Interpolation of the measure of non-compactness of bilinear operators among quasi-Banach spaces

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Abstract

Working in the setting of quasi-Banach couples, we establish a formula for the measure of non-compactness of bilinear operators interpolated by the general real method. The result applies to the real method and to the real method with a function parameter.

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1. Introduction

In recent years it has been shown that compact bilinear operators occur rather naturally in harmonic analysis. See, for example, the papers by Bényi and Torres [4], Bényi and Oh [3] and Hu [32]. In particular, it has been established in [4] that commutators of bilinear Calderón-Zygmund operators and multiplication by functions in the subspace CMO of BMO are compact bilinear operators from $L_p \times L_q \to L_r$ for $1 < p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. These results have motivated the research on interpolation properties of compact bilinear operators, a problem already considered by Calderón [7] in his pioneering paper on the complex interpolation method. The case of the real interpolation method has been studied more recently by Fernández and Silva [25], Fernández-Cabrera and Martínez [27, 28], Mastylo and Silva [37] and Cobos, Fernández-Cabrera and Martínez [12]. It is shown in [12] that commutators of bilinear Calderón-Zygmund operators and multiplication by functions in CMO are also compact for $\frac{1}{2} < r < 1$.

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Once the behaviour under interpolation of compact bilinear operators is understood, it is time to enquire for quantitative results. This leads naturally to investigate how the measure of non-compactness of a bilinear operator behaves under interpolation.

In the case of linear operators, interpolation formulae for the measure of non-compactness $\beta(T)$ have attracted the attention of a number of authors. Let us recall that $\beta(T) = \lim_{n\to\infty} e_n(T)$, where $(e_n(T))$ is the sequence of entropy numbers of the operator T. Peetre, Triebel and Pietsch (see [47, 1.16.2] and [42, 12.1]) started the study of the interpolation properties of entropy numbers. They considered the case when one of the Banach couples degenerates to a Banach space, i.e. $A_0 = A_1$ or $B_0 = B_1$. Similar results in the quasi-Banach case can be found in the book by Edmunds and Triebel [22, 1.3.2]. As for the measure of non-compactness, the first results were obtained by Edmunds and Teixeira [46]. They work with Banach spaces and assume that one of the couples degenerates to a space, or the couples are arbitrary but the target couple satisfies a certain approximation condition. For the real interpolation method $(A_0, A_1)_{\theta,q}$, these assumptions were removed in the work of Cobos, Fernández-Martínez and Martínez [13], who proved the following logarithmically convex inequality

$$\beta\left(T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}\right) \leq C\beta\left(T : A_0 \to B_0\right)^{1-\theta} \beta\left(T : A_1 \to B_1\right)^{\theta} . \tag{1.1}$$

Similar formulae to (1.1) hold for two important extensions of the real method: the real method with a function parameter $(A_0, A_1)_{\rho,q}$ and the general real method $(A_0, A_1)_\Gamma$ (definitions of these constructions are recalled in Section 2 below). See the papers by Cordeiro [16] and by Szwedek [43]. See also the papers by Cobos, Fernández-Cabrera and Martínez [10, 11]. An extension of (1.1) to linear operators between quasi-Banach couples has been done by Fernández-Martínez [29]. Other quantitative results can be found in the more recent papers by Edmunds and Netrusov [19, 20] and by Szwedek [44, 45].

Returning to bilinear operators, in a recent paper Mastyło and Silva [37] have shown an abstract approach that allows to lift (1.1) to bilinear operators between Banach couples. Among other things, they have proved that

$$\begin{aligned} &\beta\left(T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \to (E_0, E_1)_{\theta,q}\right) \\ &\qquad \leq C\beta\left(T : A_0 \times B_0 \to E_0\right)^{1-\theta} \beta\left(T : A_1 \times B_1 \to E_1\right)^{\theta} \end{aligned} \tag{1.2}$$

provided that $1 \leq q_0, q_1 < \infty$, $1 < q < \infty$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - 1$. Their arguments are based on duality and on formula (1.1).

