jets **Definitions 2.1.** Fix a semialgebraic set $M \subset \mathbb{R}^m$ and denote $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_m)$. A *semialgebraic jet on M of order $r \geq 0$* is a collection of semialgebraic functions $F := (f_\alpha)_{|\alpha| \leq r}$ on M. Here we denote $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $|\alpha| := \alpha_1 + \cdots + \alpha_m$. For each $a \in M$ write

$$T_a^r F := \sum_{|\alpha| \leq r} \frac{f_\alpha(a)}{\alpha!}(\mathtt{x} - a)^\alpha \quad \text{and} \quad R_a^r F := f_0 - T_a^r F.$$

Denote the set of all semialgebraic jets on M of order r with $\mathcal{J}^r(M)$, which has a natural structure of $\mathbb{R}$-vector space. For every $\beta \in \mathbb{N}^m$ such that $|\beta| \leq r$ we denote

$$D^\beta : \mathcal{J}^r(M) \to \mathcal{J}^{r-|\beta|}(M),\ F := ((f_\alpha)_{|\alpha| \leq r}) \mapsto F_\beta := (f_{\gamma+\beta})_{|\gamma| \leq r - |\beta|},$$

that is a linear map.

defsr diff **Definition 2.2.** Let $f \in \mathcal{S}(M)$ be a semialgebraic function. We say that f is a $\mathcal{S}^r$ *function* if there exists a semialgebraic jet $F := (f_\alpha)_{|\alpha| \leq r}$ on M of order r such that $f_0 = f$ and

2.A.1. *For each β with $|\beta| \leq r$ and every point $a \in M$ it holds $|R_x^{r-|\beta|} F_\beta(y)| = o(\|x - y\|^{r-|\beta|})$ for $x, y \in M$ when $x, y \to a$,*

that is, the function $\tau : (0, +\infty) \to \mathbb{R}$ defined by $\tau(t) := \sup_{\substack{x,y \in M,\ x \neq y \\ \|x-a\| \leq t,\ \|y-a\| \leq t}} \frac{|R_x^{r-|\beta|} F_\beta(y)|}{\|x-y\|^{r-|\beta|}}$ is an increasing function that can be continuously extended to $t = 0$ by $\tau(0) = 0$.

We denote by $\mathcal{S}^r(M)$ the collection of all the $\mathcal{S}^r$ functions on M.

Rest *Remarks* 2.3. (i) Condition 2.A.1 is equivalent by [M, I.Thm.2.2] to the following one.

2.A.2. *For every point $a \in M$ there exists an increasing, continuous and concave function $\alpha_a : [0, +\infty) \to [0, +\infty)$ such that $\alpha_a(0) = 0$ and for each $z \in \mathbb{R}^n$ it holds*

$$|T_x^r F(z) - T_y^r F(z)| \leq \alpha_a(\|x - y\|) \cdot (\|z - x\|^r + \|z - y\|^r)$$

for $x, y \in M$ when $x, y \to a$.

(ii) The definition introduced is consistent with the continuous case, that is, a function is $\mathcal{S}^0$ if and only if it is semialgebraic. Therefore the forthcoming results apply also to continuous functions.

(iii) Definition 2.2 coincides, if $M \subset \mathbb{R}^m$ is an open semialgebraic subset, with the expected one. That is, *f is an $\mathcal{S}^r$ function on M if and only if f is a $\mathcal{C}^r$ function and a semialgebraic function.* Indeed, if $f \in \mathcal{S}(M)$ is a $\mathcal{C}^r$ function the semialgebraic jet $(f_\alpha := \frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha})_{|\alpha|\leq r}$ of order r on M satisfies by Taylor's Theorem the condition 2.A.1. Conversely, condition 2.A.1 straightforward implies that $f_\alpha = \frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha}$ if $|\alpha| \leq r$, so f is a $\mathcal{C}^r$ function. In addition, $f = f_0$ is semialgebraic.

(iv) It follows directly from the definition that if $M_1 \subset M_2 \subset \mathbb{R}^m$ are semialgebraic sets and f is an $\mathcal{S}^r$ function on M_2, then $f|_{M_1}$ is an $\mathcal{S}^r$ function on M_1.

Whitney already proved in his pioneer work that jets on closed subsets satisfying condition 2.A.1 are restrictions of global differentiable functions (Whitney's extension theorem). The first result in a tame category, the subanalytic one, can be found in [KP1]. Recently, this has been generalized to o-minimal structures in [KP2, Th]. Specifically, in our setting we have:

thfact **Fact 2.4** ([Th]). *Let $M \subset \mathbb{R}^m$ be a closed semialgebraic set and let f be an $\mathcal{S}^r$ function on M. Let $(f_\alpha)_{|\alpha|\leq r}$ be a semialgebraic jet of order r on M satisfying condition* 2.A.1*. Then there exists an $\mathcal{S}^r$ function F on $\mathbb{R}^m$ such that $F|_M = f$ and $\frac{\partial^{|\alpha|} F}{\partial \mathtt{x}^\alpha}|_M = f_\alpha$ for $|\alpha| \leq r$.*

In particular, our definition of an $\mathcal{S}^r$ function on a closed subsets coincides with the one concerning extensions that can be found in the literature. However, as we point out in the introduction we will not focus our study to just closed or locally compact semialgebraic subsets of $\mathbb{R}^n$.

2.B. $\mathcal{S}^r$ maps and embeddings. We deal next with maps instead of functions. This allows to work with locally compact semialgebraic sets as closed subsets (see Corollary 2.10).

Definition 2.5. Let $M \subset \mathbb{R}^m$ be a semialgebraic set. A map

$$\varphi := (\varphi_1, \dots, \varphi_n) : M \to \mathbb{R}^n$$

is a $\mathcal{S}^r$ *map* if each component φ_i is an $\mathcal{S}^r$ function. In addition, φ is a $\mathcal{S}^r$ *embedding* if it is injective and $\varphi^{-1} : \varphi(M) \to \mathbb{R}^m$ is an $\mathcal{S}^r$ map. In such a case we say that $\varphi : M \to \varphi(M)$ is a *$\mathcal{S}^r$ diffeomorphism,* and that *M and $\varphi(M)$ are $\mathcal{S}^r$ diffeomorphic.*

The following composition result is crucial.

comp **Theorem 2.6.** *Let $M \subset \mathbb{R}^m$ be a semialgebraic set, $\varphi := (\varphi_1, \dots, \varphi_n) : M \to \mathbb{R}^n$ an $\mathcal{S}^r$ map and let $N := \varphi(M)$. Let f be an $\mathcal{S}^r$ function on N. Then $g := f \circ \varphi$ is an $\mathcal{S}^r$ function on M.*

Proof. We deal with the case $r \geq 2$; the case $r = 1$ follows with the obvious modifications. As φ is an $\mathcal{S}^r$ map, for $i = 1, \dots, n$ there exists a semialgebraic jet $\Phi_i := (\varphi_{i,\alpha})_{|\alpha|\leq r}$ on M of order r such that $\varphi_{i,0} = \varphi_i$ and for each $a \in M$ it holds $|R^r_x \Phi_i(y)| = o(\|x - y\|^r)$ for $x, y \in M$ when $x, y \to a$. Denote $\varphi_\alpha := (\varphi_{1,\alpha}, \dots, \varphi_{n,\alpha})$ for $|\alpha| \leq r$ and $\Phi := (\varphi_\alpha)_{|\alpha|\leq r}$. We also write $T^r_x \Phi := (T^r_x \Phi_1, \dots, T^r_x \Phi_n)$ and $R^r_x \Phi := (R^r_x \Phi_1, \dots, R^r_x \Phi_n)$.

As f is an $\mathcal{S}^r$ function on N, there exists a semialgebraic jet $F := (f_\alpha)_{|\alpha|\leq r}$ on N of order r such that $f_0 = f$ and for every $b \in N$ there exists an increasing, continuous, concave function $\alpha_b : [0, +\infty) \to [0, +\infty)$ such that $\alpha_b(0) = 0$ and for each $w \in \mathbb{R}^n$ and $u, v \in N$ with $u, v \to b$,

$$|T^r_u F(w) - T^r_v F(w)| \leq \alpha_b(\|u - v\|) \cdot (\|w - u\|^r + \|w - v\|^r).$$

Consequently, for every $a \in M$ there exists an increasing, continuous and concave function $\alpha_{\varphi(a)} : [0, +\infty) \to [0, +\infty)$ such that $\alpha_{\varphi(a)}(0) = 0$ and for each $z \in \mathbb{R}^m$ and $x, y \in M$ with

$x, y \to a$,

$$\begin{aligned}|T^r_{\varphi(x)}F(T^r_x\Phi(z)) - T^r_{\varphi(y)}F(T^r_x\Phi(z))| \\ \leq \alpha_{\varphi(a)}(\|\varphi(x)-\varphi(y)\|)\cdot(\|T^r_x\Phi(z)-\varphi(x)\|^r + \|T^r_x\Phi(z)-\varphi(y)\|^r) \quad (2.1)\end{aligned}$$

`ineq`

Let us construct a suitable semialgebraic jet $G := (g_\alpha)_{|\alpha|\leq r}$ on M of order r such that $g_0 = g$. Write $\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_m)$ and $\mathtt{z} := (\mathtt{z}_1, \dots, \mathtt{z}_m)$. Let $Q(x, \mathtt{y}) \in \mathcal{S}(M)[\mathtt{y}]$ be a polynomial of degree $\leq r^2$ such that for each $x \in M$ we have

$$T^r_{\varphi(x)}F(T^r_x\Phi(\mathtt{z})) = Q(x, \mathtt{z}-x).$$

Let $g_\alpha \in \mathcal{S}(M)$ be $\alpha!$ times the coefficient of Q corresponding to $\mathtt{y}^\alpha$. Note that g_α is a finite sum of finite products of some of the semialgebraic functions $f_\alpha \circ \varphi$ and $\varphi_{i,\alpha}$. Thus, $G := (g_\alpha)_{|\alpha|\leq r}$ is a semialgebraic jet on M of order r. Observe that $g_0 = f_0 \circ \varphi = f \circ \varphi = g$. Notice that

$$T^r_x G(\mathtt{z}) = \sum_{|\alpha|\leq r} \frac{g_\alpha(x)}{\alpha!}(\mathtt{z}-x)^\alpha \in \mathbb{R}[\mathtt{z}-x]$$

and let $S(x, \mathtt{y}) \in \mathcal{S}(M)[\mathtt{y}]$ be such that $T^r_x G(\mathtt{z}) = S(x, \mathtt{z}-x)$. Define $P := Q - S$. For each $x \in M$ the polynomial $P_x(\mathtt{z}) := P(x, \mathtt{z}-x) \in \mathbb{R}[\mathtt{z}-x]$ has no non-zero monomial of degree $\leq r$ and has degree $\leq r^2$. As the coefficients of P are continuous functions, for every $a \in M$ there exists a constant $K_a > 0$ such that

$$|P_x(z)| < K_a \sum_{k=r+1}^{r^2} \|z-x\|^k \quad (2.2)$$

`pasito1`

for x close to a and $z \in \mathbb{R}^m$. Consequently, there exists a constant $K'_a > 0$ such that

$$|P_x(z)| < K'_a \cdot \big(\|z-x\|^{r+1} + \|z-x\|^{r^2}\big). \quad (2.3)$$

`pasito2`

For $z \in \mathbb{R}^m$ and $x, y \in M$ with $x, y \to a$ we have, by (2.1) and (2.3),

$$\begin{aligned}|T^r_x G(z) - T^r_y G(z)| &\leq |T^r_{\varphi(x)}F(T^r_x\Phi(z)) - T^r_{\varphi(y)}F(T^r_y\Phi(z))| + |P_x(z)| + |P_y(z)| \\ &\leq \alpha_{\varphi(a)}(\|\varphi(x)-\varphi(y)\|)\cdot(\|T^r_x\Phi(z)-\varphi(x)\|^r + \|T^r_x\Phi(z)-\varphi(y)\|^r) \\ &\quad + K'_a\cdot(\|z-x\|^{r+1} + \|z-x\|^{r^2} + \|z-y\|^{r+1} + \|z-y\|^{r^2}). \quad (2.4)\end{aligned}$$

`ineq2`

As the equality

$$T^r_x\Phi(\mathtt{z}) - \varphi(x) = \sum_{1\leq|\beta|\leq r} \frac{\varphi_\alpha(x)}{\alpha!}(\mathtt{z}-x)^\beta,$$

holds for each $a \in M$ there exists a constant $L_a > 0$ such that for $z \in \mathbb{R}^m$ and x close to a,

$$\|T^r_x\Phi(z) - \varphi(x)\| \leq \sum_{1\leq|\beta|\leq r} \frac{\|\varphi_\alpha(x)\|}{\alpha!}\cdot|(z-x)^\beta| \leq L_a \sum_{1\leq j\leq r} \|z-x\|^j. \quad (2.5)$$

`paso2`

Consequently, there exists a constant $L'_a > 0$ depending on a such that

$$\|T^r_x\Phi(z) - \varphi(x)\|^r \leq L'_a \cdot \big(\|z-x\|^r + \|z-x\|^{r^2}\big). \quad (2.6)$$

`paso3`

In addition, as φ is an $\mathcal{S}^r$ map, it is straightforward to check that for every $a \in M$ there exist a constant $B_a > 0$ such that $|\varphi(x) - \varphi(y)| \leq B_a \cdot \|x-y\|$ for $x, y \in M$ close to a. By inequalities (2.4) and (2.5)

$$\begin{aligned}|T^r_x G(z) - T^r_y G(z)| &\leq \alpha_a(B_a\|x-y\|)\cdot L'_a\cdot\big(\|z-x\|^r + \|z-x\|^{r^2} + \|z-y\|^r + \|z-y\|^{r^2}\big) \\ &\quad + K'_a\cdot\big(\|z-x\|^{r+1} + \|z-x\|^{r^2} + \|z-y\|^{r+1} + \|z-y\|^{r^2}\big). \quad (2.7)\end{aligned}$$

`ineq3`

Let us prove: *For every point $a \in M$ and each $\beta \in \mathbb{N}^m$ with $|\beta| \leq r$ it holds $|R^{r-|\beta|}_x G_\beta(y)| = o(\|x-y\|^{r-|\beta|})$ for $x, y \in M$ when $x, y \to a$.*

We follow, slightly modified, the strategy employed in the proof of implication (2.2.3) $\Longrightarrow$ (2.2.2) of [M, I.Thm. 2.2]. Using [M, I.(1.5)&(1.6)] we have

$$T_x^r G(z) - T_y^r G(z) = \sum_{|\beta|\leq r} \frac{(z-x)^\beta}{\beta!} R_y^r G_\beta(x).$$

Consequently, by (2.7)

$$\Big|\sum_{|\beta|\leq r} \frac{(z-x)^\beta}{\beta!} R_y^r G_\beta(x)\Big| \leq \alpha_a(B_a\cdot\|x-y\|)\cdot L'_a\cdot\big(\|z-x\|^r+\|z-x\|^{r^2}+\|z-y\|^r+\|z-y\|^{r^2}\big)$$
$$+K'_a\cdot\big(\|z-x\|^{r+1}+\|z-x\|^{r^2}+\|z-y\|^{r+1}+\|z-y\|^{r^2}\big). \quad (2.8)$$ ineq4

Write $\lambda := \|x-y\|$ and define $z-x=\lambda(z'-x)$. Note that we are only interested in the case $x\neq y$ and therefore $\lambda\neq 0$. Consequently,

$$\|z-y\|\leq\|z-x\|+\lambda=\lambda(\|z'-x\|+1). \quad (2.9)$$ lambda

We may assume that $\lambda<1$ and from (2.8) and (2.9) there exists $C_a>0$ such that

$$\Big|\sum_{|\beta|\leq r}\frac{\lambda^{|\beta|}}{\beta!}(z'-x)^\beta R_y^r G_\beta(x)\Big|\leq C_a\cdot(\alpha_a(B_a\lambda)\lambda^r\big(1+\|z'-x\|^{r^2}\big)+\lambda^{r+1}\big(1+\|z'-x\|^{r^2}\big))$$
$$=C_a\cdot(\alpha_a(B_a\lambda)+\lambda)\lambda^r\big(1+\|z'-x\|^{r^2}\big).$$

Fixing x and y and treating the sum on the left as a polynomial in $z'-x$ (taking into account that the coefficients are linearly determined in terms of the values of the polynomial at a suitable finite collection of points) we see that there exists a constant C'_a such that

$$\Big|\frac{\lambda^{|\beta|}}{\beta!}R_y^r G_\beta(x)\Big|\leq C'_a\cdot(\alpha_a(B_a\lambda)+\lambda)\lambda^r$$
$$\rightsquigarrow\quad |R_y^r G_\beta(x)|\leq\beta!\cdot C'_a\cdot(\alpha_a(B_a\|x-y\|)+\|x-y\|)\|x-y\|^{r-|\beta|}.$$

Consequently, $|R_y^r G_\beta(x)|=o(\|x-y\|^{r-|\beta|})$ if $|\beta|\leq r$, $x,y\in M$ and $x,y\to a$. Thus, g is an $\mathcal{S}^r$ function, as required. □

Straightforward consequences of the theorem above are the following.

potabs *Remarks* 2.7. (i) Let $f\in\mathcal{S}^r(M)$ with empty zero set. Then $|f|\in\mathcal{S}^r(M)$ because $f(M)\subset\mathbb{R}\setminus\{0\}$ and $g:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $x\mapsto|x|$ is an $\mathcal{S}^r$ function.

(ii) Let $f\in\mathcal{S}^r(M)$ and $\ell\geq r+1$. Then $|f|^\ell\in\mathcal{S}^r(M)$ because $\mathbb{R}\to\mathbb{R}$, $t\mapsto|t|^\ell$ is an $\mathcal{S}^r$ function.

(iii) Let $M\subset\mathbb{R}^m$ and $N\subset\mathbb{R}^n$ be semialgebraic sets and let $\varphi:M\to\mathbb{R}^n$ and $\psi:N\to\mathbb{R}^p$ be $\mathcal{S}^r$ maps such that $\varphi(M)\subset N$. Then $\psi\circ\varphi:M\to\mathbb{R}^p$ is an $\mathcal{S}^r$ map.

Next, let us show how can we represent closed semialgebraic subsets of a semialgebraic set $M\subset\mathbb{R}^m$ as zero-sets of $\mathcal{S}^r$ functions on M.

zero **Lemma 2.8.** *If Z is a closed semialgebraic subset of M there exists $g\in\mathcal{S}^{r*}(M)$ such that $Z=Z(g)$.*

Proof. Since $N:=\operatorname{Cl}(Z)$ is a closed semialgebraic subset of $\mathbb{R}^m$ and $Z=M\cap N$, there exists by [Sh, I.4.5] or [vdDM, Thm C.11] an $\mathcal{S}^r$ function $f\in\mathcal{S}^r(\mathbb{R}^m)$ such that $N=Z(f)$. Thus, $Z=Z(f|_M)$ and $f|_M\in\mathcal{S}^r(M)$. Consequently, $g:=\frac{f|_M}{1+(f|_M)^2}\in\mathcal{S}^{r*}(M)$ and $Z=Z(g)$. □

The following is a direct consequence.

separation **Corollary 2.9** (Urysohn's separation)**.** *Let $M_1,M_2\subset M$ be closed and disjoint semialgebraic subsets of M. Then there exists $f\in\mathcal{S}^{r*}(M)$ such that $f|_{M_1}\equiv 0$ and $f|_{M_2}\equiv 1$.*

Proof. By Lemma 2.8 there exist $g, h \in \mathcal{S}^{r*}(M)$ with $M_1 = Z(g)$ and $M_2 = Z(h)$. As $M_1 \cap M_2 = \varnothing$, the sum $g^2 + h^2$ never vanishes and $f := g^2/(g^2 + h^2) \in \mathcal{S}^{r*}(M)$ satisfies the statement. □

As mentioned above, it follows from Theorem 2.6 that the $\mathcal{S}^r$ functions on a locally compact semialgebraic set M are the restrictions to M of $\mathcal{S}^r$ functions on open semialgebraic neighborhoods of M.

ext **Lemma 2.10.** *Let $M \subset \mathbb{R}^m$ be a locally compact semialgebraic set. Then M is $\mathcal{S}^r$-diffeomorphic to a closed semialgebraic subset of $\mathbb{R}^{m+1}$. In addition, if f is an $\mathcal{S}^{r\diamond}$ function on M then there exists an $\mathcal{S}^{r\diamond}$ function F on the open semialgebraic neighborhood $U := \mathbb{R}^m \setminus (\operatorname{Cl}(M) \setminus M)$ of M such that $F|_M = f$.*

Proof. By [BCR, 2.7.5] there exists $h \in \mathcal{S}(\mathbb{R}^m)$ such that $h|_U$ is Nash and strictly positive and $Z(h) = \mathbb{R}^m \setminus U = \operatorname{Cl}(M) \setminus M$. The image of the Nash map $\varphi : U \to \mathbb{R}^{m+1}$, $x \mapsto \big(x, \frac{1}{h(x)}\big)$ is the closed semialgebraic set $C := \varphi(U) = \{(x, y) \in \mathbb{R}^{m+1} : yh(x) = 1\}$ and the projection

$$\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m,\ (x, x_{m+1}) := (x_1, \ldots, x_m, x_{m+1}) \to x$$

induces by restriction a Nash map $\rho := \pi|_C : C \to U$. Observe that $\varphi : U \to C$ and $\rho : C \to U$ are mutually inverse homeomorphisms. Thus, as M is closed in U, it holds that $N := \varphi(M)$ is closed in C.