In this paper we study the behaviour of the measure of non-compactness of bilinear operators among quasi-Banach spaces interpolated by the general real method. We follow a direct approach based on properties of the vector-valued sequence spaces that come up with the construction of the general real method. These techniques have their origin in the papers by Cobos and Peetre [15] and Cobos, Kühn and Schonbek [14] on compact linear operators. They were also used by Cobos, Fernández-Cabrera and Martínez [12] to establish the result on interpolation of compact bilinear operators. We split the operator in pieces by using certain families of projections on the sequence spaces and then we proceed to estimate the measure of non-compactness of these pieces. There are important differences between the arguments in [12] and here. First we work with a more refined decomposition of the operator than in [12]. We use projections of different order which helps in computations. Most of the time, our estimates are based on the properties of the projections and the norm estimate given by the bilinear interpolation theorem, but for one of the pieces we must construct a suitable ε-net for the image of the product of the unit balls (see Step 2 in the proof of Theorem 3.3 below). For this aim we rely on the description of the general real interpolation method in terms of the J-functional and compactness in $\mathbb{R}^n$ of certain subsets connected with the sequence lattices used in the interpolation methods.

Writing down our result for the special case of the real method, we obtain an extension of (1.2) to couples of quasi-Banach spaces (A_0, A_1), (B_0, B_1), (E_0, E_1), with (E_0, E_1) being r-normed $(0 < r \leq 1)$. Moreover, parameters q_0, q_1 can now move in the interval $(0, \infty]$, with $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r}$ if $q_0, q_1 \geq r$ and $1/q = 1/\max(q_0, q_1)$ if $q_0 < r$ or $q_1 < r$. See Theorem 3.5 below. In the special case of Banach couples and $1 \leq q_0, q_1, q \leq \infty$ with $1/q = 1/q_0 + 1/q_1 - 1$, we show that (1.2) still holds in any of the cases $q_0 = \infty$, $q_1 = \infty$, $q = 1$ or $q = \infty$, cases which are not covered by the techniques based on duality of Mastyło and Silva [37].

2. Preliminaries

Let $(A, \|\cdot\|_A)$ be a quasi-Banach space with constant $c_A \geq 1$ in the quasi-triangle inequality and let $0 < p \leq 1$ be such that $c_A = 2^{1/p-1}$. It is well known that there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality (see [35, §15.10] or [34, Proposition 1.c.5]). We say that $|||\cdot|||$ is a p-norm and that A is a p-normed quasi-Banach space. Note that if $0 < r < p$ then A is also an r-normed quasi-Banach space.

We put $U_A = U_{(A,\ \|\cdot\|_A)} = \{x \in A : \ \|x\|_A \leq 1\}$.

A quasi-Banach space $(\Gamma,\ \|\cdot\|_\Gamma)$ of real valued sequences with $\mathbb{Z}$ as index set is said to be a *quasi-Banach sequence lattice* if Γ satisfies the following properties:

(i) Γ contains all sequences with only finitely many non-zero co-ordinates.

(ii) Whenever $|\xi_m| \leq |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \leq \|(\eta_m)\|_\Gamma$.

Let $A,\ B,\ E$ be quasi-Banach spaces and let $T : A\times B \to E$ be a bilinear operator. We say that T is *bounded* if

$$\|T\|_{A\times B,\ E} := \sup\{\|T(a,b)\|_E : \|a\|_A \leq 1,\ \|b\|_B \leq 1\} < \infty.$$

We put $\mathcal{B}(A\times B,\ E)$ for the set of all bounded bilinear operators from $A\times B$ into E.

The operator $T \in \mathcal{B}(A\times B,\ E)$ is said to be *compact* if for any bounded sets $V \subseteq A,\ W \subseteq B$ we have that the closure of the set $T(V,W) = \{T(a,b) : a \in V, b \in W\}$ is compact in E. This condition is equivalent to the fact that $T(U_A, U_B)$ is precompact in E.

The concept and properties of the measure of non-compactness for bounded linear operators can be seen, for example, in the books [18, 8]. We shall need the corresponding notion for bilinear operators.

The (ball) *measure of non-compactness* $\beta(T) = \beta(T : A\times B \to E)$ of $T \in \mathcal{B}(A\times B,\ E)$ is defined to be the infimum of the set of all $\sigma > 0$ for which there exists a finite subset $\{w_1, ..., w_s\} \subseteq E$ such that

$$T(U_A, U_B) \subseteq \bigcup_{j=1}^{s} \{w_j + \sigma U_E\}.$$

The following properties of the measure of non-compactness can be easily checked and will be used freely in our later computations:

(iii) If $T \in \mathcal{B}(A\times B,\ E)$, then $\beta(T : A\times B \to E) \leq \|T\|_{A\times B,E}$.

(iv) T is compact if and only if $\beta(T : A\times B \to E) = 0$.

(v) If F is another quasi-Banach space and R is a bounded linear operator $R \in \mathcal{L}(E,F)$, then for $RT = R\circ T$ we have

$$\beta(RT : A\times B \to F) \leq \|R\|_{E,F}\, \beta(T : A\times B \to E).$$

Moreover, if $\|Rv\|_F = \|v\|_E$ for any $v \in E$, then

$$\beta(T : A\times B \to E) \leq 2c_F \beta(RT : A\times B \to F).$$

(vi) If X, Y are quasi-Banach spaces and R_1, R_2 are bounded linear operators $R_1 \in \mathcal{L}(X,A)$, $R_2 \in \mathcal{L}(Y,B)$, then the operator $T\circ(R_1,R_2)(x,y) = T(R_1,R_2)(x,y) = T(R_1 x, R_2 y)$ belongs to $\mathcal{B}(X\times Y, E)$ and

$$\beta(T(R_1,R_2) : X\times Y \to E) \leq \|R_1\|_{X,A}\|R_2\|_{Y,B}\, \beta(T : A\times B \to E).$$

Moreover, if for any $a \in A, b \in B$ with $\|a\|_A < 1, \|b\|_B < 1$ there exists $x \in X, y \in Y$ with $\|x\|_X < 1$, $\|y\|_Y < 1$ and $(R_1,R_2)(x,y) = (a,b)$, then

$$\beta(T : A\times B \to E) \leq \beta(T(R_1,R_2) : X\times Y \to E).$$

(vii) If $S \in \mathcal{B}(A \times B, E)$ then

$$\beta(S+T : A \times B \to E) \leq c_E (\beta(S : A \times B \to E) + \beta(T : A \times B \to E)).$$

Let $\bar{A} = (A_0, A_1)$ be a (p-normed) quasi-Banach couple, that is, two (p-normed) quasi-Banach spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space. For $t > 0$, *Peetre's K- and J-functionals* are defined by

$$K(t,a) = K(t,a; A_0, A_1) = \inf\left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} : \ a = a_0 + a_1, \ a_j \in A_j \right\}$$

where $a \in A_0 + A_1$, and

$$J(t,a) = J(t,a; A_0, A_1) = \max\left\{ \|a\|_{A_0}, t\|a\|_{A_1} \right\}, \ a \in A_0 \cap A_1.$$

Note that $K(1,\cdot)$ coincides with the quasi-norm of $A_0 + A_1$ and $J(1,\cdot)$ with the quasi-norm of $A_0 \cap A_1$. Functionals $K(t,\cdot)$ and $J(t,\cdot)$ are equivalent quasi-norms in $A_0 + A_1$ and $A_0 \cap A_1$, respectively, and quasi-triangle inequality is satisfied with constant $c_{\bar{A}} = \max\{c_{A_0}, c_{A_1}\}$.

If $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ are p-norms then $J(t,\cdot)$ is also a p-norm on $A_0 \cap A_1$.

Let Γ be a quasi-Banach sequence lattice. We say that Γ is *K-non-trivial* if $(\min(1, 2^m)) \in \Gamma$. The lattice Γ is said to be *(p,J)-non-trivial*, $0 < p \leq 1$, if

$$\sup\left\{ \Big(\sum_{m=-\infty}^{\infty} \left(\min\left(1, 2^{-m}\right)|\xi_m|\right)^p \Big)^{1/p} : \ \|(\xi_m)\|_\Gamma \leq 1 \right\} < \infty.$$

Note that if Γ is (p,J)-non-trivial then Γ is also (r,J)-non-trivial for any $p \leq r \leq 1$.