Finally, let $f \in \mathcal{S}^{r\diamond}(M)$ and note that by Theorem 2.6 the function $g := f \circ \rho|_N$ is $\mathcal{S}^{r\diamond}$ on N. By Fact 2.4, combined with Lemma 2.9 in the bounded case, there is an $\mathcal{S}^{r\diamond}$ function G on $\mathbb{R}^{m+1}$ such that $G|_N = g$. The function $F := G \circ \varphi \in \mathcal{S}^{r\diamond}(U)$ satisfies $F|_M = f$, as required. □

Remark 2.11. If $M \subset \mathbb{R}^m$ is a Nash manifold, then $\mathcal{S}^r(M)$ coincides with the usual ring of $\mathcal{S}^r$ functions on M, which are (via Nash tubular neighborhoods) the restrictions to M of $\mathcal{S}^r$ functions on neighborhoods of M in $\mathbb{R}^m$.

lips **Lemma 2.12.** *Let $f : M \to \mathbb{R}$ be an $\mathcal{S}^r$ function with $r \geq 1$. Let $(f_\alpha)_{|\alpha| \leq r}$ be the semialgebraic jet associated to f. Then*

(i) *f_α is locally Lipschitz on M for each α with $|\alpha| < r$.*
(ii) *There exists an open semialgebraic neighborhood $N \subset \operatorname{Cl}(M)$ of M such that f_α admits a continuous extension F_α to N, which is locally Lipschitz on N for each α with $|\alpha| < r$. In particular, $f_\alpha \in \mathcal{S}^0(M)$ for each α with $|\alpha| < r$.*
(iii) *After shrinking N if necessary, we may assume that F_0 is an $\mathcal{S}^{r-1}$ extension of f.*

Proof. (i) Pick a point $a \in M$. As f is a $\mathcal{S}^1$ function, there exists $\varepsilon > 0$ such that

$$|f(x) - f(y) - \sum_{i=1}^{n} f_{\mathsf{e}_i}(y)(x_i - y_i)| \leq \|x - y\|,$$
$$|f_{\mathsf{e}_i}(y) - f_{\mathsf{e}_i}(a)| < 1$$

for each $x, y \in M \cap B(a, \varepsilon)$. Thus,

$$|f(x) - f(y)| \leq \Big(n + 1 + \sum_{i=1}^{n} |f_{\mathsf{e}_i}(a)|\Big)\|x - y\|$$

for each $x, y \in M \cap B(a, \varepsilon)$, so f is locally Lipschitz at $a \in M$. Analogously, each f_α is locally Lipschitz at $a \in M$ for each α with $|\alpha| < r$, as required.

(ii) Let $N \subset \operatorname{Cl}(M)$ be the set of points $x \in \operatorname{Cl}(M)$ such that f is locally Lipschitz in the intersection with M of a neighborhood of x, that is, there exist $\varepsilon > 0$ and $K > 0$ (depending on x) satisfying

$$|f(y_1) - f(y_2)| < K\|y_1 - y_2\|$$

for each $y_1, y_2 \in M \cap B(x, \varepsilon)$. Clearly, N is an open semialgebraic subset of $\operatorname{Cl}(M)$ that contains M. For each point $x \in N$ there exists an open neighborhood $V^x \subset N \subset \operatorname{Cl}(M)$ of x such that $f|_{V^x \cap M}$ is Lipschitz, so in particular $f|_{V^x \cap M}$ is uniformly continuous. Thus, $f|_{V^x \cap M}$ admits a unique continuous extension to $\operatorname{Cl}(V^x \cap M)$. For each $x \in N$ we have that $V^x \subset \operatorname{Cl}(V^x \cap M)$

and therefore there is a (unique) continuous extension F of f to N. Moreover, the function F is semialgebraic because for all $x \in N$ there is a unique $y \in \mathbb{R}$ such that $(x, y) \in \mathrm{Cl}(\Gamma(f))$.

Next, let us show that F is locally Lipschitz. Fix point $a \in N$ and let $K > 0$ and $\varepsilon > 0$ be such that

$$|f(x) - f(y)| \leq K\|x - y\|$$

for each $x, y \in M \cap B(a, \varepsilon)$ and $N \cap B(a, \varepsilon) = \mathrm{Cl}(M) \cap B(a, \varepsilon)$. Pick two points $u, v \in \mathrm{Cl}(M) \cap B(a, \varepsilon)$ with $u \neq v$. Since F is a continuous extension of f there is $x \in M$ such that $\max\{|f(x) - F(u)|, \|x - u\|\} < \|u - v\|$. Similarly, there is $y \in M$ such that $\max\{|f(y) - F(v)|, \|y - v\|\} < \|u - v\|$. Therefore,

$$\begin{aligned}|F(u) - F(v)| &\leq |F(u) - f(x)| + |f(x) - f(y)| + |f(y) - F(v)| \\ &\leq 2\|u - v\| + K\|x - y\| \leq 2\|u - v\| + K(\|x - u\| + \|u - v\| + \|v - y\|) \\ &= (3K + 2)\|u - v\|.\end{aligned}$$

and $F : N \to \mathbb{R}$ is locally Lipschitz at a. Consequently, the same happens for each α with $|\alpha| < r$ and statement (ii) holds, as required.

(iii) Pick a point $a \in M$. As f is an $\mathcal{S}^r$ function, there exists $\varepsilon > 0$ such that

$$\begin{aligned}&\Big|f(x) - \sum_{|\alpha| \leq r} \frac{f_\alpha(y)}{\alpha!}(x - y)^\alpha\Big| \leq \|x - y\|^r \\ &|f_\beta(y) - f_\beta(a)| < 1\end{aligned}$$

for each $x, y \in M \cap B(a, \varepsilon)$ and each $\beta \in \mathbb{N}^n$ with $|\beta| = r$. Let R be the cardinal of the set $\{\beta \in \mathbb{N}^n : \ |\beta| = r\}$. Thus,

$$\Big|f(x) - \sum_{|\alpha| \leq r-1} \frac{f_\alpha(y)}{\alpha!}(x - y)^\alpha\Big| \leq \Big(1 + \sum_{|\beta| = r} \frac{1 + |f_\beta(a)|}{\beta!}\Big)\|x - y\|^r \tag{2.10}$$

formula

for each $x, y \in M \cap B(a, \varepsilon)$.

Let N_0 be an open semialgebraic subset of $\mathrm{Cl}(M)$ containing M such that each f_α with $|\alpha| < r$ admits a locally Lipschitz (continuous) semialgebraic extension F_α to N_0, and whose existence is guaranteed by (ii). Let N be the set of points $x \in N_0$ such that there exists $\varepsilon > 0$ and $K > 0$ (depending on x) satisfying

$$\Big|f(y_1) - \sum_{|\alpha| \leq r-1} \frac{f_\alpha(y_2)}{\alpha!}(y_1 - y_2)^\alpha\Big| < K\|y_1 - y_2\|^r$$

for each $y_1, y_2 \in M \cap B(x, \varepsilon)$. Clearly, N is an open semialgebraic subset of $\mathrm{Cl}(M)$ that contains M. Let us show that $F := F_0$ is $\mathcal{S}^{r-1}$ on N. Fix a point $a \in N$ and let $\varepsilon > 0$ and $K > 0$ be such that

$$\begin{aligned}&\Big|f(x) - \sum_{|\alpha| \leq r-1} \frac{f_\alpha(y)}{\alpha!}(x - y)^\alpha\Big| < K\|x - y\|^r, \\ &|f_\gamma(y) - f_\gamma(a)| < 1\end{aligned}$$

for each $x, y \in M \cap B(a, \varepsilon)$ and each $\gamma \in \mathbb{N}^n$ with $|\gamma| < r$ and $N \cap B(a, \varepsilon) = \mathrm{Cl}(M) \cap B(a, \varepsilon)$. Note that

$$|f_\gamma(y)| < 1 + |f_\gamma(a)|$$

for all $\gamma \in \mathbb{N}^n$ with $|\gamma| < r$.

We must show that there is a constant $K' > 0$ such that for any two fixed points $u, v \in \mathrm{Cl}(M) \cap B(a, \varepsilon)$ with $u \neq v$, we have

$$\frac{\Big|F(u) - \sum_{|\alpha| \leq r-1} \frac{F_\alpha(v)}{\alpha!}(u - v)^\alpha\Big|}{\|u - v\|^{r-1}} \leq K'\|u - v\|. \tag{2.11}$$

r-1diff

To that aim, let $x, y \in M \cap B(a, \varepsilon)$ be such that

$$\begin{aligned}
&|F_\alpha(u) - F_\alpha(x)| < \|u - v\|^r, \\
&|F_\alpha(v) - F_\alpha(y)| < \|u - v\|^{r-|\alpha|}, \\
&|(u-v)^\alpha - (x-y)^\alpha| < \|u - v\|^r, \\
&\|u - x\| < \|u - v\|, \\
&\|v - y\| < \|u - v\|
\end{aligned}$$

for each $\alpha \in \mathbb{N}^n$ with $|\alpha| < r$ (to prove the existence of x, y it is enough to consider the continuous map $N^2 \mapsto \mathbb{R}^5$ whose function coordinates are the left hand expressions of the above inequalities and to note that the image of (u, v) is $0 \in \mathbb{R}^5$). In particular,

$$\|x - y\| \leq \|x - u\| + \|u - v\| + \|v - y\| \leq 3\|u - v\|.$$

We deduce

$$\begin{aligned}
\Big|F(u) - \sum_{|\alpha| \leq r-1} \frac{F_\alpha(v)}{\alpha!}(u-v)^\alpha\Big| &\leq |F(u) - f(x)| + \sum_{|\alpha| \leq r-1} \Big|\frac{F_\alpha(v) - f_\alpha(y)}{\alpha!}\Big| \|u - v\|^{|\alpha|} \\
&+ \sum_{0<|\alpha| \leq r-1} \Big|\frac{f_\alpha(y)}{\alpha!}\Big| |(u-v)^\alpha - (x-y)^\alpha| + \Big|f(x) - \sum_{|\alpha| \leq r-1} \frac{f_\alpha(y)}{\alpha!}(x-y)^\alpha\Big| \\
&\leq \|u - v\|^r + \sum_{|\alpha| \leq r-1} \frac{1}{\alpha!}\|u - v\|^r + \sum_{|\alpha| \leq r-1} \frac{1 + |f_\alpha(a)|}{\alpha!}\|u - v\|^r + K\|x - y\|^r \\
&\leq \Big(1 + 3^r K + \sum_{|\alpha| \leq r-1} \frac{2 + |f_\alpha(a)|}{\alpha!}\Big)\|u - v\|^r,
\end{aligned}$$

and therefore to get inequality (2.11) it suffices to define $K' := (1 + 3^r K + \sum_{|\alpha| \leq r-1} \frac{2+|f_\alpha(a)|}{\alpha!}\Big)$, as desired. □

LK

2.C. Rings of $\mathcal{S}^{r\diamond}$ functions as direct limits. Let M be a semialgebraic set, $r \geq 0$ an integer. The collection $\mathcal{S}^r(M)$ of all the $\mathcal{S}^r$ functions on M is a subring of $\mathcal{S}(M)$ whose units are those $f \in \mathcal{S}^r(M)$ with empty zero-set. Indeed, given $f, g \in \mathcal{S}^r(M)$ we must prove that $fg, f + g \in \mathcal{S}^r(M)$ and that $\frac{1}{f} \in \mathcal{S}^r(M)$ if $Z(f) = \varnothing$. Consider the $\mathcal{S}^r$ functions $h_1 : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto xy,\ \ h_2 : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x + y,$

$$h_3 : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\},\ x \mapsto \frac{1}{x},$$

and the $\mathcal{S}^r$ map $\varphi : M \to \mathbb{R}^2,\ x \mapsto (f(x), g(x))$. By Theorem 2.6, $fg = h_1 \circ \varphi$, $f + g = h_2 \circ \varphi$ and $\frac{1}{f} = h_3 \circ f$ are $\mathcal{S}^r$ functions (the latter if $Z(f) = \varnothing$).

The set $\mathcal{S}^{r*}(M) := \mathcal{S}^r(M) \cap \mathcal{S}^*(M)$ of bounded $\mathcal{S}^r$ functions on M is a subalgebra of $\mathcal{S}^r(M)$, that coincides with $\mathcal{S}^r(M)$ if M is compact. The multiplicatively closed subset

$$\mathcal{W}^r(M) := \{f \in \mathcal{S}^{r*}(M) : Z(f) = \varnothing\}$$

of $\mathcal{S}^{r*}(M)$ contains 1 but not 0, so $\mathcal{S}^{r*}(M)_{\mathcal{W}^r(M)} = \{f/g : f \in \mathcal{S}^{r*}(M)\ \&\ g \in \mathcal{W}^r(M)\}$ is an $\mathbb{R}$-algebra and it coincides with $\mathcal{S}^r(M)$. The inclusion $\mathcal{S}^{r*}(M)_{\mathcal{W}^r(M)} \subset \mathcal{S}^r(M)$ follows from the above, where we noted that each $g \in \mathcal{S}^r(M)$ with empty zero-set is a unit in $\mathcal{S}^r(M)$. Conversely, each $h \in \mathcal{S}^r(M)$ can be written as $h = f/g$, where $f := h/(1 + h^2) \in \mathcal{S}^{r*}(M)$ and $g := 1/(1 + h^2) \in \mathcal{W}^r(M)$.

Let us present the rings $\mathcal{S}^{r*}(M)$ and $\mathcal{S}^r(M)$ as a direct limit of rings of $\mathcal{S}^r$ functions on compact and locally compact semialgebraic sets, respectively. In the sequel we will prove some nice properties of the ring of $\mathcal{S}^r$ functions on a locally compact semialgebraic set, and we will show how some of them transfer through the direct limit (with limitations, as we already pointed out in the introduction).

An $\mathcal{S}^r$ *local compactification of* M is a pair $(E,\mathtt{j})$ where $\mathtt{j}: M\to\mathbb{R}^n$ is an $\mathcal{S}^r$ embedding such that $E=\operatorname{Cl}(\mathtt{j}(M))$, so that E is a locally compact semialgebraic set. If in addition each coordinate function of the map $\mathtt{j}$ is bounded then we say that $(E,\mathtt{j})$ is a $\mathcal{S}^{r*}$ *local compactification of* M (note that in this case E is compact). We denote by $\mathfrak{C}^r(M)$ and $\mathfrak{C}^{r*}(M)$ the collection of all $\mathcal{S}^r$ and $\mathcal{S}^{r*}$ local compactifications of M respectively. Henceforth, we will write $\mathfrak{C}^r$ and $\mathfrak{C}^{r*}$ hopefully without ambiguity. We will also write $\mathfrak{C}^{r\diamond}$ to refer indistinctly to the previous two collection of maps, when a result is valid for both of them.

For the elements of $\mathfrak{C}^{r\diamond}$ we define $(E_1,\mathtt{j}_1)\preceq(E_2,\mathtt{j}_2)$ if there exists an $\mathcal{S}^r$ map $\rho_{21}:E_2\to E_1$ with $\mathtt{j}_1=\rho_{21}\circ\mathtt{j}_2$. As $\mathtt{j}_i(M)$ is dense in E_i the map ρ_{21} is determined by $\rho_{21}|_{\mathtt{j}_2(M)}=\mathtt{j}_1\circ\mathtt{j}_2^{-1}:\mathtt{j}_2(M)\to\mathtt{j}_1(M)$ and $\rho_{21}(E_2)$ is dense in E_1. The $\mathbb{R}$-homomorphisms $\rho_{21}^*:\mathcal{S}^r(E_1)\to\mathcal{S}^r(E_2)$, $f\mapsto f\circ\rho_{21}$ and $\mathtt{j}^*:\mathcal{S}^r(E)\to\mathcal{S}^r(M)$, $f\mapsto f\circ\mathtt{j}$ are always injective. In addition, if $(E_1,\mathtt{j}_1)\preceq(E_2,\mathtt{j}_2)\preceq(E_3,\mathtt{j}_3)$, then $\rho_{31}=\rho_{21}\circ\rho_{32}$. We are ready to prove equation (1.1).

LD **Lemma 2.13.** *The family* $(\mathfrak{C}^{r\diamond},\preceq)$ *is a directed set and* $\mathcal{S}^{r\diamond}(M)$ *coincides with the direct limit* $\varinjlim\mathcal{S}^{r\diamond}(E)$ *where* $(E,\mathtt{j})\in\mathfrak{C}^{r\diamond}$.

Proof. Let $(E_1,\mathtt{j}_1),(E_2,\mathtt{j}_2)\in\mathfrak{C}^{r\diamond}$ with $E_i\subset\mathbb{R}^{n_i}$, $n:=n_1+n_2$, the $\mathcal{S}^{r\diamond}$ embedding $(\mathtt{j}_1,\mathtt{j}_2): M\to\mathbb{R}^n$ and the projections $\pi_i:E_1\times E_2\to E_i$. Define $E_3:=\operatorname{Cl}((\mathtt{j}_1,\mathtt{j}_2)(M))\cap(E_1\times E_2)$ and let $\mathtt{j}_3:M\to E_3,\ x\mapsto(\mathtt{j}_1(x),\mathtt{j}_2(x))$, which is an $\mathcal{S}^{r\diamond}$ embedding. Then the restriction $\rho_{3i}:=\pi_i|_{E_3}$ is an $\mathcal{S}^r$ map and $\rho_{3i}\circ\mathtt{j}_3=\mathtt{j}_i$. Thus $(E_3,\mathtt{j}_3)\in\mathfrak{C}^{r\diamond}$ and $(E_i,\mathtt{j}_i)\preceq(E_3,\mathtt{j}_3)$ for $i=1,2$. Consequently, $(\mathfrak{C}^{r\diamond},\preceq)$ is a directed set.

To prove that $\mathcal{S}^{r\diamond}(M)=\varinjlim\mathcal{S}^{r\diamond}(E)$ it is enough to show: *For each* $f\in\mathcal{S}^{r\diamond}(M)$*, there exist* $(E,\mathtt{j})\in\mathfrak{C}^{r\diamond}$ *and* $F\in\mathcal{S}^{r\diamond}(E)$ *such that* $f=F\circ\mathtt{j}$. We may assume that $M\subset\mathbb{R}^m$ is bounded. Indeed, let $N:=\varphi^{-1}(M)$, where B is the open ball of radius 1 in $\mathbb{R}^m$ centered at the origin and φ is the Nash diffeomorphism

$$\varphi:B\to\mathbb{R}^m,\ x\mapsto\frac{x}{\sqrt{1-\|x\|^2}}. \tag{2.12}$$ nshdiff

It induces an $\mathbb{R}$-algebra isomorphism $\mathcal{S}^{r\diamond}(M)\to\mathcal{S}^{r\diamond}(N)$, $f\mapsto f\circ\varphi$. So let $f\in\mathcal{S}^{r\diamond}(M)$ and consider the closure E in $\mathbb{R}^{m+1}$ of the graph of f. Notice that $\mathtt{j}:M\to\mathbb{R}^{m+1}$, $x\mapsto(x,f(x))$ is an $\mathcal{S}^r$ embedding with $E=\operatorname{Cl}(\mathtt{j}(M))$. The function $F:=\pi|_E$, where $\pi:\mathbb{R}^{m+1}\to\mathbb{R},\ (x,x_{m+1})\mapsto x_{m+1}$, belongs to $\mathcal{S}^{r\diamond}(E)$ and $f=F\circ\mathtt{j}$, as required. □

s6

2.D. Comparison between $\mathcal{S}^0(M)$ and $\mathcal{S}(M)$. A consequence of one of the main results of [Fe2] is Corollary 6 which establishes that if M is a 2-dimensional semialgebraic set such that the germ M_x is connected for each $x\in\operatorname{Cl}(M)$ then $\mathcal{S}(M)=\mathcal{S}^0(M)$. The following result, which is the 2-dimensional version of Theorem 1.1, is a generalization of the latter:

main2:2dim **Proposition 2.14.** *Let* $M\subset\mathbb{R}^m$ *be a* 2*-dimensional semialgebraic set. If* M *is non problematic then* $\mathcal{S}^\diamond(M)=\mathcal{S}^{0\diamond}(M)$*. Conversely, if the map* $\varphi:\operatorname{Spec}^\diamond(M)\to\operatorname{Spec}^{0\diamond}(M)$, $\mathfrak{p}\mapsto\mathfrak{p}\cap\mathcal{S}^{0\diamond}(M)$ *is injective then* M *is non-problematic.*

Proof. We give the proof for the $\mathcal{S}$ case, the bounded one is the same mutatis mutandis. Suppose that M is not problematic. Then for each $x\in M$ there is an open ball B of $\mathbb{R}^m$ such that for all $y\in\operatorname{Cl}(M)\cap B$ the germ M_y is connected. By [Fe2, Cor.6] there is $\varepsilon>0$ such that for $N^x:=B(x,\varepsilon)$ there is an $\mathcal{S}$-function $F^x:N^x\to\mathbb{R}$ that extends $f|_{M\cap B(x,\varepsilon)}$. Note that $M\cap B(x,\varepsilon)$ is dense in N^x and so F^x is unique. In particular, this is a first order statement and therefore we can choose ε semialgebraically uniform on x. Thus, the set $N:=\bigcup_{x\in M}N^x$ is an open semialgebraic neighborhood of M in $\operatorname{Cl}(M)$ and the function $F:N\to\mathbb{R}$ given by $F(y)=F^x(y)$ if $y\in N^x$ is an $\mathcal{S}$ extension of f, as required.

Now, let us assume that φ is injective, and suppose that M is problematic at a point $p\in M$. In particular, M is not locally compact at p. Without loss we can assume that M is bounded. Denote $X:=\operatorname{Cl}(M)$, which is a compact set. Let (K,Φ) be a semialgebraic triangulation of X compatible with M and $\{p\}$, i.e., K is a finite simplicial complex and $\Phi:|K|\to X$ is a semialgebraic homeomorphism such that both M and $\{p\}$ are union of images of simplices.