Let Γ be a K-non-trivial quasi-Banach sequence lattice and let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple. *The general real interpolation space realized by means of the K-functional* $\bar{A}_{\Gamma;K} = (A_0, A_1)_{\Gamma;K}$ consists of all $a \in A_0 + A_1$ such that $(K(2^m, a)) \in \Gamma$. The quasi-norm on $\bar{A}_{\Gamma;K}$ is $\|a\|_{\bar{A}_{\Gamma;K}} = \|(K(2^m, a))\|_\Gamma$.

If Γ is a (p,J)-non-trivial quasi-Banach sequence lattice and $\bar{A} = (A_0, A_1)$ is a p-normed quasi-Banach couple, *the general real interpolation space realized by means of the J-functional* $\bar{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is defined as the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) where $(u_m) \subseteq A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. The quasi-norm on $\bar{A}_{\Gamma;J}$ is given by

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\left\{ \|(J(2^m, u_m))\|_\Gamma : \ a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

We have

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\Gamma;K} \hookrightarrow (A_0, A_1)_{\Gamma;J} \hookrightarrow A_0 + A_1,$$

where $\hookrightarrow$ means continuous inclusion. Embedding $(A_0, A_1)_{\Gamma;J} \hookrightarrow (A_0, A_1)_{\Gamma;K}$ holds provided that the Calderón transform

$$\Lambda_p(\xi_m) = \left(\Big(\sum_{k=-\infty}^{\infty} \Big(\min(1, 2^{m-k}) \, |\xi_k| \Big)^p \Big)^{1/p} \right)_{m \in \mathbb{Z}}$$

is bounded in Γ.

Hence, if

$$\Gamma \text{ is } K\text{-non-trivial, } (p, J)\text{-non-trivial and } \Lambda_p \text{ is bounded in } \Gamma, \tag{2.1}$$

then for any p-normed quasi-Banach couple $\bar{A}$ we have that $\bar{A}_{\Gamma;K} = \bar{A}_{\Gamma;J}$ with equivalence of quasi-norms. In this case we write $\bar{A}_\Gamma$ for any of the spaces $\bar{A}_{\Gamma;K}$ or $\bar{A}_{\Gamma;J}$ and we put $\|\cdot\|_{\bar{A}_\Gamma}$ for any of the two quasi-norms. This will not cause any confusion.

We refer to the books by Peetre [40] and Brudnyĭ and Krugljak [6] and the paper by Nilsson [38] for the basic theory on the general real interpolation method. Other properties of this method can be found, for example, in the papers by Cwikel and Peetre [17], Nilsson [39], Cobos, Fernández-Cabrera, Manzano and Martínez [9], Fernández-Cabrera and Martínez [26] or Cobos, Fernández-Cabrera and Martínez [10].

For $k \in \mathbb{Z}$, *the shift operator* τ_k is defined by $\tau_k \xi = (\xi_{m+k})_{m \in \mathbb{Z}}$ for $\xi = (\xi_m)_{m \in \mathbb{Z}}$. Assume that the quasi-Banach sequence lattice Γ satisfies that τ_k is bounded in Γ for all $k \in \mathbb{Z}$ and

$$\lim_{n \to \infty} 2^{-n} \|\tau_n\|_{\Gamma,\Gamma} = 0 \text{ and } \lim_{n \to \infty} \|\tau_{-n}\|_{\Gamma,\Gamma} = 0. \tag{2.2}$$

We put

$$f_\Gamma(t) = \left\| \tau_{[\log_2 t]} \right\|_{\Gamma,\Gamma}, \ t > 0,$$

where the logarithm is taken in base 2 and $[\cdot]$ is the greatest integer function. Let $M_1 = \max(1, \|\tau_1\|_{\Gamma,\Gamma})$, $M_2 = \sup\{f_\Gamma(t) : \ 0 < t \le 1\} = \sup\{\|\tau_{-n}\|_{\Gamma,\Gamma} : n \ge 0\}$ and $M_3 = \sup\{f_\Gamma(t)/t : \ 1 \le t < \infty\} = \sup\{2^{-n} \|\tau_n\|_{\Gamma,\Gamma} : \ n \ge 0\}$. The following properties hold for the function f_Γ:

$$f_\Gamma(t) = o(\max(1, t)) \text{ as } t \to 0 \text{ and } t \to \infty. \tag{2.3}$$

$$\text{For any } s, t > 0, \ f_\Gamma(st) \le M_1 f_\Gamma(s) f_\Gamma(t). \text{ Hence, if } s < t \text{ then}$$
$$f_\Gamma(s) \le M_1 M_2 f_\Gamma(t) \text{ and } f_\Gamma(t)/t \le M_1 M_3 f_\Gamma(s)/s. \tag{2.4}$$

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples. By $T : \bar{A} \times \bar{B} \to \bar{E}$ we mean that T is a bounded bilinear operator $T \in \mathcal{B}((A_0 + A_1) \times (B_0 + B_1) \to E_0 + E_1)$ such that the restriction of T to $A_j \times B_j$ defines a bounded bilinear operator $T \in \mathcal{B}(A_j \times B_j, E_j)$, for $j = 0$ and $j = 1$. We put $\|T\|_j = \|T\|_{A_j \times B_j, E_j}$, $j = 0, 1$.

Next we recall an interpolation property for bilinear operators which has been established in [12, Theorem 3.1].

If $\xi = (\xi_m)_{m\in\mathbb{Z}}$ and $\eta = (\eta_m)_{m\in\mathbb{Z}}$ are sequences of non-negative scalars, we write $\xi \star \eta = \big(\sum_{k=-\infty}^{\infty} \xi_k \eta_{m-k}\big)_{m\in\mathbb{Z}}$ for their convolution. If $r > 0$, we put $\xi^r = (\xi_m^r)_{m\in\mathbb{Z}}$.

Theorem 2.1. *Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, let $\bar{B} = (B_0, B_1)$ be a p-normed quasi-Banach couple and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < p, r \leq 1)$. Assume that Γ_0 and Γ_2 are K-non-trivial quasi-Banach sequence lattices and Γ_1 is a (p, J)-non-trivial quasi-Banach sequence lattice satisfying (2.2). Assume in addition that there is a constant $M > 0$ such that for all non-negative scalar sequences $\xi \in \Gamma_0$ and $\eta \in \Gamma_1$ we have*

$$\left\| (\xi^r \star \eta^r)^{1/r} \right\|_{\Gamma_2} \leq M \,\|\xi\|_{\Gamma_0} \|\eta\|_{\Gamma_1}. \tag{2.5}$$

Then, for each $T : \bar{A} \times \bar{B} \to \bar{E}$ the restriction of T to $\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}$ defines a bounded bilinear operator $T : \bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J} \to \bar{E}_{\Gamma_2;K}$ with

$$\|T\|_{\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K}} \leq \begin{cases} 0 & \text{if } \;\|T\|_j = 0 \text{ for } j = 0 \text{ or } 1, \\ C\|T\|_0 f_{\Gamma_1}(\|T\|_1/\|T\|_0) & \text{otherwise.} \end{cases}$$

Here C is a constant independent of T.

We close this section with some examples. For $0 < q \leq \infty$ let ℓ_q be the usual space of q-summable real valued sequences with $\mathbb{Z}$ as index set. Given any sequence (w_m) of positive numbers, we put $\ell_q(w_m)$ for the corresponding weighted space of those sequences (ξ_m) for which $(w_m \xi_m) \in \ell_q$.

In what follows, (Ω, μ) is a measure space. For $0 < p \leq \infty$, we put $L_p(\Omega)$ for the usual Lebesgue space. If $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$, we recall that the Lorentz-Zygmund space $L_{p,q}(\log L)_\alpha(\Omega)$ is formed by all (equivalence classes of) measurable functions f on Ω having a finite quasi-norm

$$\|f\|_{L_{p,q}(\log L)_\alpha(\Omega)} = \Big(\int_0^\infty \big(t^{1/p}(1 + |\log t|)^\alpha f^*(t)\big)^q \frac{dt}{t} \Big)^{1/q}.$$

Here f^* is the non-increasing rearrangement of f and the integral should be replaced by the supremum if $q = \infty$ (see [1]). When $\alpha = 0$ we get the Lorentz spaces $L_{p,q}(\Omega)$.