Henceforth, we identify X with $|K|$ and the involved objects M and $\{p\}$ with their inverse images under Φ.

Since M is problematic at p there is a sequence of points $\{x_k\}_{k\geq 1} \subset X \setminus M$ converging to p such that the germ M_{x_k} is disconnected. Since $\dim(X \setminus M) = 1$, there are two 2-simplices $\sigma_1, \sigma_2 \in K$ such that $\sigma_j^0 \subset M$ for $j = 1, 2$ with a common 1-simplex face τ which satisfies that $\tau^0 \subset X \setminus M$ and p is a vertex of τ.

Next, consider the closed semialgebraic subset $T_j := M \cap \sigma_j$ of M for $j = 1, 2$. We define $\mathfrak{p}_j$ as the set of functions $f \in \mathcal{S}(M)$ satisfying that there is a semialgebraic neighborhood U of p in τ and a continuous semialgebraic function $F : T_j \cup U \to \mathbb{R}$ extending $f|_{T_j}$ and with $F|_U = 0$.

Claim. $\mathfrak{p}_j \in \operatorname{Spec}(M)$ for $j = 1, 2$.

Proof. Fix $j = 1, 2$. Let R be a real closed field extension of $\mathbb{R}$ such that there is a positive element $\varepsilon \in R$ with $\varepsilon < x$ for all positive $x \in \mathbb{R}$. Let us construct a morphism

$$\phi_j : \mathcal{S}(M) \to R$$

whose kernel is $\mathfrak{p}_j$. Denote by τ^R the 2-simplex in R^m that defines the vertices of τ. Pick a point $p_0 \in \tau^R$ such that $|p - p_0| < \varepsilon$. By [Fe2, Cor.6] we have that $\mathcal{S}(T_j) = \mathcal{S}^0(T_j)$ and therefore for any $\mathcal{S}$ function $f : T_j \to \mathbb{R}$ there is a neighborhood N of T_j in $\operatorname{Cl}(T_j)$ and an $\mathcal{S}$ extension $F : N \to \mathbb{R}$ of f. Take the realization of N and F in R, which we denote by N^R and $F^R : N^R \to R$ respectively. Since $p_0 \in N^R$ we can consider the evaluation morphism

$$\psi_j : \mathcal{S}(T_j) \to R : f \mapsto F^R(p_0).$$

Note that it is well-defined: if we pick another extension of f then both extension will coincide in a neighborhood of p because M is dense in $\operatorname{Cl}(M)$. Finally, as T_j is closed in M, the restriction map $\mathcal{S}(M) \to \mathcal{S}(T_j)$ is surjective and therefore the kernel of the morphism

$$\phi_j : \mathcal{S}(M) \to R : f \mapsto \psi_j(f|_{T_j})$$

is exactly $\mathfrak{p}_j$, as required. □

Let us show that $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Denote by $v_0 = p, v_1, v_2 \in \sigma_1$ the vertices of σ_1. Any point $x \in \mathbb{R}^m$ is of the form $x = t_0 v_0 + t_1 v_1 + t_2 v_2$ for some $t_0, t_1, t_2 \in \mathbb{R}$ with $t_0 + t_1 + t_2 = 1$. Consider the $\mathcal{S}$ function

$$f_1 : T_1 \to \mathbb{R} : x \mapsto 1 - t_0.$$

On the other hand, by Urysohn's separation lemma 2.9 there is an $\mathcal{S}$-function $f_2 : T_2 \to \mathbb{R}$ such that $f_2 = 0$ in a neighborhood of p and $f_2 = 1$ in a neighborhood of the other vertex of τ. Consider the $\mathcal{S}$ function $f_0 : T_1 \cup T_2 \to \mathbb{R}$ such that $f_0(x) = f_1(x)$ if $x \in T_1$ and $f_0(x) = f_2(x)$ if $x \in T_2$. Since $T_1 \cup T_2$ is closed in M, there is an $\mathcal{S}$ function $f : M \to \mathbb{R}$ such that $f|_{T_1 \cup T_2} = f_0$. Clearly, we have that $f \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$, as desired.

Finally, we note that $f \in \mathfrak{p}_j \cap \mathcal{S}^0(M)$ if and only if there is an open neighborhood N of M in $\operatorname{Cl}(M)$ and a semialgebraic $\mathcal{S}$-extension $F : N \to \mathbb{R}$ of f such that $f|_{\tau \cap N} = 0$. Therefore $\mathfrak{p}_1 \cap \mathcal{S}^0(M) = \mathfrak{p}_2 \cap \mathcal{S}^0(M)$ and so φ is not injective, a contradiction. □

Essentially, the idea of the proof of Theorem 1.1 is to reduce the problem to the 2-dimensional case. For that aim, we recall from [Fe1, Prop. 3.2] the following decomposition of M as an irredundant finite union of closed pure dimensional semialgebraic subsets of M as well as some of its main properties. There exists a unique finite family $\{M_1, \dots, M_r\}$ of semialgebraic subsets of M satisfying the following properties:

(i) Each M_i is the closure in M of the set of points of M whose local dimension is equal to some fixed value. In particular, M_i is pure dimensional and closed in M.
(ii) $M = \bigcup_{i=1}^r M_i$.
(iii) $M_i \setminus \bigcup_{j\neq i} M_j$ is dense in M_i.
(iv) $\dim(M_i) > \dim(M_{i+1})$ for $i = 1, \dots, r-1$. In particular, $\dim(M_1) = \dim(M)$.

We call the sets M_i the bricks of M and denote the *family of bricks* of M.

Proof of Theorem 1.1. Once again, we give the proof for the $\mathcal{S}$ case, the bounded one is similar.

To show that (i) implies (ii), pick $f \in \mathcal{S}(M)$ and let us show that $f \in \mathcal{S}^0(M)$. If M is locally compact then it is clear. Assume that M is not locally compact and denote M_i the 2-dimensional brick of M. Since M_i and M' are closed in M, we have $\operatorname{Cl}(M_i) \cap M' = M_i \cap M'$ and $M_i \cap \operatorname{Cl}(M') = M_i \cap M'$. By hypothesis M_i is not locally compact and it is not problematic. Therefore, by Proposition 2.14 there is an open semialgebraic neighborhood N_i of M_i in $\operatorname{Cl}(M_i)$ and an $\mathcal{S}$-extension $F_i : N_i \to \mathbb{R}$ of $f|_{M_i}$. We can assume that N_i satisfies $N_i \cap \operatorname{Cl}(M') = M_i \cap M'$. Indeed, it suffices to replace N_i by $\widetilde{N}_i := N_i \setminus (\operatorname{Cl}(M') \setminus M')$. Since M' is locally compact, the set $\operatorname{Cl}(M') \setminus M'$ is closed $\mathbb{R}^n$ and so $\widetilde{N}_i$ is open in $\operatorname{Cl}(M_i)$. Note that $\widetilde{N}_i$ also contains M_i. For, if $x \in M_i$ belongs to $\operatorname{Cl}(M') \setminus M'$ then $x \in \operatorname{Cl}(M') \cap M_i = M' \cap M_i \subset M'$, a contradiction. On the other hand,

$$M_i \cap M' \subset \widetilde{N}_i \cap \operatorname{Cl}(M') = \widetilde{N}_i \cap M' \subset \operatorname{Cl}(M_i) \cap M' = M_i \cap M'$$

and so $\widetilde{N}_i \cap \operatorname{Cl}(M') = M_i \cap M'$.

Next, consider the semialgebraic set $N := M' \cup N_i$. Note that

$$\operatorname{Cl}(N_i) \cap N = (\operatorname{Cl}(N_i) \cap M') \cup N_i) \subset (\operatorname{Cl}(M_i) \cap M') \cup N_i = M_i \cup N_i = N_i$$

and therefore N_i is closed in N. Moreover,

$$\operatorname{Cl}(M') \cap N = M' \cup (\operatorname{Cl}(M') \cap N_i) = M' \cup (M' \cap M_i) = M'$$

and so M' is also closed in N. Hence, the function

$$F : N \to \mathbb{R},\ x \mapsto \begin{cases} f(x) & \text{if } x \in M', \\ F_i(x) & \text{if } x \in N_i \end{cases}$$

is a semialgebraic extension of f to the locally compact set N, so that $f \in \mathcal{S}^0(M)$, as desired.

We prove that (iii) implies (i), the fact that (ii) implies (iii) is trivial. Suppose first that there exists a brick M_ℓ of dimension $k \geq 3$ that is not locally compact. Note that the canonical diagram

$$\begin{array}{ccc} \operatorname{Spec}(M_\ell) & \longrightarrow & \operatorname{Spec}(M) \\ \Big\downarrow{\scriptstyle \varphi_\ell} & & \Big\downarrow{\scriptstyle \varphi} \\ \operatorname{Spec}^0(M_\ell) & \longrightarrow & \operatorname{Spec}^0(M) \end{array}$$

commutes. Moreover, the map $\operatorname{Spec}(M_\ell) \to \operatorname{Spec}(M)$ is injective because the restriction homomorphism $\mathcal{S}(M) \to \mathcal{S}(M_\ell)$ is surjective [DK1]. By hypothesis φ is injective and therefore we deduce that φ_ℓ is also injective. Thus, we can assume that M is not locally compact and pure dimensional with $\dim(M) = k \geq 3$.

Pick a point $p \in M$ such that M is not locally compact at p. Let (K, Φ) be a semialgebraic triangulation of $X := \operatorname{Cl}(M)$ compatible with M and $\{p\}$. For the sake of simplicity, we identify X with $|K|$ and the involved objects M and $\{p\}$ with their inverse images under Φ. Let τ be a simplex of K such that $p \in \tau$ and $\tau \subset X \setminus M$. The existence of τ is guaranteed because M is not locally compact at p (otherwise, the star of p in X would be contained in M). Since M is pure dimensional, there is a k-simplex $\sigma \subset M$ such that τ is a face of σ. Denote $v_0 := p$ and take a point $v_1 \in \tau^0$. Next, pick a point $v_2 \in \sigma^0$ and consider the 2-simplex σ_1 spanned by the affinely independent points v_0, v_1, v_2. Since $\dim(\sigma) \geq 3$, there is a point $v_3 \in \sigma^0$ such that v_3 is not contained in the plane spanned by v_0, v_1, v_2. Consider the 2-simplex σ_2 spanned by v_0, v_1, v_3 and note that $\sigma_1 \cap \sigma_2$ is the 1-simplex spanned by v_0 and v_1. Let us define the closed semialgebraic subset $T := (\sigma_1 \cup \sigma_2) \cap M$ of M. As above, the canonical map $\operatorname{Spec}(T) \to \operatorname{Spec}^0(T)$ is injective and therefore by Proposition 2.14 the 2-dimensional set T is not problematic. This is a contradiction because T is clearly problematic at p.

Hence, we can assume that the only non-locally compact brick of M is the 2-dimensional M_{i_0}. As before, since M_{i_0} is closed in M, we deduce that $\phi_{i_0} : \operatorname{Spec}(M_{i_0}) \to \operatorname{Spec}^0(M_{i_0})$ is injective. By Proposition 2.14 it follows that M_{i_0} is non-problematic, as required. □

s3 3. Zariski spectra of rings of $\mathcal{S}^{r\diamond}$ functions

In this chapter we compare the Zariski and maximal spectra of the ring of the $\mathcal{S}^r$ functions with that of the $\mathcal{S}^0$ functions of a semialgebraic set M. In Section 3.A we first prove the Łojasiewicz's Nullstellensatz in the locally compact case, and a weak version of the latter in the non-locally compact case. Both results will be crucial to prove the comparition Theorem 1.2 in Section 3.B for the $\mathcal{S}^r$ case and 3.C for the $\mathcal{S}^{r*}$ case.

LNull **3.A. Łojasiewicz's Nullstellensatz.** First, we analyze the Nullstelensatz for $\mathcal{S}^r$ functions. We need some preliminary results. Along this section we fix integers r, m with $r \geq 0$ and $m \geq 1$ and a semialgebraic set $M \subset \mathbb{R}^m$. For our convenience we introduce more notation. For each $f \in \mathcal{S}^r(M)$ we denote

$$D(f) := M \setminus Z(f) \text{ and } U(f) := \{x \in M : \, f(x) > 0\}$$

The following result is a generalization of [BCR, 2.6.4].

suboclase **Proposition 3.1.** *Let $G \in \mathcal{S}^r(\mathbb{R}^m)$ and let $F \in \mathcal{S}^r(D(G))$. Then there exists an integer $\ell \geq 1$ such that the function*

$$H_\ell : \mathbb{R}^m \to \mathbb{R},\, x \mapsto \begin{cases} G^\ell(x)F(x) & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases}$$

belongs to $\mathcal{S}^r(\mathbb{R}^m)$.

Proof. Using a recursive argument it is enough to study the case $r = 1$. For simplicity we will prove the existence and continuity of the partial derivative $d_{e_1}H_\ell$ of H_ℓ with respect to the first variable at every point $a \in Z(G)$. By [BCR, 2.6.4] we can apply the case $r = 0$ of the statement simultaneously to the functions

$$F|_{D(G)} : D(G) \to \mathbb{R} \quad \text{and} \quad d_{e_1}F|_{D(G)} : D(G) \to \mathbb{R}.$$

Hence, there exists a positive integer k such that the function H_k in the statement and the function

$$\widehat{H}_k : \mathbb{R}^m \to \mathbb{R},\, x \mapsto \begin{cases} G^k(x)d_{e_1}F|_{D(G)}(x) & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases}$$

are continuous. We claim that $H_{k+1} \in \mathcal{S}^1(\mathbb{R}^m)$. Clearly H_{k+1} is a semialgebraic function. Let us show first that $d_{e_1}H_{k+1}(a) = 0$ for each $a \in Z(G)$. Indeed,

$$\begin{aligned} d_{e_1}H_{k+1}(a) &= \lim_{t\to 0} \frac{H_{k+1}(a+te_1) - H_{k+1}(a)}{t} = \lim_{t\to 0} \frac{G^{k+1}(a+te_1)F(a+te_1)}{t} \\ &= \Big(\lim_{t\to 0} \frac{G(a+te_1) - G(a)}{t}\Big) \cdot \big(\lim_{t\to 0} G^k(a+te_1)F(a+te_1)\big) \\ &= d_{e_1}G(a) \cdot \big(\lim_{t\to 0} H_k(a+te_1)\big) = d_{e_1}G(a) \cdot H_k(a) = 0. \end{aligned}$$

On the other hand, for each point $a \in D(G)$ we have

$$\begin{aligned} d_{e_1}H_{k+1}(a) &= (k+1)G(a)^k d_{e_1}G(a)F(a) + G^{k+1}(a)d_{e_1}F(a) \\ &= (k+1)H_k(a)d_{e_1}G(a) + G(a)\widehat{H}_k(a), \end{aligned}$$

and so $H_{k+1} \in \mathcal{S}^1(D(G))$ because $H_k, \widehat{H}_k, d_{e_1}G$ and G are continuous on $D(G)$. Since both H_k and $\widehat{H}_k$ vanish on $Z(G)$ it also follows that $H_{k+1} \in \mathcal{S}^1(\mathbb{R}^n)$, as required. □

We are now ready to prove the Łojasiewicz's Nullstellensatz in the locally compact case for $\mathcal{S}^r$ functions.

null20 **Theorem 3.2.** *Let M be a locally compact semialgebraic set and let $f_1, f_2 \in \mathcal{S}^r(M)$ with $Z(f_1) \subset Z(f_2)$. Then there exist an integer $\ell \geq 1$ and $g \in \mathcal{S}^r(M)$ such that $f_2^\ell = gf_1$ and $Z(f_2) = Z(g)$.*

Proof. By Corollary 2.10 we may assume that M is closed in some $\mathbb{R}^m$, so by Fact 2.4 there exist $F_i \in \mathcal{S}^r(\mathbb{R}^m)$ with $F_i|_M = f_i$ for $i = 1, 2$. Moreover, by Lemma 2.8 there is $L \in \mathcal{S}^r(\mathbb{R}^m)$ such that $M = Z(L)$. Then

$$Z(L^2 + F_1^2) = Z(L^2) \cap Z(F_1^2) = M \cap Z(F_1) = Z(f_1) \subset Z(f_2) = Z(L^2 + F_2^2).$$

Therefore

$$\Phi : D(L^2 + F_2^2) \to \mathbb{R},\ x \mapsto \frac{1}{L^2(x) + F_1^2(x)}$$

is an $\mathcal{S}^r$ function and by Proposition 3.1 there exists a positive integer k such that

$$\Psi : \mathbb{R}^m \to \mathbb{R},\ x \mapsto \begin{cases} \big(L^2(x) + F_2^2(x)\big)^k \Phi(x) & \text{if } L^2(x) + F_2^2(x) \neq 0 \\ 0 & \text{if } L^2(x) + F_2^2(x) = 0 \end{cases}$$

belongs to $\mathcal{S}^r(\mathbb{R}^m)$. Since $(L^2 + F_2^2)^k = \Psi \cdot (L^2 + F_1^2)$ and $L|_M \equiv 0$ we get

$$f_2^{2k} = (\Psi|_M) \cdot f_1^2.$$

Therefore, $\ell := 2k$ and $g := (\Psi|_M) f_1 \in \mathcal{S}^r(M)$ do the job because

$$Z(g) = \big(Z(L^2 + F_2^2) \cap M\big) \cup Z(f_1) = Z(f_2) \cup Z(f_1) = Z(f_2)$$

as required. □

counterloja *Example* 3.3. The local compactness of M is essential in Łojasiewicz's Nullstellensatz (see Remark [FG6, 1.2]).

The Nullstelensatz has a useful consequence in the study of the ideal structure of rings of $\mathcal{S}^r$ functions. Recall that a classical concept to analyze this structure is the following:

primez **Definition 3.4.** Let M be a semialgebraic set. An ideal $\mathfrak{a}$ in $\mathcal{S}^r(M)$ is a *z-ideal* if for every $g \in \mathcal{S}^r(M)$ and $f \in \mathfrak{a}$ such that $Z(f) \subset Z(g)$ it follows that $g \in \mathfrak{a}$.

It is easy to prove that in general a z-ideal of $\mathcal{S}^r(M)$ is a real radical ideal. Reciprocally, in the locally compact case we have:

zloc **Lemma 3.5.** *Let M be a locally compact semialgebraic set. Each radical ideal $\mathfrak{a}$ in $\mathcal{S}^r(M)$ is a z-ideal and a real ideal. In particular, each prime ideal of $\mathcal{S}^r(M)$ is a z-ideal and a real ideal.*

Proof. Let $g \in \mathcal{S}^r(M)$ and $f \in \mathfrak{a}$ such that $Z(f) \subset Z(g)$. By Theorem 3.2 there is a positive integer ℓ and $h \in \mathcal{S}^r(M)$ such that $g^\ell = hf \in \mathfrak{a}$ and so $g \in \mathfrak{a}$, as required. □

nzifd *Example* 3.6. Again local compactness is crucial in Lemma 3.5, see [FG5, 3.4.1].

We finish with a weak version of the Nullstelensatz in the non-locally compact case which will be crucial in the next sections.

crucial **Proposition 3.7.** *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and $f_1, f_2 \in \mathcal{S}^r(M)$. Consider the semialgebraic set $S := \{x \in \mathbb{R}^n : (x, 0) \in \operatorname{Cl}(\Gamma(f_1)) \setminus \operatorname{Cl}(\Gamma(f_2))\}$. The following assertions are equivalent:*

(i) $M \cap \operatorname{Cl}(S) = \varnothing$.
(ii) *There exists an open semialgebraic neighborhood N of M in $\operatorname{Cl}(M)$ and semialgebraic functions $F_1, F_2 \in \mathcal{S}^{r-1}(N)$ such that $Z(F_1) \subset Z(F_2)$ and $F_i|_M = f_i$ for $i = 1, 2$.*
(iii) *There exists an integer $\ell \geq 1$ and $g \in \mathcal{S}^{r-1}(M)$ such that $f_2^\ell = f_1 g$ and $Z(f_2) = Z(g)$.*

Proof. (i) $\Longrightarrow$ (ii) By Lemma 2.12 there is an open semialgebraic neighborhood $N \subset \operatorname{Cl}(M)$ of M in $\operatorname{Cl}(M)$ such that f_i extends to semialgebraic functions $F_i \in \mathcal{S}^{r-1}(N)$ for $i = 1, 2$. As F_i is a continuous extension of f_i and N is dense in $\operatorname{Cl}(M)$, we have $\Gamma(F_i) \subset \operatorname{Cl}(\Gamma(f_i))$ and $Z(F_i) \subset \{x \in \mathbb{R}^n : (x, 0) \in \operatorname{Cl}(\Gamma(f_i))\}$. We claim:

$$Z(F_1) \setminus Z(F_2) \subset S.$$

Pick a point $x \in Z(F_1)\setminus Z(F_2)$. Assume $c := F_2(x) > 0$ and let $U \subset \mathbb{R}^n$ be an open neighborhood of x such that $F_2(y) > \frac{c}{2}$ for each $y \in U \cap N$. Thus,

$$\Gamma(F_2) \cap \Big(U \times \Big(-\frac{c}{3}, \frac{c}{3}\Big)\Big) = \varnothing,$$

so $(x, F_1(x)) = (x, 0) \in \operatorname{Cl}(\Gamma(f_1)) \setminus \operatorname{Cl}(\Gamma(f_2))$, as claimed.

Thus, by hypothesis $M \cap \operatorname{Cl}(Z(F_1) \setminus Z(F_2)) = \varnothing$. Substitute N by $N \setminus \operatorname{Cl}(Z(F_1) \setminus Z(F_2))$.

(ii) $\Longrightarrow$ (iii) As N is locally compact and $Z(F_1) \subset Z(F_2)$, by Theorem 3.2 there exists $G \in \mathcal{S}^{r-1}(N)$ and $\ell \geq 1$ such that $F_2^\ell = F_1 G$ and $Z(F_2) = Z(G)$. Observe that $g := G|_M \in \mathcal{S}^{r-1}(M)$ satisfies $Z(f_2) = Z(g)$ and $f_2^\ell = f_1 g$.

(iii) $\Longrightarrow$ (i) Let N be an open semialgebraic neighborhood N of M in $\operatorname{Cl}(M)$ where f_1, f_2, g have semialgebraic extensions $F_1, F_2, G \in \mathcal{S}^{r-1}(N)$ by Lemma 2.12. Since M is dense in N we have $F_2^\ell = F_1 G$, and so $Z(F_1) \subset Z(F_2)$. Consider the closed set $C := \operatorname{Cl}(M) \setminus N$ and let us show that

$$S \subset C.$$

This finishes the proof since $M \cap \operatorname{Cl}(S) \subset M \cap C = \varnothing$. Pick $x \in S \subset \operatorname{Cl}(M)$ and suppose that $x \in N$. Then $(x, 0) \in \operatorname{Cl}(\Gamma(f_1)) \setminus \operatorname{Cl}(\Gamma(f_2))$, so $F_1(x) = 0$ and $F_2(x) \neq 0$, which is a contradiction because $Z(F_1) \subset Z(F_2)$, as required. □

s4

3.B. Zariski and maximal spectra of rings of $\mathcal{S}^r$ functions. In this section we prove Theorem 1.2 in the $\mathcal{S}^r$ case and we give some consequences.

Proof of Theorem 1.2 *in the $\mathcal{S}^r$ case.* Let us prove first that the map

$$\varphi : \operatorname{Spec}^0(M) \to \operatorname{Spec}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^r(M)$$

is bijective. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}^0(M)$ be distinct prime ideals. We may assume that there exists $f \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Let $N \subset \operatorname{Cl}(M)$ be an open neighborhood of M such that f extends to a semialgebraic function F on N. By Lemma 2.8 there is $G \in \mathcal{S}^r(N)$ such that $Z(F) = Z(G)$ and denote $g := G|_M$. By Proposition 3.7 there exist $\ell_1, \ell_2 \geq 1$ and $h_1, h_2 \in \mathcal{S}(M)$ such that $g^{\ell_1} = f h_1$ and $f^{\ell_2} = g h_2$. Consequently, $g \in (\mathfrak{p}_2 \cap \mathcal{S}^r(M)) \setminus (\mathfrak{p}_1 \cap \mathcal{S}^r(M))$ and $\varphi(\mathfrak{p}_1) \neq \varphi(\mathfrak{p}_2)$.

Let $\mathfrak{q}$ be a prime ideal of $\mathcal{S}^r(M)$. Let $\mathfrak{p}$ be the set of all $f \in \mathcal{S}^0(M)$ for which there is $g \in \mathfrak{q}$ such that there exist extensions $F \in \mathcal{S}^0(N)$ and $G \in \mathcal{S}^0(N)$ of f and g respectively to some open neighborhood N of M in $\operatorname{Cl}(M)$ with $Z(G) \subset Z(F)$. Next, we show that $\mathfrak{p}$ is a prime ideal of $\mathcal{S}^0(M)$ equal to $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ and such that $\mathfrak{p} \cap \mathcal{S}^r(M) = \mathfrak{q}$.

Let $f_1, f_2 \in \mathcal{S}^0(M)$ be such that $f_1 f_2 \in \mathfrak{p}$. We show that $f_1 \in \mathfrak{p}$ or $f_2 \in \mathfrak{p}$. Let $N \subset \operatorname{Cl}(M)$ be an open semialgebraic neighborhood of M such that there exists semialgebraic extensions F_1, F_2 to N of f_1, f_2. We may assume that there exists $H \in \mathcal{S}^r(N)$ such that $Z(H) \subset Z(F_1 F_2)$ and $h := H|_M \in \mathfrak{q}$. Let $G_1, G_2 \in \mathcal{S}^r(N)$ be such that $Z(F_i) = Z(G_i)$ for $i = 1, 2$. Thus,

$$Z(H) \subset Z(F_1 F_2) = Z(F_1) \cup Z(F_2) = Z(G_1) \cup Z(G_2) = Z(G_1 G_2)$$

and by Theorem 3.2 there exist $\ell \geq 1$ and $A \in \mathcal{S}^r(N)$ such that $(G_1 G_2)^\ell = HA$. Denote $g_i := G_i|_M \in \mathcal{S}^r(M)$ and $a := A|_M \in \mathcal{S}^r(M)$. We have $(g_1 g_2)^\ell = ha \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime we may assume that $g_1 \in \mathfrak{q}$ and so $f_1 \in \mathfrak{p}$, as required.

Now, let us prove that $\mathfrak{p} = \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$, so in particular $\mathfrak{p}$ is a prime ideal. Pick $f \in \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$. Then for some $\ell \geq 1$ there are $g_1, \ldots, g_s \in \mathfrak{q}$ and $h_1, \ldots, h_s \in \mathcal{S}^0(M)$ such that $f^\ell = g_1 h_1 + \cdots + g_s h_s$. Let N be an open neighborhood of M in $\operatorname{Cl}(M)$ and let $F, G_i, H_i \in \mathcal{S}^0(N)$ be extensions of f, g_i, h_i respectively for each $i = 1, \ldots, s$. Note that $F^\ell = G_1 H_1 + \cdots + G_s H_s$ and define $G := G_1^2 + \cdots + G_s^2$. We clearly have that $G|_M \in \mathfrak{q}$ and $Z(G) \subset Z(F^\ell)$, so that $f^\ell \in \mathfrak{p}$. In particular, $f \in \mathfrak{p}$, as required. To prove the other inclusion, let $f \in \mathfrak{p}$. Then there exists $g \in \mathfrak{q}$ and there is an open semialgebraic neighborhood $N \subset \operatorname{Cl}(M)$ of M and extensions $F, G \in \mathcal{S}^0(N)$ of f, g such that $Z(G) \subset Z(F)$. Thus, there exists $\ell \geq 1$ and $H \in \mathcal{S}(N)$ such that $F^\ell = GH$. Denote $h := H|_M \in \mathcal{S}^0(M)$ and observe that $f^\ell = gh \in \mathfrak{q}\mathcal{S}^0(M)$, so $f \in \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$.

To finish let us check that $\mathfrak{p}\cap\mathcal{S}^r(M)=\mathfrak{q}$. The inclusion right to left is clear. Let $f\in\mathfrak{p}\cap\mathcal{S}^r(M)$. Then there exists $g\in\mathfrak{q}$ and an open neighborhood $N\subset \operatorname{Cl}(M)$ of M with extensions $F,G\in\mathcal{S}^0(N)$ of f,g such that $Z(G)\subset Z(F)$. By Theorem 3.2 there exist $\ell\geq 1$ and $H\in\mathcal{S}^r(M)$ such that $F^\ell=GH\in\mathfrak{q}$, so $F\in\mathfrak{q}$, as required.

Finally, the map φ is continuous because it is induced by the ring inclusion $\mathtt{j}:\mathcal{S}^r(M)\hookrightarrow\mathcal{S}^0(M)$. It suffices to check that φ is an open map. Let $\mathcal{D}:=\{\mathfrak{p}\in\operatorname{Spec}_s^0(M):\ f\notin\mathfrak{p}\}$ for some $f\in\mathcal{S}^0(M)$. Let $N\subset\operatorname{Cl}(M)$ be an open semialgebraic neighborhood of M such that there exists a continuous semialgebraic extension F of f to N. Take $G\in\mathcal{S}^r(N)$ be such that $Z(F)=Z(G)$. Define $g:=G|_M\in\mathcal{S}^r(M)$ and let us prove that

$$\varphi(\mathcal{D})=\{\mathfrak{q}\in\operatorname{Spec}^r(M):g\notin\mathfrak{q}\}.$$

Note that there are $\ell_1,\ell_2\geq 1$ and $h_1,h_2\in\mathcal{S}^0(M)$ such that $f^{\ell_1}=gh_1$ and $g^{\ell_2}=fh_2$. Thus, if $\mathfrak{p}\in\mathcal{D}$ then $g\notin\mathfrak{p}\cap\mathcal{S}^r(M)=\varphi(\mathfrak{p})$. Conversely, let $\mathfrak{q}\in\operatorname{Spec}^r(M)$ be such that $g\notin\mathfrak{q}$ and let us show that $f\notin\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$. Note that $g\notin\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ because $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}\cap\mathcal{S}^r(M)=\mathfrak{q}$. Since $g^{\ell_2}=fh_2$ and $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ is prime, we deduce $f\notin\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$, as required. □

radONE *Remark* 3.8. Along the proof above we have shown that for any semialgebraic set M and $\mathfrak{q}\in\operatorname{Spec}^r(M)$, the prime ideal $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ equals the set of functions $f\in\mathcal{S}^0(M)$ for which there is an integer $\ell\geq 1$, $g\in\mathfrak{q}$ and $h\in\mathcal{S}^0(M)$ such that $f^\ell=gh$. Moreover, $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ is exactly the set of all $f\in\mathcal{S}^0(M)$ for which there is $g\in\mathfrak{q}$ such that there exist extensions $F\in\mathcal{S}^0(N)$ and $G\in\mathcal{S}^0(N)$ of f and g respectively to some open neighborhood N of M in $\operatorname{Cl}(M)$ with $Z(G)\subset Z(F)$.

In particular, if $\mathfrak{q}$ is a z-ideal then $\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ is a z-ideal. Indeed, let $f_1,f_2\in\mathcal{S}^0(M)$ be such that $f_1\in\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ and $Z(f_1)\subset Z(f_2)$. Then, there is $g_1\in\mathfrak{q}$ such that there exist extensions $F_1\in\mathcal{S}^0(N)$ and $G_1\in\mathcal{S}^0(N)$ of f_1 and g_1 respectively to some open neighborhood N of M in $\operatorname{Cl}(M)$ with $Z(G_1)\subset Z(F_1)$. Without loss of generality, we can assume that there is an $\mathcal{S}$-extension F_2 of f_2 to N. Moreover, by Lemma 2.8 we can pick a function $G_2\in\mathcal{S}^r(N)$ such that $Z(G_2)=Z(F_2)$. Let us denote $g_2:=G_2|_M$, so that

$$Z(g_1)\subset Z(f_1)\subset Z(f_2)=Z(g_2)$$

and therefore we deduce that g_2 belongs to the z-ideal $\mathfrak{q}$. In turn, we get that $f_2\in\sqrt{\mathfrak{q}\mathcal{S}^0(M)}$, as desired.

s5

3.C. Zariski and maximal spectra of rings of $\mathcal{S}^{r*}$ functions.

Łojasiewicz's Nullstellensatz 3.2 has played a crucial role in the proof of Theorem 1.2 in the $\mathcal{S}^r$ case. Since Theorem 3.2 does not have a bounded counterpart, we need to develop another tool in order to give a proof of Theorem 1.2 in the $\mathcal{S}^{r*}$ case.

gorro **Lemma 3.9.** *Let $M\subset\mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{a}$ be a (proper) ideal of $\mathcal{S}^{r*}(M)$. Then the set*

$$\widehat{\mathfrak{a}}:=\{f\in\mathcal{S}^{0*}(M):\ \forall\varepsilon\in\mathcal{S}^{0*}(M),\ \varepsilon>0\ \exists g\in\mathfrak{a}\text{ such that }|f-g|<\varepsilon\}$$

is a (proper) ideal of $\mathcal{S}^{0*}(M)$.

Proof. Let $f_1,f_2\in\widehat{\mathfrak{a}}$ and $\varepsilon\in\mathcal{S}^{0*}(M)$ strictly positive. Then there exist $g_1,g_2\in\mathfrak{a}$ with $|f_i-g_i|<\varepsilon/2$ for $i=1,2$ and so $g:=g_1+g_2\in\mathfrak{a}$ and $|(f_1+f_2)-g|\leq|f_1-g_1|+|f_2-g_2|<\varepsilon$. Thus, $f_1+f_2\in\widehat{\mathfrak{a}}$.

Let $f\in\widehat{\mathfrak{a}}$ and $a\in\mathcal{S}^{0*}(M)$. We want to show $fa\in\widehat{\mathfrak{a}}$. Fix $\varepsilon\in\mathcal{S}^{0*}(M)$ strictly positive. The exists an open semialgebraic neighborhood $N\subset\operatorname{Cl}(M)$ of M and semialgebraic extensions F,A,E of f,a,ε to N. As $M\subset\{E>0\}$, we may assume shrinking N that E is strictly positive on N. Next, by Lemma 2.10 there is an open subset U of $\mathbb{R}^n$ such that N is closed in U and there are F',A',E' bounded semialgebraic extensions of F,A,E to U. We may assume in addition that E' is strictly positive on U. Let $L>0$ be such that $|F'|,|A'|,|E'|<L$. By [BCR, Thm.8.8.4] there exists $H\in\mathcal{N}(U)$ such that $|H-A'|<\frac{E'}{2L}$. Since H is bounded by $L+\frac{1}{2}$

we get that $h := H|_M \in \mathcal{S}^{r*}(M)$. Since $f \in \widehat{\mathfrak{a}}$, there exists $g \in \mathfrak{a}$ such that $|f-g| < \frac{\varepsilon}{2L+1}$. Consequently, $gh \in \mathfrak{a}$ and

$$|fa - gh| \leq |f||a-h| + |h||f-g| < L\frac{\varepsilon}{2L} + \Big(L + \frac{1}{2}\Big)\frac{\varepsilon}{2L+1} = \varepsilon.$$

We conclude $fa \in \widehat{\mathfrak{a}}$.

Finally, if $1 \in \widehat{\mathfrak{a}}$, there exists $g \in \mathfrak{a}$ such that $|g-1| < 1/2$, so $1/2 < g < 3/2$. Thus g is a unit in $\mathcal{S}^{r*}(M)$, which is a contradiction. □

roproperties

Proposition 3.10. *Let $\mathfrak{q}$ be a prime ideal of $\mathcal{S}^{r*}(M)$ that contains a function with empty zero-set. Then $\widehat{\mathfrak{q}}$ is a prime ideal of $\mathcal{S}^{0*}(M)$ such that $\mathfrak{q} = \widehat{\mathfrak{q}} \cap \mathcal{S}^{r*}(M)$ and $\widehat{\mathfrak{q}} = \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$.*

Proof. We fix $f_0 \in \mathfrak{q}$ with $Z(f_0) = \varnothing$. By Remark 2.7 the function $\varepsilon_0 := |f_0|$ is $\mathcal{S}^{r*}$ on M. Since $\varepsilon_0^2 = f_0^2 \in \mathfrak{q}$, we have that $\varepsilon_0 \in \mathfrak{q}$.

Let us prove that $\widehat{\mathfrak{q}} \cap \mathcal{S}^{r*}(M) \subset \mathfrak{q}$, the other inclusion is clear. Pick $h \in \widehat{\mathfrak{q}} \cap \mathcal{S}^{r*}(M)$ and let $g \in \mathfrak{q}$ be such that $|h-g| < \varepsilon_0$. Thus $a := (h-g)/\varepsilon_0 \in \mathcal{S}^{r*}(M)$ and $h = g + a\varepsilon_0 \in \mathfrak{q}$, as required.

Next, we show that $\widehat{\mathfrak{q}}$ is prime. First, we need the following:

Claim. Let $\varepsilon \in \mathcal{S}^{0*}(M)$ be strictly positive and let $f \in \mathcal{S}^{0*}(M) \setminus \widehat{\mathfrak{q}}$. Then there exists $g \in \mathcal{S}^{r*}(M) \setminus \mathfrak{q}$ such that $|f-g| < \varepsilon$.

Proof of the Claim. As $f \in \mathcal{S}^{0*}(M) \setminus \widehat{\mathfrak{q}}$, there exists $\varepsilon_1 \in \mathcal{S}^{0*}(M)$ strictly positive such that for each $h \in \mathfrak{q}$ there exists $x_0 \in M$ satisfying $|f(x_0) - h(x_0)| \geq \varepsilon_1(x_0)$. Let $N \subset \mathrm{Cl}(M)$ be an open semialgebraic neighborhood of M and semialgebraic extensions F, E_1, E of $f, \varepsilon_1, \varepsilon$ to N such that E_1, E are strictly positive on N. By Lemma 2.10 there exists an open subset U of $\mathbb{R}^n$ such that N is closed in U and there exist extensions $F', E'_1, E' \in \mathcal{S}^*(U)$ of F, E_1, E with E'_1 and E strictly positive on U. Consider the strictly positive function $E'_2 := \min\{E', E'_1\}/2 \in \mathcal{S}^*(U)$. By [BCR, Thm.8.8.4] there exists $G \in \mathcal{N}(U)$ such that $|F' - G| < E'_2$, so in particular G is bounded. Note that $g := G|_M \in \mathcal{S}^{r*}(M)$ satisfies $|f-g| < \varepsilon_1 < \varepsilon$ and so $g \notin \mathfrak{q}$, as required. □

Suppose that there exist $f_1, f_2 \in \mathcal{S}^{0*}(M) \setminus \widehat{\mathfrak{q}}$ such that $f_1 f_2 \in \widehat{\mathfrak{q}}$. By the claim there exist $g_1, g_2 \in \mathcal{S}^{r*}(M) \setminus \mathfrak{q}$ such that $|f_i - g_i| < \varepsilon_0$ for $i = 1, 2$. Let $L > 0$ be such that $|\varepsilon_0|, |f_i| < L$. We have

$$|f_1 f_2 - g_1 g_2| \leq |f_1||f_2 - g_2| + |g_2||f_1 - g_1| \leq L\varepsilon_0 + (L + \varepsilon_0)\varepsilon_0 = (2L + \varepsilon_0)\varepsilon_0.$$

Since $f_1 f_2 \in \widehat{\mathfrak{q}}$, there exists $g \in \mathfrak{q}$ with $|f_1 f_2 - g| < \varepsilon_0$. Thus,

$$|g - g_1 g_2| \leq |g - f_1 f_2| + |f_1 f_2 - g_1 g_2| < (1 + 2L + \varepsilon_0)\varepsilon_0.$$

Define $h := g_1 g_2 \in \mathcal{S}^{r*}(M) \setminus \mathfrak{q}$. We have $|g - h| < (1 + 3L)\varepsilon_0$, so

$$a := \frac{h-g}{\varepsilon_0} \in \mathcal{S}^{r*}(M) \quad \text{and} \quad h - g = \varepsilon_0 a \in \mathfrak{q}.$$

Thus, $h = g + \varepsilon_0 a \in \mathfrak{q}$, which is a contradiction.

Finally, we show that $\widehat{\mathfrak{q}} = \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$. Since $\widehat{\mathfrak{q}}$ is a prime ideal which contains $\mathfrak{q}$, it follows that $\sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)} \subset \widehat{\mathfrak{q}}$. Conversely, given $f \in \widehat{\mathfrak{q}}$ there exists $g \in \mathfrak{q}$ such that $|f-g| < \varepsilon_0$. Thus $h := |f-g|/\varepsilon_0 \in \mathcal{S}^{0*}(M)$ and $|f-g| = \varepsilon h \in \mathfrak{q}\mathcal{S}^{0*}(M)$. Then $f^2 = |f-g|^2 - g(g-2f) \in \mathfrak{q}\mathcal{S}^{0*}(M)$ and therefore $f \in \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$. □

cor:Xbounded

Corollary 3.11. *Let $M \subset \mathbb{R}^m$ be a semialgebraic subset. Denote $\mathcal{W}^r(M) := \{f \in \mathcal{S}^{r*}(M) : Z(f) = \varnothing\}$ and consider the space $\mathfrak{X}^r(M) := \{\mathfrak{p} \in \mathrm{Spec}^{r*}(M) : \mathfrak{p} \cap \mathcal{W}^r(M) \neq \varnothing\}$. Then the maps*

$$\Psi_M : \mathfrak{X}^0(M) \to \mathfrak{X}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{r*}(M)$$

$$\Phi_M : \mathfrak{X}^r(M) \to \mathfrak{X}^0(M),\ \mathfrak{q} \mapsto \widehat{\mathfrak{q}} = \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$$

are mutually inverse.

Proof. By Proposition 3.10 it suffices to check that if $\mathfrak{p}$ is a prime ideal of $\mathcal{S}^{0*}(M)$ which contains a function with empty zero-set then $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}^{r*}(M)$ contains a function with empty zero-set and $\widehat{\mathfrak{q}} = \mathfrak{p}$. Let $f_0 \in \mathfrak{p}$ be such that $Z(f_0) = \varnothing$, so $\varepsilon_0 := f_0^2 \in \mathfrak{p}$ is strictly positive.

We prove first $\widehat{\mathfrak{q}} \subset \mathfrak{p}$. Given $f \in \widehat{\mathfrak{q}}$ there exists $g \in \mathfrak{q} \subset \mathfrak{p}$ with $|f-g| < \varepsilon_0$, so $h := |f-g|/\varepsilon_0 \in \mathcal{S}^{0*}(M)$. Since $f^2 = (f-g)^2 + g(2f-g) = h^2\varepsilon_0^2 + g(2f-g) \in \mathfrak{p}$, we deduce that $f \in \mathfrak{p}$, as required. For the converse inclusion suppose that there exists $f \in \mathfrak{p} \setminus \widehat{\mathfrak{q}}$. By the claim of the proof of Lemma 3.10 there exists $g \in \mathcal{S}^{r*}(M) \setminus \mathfrak{q}$ with $|f-g| < \varepsilon_0$. Thus $h := |f-g|/\varepsilon_0 \in \mathcal{S}^{0*}(M)$, so $|f-g| = h\varepsilon_0 \in \mathfrak{p}$. Consequently, $g^2 = |f-g|^2 + 2fg - f^2 \in \mathfrak{p}$ and therefore $g \in \mathfrak{p} \cap \mathcal{S}^{r*}(M) = \mathfrak{q}$, which is a contradiction.