Example 2.2. *For $\Gamma = \ell_q(2^{-\theta m})$ with $0 < q \leq \infty$ and $0 < \theta < 1$, K- and J-spaces coincide and they are equal to the* real interpolation space $(A_0, A_1)_{\theta,q}$ *(see [36, 5, 47, 2]). When we interpolate a couple of Lebesgue spaces by this method, we obtain Lorentz spaces: If $0 < p_0 \neq p_1 < \infty, 0 < q \leq \infty, 0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ then $(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,q} = L_{p,q}(\Omega)$ (see [31, Theorem 4.3]).*

Example 2.3. *Let $\rho : (0,\infty) \to (0,\infty)$ be a* function parameter*, that is to say, $\rho(t)$ increases from 0 to ∞, $\rho(t)/t$ decreases from ∞ to 0 and, for every $t > 0$, $s_\rho(t) = \sup\{\rho(ts)/\rho(s) : \ s > 0\}$ is finite and $s_\rho(t) = o(\max\{1,t\})$ as $t \to 0$ and $t \to \infty$. For $0 < q \le \infty$ and $\Gamma = \ell_q(1/\rho(2^m))$, K- and J-spaces also agree and they are equal now to the* real interpolation method with function parameter $(A_0, A_1)_{\rho,q} = \bar{A}_{\rho,q}$ *(see [30, 33, 41]). Shift operators in $\ell_q(1/\rho(2^m))$ satisfy (2.2) because $\|\tau_k\|_{\ell_q(1/\rho(2^m)),\ell_q(1/\rho(2^m))} \le s_\rho(2^k)$. This inequality allows to replace $f_{\ell_q(1/\rho(2^m))}$ by s_ρ in Theorem 2.1. It follows from the properties of ρ and definition of s_ρ that $s_\rho(t)$ is submultiplicative, non-decreasing and $s_\rho(t)/t$ is non-increasing. Hence s_ρ satisfies (2.4) with $M_1 = M_2 = M_3 = 1$. Interpolating a couple of Lebesgue spaces by this method we obtain*

$$(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\rho,q} = \left\{ f : \|f\| = \Big(\int_0^\infty \big(\varphi(t) f^*(t)\big)^q \frac{dt}{t} \Big)^{1/q} < \infty \right\}.$$

Here $0 < p_0 \ne p_1 < \infty, 0 < q \le \infty$ and $\varphi(t) = t^{1/p_0}/\rho(t^{1/p_0 - 1/p_1})$ (see [41, Proposition 6.2]).

Example 2.4. *Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and*

$$(1 + |\log t|)^{\mathbb{A}} = \begin{cases} (1 - \log t)^{\alpha_0} & \text{if} \quad 0 < t \le 1, \\ (1 + \log t)^{\alpha_\infty} & \text{if} \quad 1 < t < \infty. \end{cases}$$

For $0 < \theta < 1$ and $-\mathbb{A} = (-\alpha_0, -\alpha_\infty)$, put $g(t) = t^\theta (1 + |\log t|)^{-\mathbb{A}}$. The function g is equivalent to a function parameter ρ, meaning that there are positive constants c_1, c_2 such that

$$c_1 g(t) \le \rho(t) \le c_2 g(t) \text{ for all } t > 0.$$

Let $0 < q \le \infty$ and $\Gamma = \ell_q(1/g(2^m))$, then K- and J- spaces agree again, being now equal to logarithmic interpolation spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ *(see [23, 24, 21]). Interpolating a couple of Lebesgue spaces by this method we obtain Lorentz-Zygmund spaces: $(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,q,(\alpha,\alpha)} = L_{p,q}(\log L)_\alpha(\Omega)$ where $0 < p_0 \ne p_1 < \infty, 0 < q \le \infty, 0 < \theta < 1, 1/p = (1-\theta)/p_0 + \theta/p_1$ and $\alpha \in \mathbb{R}$ (see [41, Proposition 6.2/(c)]).*

3. Interpolation of the measure of non-compactness

We start with two auxiliary results. The first one correspond to [12, Lemma 3.2] but dispensing the operator T with the compactness assumption used there.