Finally, note that since $\varepsilon_0 \in \mathfrak{p} = \widehat{\mathfrak{q}}$, there exists $g \in \mathfrak{q}$ with $|\varepsilon_0 - g| < \varepsilon_0$, and so $Z(g) = \varnothing$. $\square$

We are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 *in the* $\mathcal{S}^{r*}$ *case.* By Corollary 3.11 it is enough to show that the maps

$$\Phi : \operatorname{Spec}^{0*}(M) \setminus \mathfrak{X}^0(M) \to \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{r*}(M)$$
$$\Psi : \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M) \to \operatorname{Spec}^{0*}(M) \setminus \mathfrak{X}^0(M),\ \mathfrak{q} \mapsto \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$$

are mutually inverse. Since $\mathcal{S}^0(M) = \mathcal{S}^{0*}(M)_{\mathcal{W}^0(M)}$ and $\mathcal{S}^r(M) = \mathcal{S}^{r*}(M)_{\mathcal{W}^r(M)}$, by [AM, 3.11 & Ch.3, Ex. 21] the map

$$\gamma^r : \operatorname{Spec}^r(M) \to \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{r*}(M)$$

is a homeomorphism whose inverse is given by $\mathfrak{q} \mapsto \mathfrak{q}\mathcal{S}^r(M)$. On the other hand, by Theorem 1.2 for $\mathcal{S}^r$ functions we have that the maps

$$\rho : \operatorname{Spec}^0(M) \to \operatorname{Spec}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^r(M)$$
$$\mathtt{i} : \operatorname{Spec}^r(M) \to \operatorname{Spec}^0(M),\ \mathfrak{q} \mapsto \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$$

are mutually inverse homeomorphisms. Thus, the map

$$\eta := \gamma^r \circ \rho \circ (\gamma^0)^{-1} : \operatorname{Spec}^{0*}(M) \setminus \mathfrak{X}^0(M) \to \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M),\ \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^{r*}(M)$$

is a homeomorphism and the diagram

$$\begin{array}{ccc} \operatorname{Spec}^r(M) & \xrightarrow{\ \gamma^r\ } & \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M) \\ \rho \uparrow & & \uparrow \eta \\ \operatorname{Spec}^0(M) & \xrightarrow{\ \gamma^0\ } & \operatorname{Spec}^{0*}(M) \setminus \mathfrak{X}^0(M) \end{array}$$

commutes. For each $\mathfrak{q} \in \operatorname{Spec}^{r*}(M) \setminus \mathfrak{X}^r(M)$ we have $\eta^{-1}(\mathfrak{q}) = \sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{0*}(M)$, so it only remains to show that

$$\sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)} = \sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{0*}(M).$$

First note that by Theorem 1.2 for $\mathcal{S}^r$ functions and since $\mathfrak{q}\mathcal{S}^r(M) \in \operatorname{Spec}^r(M)$, we get

$$\sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{0*}(M) = \mathfrak{q}\mathcal{S}^r(M), \text{ and so } \sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{r*}(M) = \mathfrak{q}.$$

Now, pick $f \in \sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{0*}(M)$. Let $N \subset \operatorname{Cl}(M)$ be an open semialgebraic neighborhood of M such that there is a semialgebraic extension F of f to N. We show that there exists $Q \in \mathcal{S}^{r*}(N)$ and $k \geq 1$ such that $f^{2k+2} \in \mathfrak{q}\mathcal{S}^0(M)$, $F^{2k+2} \leq Q$ and $Z(Q) = Z(F)$. Indeed, by Lemma 2.8 and Theorem 3.2 there is $k \geq 1$, $G \in \mathcal{S}^{r*}(N)$ and $H \in \mathcal{S}^0(N)$ such that $F^k = GH$ and $Z(G) = Z(F)$. Without loss we can assume that $f^k \in \mathfrak{q}\mathcal{S}^0(M)$. Consider the open semialgebraic set $W := \{|FH| < 1\}$ of N that contains $Z(F)$. As F is bounded, there exists $L > 0$ such that $F^{2k+2} < L$ and $G^2 < L$. Let $\{\sigma, 1-\sigma\}$ be an $\mathcal{S}^r$ partition of unity subordinated to $\{W, N \setminus Z(F)\}$ and define

$$Q := G^2\sigma + L(1-\sigma) \in \mathcal{S}^{r*}(M).$$

If $x \in W$, then $F^{2k+2}(x) \leq G^2(x)$ and so $F^{2k+2}(x) \leq G^2(x) + (L - G^2(x))(1-\sigma(x)) = Q(x)$. If $x \in N \setminus W$, then $\sigma(x) = 0$ and so $Q(x) = L \geq F(x)^{2k+2}$. Clearly, $Z(Q) = Z(G) = Z(F)$.

Finally, again by Theorem 3.2 there exist $\ell \geq 1$ and $A \in \mathcal{S}^0(N)$ such that $Q^\ell = F^{2k+2}A$ and so $q^\ell \in \mathfrak{q}\mathcal{S}^0(M)$ for $q := Q|_M$. Thus, $q \in \sqrt{\mathfrak{q}\mathcal{S}^0(M)} \cap \mathcal{S}^{r*}(M) = \mathfrak{q}$. On the other hand, since $F^{2k+2} \leq Q$ and $Z(F) = Z(Q)$, we have that $B := \frac{F^{2k+3}}{Q}$ is a $\mathcal{S}^{0*}$ function on N. Therefore $f^{2k+3} \in \mathfrak{q}\mathcal{S}^{0*}(M)$ and $f \in \sqrt{\mathfrak{q}\mathcal{S}^{0*}(M)}$, as required. □

dim **Corollary 3.12.** *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then*

(i) $\dim(\mathcal{S}^{r\diamond}(M)) = \dim(M)$.
(ii) $\mathcal{S}^{r\diamond}(M)$ *is a Gelfand ring.*

Proof. By Theorem 1.2 it is enough to prove the result for $r = 0$. We write down the general case, the bounded one is similar.

(i) Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s$ be a chain of prime ideals in $\mathcal{S}^0(M)$. Let $f_i \in \mathfrak{p}_i \setminus \mathfrak{p}_{i-1}$ for $i = 1, \dots, s$. Let $N \subset \mathrm{Cl}(M)$ and an open semialgebraic neighborhood of M and let F_i be semialgebraic extensions of f_i to N. Let $\mathfrak{q}_i := \mathfrak{p}_i \cap \mathcal{S}(N)$ and observe that $F_i \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1}$ for $i = 1, \dots, s$. The equality $\dim(\mathcal{S}(N)) = \dim(N) = \dim(M)$ was proved in [CC, S3, S5, FG6]. Therefore $\dim(\mathcal{S}^0(M)) \leq \dim(M)$. For the other inequality, by cylindrical decomposition of semialgebraic sets there is a closed ball B such that

$$M \cap B = \mathrm{Cl}(M) \cap B$$

is compact and $\dim(M) = \dim(M \cap B)$. Since $M \cap B$ is closed in $\mathbb{R}^n$, the restriction map $\mathcal{S}^0(M) \to \mathcal{S}^0(M \cap B)$ is surjective. So $\dim(\mathcal{S}^0(M)) \geq \dim(\mathcal{S}^0(M \cap B)) = \dim(M \cap B) = \dim(M)$, as required.

(ii) Let $\mathfrak{p}$ be a prime ideal in $\mathcal{S}^0(M)$. We show that the set of prime ideals of $\mathcal{S}^0(M)$ that contain $\mathfrak{q}$ is a spear, i.e., is totally ordered by the inclusion. This clearly implies that $\mathfrak{p}$ is contained in a unique maximal ideal, as desired. Assume there exists prime ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ containing $\mathfrak{p}$ for which there are functions $f_1 \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and $f_2 \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$. Let $N \subset \mathrm{Cl}(M)$ be an open semialgebraic neighborhood of M such that there exist semialgebraic extensions F_1, F_2 to N of f_1, f_2. Consider the prime ideals $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}(N)$ and $\mathfrak{q}_i := \mathfrak{m}_i \cap \mathcal{S}(N)$ for $i = 1, 2$. Observe that $\mathfrak{q} \subset \mathfrak{q}_1 \cap \mathfrak{q}_2$, $F_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and $F_2 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. But this is a contradiction because N is locally compact and therefore the set of prime ideals of $\mathcal{S}(N)$ that contain $\mathfrak{q}$ is a spear, as required. □

spcmaxh **Corollary 3.13.** *Let $\beta^{0\diamond}M$ and $\beta^{r\diamond}M$ be, respectively, the subspaces of* $\mathrm{Spec}^{0\diamond}(M)$ *and* $\mathrm{Spec}^{r\diamond}(M)$ *consisting of the maximal ideals in $\mathcal{S}^{0\diamond}(M)$ and $\mathcal{S}^{r\diamond}(M)$. Then the map*

$$\beta^{0\diamond}M \to \beta^{r\diamond}M,\ \mathfrak{m} \mapsto \mathfrak{m} \cap \mathcal{S}^{r\diamond}(M)$$

is a homeomorphism.

Proof. This is a straightforward consequence of Theorem 1.2 because $\beta^{0\diamond}M$ and $\beta^{r\diamond}M$ are respectively the subsets of closed points of $\mathrm{Spec}^{0\diamond}(M)$ and $\mathrm{Spec}^{r\diamond}(M)$. □

tilde *Remark* 3.14. It was proved in [FG4, 4.4] that βM is the semialgebraic Stone–Čech compactification of M and so Corollary 3.13 states that the same holds true for the space $\beta^r M$ of maximal ideals in $\mathcal{S}^r(M)$. This can be seen as the semialgebraic counterpart of the analogous result by Bkouche [B] for the ring of differentiable functions on a differentiable manifold.

s7

4. Residue fields of rings of $\mathcal{S}^{r\diamond}$ functions

Our goal in this section is to prove Theorem 1.4, that says that if M is a semialgebraic set and $\mathfrak{p}$ is a prime ideal of $\mathcal{S}^{r\diamond}(M)$ the fraction field $\kappa(\mathfrak{p})$ of $\mathcal{S}^{r\diamond}(M)/\mathfrak{p}$ is a real closed field. In Subsection 4.A we prove it for the ring $\mathcal{S}^r(M)$ with M locally compact and in Subsection 4.B we show it for the ring $\mathcal{S}^{r\diamond}(M)$ for any semialgebraic set M.

ocallcompact **4.A. The locally compact case of Theorem 1.4.** We will use the notion of semialgebraic depth of an ideal of a ring of semialgebraic functions firstly introduced in [FG6]. Let M be a semialgebraic set. The *semialgebraic depth* of $\mathfrak{q} \in \operatorname{Spec}^r(M)$ is the nonnegative integer

$$\mathtt{d}(\mathfrak{q}) := \min\{\dim(Z(f)) : f \in \mathfrak{q}\}.$$

ht00 **Lemma 4.1.** Let M be a semialgebraic set and let $\mathfrak{q}$ be a prime ideal in $\mathcal{S}^r(M)$.

(1) $\mathtt{d}(\mathfrak{q}) = \mathtt{d}(\sqrt{\mathfrak{q}\mathcal{S}^0(M)})$.
(2) If $\mathfrak{q}_1 \in \operatorname{Spec}^r(M)$ is a z-ideal such that $\mathfrak{q}_1 \subsetneq \mathfrak{q}$ then $\mathtt{d}(\mathfrak{q}) < \mathtt{d}(\mathfrak{q}_1)$.
(3) If M is locally compact then $\mathtt{d}(\mathfrak{q}) + \operatorname{ht}(\mathfrak{q}) \leq \dim(M)$.

Proof. (1) Clearly $\mathtt{d}(\mathfrak{q}) \geq \mathtt{d}(\sqrt{\mathfrak{q}\mathcal{S}^0(M)})$. Let $f \in \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ such that $\dim(Z(f)) = \mathtt{d}(\sqrt{\mathfrak{q}\mathcal{S}^0(M)})$. By Remark 3.8 there is $g \in \mathfrak{q}$ such that $Z(g) \subset Z(f)$ and therefore $\mathtt{d}(\mathfrak{q}) \leq \dim(Z(g)) \leq \dim(Z(f)) = \mathtt{d}(\sqrt{\mathfrak{q}\mathcal{S}^0(M)})$, as required.

Once we have proved (1), by Theorem 1.2 and Remark 3.8 it suffices to prove (2) and (3) for $r = 0$. This is the content of Lemma 2.1 and Theorem 2.2 in [FG6] respectively. □

We point out that Lemma 4.1(2) is false if $\mathfrak{q}_1$ is not a z-ideal, see [FG6, Rmk.2.3]. We will reduce Theorem 1.4 in the locally compact case to a pure-dimensional setting via the following technical result:

ExTDiFF **Lemma 4.2.** Let $M \subset \mathbb{R}^m$ be a pure dimensional semialgebraic set and $f \in \mathcal{S}(M)$. Then there exists $g \in \mathcal{S}^{r*}(M)$ such that $\dim(Z(g)) < \dim(M)$ and fg extends by zero to an $\mathcal{S}^r$ function on $\mathbb{R}^m$.

Proof. By [FGR, 2.4.2], M is the disjoint union of finitely many Nash affine manifolds $N_1, \ldots, N_s$ such that each restriction $f|_{N_i}$ is a Nash function and where those N_i with $\dim(N_i) = \dim(M)$ are open in M. We can assume that $\dim(N_i) = \dim(M)$ if $1 \leq i \leq k$ and $\dim(N_i) < \dim(M)$ if $k+1 \leq i \leq s$. Since each N_i, $1 \leq i \leq k$, is open in the pure dimensional set M the union $N := \bigcup_{i=1}^k N_i$ is an affine Nash manifold and a dense open subset of M. In addition, the restriction $f|_N$ is a Nash function.

By [BCR, 8.9.5] there exists a Nash tubular neighborhood (V, ρ) of N in $\mathbb{R}^m$. Let us define $F := f \circ \rho \in \mathcal{S}^r(V)$. Since V is an open semialgebraic subset of $\mathbb{R}^m$ there exists $G \in \mathcal{S}^{r*}(\mathbb{R}^m)$ with $Z(G) = \mathbb{R}^m \setminus V$. By Proposition 3.1 there exists $\ell \geq 1$ such that $G^\ell F$ extends by zero to an $\mathcal{S}^r$ function on $\mathbb{R}^m$. As $Z(G) \cap M = M \setminus V$ has dimension $\leq \dim(M) - 1$ and $F|_{M\setminus V} = f|_{M\setminus V}$, the function $g := G|_M \in \mathcal{S}^{r*}(M)$ does the job. □

Proof of Theorem 1.4 in the non-bounded locally compact case. We prove Theorem 1.4 for a prime $\mathfrak{p} \in \operatorname{Spec}^0(M)$ where M is locally compact. Denote $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}^r(M)$. Let $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$ be the quotient fields of $\mathcal{S}^0(M)/\mathfrak{p}$ and $\mathcal{S}^r(M)/\mathfrak{q}$ respectively. We have to show that the natural inclusion

$$\mathtt{j} : \mathcal{S}^r(M)/\mathfrak{q} \to \mathcal{S}^0(M)/\mathfrak{p}.$$

induces an isomorphism between $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{p})$. We will make several reductions:

First reduction. We can assume that $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^0(M)$.

In the general case, pick $f \in \mathfrak{p}$ with $\mathtt{d}(\mathfrak{p}) = \dim(N)$, where $N := Z(f)$ is locally compact since it is closed in the locally compact set M. By Fact 2.4 the epimorphism $\varphi_0 : \mathcal{S}^0(M) \to \mathcal{S}^0(N)$, $g \mapsto g|_N$ induces an isomorphism $\mathcal{S}^0(M)/\ker\varphi_0 \to \mathcal{S}^0(N)$. As $\mathfrak{p}$ is a z-ideal we have that $\ker\varphi_0 \subset \mathfrak{p}$ and therefore the image $\mathfrak{p}_1$ of $\mathfrak{p}/\ker\varphi_0$ by the latter map is a prime ideal of $\mathcal{S}^0(N)$. In particular, we have an isomorphism

$$\overline{\varphi}_0 : \mathcal{S}^0(M)/\mathfrak{p} \to \mathcal{S}^0(N)/\mathfrak{p}_1.$$

Similarly, the restriction map $\varphi_r : \mathcal{S}^r(M) \to \mathcal{S}^r(N)$ induces an isomorphism $\mathcal{S}^r(M)/\ker\varphi_r \to \mathcal{S}^r(N)$. Since $\mathfrak{q}$ is clearly a z-ideal, we can consider the image $\mathfrak{q}_1 \in \operatorname{Spec}^r(N)$ of $\mathfrak{q}/\ker\varphi_r$ by the

latter map. Then we get an isomorphism
$$\overline{\varphi}_r : \mathcal{S}^r(M)/\mathfrak{q} \to \mathcal{S}^r(N)/\mathfrak{q}_1.$$
As the following diagram
$$\begin{array}{ccc} \mathcal{S}^r(M)/\mathfrak{q} & \xrightarrow{\ \mathtt{j}\ } & \mathcal{S}^0(M)/\mathfrak{p} \\ \downarrow{\scriptstyle \overline{\varphi}_r} & & \downarrow{\scriptstyle \overline{\varphi}_0} \\ \mathcal{S}^r(N)/\mathfrak{q}_1 & \xrightarrow{\ \mathtt{j}_N\ } & \mathcal{S}^0(N)/\mathfrak{p}_1 \end{array}$$
commutes, to finish this reduction step it suffices to see that $\mathfrak{p}_1$ is a minimal prime ideal of $\mathcal{S}^0(N)$. Otherwise there would exist a prime ideal $\mathfrak{p}'_1 \subsetneq \mathfrak{p}_1$ in $\mathcal{S}^r(N)$ and so $\mathfrak{p}' := \varphi_0^{-1}(\mathfrak{p}'_1)$ is a prime ideal of $\mathcal{S}^0(M)$ such that $\ker\varphi_0 \subset \mathfrak{p}' \subsetneq \mathfrak{p}$. Since M is locally compact $\mathfrak{p}'$ is a z-ideal and so $\mathtt{d}(\mathfrak{p}) < \mathtt{d}(\mathfrak{p}')$ by Lemma 4.1. However, since $f \in \ker\varphi_0 \subset \mathfrak{p}'$, we have $\mathtt{d}(\mathfrak{p}') \leq \dim(Z(f)) = \mathtt{d}(\mathfrak{p})$, a contradiction.

Second reduction. We can assume that M is pure dimensional.

Let M be a semialgebraic case and let $\{M_1, \ldots, M_s\}$ be the bricks of M. By Lemma 2.8 there exist $f_1, \ldots, f_s \in \mathcal{S}^{r*}(M)$ with $M_i = Z(f_i)$ for $1 \leq i \leq s$. Since $f := \prod_{i=1}^s f_i$ is the zero function there exists an index $i \in \{1, \ldots, s\}$ such that $f_i \in \mathfrak{p}$. The epimorphism $\varphi_0 : \mathcal{S}^0(M) \to \mathcal{S}^0(M_i)$, $g \mapsto g|_{M_i}$ induces an isomorphism $\mathcal{S}^0(M)/\ker\varphi_0 \to \mathcal{S}^0(M_i)$. Since $\mathfrak{p}$ is a z-ideal we get that $\ker\varphi_0 \subset \mathfrak{p}$. Denote by $\mathfrak{p}_i \in \operatorname{Spec}^0(M_i)$ the image of $\mathfrak{p}/\ker\varphi_0$ by the latter map. Arguing as in the first reduction, it is enough to prove the statement for the prime $\mathfrak{p}_i$ of the ring $\mathcal{S}^0(M_i)$. As M_i is pure dimensional, this follows by assumption.

Final step. All in all, we may assume from the beginning that M is pure dimensional and $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^0(M)$. We have to prove that the homomorphism $\mathtt{j} : \mathcal{S}^r(M)/\mathfrak{q} \to \mathcal{S}^0(M)/\mathfrak{p}$ induces an isomorphism $\kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$, which it will be denoted by $\mathtt{j}$ again. To show the surjectivity, pick $f \in \mathcal{S}^0(M)$. By Lemma 4.2 there are $g \in \mathcal{S}^{r*}(M)$ and $H \in \mathcal{S}^r(\mathbb{R}^m)$ such that $\dim(Z(g)) < \dim(M)$ and $h := H|_M = fg$. Let us show that $g \notin \mathfrak{p}$. Otherwise, as $\mathfrak{p}$ is a minimal prime ideal and by [HJ, 1.1] there would exist $v \in \mathcal{S}^r(M) \setminus \mathfrak{p}$ such that $gv = 0$. Thus, $D(v) \subset Z(g)$ and so
$$\dim(D(v)) \leq \dim(Z(g)) < \dim(M).$$
Since M is pure dimensional this implies that $v = 0$, which is a contradiction because $v \notin \mathfrak{p}$. Hence $g + \mathfrak{p} \neq 0$ and so $u := (h + \mathfrak{p})/(g + \mathfrak{p}) \in \kappa(\mathfrak{p})$. Therefore $\mathtt{j}(u) = f + \mathfrak{p}$, as wanted.

Finally, it was proved in [S3, Cor. 3.26 §1 & Thm. 1.1] that $\kappa(\mathfrak{p})$ is a real closed ring, and so is $\kappa(\mathfrak{q})$ as well. □

trdeg **Corollary 4.3.** *Let M be a locally compact semialgebraic set and let $\mathfrak{q} \in \operatorname{Spec}^r(M)$. Then:*

(i) $\operatorname{tr\,deg}_{\mathbb{R}}(\kappa(\mathfrak{q})) = \mathtt{d}(\mathfrak{q})$.

(ii) *Let $\rho : M \to M'$ be a $\mathcal{S}^r$ embedding such that $\rho(M)$ is dense in the locally compact semialgebraic set M', and let $\mathfrak{q}' := (\rho^*)^{-1}(\mathfrak{q})$ where $\rho^* : \mathcal{S}^r(M') \to \mathcal{S}^r(M)$ is the natural injective homomorphism induced by ρ. Then the natural inclusion $\mathcal{S}^r(M')/\mathfrak{q}' \to \mathcal{S}^r(M)/\mathfrak{q}$ induces an isomorphism on the quotient fields $\kappa(\mathfrak{q}')$ and $\kappa(\mathfrak{q})$.*

Proof. (i) By Theorem 1.2, the ideal $\mathfrak{p} := \sqrt{\mathfrak{q}\mathcal{S}^0(M)}$ belongs to $\operatorname{Spec}^0(M)$ and by Theorem 1.4 the fields $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{p})$ are $\mathbb{R}$-isomorphic. Since $\mathtt{d}(\mathfrak{p}) = \mathtt{d}(\mathfrak{q})$ by Lemma 4.1(1), it only remains to prove the result for $\mathfrak{p}$, which follows from [FG6, Thm. 1.1].

(ii) It suffices to show that the inclusion $\kappa(\mathfrak{q}') \to \kappa(\mathfrak{q})$ is surjective. Since both $\kappa(\mathfrak{q}')$ and $\kappa(\mathfrak{q})$ are real closed fields, it is enough if we prove that $\operatorname{tr\,deg}_{\mathbb{R}}(\kappa(\mathfrak{q}')) \geq \operatorname{tr\,deg}_{\mathbb{R}}(\kappa(\mathfrak{q}))$, or equivalently, that $\mathtt{d}(\mathfrak{q}') \geq \mathtt{d}(\mathfrak{q})$. Since ρ is injective, for any $f \in \mathfrak{q}'$ we have that $\dim(Z(f)) \geq \dim(Z(f \circ \rho)) \geq \mathtt{d}(\mathfrak{q})$, as required. □

generalnonbd

4.B. General case of Theorem 1.4. Along this section we fix a semialgebraic set $M \subset \mathbb{R}^m$ and an integer $r \geq 0$. We denote $\mathfrak{C}^{r\diamond}$ as in 2.C the collection of pairs $(E, \mathtt{j})$ where $\mathtt{j} : M \to \mathbb{R}^n$ is an $\mathcal{S}^{r\diamond}$ embedding such that $E = \operatorname{Cl}(\mathtt{j}(M))$, so that E is a locally compact semialgebraic set (even compact in the bounded case). By Theorem 2.6 we have a homomorphism $\mathtt{j}^* : \mathcal{S}^{r\diamond}(E) \to \mathcal{S}^{r\diamond}(M)$, $f \mapsto f \circ \mathtt{j}$ and for every ideal $\mathfrak{a}$ in $\mathcal{S}^{r\diamond}(M)$ we will denote $\mathfrak{a} \cap \mathcal{S}^{r\diamond}(E) := (\mathtt{j}^*)^{-1}(\mathfrak{a})$ and we say that $\mathfrak{a}$ *lies over* $\mathfrak{a} \cap \mathcal{S}^{r\diamond}(E)$.

citalc1 **Proposition 4.4.** (1) *Given a subset* $\mathfrak{F} := \{f_1, \dots, f_k\} \subset \mathcal{S}^{r\diamond}(M)$ *there exist* $(E_{\mathfrak{F}}, \mathtt{j}_{\mathfrak{F}}) \in \mathfrak{C}^{r\diamond}$ *and* $F_1, \dots, F_k \in \mathcal{S}^{r\diamond}(E_{\mathfrak{F}})$ *with* $F_i \circ \mathtt{j}_{\mathfrak{F}} = f_i$.

(2) *Given a chain of prime ideals* $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_k$ *in* $\mathcal{S}^{r\diamond}(M)$ *there exists* $(E, \mathtt{j}) \in \mathfrak{C}^{r\diamond}$ *such that the prime ideals* $\mathfrak{Q}_i := \mathfrak{q}_i \cap \mathcal{S}^{r\diamond}(E)$ *constitute a chain* $\mathfrak{Q}_0 \subsetneq \cdots \subsetneq \mathfrak{Q}_k$ *in* $\mathcal{S}^{r\diamond}(E)$.

(3) *There exists* $(E, \mathtt{j}) \in \mathfrak{C}^{r\diamond}$ *such that* $\dim(\mathcal{S}^{r\diamond}(M)) \leq \dim(\mathcal{S}^{r\diamond}(E)) = \dim(M)$.

Proof. (1) Consider the $\mathcal{S}^{r\diamond}$ embedding $\mathtt{j}_{\mathfrak{F}} : M \to \mathbb{R}^{m+k},\ x \mapsto (x, f_1(x), \dots, f_k(x))$ and define $E_{\mathfrak{F}} := \operatorname{Cl}(\mathtt{j}_{\mathfrak{F}})$. Define $F_i := \pi_{m+i}|_{E_{\mathfrak{F}}}$, where

$$\pi_{m+i} : \mathbb{R}^{m+k} \to \mathbb{R},\ x := (x_1, \dots, x_{m+k}) \mapsto x_{m+i}$$

for $i = 1, \dots, k$. Each $F_i \in \mathcal{S}^{r\diamond}(E_{\mathfrak{F}})$ and $F_i \circ \mathtt{j}_{\mathfrak{F}} = f_i$, as required.

(2) For $1 \leq i \leq k$ pick a function $f_i \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1}$ and consider the pair $(E_{\mathfrak{F}}, \mathtt{j}_{\mathfrak{F}}) \in \mathfrak{C}^{r\diamond}$ provided by part (1) for the family $\mathfrak{F} := \{f_1, \dots, f_k\}$. Clearly, $\mathfrak{Q}_{i-1} \subset \mathfrak{Q}_i$ and each $F_i \in \mathfrak{Q}_i \setminus \mathfrak{Q}_{i-1}$.

(3) By part (2) there exists $(E, \mathtt{j}) \in \mathfrak{C}^{r\diamond}$ such that $\dim(\mathcal{S}^{r\diamond}(M)) \leq \dim(\mathcal{S}^{r\diamond}(E))$. Finally, the equalities $\dim(\mathcal{S}^{r\diamond}(E)) = \dim(E) = \dim(M)$ follow from Corollary 3.12 and [BCR, 2.8.2]. □

RCf **Corollary 4.5.** *Let* $\mathfrak{q} \in \operatorname{Spec}^{r\diamond}(M)$. *Then* $\kappa(\mathfrak{q}) := \operatorname{qf}(\mathcal{S}^{r\diamond}(M)/\mathfrak{q})$ *is a real closed field.*

Proof. For each $(E, \mathtt{j}) \in \mathfrak{C}^{r\diamond}$ and since $\mathtt{j}(M)$ is dense in E, the homomorphism $\mathtt{j}^* : \mathcal{S}^r(E) \to \mathcal{S}^{r\diamond}(M)$ is injective. Let $\mathfrak{Q}_{\mathtt{j}} := (\mathtt{j}^*)^{-1}(\mathfrak{q}) \in \operatorname{Spec}^{r\diamond}(E)$ and note that $\overline{\mathtt{j}^*} : \mathcal{S}^r(E)/\mathfrak{Q}_{\mathtt{j}} \to \mathcal{S}^{r\diamond}(M)/\mathfrak{q}$ is also an injective homomorphism. Thus, by Lemma 2.13 it follows that $\kappa(\mathfrak{q}) = \varinjlim \kappa(\mathfrak{Q}_{\mathtt{j}})$. Since each field $\kappa(\mathfrak{Q}_{\mathtt{j}}) := \operatorname{qf}(\mathcal{S}^{r\diamond}(E)/\mathfrak{Q}_{\mathtt{j}})$ is a real closed ring, by [S3, §1 Thm 4.8] the field $\kappa(\mathfrak{q})$ is also a real closed ring, and so it is a real closed field, as required. □

Remark 4.6. If $\mathfrak{n}$ is a maximal ideal of $\mathcal{S}^{r*}(M)$ then the field homomorphism $\mathtt{j} : \mathbb{R} \hookrightarrow \kappa(\mathfrak{n}) := \mathcal{S}^{r*}(M)/\mathfrak{n}$, $r \mapsto r + \mathfrak{n}$ is an isomorphism. Indeed, note that the extension of real closed fields $\kappa(\mathfrak{n})|\mathbb{R}$ is archimedean because each function in $\mathcal{S}^{r*}(M)$ is bounded by a real number. This proves the surjectivity of the embedding $\mathtt{j}$ since $\mathbb{R}$ does not admit proper archimedean extensions.

brimming1 **Lemma 4.7.** *Let* $\mathfrak{q} \in \operatorname{Spec}^{r\diamond}(M)$. *Then there exist* $(E, \mathtt{j}) \in \mathfrak{C}^{r\diamond}$ *and* $\mathfrak{q}_E \in \operatorname{Spec}^{r\diamond}(E)$ *such that* $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}_E)$ *and* $\mathfrak{q}$ *lies over* $\mathfrak{q}_E$. *We refer to* $(E, \mathtt{j})$ *as a* brimming $\mathcal{S}^{r\diamond}$ local compactification *of* M *for* $\mathfrak{q}$.

Proof. We write the proof for the $\mathcal{S}^r$ case, the bounded one is similar. Let $\varphi : \mathcal{S}^r(M) \to \mathcal{S}^r(M)/\mathfrak{q} \hookrightarrow \kappa(\mathfrak{q})$. For each finite subset $\mathfrak{F}$ of $\mathcal{S}^r(M)$ let $(E_{\mathfrak{F}}, \mathtt{j}_{\mathfrak{F}}) \in \mathfrak{C}^r$ as in Proposition 4.4 and denote

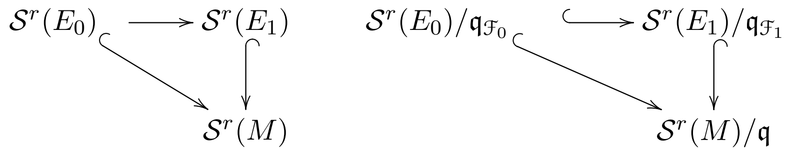
$$\mathtt{j}_{\mathfrak{F}}^* : \mathcal{S}^r(E_{\mathfrak{F}}) \to \mathcal{S}^r(M),\ F \mapsto F \circ \mathtt{j}_{\mathfrak{F}} \quad \text{and} \quad \varphi_{\mathfrak{F}} := \varphi \circ \mathtt{j}_{\mathfrak{F}}^* : \mathcal{S}^r(E_{\mathfrak{F}}) \to \kappa(\mathfrak{q}).$$

Define $\mathfrak{q}_{\mathfrak{F}} := \ker \varphi_{\mathfrak{F}} = \mathfrak{q} \cap \mathcal{S}^r(E_{\mathfrak{F}})$ and $\mathtt{d} := \max_{\mathfrak{F}}\{\mathtt{d}(\mathfrak{q}_{\mathfrak{F}})\}$ where $\mathfrak{F}$ runs over all finite subsets of $\mathcal{S}^r(M)$. Fix a finite subset $\mathfrak{F}_0$ of $\mathcal{S}^r(M)$ such that $\mathtt{d}(\mathfrak{q}_{\mathfrak{F}_0}) = \mathtt{d}$. Denote

$$E_0 := E_{\mathfrak{F}_0}, \quad \mathfrak{q}_{\mathfrak{F}_0} := \ker \varphi_{\mathfrak{F}_0} = \mathcal{S}^r(E_0) \cap \mathfrak{q} \quad \text{and} \quad \kappa_0 := \operatorname{qf}(\mathcal{S}^r(E_0)/\mathfrak{q}_{\mathfrak{F}_0}) \subset \kappa(\mathfrak{q}).$$

Let us prove that $\kappa(\mathfrak{q}) = \kappa_0$ and so $(E_0, \mathtt{j}_{\mathfrak{F}_0})$ does the job. Since both are real closed fields, it is enough to show that $\kappa(\mathfrak{q})$ is an algebraic extension of κ_0. For that, it suffices to see that $f + \mathfrak{q}$ is algebraic over κ_0 for each $f \in \mathcal{S}^r(M) \setminus \mathfrak{q}$. Let $\mathfrak{F}_1 := \mathfrak{F}_0 \cup \{f\}$, $E_1 := E_{\mathfrak{F}_1}$ and $(E_1, \mathtt{j}_{\mathfrak{F}_1}) \in \mathfrak{C}^r$. The projection onto all the coordinates except the last one induces an $\mathcal{S}^r$ map $\rho : E_1 \to E_0$ and so $\rho \circ \mathtt{j}_{\mathfrak{F}_1} = \mathtt{j}_{\mathfrak{F}_0} : M \to \mathtt{j}_{\mathfrak{F}_0}(M)$ is an $\mathcal{S}^r$ diffeomorphism. Consider the $\mathbb{R}$-homomorphism $\mathcal{S}^r(E_0) \to \mathcal{S}^r(E_1)$, $h \mapsto h \circ \rho$. Denote $\mathfrak{q}_{\mathfrak{F}_1} := \ker \varphi_{\mathfrak{F}_1} = \mathcal{S}^r(E_1) \cap \mathfrak{q}$ and the real closed field

$\kappa_1 := \mathrm{qf}(\mathcal{S}^r(E_1)/\mathfrak{q}_{\mathcal{F}_1})$. We have the following diagrams of ring homomorphisms



and so $\kappa_0 \subset \kappa_1 \subset \kappa(\mathfrak{q})$. Since $f \in \mathcal{F}_1$ there exists $F \in \mathcal{S}^r(E_1)$ such that $F \circ \mathtt{j}_{\mathcal{F}_1} = f$ and to see that $f + \mathfrak{q}$ is algebraic over κ_0 it is enough to prove that $F + \mathfrak{q}_{\mathcal{F}_1}$ is algebraic over κ_0. To that end it suffices to see that the transcendence degree over $\mathbb{R}$ of κ_0 and κ_1 coincide. This follows from Corollary 4.3 because

$$\operatorname{tr}\deg_{\mathbb{R}}(\kappa_0) \leq \operatorname{tr}\deg_{\mathbb{R}}(\kappa_1) = \mathtt{d}(\mathfrak{q}_{\mathcal{F}_1}) \leq \mathtt{d} = \mathtt{d}(\mathfrak{q}_{\mathcal{F}_0}) = \operatorname{tr}\deg_{\mathbb{R}}(\kappa_0).$$

Hence, $\operatorname{tr}\deg_{\mathbb{R}}(\kappa_0) = \operatorname{tr}\deg_{\mathbb{R}}(\kappa_1)$, as wanted. □

Example 4.8. Not all pairs $(E, \mathtt{j}) \in \mathfrak{C}^r$ are brimming $\mathcal{S}^r$ local compactifications of M for a given $\mathfrak{q} \in \operatorname{Spec}^r(M)$, see [FG6, Rmk.3.1].

cor:brimming **Corollary 4.9.** *Let $\mathfrak{p} \in \operatorname{Spec}^{0\diamond}(M)$ and $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}^{r\diamond}(M) \in \operatorname{Spec}^{r\diamond}(M)$. Then there exist $(E_1, \mathtt{j}_1) \in \mathfrak{C}^{0\diamond}$ and $(E_2, \mathtt{j}_2) \in \mathfrak{C}^{r\diamond}$ brimming local compactifications of $\mathfrak{p}$ and $\mathfrak{q}$ respectively, and a $\mathcal{S}^r$ map $\rho : E_1 \to E_2$ such that $\mathtt{j}_2 = \rho \circ \mathtt{j}_1$.*

Proof. By the proof of Lemma 4.7, there is a finite subset $\mathcal{F}_1$ of $\mathcal{S}^{0\diamond}(M)$ such that $(E_{\mathcal{F}_1}, \mathtt{j}_{\mathcal{F}_1})$ is a brimming $\mathcal{S}^{0\diamond}$ local compactifications of $\mathfrak{p}$. Similarly, there is a finite subset $\mathcal{F}_2$ of $\mathcal{S}^{r\diamond}(M)$ such that $(E_{\mathcal{F}_2}, \mathtt{j}_{\mathcal{F}_2})$ is a brimming $\mathcal{S}^{r\diamond}$ local compactifications of $\mathfrak{q}$. Define $\mathcal{F}_1' := \mathcal{F}_1 \cup \mathcal{F}_2$, and note that by the proof of Lemma 4.7 we have that $(E_{\mathcal{F}_1'}, \mathtt{j}_{\mathcal{F}_1'})$ is also a brimming $\mathcal{S}^{r\diamond}$ local compactification of $\mathfrak{p}$. Moreover, the projection onto all the coordinates except the last $|\mathcal{F}_2|$ coordinates induces a $\mathcal{S}^r$ map $\rho : E_{\mathcal{F}_1'} \to E_{\mathcal{F}_2}$ such that $\mathtt{j}_{\mathcal{F}_2} = \rho \circ \mathtt{j}_{\mathcal{F}_1'}$, as required. □

We have all the ingredients to give a proof of Theorem 1.4 in the general case.

Proof of Theorem 1.4 in the general case. Let M be a semialgebraic set. We first prove that given $\mathfrak{p} \in \operatorname{Spec}^0(M)$, if we denote $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}^r(M) \in \operatorname{Spec}^r(M)$ then the natural homomorphism $\mathcal{S}^r(M)/\mathfrak{q} \to \mathcal{S}^0(M)/\mathfrak{p}$ induces an isomorphism between the quotient fields $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{p})$.

Pick $(E_1, \mathtt{j}_1) \in \mathfrak{C}^0$ and $(E_2, \mathtt{j}_2) \in \mathfrak{C}^r$ brimming $\mathcal{S}^r$ local compactifications of M for $\mathfrak{p}$ and $\mathfrak{q}$ as in Corollary 4.9, and let $\rho : E_1 \to E_2$ an $\mathcal{S}^r$ map such that $\mathtt{j}_2 = \rho \circ \mathtt{j}_1$. Denote $\mathfrak{p}_{E_1} := \mathfrak{p} \cap \mathcal{S}^0(E_1)$ and $\mathfrak{q}_{E_2} := \mathfrak{q} \cap \mathcal{S}^r(E_2)$, so that

$$\mathcal{S}^0(E_1)/\mathfrak{p}_{E_1} \to \mathcal{S}^0(M)/\mathfrak{p} \quad \& \quad \mathcal{S}^r(E_2)/\mathfrak{q}_{E_2} \to \mathcal{S}^r(M)/\mathfrak{q}$$

induce isomorphisms in the quotient fields. Moreover, since $\mathtt{j}_2$ is also a $\mathcal{S}^0$ map we can consider $\mathfrak{p}_{E_2} := \mathfrak{p} \cap \mathcal{S}^0(E_2)$, Note that $\mathfrak{q}_2 = \mathfrak{p}_2 \cap \mathcal{S}^r(E_2)$ and therefore by the locally compact version of Theorem 1.4 proved in subsection 4.A we get that

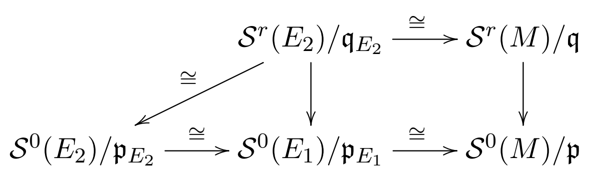
$$\mathcal{S}^r(E_2)/\mathfrak{q}_{E_2} \to \mathcal{S}^0(E_2)/\mathfrak{p}_{E_2}$$

induces an isomorphism in the quotient fields.

On the other hand, note that for the $\mathbb{R}$-homomorphism $\rho^* : \mathcal{S}^0(E_2) \to \mathcal{S}^0(E_1)$ we have that $\mathfrak{p}_{E_2} = (\rho^*)^{-1}(\mathfrak{p}_{E_1})$. By Corollary 4.3(ii) we get that

$$\mathcal{S}^0(E_2)/\mathfrak{p}_{E_2} \to \mathcal{S}^0(E_1)/\mathfrak{p}_{E_1}$$

induces an isomorphism of the quotient fields. All in all, we have the commuting diagram



where the symbol $\cong$ means that the map induces an isomorphism of the quotient fields. We deduce that $\mathcal{S}^r(M)/\mathfrak{q} \to \mathcal{S}^0(M)/\mathfrak{p}$ also induces an isomorphism of the quotient fields, as required.

Similarly, we prove that given $\mathfrak{p} \in \operatorname{Spec}^{0*}(M)$, if we denote $\mathfrak{q} := \mathfrak{p} \cap \mathcal{S}^{r*}(M) \in \operatorname{Spec}^{r*}(M)$ then the natural homomorphism $\mathcal{S}^{r*}(M)/\mathfrak{q} \to \mathcal{S}^{0*}(M)/\mathfrak{p}$ induces an isomorphism between the quotient fields $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{p})$. We point out that along the adapted proof we will need the fact that

$$\mathcal{S}^{r*}(E_2)/\mathfrak{q}_{E_2} \to \mathcal{S}^{0*}(E_2)/\mathfrak{p}_{E_2}$$

induces an isomorphism in the quotient fields, where $(E_2, \mathtt{j}_2)$ is a brimming $\mathcal{S}^{r*}$ local compactifications of M. This follows again by the locally compact version of Theorem 1.4 in subsection 4.A because E_2 is compact and therefore $\mathcal{S}^{0*}(E_2) = \mathcal{S}^0(E_2)$ and $\mathcal{S}^{r*}(E_2) = \mathcal{S}^r(E_2)$. □

There exist examples of non locally compact semialgebraic sets M for which there is $\mathfrak{p} \in \operatorname{Spec}^r(M)$ with $\mathtt{d}(\mathfrak{q}) < \operatorname{tr}\deg_{\mathbb{R}}(\kappa(\mathfrak{q}))$, see a counterexample in [FG5, 3.4.1]. In general, we have the following result:

trdgz **Corollary 4.10.** *Let $\mathfrak{q}$ be a prime ideal of $\mathcal{S}^r(M)$. Then $\mathtt{d}(\mathfrak{q}) \leq \operatorname{tr}\deg_{\mathbb{R}}(\kappa(\mathfrak{q}))$. If in addition $\mathfrak{q}$ is a z-ideal then the equality holds.*

Proof. Pick a brimming $\mathcal{S}^r$ local compactification $(E, \mathtt{j})$ of M for $\mathfrak{q}$ and denote $\mathfrak{q}_E := \mathfrak{q} \cap \mathcal{S}^r(E)$. By Corollary 4.3 and Lemma 4.7, $\mathtt{d}(\mathfrak{q}_E) = \operatorname{tr}\deg_{\mathbb{R}}(\kappa(\mathfrak{q}_E)) = \operatorname{tr}\deg_{\mathbb{R}}(\kappa(\mathfrak{q}))$ and so it is enough to check that $\mathtt{d}(\mathfrak{q}_E) = \mathtt{d}(\mathfrak{q})$. Let $F \in \mathfrak{q}_E$ with $\mathtt{d}(\mathfrak{q}_E) = \dim(Z(F))$. Then $f := F \circ \mathtt{j} \in \mathfrak{q}$ and

$$\mathtt{d}(\mathfrak{q}) \leq \dim(Z(f)) \leq \dim(Z(F)) = \mathtt{d}(\mathfrak{q}_E).$$

For the converse inequality, let $h \in \mathfrak{q}$ be such that $\mathtt{d}(\mathfrak{q}) = \dim(Z(h))$. By Lemma 2.8 there exists $g \in \mathcal{S}^r(E)$ such that $\operatorname{Cl}_E(\mathtt{j}(Z(h))) = Z(g)$. Hence, $Z(h) \subset Z(g \circ \mathtt{j})$ and, since $\mathfrak{q}$ is a z-ideal, $g \circ \mathtt{j} \in \mathfrak{q}$, that is, $g \in \mathfrak{q}_E$. Therefore,

$$\mathtt{d}(\mathfrak{q}_E) \leq \dim(Z(g)) = \dim(\operatorname{Cl}_E(\mathtt{j}(Z(h)))) = \dim(Z(h)) = \mathtt{d}(\mathfrak{q}),$$

as required. □

We finish this section by finding two important classes of ideals which are z-ideals.

zdl **Proposition 4.11.** *Let M be a semialgebraic set.*

(1) *Let $\mathfrak{a}$ be a z-ideal of $\mathcal{S}^r(M)$. Every prime ideal of $\mathcal{S}^r(M)$ that is minimal among the prime ideals containing $\mathfrak{a}$ is a z-ideal. In particular, minimal prime ideals of $\mathcal{S}^r(M)$ are z-ideals.*

(2) *Let $\mathfrak{b}$ be a proper ideal in $\mathcal{S}^r(M)$. Then there exists a proper z-ideal $\mathfrak{b}^z$ in $\mathcal{S}^r(M)$ that contains $\mathfrak{b}$. Every proper prime ideal which is maximal among the proper ideals containing $\mathfrak{b}$ is a z-ideal. In particular, maximal ideals of $\mathcal{S}^r(M)$ are z-ideals.*

Proof. (1) Let $\mathfrak{q} \in \operatorname{Spec}^r(M)$ be minimal among the prime ideals containing $\mathfrak{a}$. Suppose there exist $f, g \in \mathcal{S}^r(M)$ with $Z(f) \subset Z(g)$ and $f \in \mathfrak{q}$ but $g \notin \mathfrak{q}$. Let $T := \mathcal{S}^r(M) \setminus \mathfrak{q}$ and consider the multiplicatively closed subset $S := \{hf^\ell : h \in T, \ell \in \mathbb{Z}, \ell \geq 0\}$ of $\mathcal{S}^r(M)$. Note that $S \cap \mathfrak{a} = \varnothing$. Otherwise there would exist $h \in T$ and $\ell \geq 0$ such that $hf^\ell \in \mathfrak{a}$. But $\mathfrak{a}$ is a z-ideal and

$$Z(hf^\ell) = Z(h) \cup Z(f) \subset Z(h) \cup Z(g) = Z(hg).$$

Thus $hg \in \mathfrak{a} \subset \mathfrak{q}$ and this is false. Therefore $S^{-1}\mathfrak{a}$ is a proper ideal of the localization $S^{-1}\mathcal{S}^r(M)$, so there is an ideal $\mathfrak{q}_0$ of $\mathcal{S}^r(M)$ such that $S^{-1}\mathfrak{q}_0$ is maximal and contains $S^{-1}\mathfrak{a}$. In particular, $\mathfrak{q}_0$ is a prime ideal such that $S \cap \mathfrak{q}_0 = \varnothing$ and $\mathfrak{a} \subset \mathfrak{q}_0$. Since $S \cap \mathfrak{q}_0 = \varnothing$ and $f \in S$, we have that $\mathfrak{q}_0 \subsetneq \mathfrak{q}$, a contradiction with the minimality of $\mathfrak{q}$.

(2) Let us check that the subset

$$\mathfrak{b}^z := \{f \in \mathcal{S}^r(M) : \exists\, g \in \mathfrak{b} \text{ such that } Z(g) \subset Z(f)\}$$

is a z-ideal of $\mathcal{S}^r(M)$ containing $\mathfrak{b}$. First, given $f_1, f_2 \in \mathfrak{b}^z$ and $h \in \mathcal{S}^r(M)$ there exist $g_1, g_2 \in \mathfrak{b}$ such that $Z(g_i) \subset Z(f_i)$ for $i = 1, 2$. Notice that $g_1^2 + g_2^2 \in \mathfrak{b}$ and $hg_1 \in \mathfrak{b}$ and

$$Z(g_1^2 + g_2^2) \subset Z(f_1 + f_2) \quad \text{and} \quad Z(hg_1) \subset Z(hf_1).$$

This proves that $\mathfrak{b}^z$ is an ideal and clearly it is a z-ideal containing $\mathfrak{b}$. Finally, if $\mathfrak{b}^z = \mathcal{S}^r(M)$ there exists $g \in \mathfrak{b}$ such that $Z(g) \subset Z(1) = \varnothing$. Thus g is a unit in $\mathcal{S}^r(M)$, a contradiction since $\mathfrak{b}$ is proper.

Finally, if $\mathfrak{m}$ is a proper prime ideal which is maximal among the proper ideals containing $\mathfrak{b}$, then $\mathfrak{m}^z = \mathfrak{m}$, as required. □

s8

5. The ring of $\mathcal{S}^\infty$ functions

The main purpose of this final section is to study the ring $\mathcal{S}^\infty(M)$ of a semialgebraic set M. To that aim, we need to recall the machinery of real closed rings.

5.A. Real closed rings and real closure. Let A be a commutative ring with unity. The set $\operatorname{Sper}(A)$ is the collection of *prime cones* of A, i.e., subsets α of A such that $\mathfrak{p}_\alpha := \alpha \cap -\alpha$ is a prime ideal of A and $\alpha/\mathfrak{p}_\alpha$ is the positive cone of a total order of $A/\mathfrak{p}_\alpha$. We denote by $\rho(\alpha)$ the real closure of the quotient field of $A/\mathfrak{p}_\alpha$. Let $\rho_\alpha : A \to \rho(\alpha)$ be the natural homomorphism and denote $a(\alpha) := \rho_\alpha(a)$ for each $a \in A$. The subsets

$$\mathcal{D}_r(a) := \{\alpha \in \operatorname{Sper}(A) : a(\alpha) > 0\}$$

for $a \in A$, called basic open subsets, form the basis of the *spectral topology* on $\operatorname{Sper}(A)$. A boolean combination of sets of the latter form is called a *constructible* set. We refer the reader to [BCR, §7.1] for further details concerning the real spectrum of a ring A and its constructible subsets.

def:rcr **Definition 5.1.** [S2] A commutative ring with unity A is *real closed* if it satisfies the following conditions:

(i) A is a reduced ring,
(ii) The support map $\operatorname{supp} : \operatorname{Sper}(A) \to \operatorname{Spec}(A),\ \alpha \mapsto \mathfrak{p}_\alpha = \alpha \cap (-\alpha)$ is *identifying*, that is, it is a homeomorphism, which induces a bijection between the constructible subsets of $\operatorname{Sper}(A)$ and those of $\operatorname{Spec}(A)$,
(iii) For each $\mathfrak{p} \in \operatorname{Spec}(A)$ we have:
  (a) The quotient field $R := \operatorname{qf}(A/\mathfrak{p})$ is a real closed field and $A/\mathfrak{p}$ is integrally closed in R and
  (b) Each $\mathfrak{Q} \in \operatorname{Spec}(A/\mathfrak{p})$ is convex with respect to the unique ordering of $A/\mathfrak{p}$,
(iv) A finite sum of radical ideals of A is a radical ideal of A.

We will need a number of well-known facts by the experts. For the sake of completeness, we give the proof of those which do not appear in the literature.

directlimit **Fact 5.2.** *If $\{A_i\}_{i\in I}$ is a direct system of rings, then the real closure of $A := \varinjlim A_i$ equals $\varinjlim$ rcl(A_i) where rcl(A_i) denotes the real closure of A_i.*

Proof. Indeed, since we have the canonical homomorphisms $A_i \to A$ for each $i \in I$, we obtain a homomorphism $\operatorname{rcl}(A_i) \to \operatorname{rcl}(A)$ and so a homomorphism $\varinjlim \operatorname{rcl}(A_i) \to \operatorname{rcl}(A)$. On the other hand, we also have a natural homomorphism from A to $\varinjlim \operatorname{rcl}(A_i)$ and since the latter is real closed, by [S3, Ch. I, Thm. 4.8] we obtain a homomorphism $\operatorname{rcl}(A) \to \varinjlim \operatorname{rcl}(A_i)$ as desired. □

Recall that given an open semialgebraic set U of $\mathbb{R}^n$, a Nash function $f : U \to \mathbb{R}$ is by definition a $\mathcal{C}^\infty$ semialgebraic function. If M is a semialgebraic subset of $\mathbb{R}^n$, we denote by $\mathcal{N}(M)$ the collection of all functions $f : M \to \mathbb{R}$ that admit a Nash extension to an open semialgebraic neighborhood of M in $\mathbb{R}^n$. Recall that a semialgebraic set $M \subset \mathbb{R}^m$ is a Nash set if it is locally compact and there exists a Nash function f on the open semialgebraic set $U := \mathbb{R}^m \setminus (\operatorname{Cl}(M) \setminus M)$ such that $M = \mathcal{Z}(f)$, see [FG2, 2.12].

f:rcl2 **Fact 5.3.** *Let X be a closed Nash subset of $\mathbb{R}^n$. Then $\mathcal{S}^{0\diamond}(X)$ is the real closure of $\mathcal{N}^\diamond(X)$.*

Proof. It suffices to prove it for $X = \mathbb{R}^n$. For, consider the surjective homomorphism $\varphi : \mathcal{S}^{0\diamond}(\mathbb{R}^n) \to \mathcal{S}^{0\diamond}(X)$ given by the restriction to X. Then by [Sh, Thm II.5.2] and Lemma 2.10 we have $\mathcal{N}^\diamond(X) = \mathcal{N}^\diamond(\mathbb{R}^n)/I(X)$ and $\mathcal{S}^{0\diamond}(X) = \mathcal{S}^{0\diamond}(\mathbb{R}^n)/\ker(\varphi)$ where $I(X) = \ker(\varphi) \cap \mathcal{N}^\diamond(\mathbb{R}^n)$. Thus, by [S3, Ch. I, Lemma 4.5] we get the result.

Now, the ring of polynomial functions $\mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$ is contained in $\mathcal{N}(\mathbb{R}^n)$ and so $\mathcal{S}^0(\mathbb{R}^n)$ is the real closure of $\mathcal{N}(\mathbb{R}^n)$ by [S3, Ch. III §1]. It only remains to prove the bounded case. Since $\mathcal{S}^{0*}(\mathbb{R}^n)$ is convex in the real closed ring $\mathcal{S}^0(\mathbb{R}^n)$, it is also real closed. Thus, the real closure A of the ring $\mathcal{N}^*(\mathbb{R}^n)$ is contained in $\mathcal{S}^*(\mathbb{R}^n)$. Note that $g := \frac{1}{1+\|x\|^2} \in \mathcal{N}^*(\mathbb{R}^n) \subset A$ and the localization $A_g \subset \mathcal{S}^0(\mathbb{R}^n)$ is a real closed ring which contains $\mathcal{N}(\mathbb{R}^n) = \mathcal{N}^*(\mathbb{R}^n)_g$, so that $A_g = \mathcal{S}^0(\mathbb{R}^n)$. Therefore by [S6, Prop. 29] it suffices to show that for every $n \in \mathbb{N}$ and $a \in A$ with $0 \leq a \leq g^n$ we have that $a \in g^n A$. Indeed, since $\frac{1}{2}g^{2n} > 0$ there is a Nash function $\mathcal{N}^*(\mathbb{R}^n)$ such that $|h - (a + \frac{1}{2}g^{2n})| < \frac{1}{2}g^{2n}$, and so $0 < h - a < g^{2n}$. Since $h < g^{2n} + a < 2g^n$ we get $h/g^n \in \mathcal{N}^*(\mathbb{R}^n) \subset A$. On the other hand, since A is real closed, by [PS, Thm] there is $b \in A$ such that $(h-a)^2 = g^{2n}b$, and therefore $h - a = g^n\sqrt{b}$. We deduce that $a \in g^n A$, as required. □

The following is a direct consequence of the fact that $\mathcal{S}^{0*}(\mathbb{R})$ is the real closure of $\mathcal{N}^*(\mathbb{R}^n)$. However, since it is a relevant effect that concerns the real spectra of both rings, we also give a direct proof with explicit computations.

Nash*sper **Fact 5.4.** *The inclusion* $\mathtt{j} : \mathcal{N}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$ *induces a homeomorphism*

$$\operatorname{Sper}(\mathtt{j}) : \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) \to \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)), \ \alpha \mapsto \alpha \cap \mathcal{N}^*(\mathbb{R}^n).$$

In addition, for each $\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n))$ *the quotient field* $\kappa(\alpha) = \operatorname{qf}(\mathcal{S}^*(\mathbb{R}^n)/\mathfrak{p}_\alpha)$ *is the real closure of the quotient field* $\kappa(\beta) := \operatorname{qf}(\mathcal{N}^*(\mathbb{R}^n)/\mathfrak{q}_\beta)$ *where* $\beta := \operatorname{Spec}_r(\mathtt{j})(\alpha)$, $\mathfrak{p}_\alpha := \alpha \cap (-\alpha)$ *and* $\mathfrak{q}_\beta := \beta \cap (-\beta)$.

Proof. Consider the Nash function $g := \frac{1}{1+\|x\|^2} \in \mathcal{N}^*(\mathbb{R}^n)$. Recall that for any $f \in \mathcal{N}(\mathbb{R}^n)$ there exists a non negative integer ℓ such that $h := fg^\ell \in \mathcal{N}^*(\mathbb{R}^n)$, see [BCR, Prop.2.6.2]. Thus, $\mathcal{N}(\mathbb{R}^n) = \mathcal{N}^*(\mathbb{R}^n)_g$. Analogously, $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}^*(\mathbb{R}^n)_g$. Define the open constructible sets $\mathcal{U} := \{\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) : g(\alpha) > 0\}$ and $\mathcal{V} := \{\beta \in \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)) : g(\beta) > 0\}$. Similarly as in [AM, 3.11 & Ch.3, Ex. 21] , we have the following commutative diagram,

$$\begin{array}{ccccc}
\operatorname{Sper}(\mathcal{S}(\mathbb{R}^n)) = \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)_g) & \overset{\cong}{\longleftrightarrow} & \mathcal{U} & \hookrightarrow & \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) \\
\cong \updownarrow & & \updownarrow \operatorname{Sper}(\mathtt{j})|_{\mathcal{V}} & & \updownarrow \operatorname{Sper}(\mathtt{j}) \\
\operatorname{Sper}(\mathcal{N}(\mathbb{R}^n)) = \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)_g) & \overset{\cong}{\longleftrightarrow} & \mathcal{V} & \hookrightarrow & \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)).
\end{array} \tag{5.1}$$

cd

Now, pick $\alpha \in \mathcal{U}$ and let us show that $\kappa(\alpha)$ is the real closure of $\kappa(\beta)$ where $\beta := \alpha \cap \mathcal{N}^*(\mathbb{R}^n)$. Indeed, note that the injective ring homomorphism

$$\mathcal{N}^*(\mathbb{R}^n)/\mathfrak{q}_\beta \to \mathcal{S}^*(\mathbb{R}^n)/\mathfrak{p}_\alpha$$

induces a homomorphism $\kappa(\beta) \hookrightarrow \kappa(\alpha)$. Since $\mathcal{S}^*(\mathbb{R}^n)$ is a real closed ring, the field $\kappa(\alpha)$ is real closed. Thus, it only remains to prove that any other homomorphism $\mathtt{i} : \kappa(\beta) \to R$ into a real closed field R extends to $\kappa(\alpha)$. To that aim, consider the ring homomorphism $\psi := \mathtt{i} \circ \pi : \mathcal{N}^*(\mathbb{R}^n) \to R$ where $\pi : \mathcal{N}^*(\mathbb{R}^n) \to \mathcal{N}^*(\mathbb{R}^n)/\mathfrak{q}_\beta$ is the canonical projection. Since $g \notin \mathfrak{q}_\beta$, the previous homomorphism extends to $\Psi : \mathcal{N}(\mathbb{R}^n) = \mathcal{N}^*(\mathbb{R}^n)_g \to R$. Finally, the ring $\mathcal{S}(\mathbb{R}^n)$ is the real closure of $\mathcal{N}(\mathbb{R}^n)$, so we obtain a homomorphism $\widehat{\Phi} : \mathcal{S}(\mathbb{R}^n) \to R$ extending Φ, which in turn induces a homomorphism $\kappa(\beta) \to R$ extending $\mathtt{i}$, as required.

To show that $\operatorname{Sper}(\mathtt{j})$ is surjective it suffices to find a preimage of $\beta \in \operatorname{Spec}(\mathcal{N}^*(\mathbb{R}^n))$ with $g(\beta) \leq 0$. Since $g > 0$ in $\mathbb{R}^n$, we must have $g(\beta) = 0$ and so $g \in \mathfrak{q}_\beta$. Denote $\mathfrak{q} := \mathfrak{q}_\beta$ and define

$$\widehat{\mathfrak{q}} := \{f \in \mathcal{S}^{0*}(\mathbb{R}^n) : \ \forall \varepsilon \in \mathcal{S}^{0*}(\mathbb{R}^n), \ \varepsilon > 0 \ \exists h \in \mathfrak{q} \text{ such that } |f - h| < \varepsilon\}.$$

Similarly as in Lemma 3.9 and Proposition 3.10, we obtain that $\widehat{\mathfrak{q}}$ is a prime ideal of $\mathcal{S}^*(\mathbb{R}^n)$ and $\widehat{\mathfrak{q}} \cap \mathcal{N}^*(\mathbb{R}^n) = \mathfrak{q}$. Moreover, the injective homomorphism

$$\psi : \mathcal{N}^*(\mathbb{R}^n)/\mathfrak{q} \hookrightarrow \mathcal{S}^*(\mathbb{R}^n)/\widehat{\mathfrak{q}}, \ h + \mathfrak{q}_\beta \mapsto h + \widehat{\mathfrak{q}}$$

is an isomorphism. For, pick $f \in \mathcal{S}^*(\mathbb{R}^n)$ and let $h \in \mathcal{N}^*(\mathbb{R}^n)$ be an approximation of f such that $|f-h|<g$, see [BCR, Thm.8.8.4]. As $Z(g)=\varnothing$, the function

$$a := \frac{f-h}{g} \in \mathcal{N}^*(\mathbb{R}^n),$$

so $f-h = ga \in \mathfrak{q}\mathcal{S}^*(\mathbb{R}^n) \subset \widehat{\mathfrak{q}}$ and $\psi(h+\mathfrak{q}) = f+\widehat{\mathfrak{q}}$. Therefore, $\alpha := \psi(\beta) \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n))$ and $\mathfrak{p}_\alpha = \widehat{\mathfrak{q}}$, as required.

Next, let us show that $\operatorname{Sper}(\mathtt{j})$ is injective. Take $\alpha_1, \alpha_2 \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n))$ with $g(\alpha_1) = g(\alpha_2) = 0$ and such that $\alpha_1 \cap \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)) = \alpha_2 \cap \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)$. Assume that there exists $f \in \alpha_1 \setminus \alpha_2$. Let $h \in \mathcal{N}^*(\mathbb{R}^n)$ be such that $|f-h|<g$. Then

$$a := \frac{h-f}{g} \in \mathcal{S}^*(\mathbb{R}^n),$$

and so $h-f = ag \in \mathfrak{p}_{\alpha_i}$ for $i=1,2$. Thus, $h \in \mathfrak{p}_{\alpha_1} \cap \mathcal{N}^*(\mathbb{R}^n) = \mathfrak{p}_{\alpha_2} \cap \mathcal{N}^*(\mathbb{R}^n)$ and so $f = h - ag \in \mathfrak{p}_{\alpha_2}$, a contradiction.

All in all, we have that the continuous real spectral map (see [BCR, Prop.7.1.7])

$$\operatorname{Spec}_r(\mathtt{j}) : \operatorname{Spec}_r(\mathcal{S}^*(\mathbb{R}^n)) \to \operatorname{Spec}_r(\mathcal{N}^*(\mathbb{R}^n)),\ \beta \mapsto \beta \cap \mathcal{N}^*(\mathbb{R}^n).$$

is bijective, and it only remains to prove that $\operatorname{Sper}(\mathtt{j})$ is an open map. We already know that the restriction $\operatorname{Sper}(\mathtt{j})|_{\mathcal{V}}$ is open. As $\mathcal{V}$ is an open subset of $\operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n))$, it only remains to show that: *if* $\alpha_0 \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) \setminus \mathcal{V} = \{\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) :\ g(\alpha)=0\}$ *then the image by* $\operatorname{Sper}(\mathtt{j})$ *of an open neighborhood of* α_0 *contains an open neighborhood of* $\operatorname{Sper}(\mathtt{j})(\alpha_0)$ *in* $\operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n))$.

Let $f \in \mathcal{S}^*(\mathbb{R}^n)$ such that $\alpha_0 \in \mathcal{D}_0 := \{\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) :\ f(\alpha)>0\}$, and let us show that

$$\operatorname{Sper}(\mathtt{j})(\mathcal{D}_0)$$

is a neighborhood of $\beta_0 := \operatorname{Sper}(\mathtt{j})(\alpha_0)$ in $\operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n))$. Note that

$$\alpha_0 \in \mathcal{D}_1 := \{\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) :\ (f-2g)(\alpha)>0\} \subset \mathcal{D}_0.$$

By [BCR, Thm.8.8.4] there exists $h \in \mathcal{N}^*(\mathbb{R}^n)$ such that $|h-(f-g)|<g$, that is, $f-2g<h<f$ on $\mathbb{R}^n$. Define $\mathcal{D}_2 := \{\alpha \in \operatorname{Sper}(\mathcal{S}^*(\mathbb{R}^n)) :\ h(\alpha)>0\}$, so that

$$\alpha_0 \in \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_0.$$

Since $\beta_0 \in \operatorname{Sper}(\mathcal{D}_2) = \{\beta \in \operatorname{Sper}(\mathcal{N}^*(\mathbb{R}^n)) :\ h(\beta)>0\} \subset \operatorname{Sper}(\mathcal{D}_0)$, we are done. □

We would like to stress that in turn, the fact that $\mathcal{S}^{0*}(\mathbb{R})$ is the real closure of $\mathcal{N}^*(\mathbb{R}^n)$ could be deduce from Fact 5.4 and the following:

rclc **Fact 5.5.** [SM, Thm. 5.2] *Let A be a commutative ring with unity and let B be a real closed ring. Let $\mathtt{i} : A \to B$ be an injective homomorphism of rings such that*

(i) *the map* $\mathtt{i}^* : \operatorname{Sper}(B) \to \operatorname{Sper}(A),\ \beta \mapsto \mathtt{i}^{-1}(\beta)$ *is a homeomorphism, and*
(ii) *for each* $\beta \in \operatorname{Sper}(B)$ *the field* $\operatorname{qf}(B/\mathfrak{q}_\beta)$ *is the real closure of* $\operatorname{qf}(A/\mathfrak{p}_\alpha)$*, where* $\alpha := \mathtt{i}^*(\beta)$*,* $\mathfrak{p}_\alpha := \alpha \cap (-\alpha)$ *and* $\mathfrak{q}_\beta := \beta \cap (-\beta)$*.*

Then B is the real closure of A.

We finish this section with a result the involves the central object of our work:

f:rcl1 **Proposition 5.6.** *Let $r \geq 1$ be an integer and let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then the ring $\mathcal{S}^{r\diamond}(M)$ is not a real closed ring and the natural inclusion $\mathcal{S}^{r\diamond}(M) \hookrightarrow \mathcal{S}^{0\diamond}(M)$ provides its real closure.*

Proof. By Theorems 1.2 and 1.4 and Fact 5.5 it is enough to show that $\mathcal{S}^{0\diamond}(M)$ is a real closed ring for every semialgebraic set M. The case $M=\mathbb{R}^n$ is well-known, see [S3, Ch. III §1] and [T2, Thm. 10.5]. If M is a closed subset of $\mathbb{R}^n$, then the restriction map $\mathcal{S}^{0\diamond}(\mathbb{R}^n) \to \mathcal{S}^{0\diamond}(M)$ is an epimorphism and therefore $\mathcal{S}^{0\diamond}(M)$ is a real closed ring by [S5, Thm. 3.8]. If $M \subset \mathbb{R}^n$ is locally compact, then it follows from Lemma 2.10. Finally, for any arbitrary semialgebraic set $M \subset \mathbb{R}^n$, it follows from Lemma 2.13 that

$$\mathcal{S}^{0\diamond}(M) \cong \varinjlim(\mathcal{S}^{0\diamond}(E), \mathtt{j})$$

where $(E, \mathtt{j}) \in \mathfrak{C}^{0\diamond}$. Since the direct limit of real closed rings is real closed [S3, Ch. I, Thm 4.8], we are done. □

Remark 5.7. Some particular cases of the above lemma can be deduced from the general theory of real closed ring developed in [S3] and [T2]. Indeed, if M is a semialgebraic set, then clearly the real closure of $\mathcal{S}^r(M)$ is contained in $\mathcal{S}^0(M)$. To show that they are equal, pick $f \in \mathcal{S}^0(M)$, so that f can be extended to an open semialgebraic neighborhood U of M in $\operatorname{Cl}(M)$. By Lemma 2.10, we can assume that U is closed in $\mathbb{R}^n$. Thus, there is an extension $F \in \mathcal{S}^0(\mathbb{R}^n)$ of f. Since $x_i|_M \in \mathcal{S}^r(M)$ for each $i = 1, \dots, n$, by [T2, Lem. 2.10] we have that $F \circ (x_1|_M, \dots, x_n|_M) = f$ is in the real closure of $\mathcal{S}^r(M)$, as desired.

If M is a locally compact semialgebraic subset of $\mathbb{R}^n$ then it could be possible to deduce that $\mathcal{S}^{0*}(M)$ is the real closure of $\mathcal{S}^{r*}(M)$ using [S6, Prop. 29] as we did in the proof of Fact 5.3. However, for an arbitrary semialgebraic set M the only proof we know is the one we wrote above and which depends on our results in this paper.

5.B. The proof of Theorem 1.5. The first natural question we must answer concerning the ring of $\mathcal{S}^\infty$ functions on a semialgebraic set M, $\mathcal{S}^\infty(M) := \bigcap_{r\geq 1} \mathcal{S}^r(M)$, is whether it coincides with the ring $\mathcal{N}(M)$ of Nash functions on M. Of course, this is false even if M is compact or Nash.

cex:nash *Example* 5.8. 1) Let $M \subset \mathbb{R}^2$ be the (compact) semialgebraic set

$$([-2,-1] \times [-1,1]) \cup ([1,2] \times [-1,1]) \cup \{\mathtt{y} = 0, -1 \leq \mathtt{x} \leq 1\}$$

and consider the semialgebraic function $f : M \to \mathbb{R},\ (x,y) \mapsto y\|(x,y)\|$. The function f is $\mathcal{S}^\infty$ because f is the restriction to M of a continuous semialgebraic function which is Nash on $\mathbb{R}^2 \setminus \{(0,0)\}$ and $f|_{M\cap B((0,0),1/2)} \equiv 0$. However, by the identity principle f does not admit a Nash extension to an open neighborhood of M in $\mathbb{R}^2$.

2) For the umbrella $X := \{z(x^2+y^2) - x^3 = 0\} \subset \mathbb{R}^3$ it turns out that the function $f : X \to \mathbb{R}$ defined by $f(x,y,z)_= (z-1)^2/[(z-1)^2 + x^2 + y^2]$ if $xy \neq 0$ and $f(x,y,z) = 1$ otherwise, is a $\mathcal{S}^\infty$ function but not a Nash one, see [CRS, §3].

However, as we now show, it is true that $\mathcal{S}^\infty$ functions are locally Nash functions:

local **Lemma 5.9.** *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $f \in \mathcal{S}^{\infty\diamond}(M)$. For each $x \in M$ there exists an open semialgebraic neighborhood $V^x \subset \mathbb{R}^n$ of x and a Nash function $F_x \in \mathcal{N}^\diamond(V^x)$ such that $F_x|_{M\cap V^x} = f|_{M\cap V^x}$.*

Proof. It suffices to prove it for $f \in \mathcal{S}^\infty(M)$, and we can assume that $x = 0 \in M$. For each r there is a semialgebraic jet $F^r = (f^r_\alpha)$ of order r such that $f_0 = f$. In fact, we can assume that $f^r_\alpha(0) = f^k_\alpha(0) =: f_\alpha(0)$ for any pair of numbers $k > r$ and any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$. For, it is enough to note that recursively we can replace F_{r+1} by $\widetilde{F}_{r+1} = (\widetilde{f}^{r+1}_\alpha)$ where $\widetilde{f}^{r+1}_\alpha(x) := \widetilde{f}^{r+1}_\alpha(x) - (f^{r+1}_\alpha(0) - f^r_\alpha(0))$ for any $|\alpha| \leq r$. In particular, we can consider the formal series $h_0 := \sum_\alpha \frac{1}{\alpha!} f_\alpha(0)\mathtt{x}^\alpha \in \mathbb{R}[[\mathtt{x}]]$. Next, consider the Nash closure Z of the germ M_0 of M at the point 0. Let $I(Z) := \{\eta \in \mathcal{N}(\mathbb{R}^n_0) : \eta|_Z = 0\}$. Henceforth, we identify $\mathcal{N}(\mathbb{R}^n_0)$ with $\mathbb{R}[[\mathtt{x}]]_{\rm alg}$. Let X also be the Nash closure of $\Gamma(f|_{M_0}) = \Gamma(f)_0$, and note that $\dim(X) = \dim(M)$. Let $g(\mathtt{x}, \mathtt{y}) \in \mathbb{R}[[\mathtt{x}, \mathtt{y}]]_{\rm alg}$ be such that its zero set is X.

Claim. We have that $g(\mathtt{x}, h_0(\mathtt{x})) \in I(Z)\mathbb{R}[[\mathtt{x}]]$.

Proof. Let Γ be the collection of all continuous semialgebraic curves $\gamma : [0,1) \to M$ such that $\gamma(0) = 0$. Recall that we can identify each germ γ_0 with a Puiseux series $\mathbb{R}[[t^*]]$. For each $\gamma \in \Gamma$, define the homomorphism

$$\gamma^* : \mathbb{R}[[\mathtt{x}]] \to \mathbb{R}[[t^*]],\ \zeta \mapsto \zeta \circ \gamma.$$

Since $M_0 = \bigcup_{\gamma\in\Gamma} \operatorname{Im}(\gamma)_0$, it holds $I(Z) = \bigcap_{\gamma\in\Gamma} \ker(\gamma^*) \cap \mathcal{N}(\mathbb{R}^n_0)$. The completion of the local noetherian ring $\mathcal{N}(\mathbb{R}^n_0)$ is $\mathbb{R}[[x]]$ and therefore

$$I(Z)\mathbb{R}[[x]] = \Big(\bigcap_{\gamma\in\Gamma} \ker(\gamma^*) \cap \mathcal{N}(\mathbb{R}^n_0)\Big)\mathbb{R}[[x]] = \bigcap_{\gamma\in\Gamma} \big(\ker(\gamma^*) \cap \mathcal{N}(\mathbb{R}^n_0)\big)\mathbb{R}[[x]] = \bigcap_{\gamma\in\Gamma} \ker(\gamma^*).$$

Assume $\xi := g(\mathtt{x}, h_0(\mathtt{x})) \notin I(Z)\mathbb{R}[[\mathtt{x}]]$, so that there is $\gamma \in \Gamma$ such that $\xi \notin \ker(\gamma^*)$. Without loss, we can identify γ with an element in $\mathbb{R}[[t^*]]^n_{\mathrm{alg}}$. Moreover, if we reparametrise the variable t we can assume that $\gamma \in \mathbb{R}[[t]]^n_{\mathrm{alg}}$. Let $k \in \mathbb{N}$ be the order of the series $\xi(\gamma)$. Write $h_0 = a + b$ where $a := \sum_{|\alpha|\le k} \frac{1}{\alpha!} f_\alpha(0)\mathtt{x}^\alpha \in \mathbb{R}[\mathtt{x}]$ has degree $\le k$ and b has degree $> k$. We have

$$g(\mathtt{x}, h_0) = g(\mathtt{x}, a) + bs(\mathtt{x}, a, b)$$

for some $s \in \mathbb{R}[\mathtt{x}, \mathtt{z}_1, \mathtt{z}_2]$. Thus,

$$\xi(\gamma) = g(\mathtt{x}, a)(\gamma) + bs(\mathtt{x}, a, b)(\gamma).$$

As $\omega(\xi(\gamma)) = k$ and $\omega(bs(\mathtt{x}, a, b)) \ge k + 1$, we deduce $\omega(g(\mathtt{x}, a)(\gamma)) = k$. Next, consider the semialgebraic curve $f(\gamma) \in \mathbb{R}[[t^*]]$. Note that if $q \in \mathbb{Q}_{\ge 0}$ satisfies that the limit of $f(\gamma)/t^q$ when $t \to 0^+$ is zero then the order of $f(\gamma)$ is $> q$. Now, for any r we have that

$$|f(x) - \sum_{|\alpha|\le r} \frac{1}{\alpha!} f_\alpha(0)x^\alpha| = o(\|x\|^r)$$

for $x \in M$ when $x \to 0$. Thus, $|f(\gamma) - a(\gamma)|/t^k \to 0$ when $t \to 0^+$ and therefore the order of $f(\gamma) - a(\gamma)$ is greater than k. In particular,

$$\omega(g(\mathtt{x}, a)(\gamma) - g(\mathtt{x}, f)(\gamma)) \ge k + 1,$$

and so $\omega(g(\mathtt{x}, f)(\gamma)) = k$, which is a contradiction because $g(\gamma(t), f(\gamma(t))) = 0$ for each $t \in [0, 1)$, as required. □

Let $f_1, \dots, f_m$ be Nash germs at the origin that generate $I(Z)$, so there are $g_1, \dots, g_m \in \mathbb{R}[[\mathtt{x}]]$ such that $g(\mathtt{x}, h_0) = g_1 f_1 + \cdots g_m f_m$. By Artin's approximation theorem [BCR, Thm. 8.3.1] there are Nash functions $h, \widetilde{g}_1, \dots, \widetilde{g}_m \in \mathcal{N}(\mathbb{R}^n_0)$ such that $g(\mathtt{x}, h) = \widetilde{g}_1 f_1 + \cdots \widetilde{g}_m f_m$. Let us prove that $h|_{M_0} = f|_{M_0}$.

Let $Z_1, \dots, Z_s$ be the irreducible components of Z, and let $Y_i \subset Z_i \times \mathbb{R}$ be the Nash closure of $\Gamma(f|_{M_0 \cap Z_i})$ for each $i = 1, \dots, s$. Since $X = \bigcup_{i=1}^s Y_i$, we deduce

$$X \cap (Z_i \times \mathbb{R}) = \Big(\bigcup_{j=1}^s Y_j\Big) \cap (Z_i \times \mathbb{R}) = \Big(\bigcup_{j\ne i} (Z_j \cap Z_i) \times \mathbb{R}\Big) \cup Y_i.$$

We have that $\dim(Z_j \cap Z_i) < \dim(Z_i)$ for $i \ne j$ and $\dim(Y_i) = \dim(Z_i)$, so that

$$\dim(X \cap (Z_i \times \mathbb{R})) \le \dim(Z_i).$$

On the other hand, by the Claim it holds $g(x, h)|_{M_0} = 0$ and therefore $\Gamma(h) \subset Z(g) = X$. Since h is Nash and Z_i is irreducible we get that $\Gamma(h|_{Z_i}) \subset X \cap (Z_i \times \mathbb{R})$ is a Nash irreducible set of dimension $\dim(Z_i)$. Therefore the irreducible components of X are exactly $\Gamma(h|_{Z_1}), \dots, \Gamma(h|_{Z_s})$. In particular, since $\Gamma(f) \subset X = \bigcup_{j=1}^s \Gamma(h|_{Z_i})$, we get that $h|_{M_0} = f|_{M_0}$, as required. □

The following is a direct consequence of Lemma 5.9 and Serre's coherence condition (see [BFR, §2.B]).

cohNash **Corollary 5.10.** *Let $U \subset \mathbb{R}^n$ be an open semialgebraic set, and let $X \subset U$ be a coherent Nash subset of U. Then $\mathcal{N}^\diamond(X) = \mathcal{S}^{\infty\diamond}(X)$.*

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Assume first that there exists an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ such that $X := \mathrm{Cl}(M) \cap U$ is a coherent Nash set. By Lemma 5.9 and since X is coherent, we have that $\mathcal{S}^\infty(M) = \mathcal{N}(X)$. Thus, it suffices to show that the real closure of $\mathcal{N}(X)$ equals $\mathcal{S}^0(M)$.

We may assume that X is a closed Nash subset of $\mathbb{R}^n$ and $U = \mathbb{R}^n$. For, by [FG2, §2.12] we can assume that $U = \mathbb{R}^n \setminus (\mathrm{Cl}(X) \setminus X)$. Thus, by [Sh, Thm. II.5.2] it holds $\mathcal{N}(X) = \mathcal{N}(U)/I(X)$ where

$$I(X) := \{f \in \mathcal{N}(U) : \ f|_X = 0\} = \{f \in \mathcal{N}(U) : \ f|_M = 0\}.$$

On the other hand, there exists $h \in \mathcal{S}^0(\mathbb{R}^n)$ such that $Z(h) = \mathrm{Cl}(X) \setminus X$ and $h|_U$ is Nash (see [BCR, 2.7.5]). Consider the embedding $\varphi : U \to \mathbb{R}^{n+1}, \ x \mapsto (x, \frac{1}{h(x)})$ whose image is the closed semialgebraic set $C := \{(x,y) \in \mathbb{R}^{n+1} : \ yh(x) = 1\}$. Note that $\varphi(X)$ is a closed semialgebraic subset of $\mathbb{R}^{n+1}$, as desired.

Finally, let us show that $\mathcal{S}^0(M)$ is the real closure of $\mathcal{N}(X)$. Let $\mathcal{V}$ be the collection of open semialgebraic neighborhoods of M. Note that for each $V \in \mathcal{V}$ the restriction map $\mathcal{N}(X \cap V) \to \mathcal{N}(M)$ is injective because $\mathrm{Cl}(M) = X$. On the other hand, for each $f \in \mathcal{N}(M)$ by definition there is $V \in \mathcal{V}$ and a Nash function $F \in \mathcal{N}(V)$ such that $F|_V = f$. We deduce that $\mathcal{N}(M) = \varinjlim_{V \in \mathcal{V}} \mathcal{N}(X \cap V)$. Since $X \cap V$ is a Nash subset of $V \in \mathcal{V}$, by Fact 5.3 the real closure of $\mathcal{N}(X \cap V)$ is $\mathcal{S}^0(X \cap V)$. In particular, by Lemma 5.2 the real closure of $\mathcal{N}(M)$ is $\varinjlim_{V \in \mathcal{V}} \mathcal{S}^0(X \cap V)$. Since again the map $\mathcal{S}^0(X \cap V) \to \mathcal{S}^0(M)$ is injective for each $V \in \mathcal{V}$, by the definition of $\mathcal{S}^0$ function it follows that $\mathcal{S}^0(M) = \varinjlim_{V \in \mathcal{V}} \mathcal{S}^0(X \cap V)$, as required.

Assume next that there is $x \in M$ such that the germ $\mathrm{Cl}(M)_x$ is not a germ of a Nash set. Let $\overline{M_x}^{\mathrm{an}}$ be the smallest Nash set germ that contains M_x. By the curve selection lemma there exists a Nash arc $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma((0,1])_x \subset \overline{M_x}^{\mathrm{an}} \setminus \mathrm{Cl}(M)_x$. Let $f \in \mathcal{S}^\infty(M)$. By Lemma 5.9 there exist an open neighborhood $V^x \subset \mathbb{R}^n$ of x and a Nash extension F_x of $f|_{V^x \cap M}$ to V^x. Note that $F_x|_{\overline{M_x}^{\mathrm{an}}}$ is completely determined by f. Thus, we get a well-defined homomorphism

$$\varphi : \mathcal{S}^\infty(M) \to \mathcal{S}^\infty(\mathrm{Im}(\gamma)_x) : f \mapsto F_x|_{\mathrm{Im}(\gamma)_x}.$$

Pick an open neighbourhood U of $\mathrm{Cl}(M)$, and let us consider the restriction homomorphism $\mathcal{S}^\infty(U) \to \mathcal{S}^\infty(M) : f \mapsto f|_M$. By the case above, the real closures of $\mathcal{S}^\infty(U)$ and $\mathcal{S}^\infty(\mathrm{Im}(\gamma)_x)$ are $\mathcal{S}^0(U)$ and $\mathcal{S}^0(\mathrm{Im}(\gamma)_x)$ respectively. Thus, by [S3] we obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}^0(U) & & \\ \downarrow & \searrow & \\ \mathcal{S}^0(M) & \longrightarrow & \mathcal{S}^0(\mathrm{Im}(\gamma)_x) \end{array}$$

where the map from $\mathcal{S}^0(U)$ to $\mathcal{S}^0(\mathrm{Im}(\gamma)_x)$ is the restriction map. The existence of this diagram is absurd: take $f \in \mathcal{S}^0(U)$ whose zero set is $\mathrm{Cl}(M)$, so that $f|_M = 0$ and therefore by the diagram above $f|_{\mathrm{Im}(\gamma)_x} = 0$, a contradiction since then $\mathrm{Im}(\gamma)_x \subset Z(f)_x = \mathrm{Cl}(M)_x$. □

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