Lemma 3.1. *Let A, B, E, Z be quasi-Banach spaces, let D be a dense subspace of A and let V be a dense subspace of B. Let $T \in \mathcal{B}(A \times B, E)$, put $\beta = \beta(T : A \times B \to E)$ and assume that there exists $(S_n) \subseteq \mathcal{L}(E, Z)$ with $\sup_{n\in\mathbb{N}} \|S_n\|_{E,Z} = M < \infty$ and $\lim_{n\to\infty} \|S_n T(u,v)\|_Z = 0$ for all $(u,v) \in D \times V$. Then the following holds.*

a) If $\beta = 0$*, then* $\lim_{n\to\infty} \|S_n T\|_{A\times B,Z} = 0$*.*

b) If $\beta > 0$*, then there is a constant* C *independent of* T *and there is* $N \in \mathbb{N}$ *such that*

$$\|S_n T\|_{A\times B,Z} \leq C\beta \text{ for all } n \geq N.$$

PROOF. Take $\sigma > \beta$. There exists a finite set $\{w_1, ..., w_s\} \subseteq E$ such that

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{w_k + \sigma U_E\}.$$

If $\{w_k + \sigma U_E\} \cap T(U_A, U_B) \neq \emptyset$, choose $a_k \in U_A$, $b_k \in U_B$ such that $T(a_k, b_k) \in w_k + \sigma U_E$. Then

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{T(a_k, b_k) + 2c_E \sigma U_E\}.$$

By the density assumption, there are $u_k \in D$, $v_k \in V$ such that

$$\begin{aligned} \|a_k - u_k\|_A &\leq \frac{\sigma}{2c_E \|T\|_{A\times B,E}} \text{ and} \\ \|b_k - v_k\|_B &\leq \frac{\sigma}{2c_E c_A \|T\|_{A\times B,E} \left(1 + \frac{\sigma}{2c_E \|T\|_{A\times B,E}}\right)}. \end{aligned}$$

Hence

$$\|u_k\|_A \leq c_A(\|u_k - a_k\|_A + \|a_k\|_A) \leq c_A \left(\frac{\sigma}{2c_E \|T\|_{A\times B,E}} + 1 \right)$$

and so

$$\begin{aligned} \|T(a_k, b_k) - T(u_k, v_k)\|_E &= \|T(a_k - u_k, b_k) + T(u_k, b_k - v_k)\|_E \\ &\leq c_E \|T\|_{A\times B,E} (\|a_k - u_k\|_A \|b_k\|_B + \|u_k\|_A \|b_k - v_k\|_B) \\ &\leq \sigma. \end{aligned}$$

It follows that

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{T(u_k, v_k) + c_E(2c_E + 1)\sigma U_E\}.$$

Let $C = 2c_Z(Mc_E(2c_E + 1) + 1)$ and let $N \in \mathbb{N}$ such that for any $n \geq N$ we have $\|S_n T(u_k, v_k)\|_Z \leq \sigma$ for any $1 \leq k \leq s$. Given any $(a, b) \in U_A \times U_B$, we can find k such that $\|T(a, b) - T(u_k, v_k)\|_E \leq c_E(2c_E + 1)\sigma$. Therefore, we obtain

$$\begin{aligned} \|S_n T(a, b)\|_Z &\leq c_Z (\|S_n(T(a, b) - T(u_k, v_k))\|_Z + \|S_n T(u_k, v_k)\|_Z) \\ &\leq c_Z (Mc_E(2c_E + 1) + 1)\, \sigma \\ &= C\sigma/2. \end{aligned}$$

This yields that $\|S_n T\|_{A\times B,Z} \le C\sigma/2$ for $n \ge N$.

If $\beta = 0$, we derive that $\lim_{n\to\infty} \|S_n T\|_{A\times B,E} = 0$. If $\beta > 0$, the choice $\sigma = 2\beta$ gives that $\|S_n T\|_{A\times B,Z} \le C\beta$ for any $n \ge N$. $\square$

When $\beta\,(T : A \times B \to E) = 0$ then Lemma 3.1 coincides with [12, Lemma 3.2].

In what follows, we shall work with spaces of vector-valued sequences. Let $0 < q \le \infty$, let (λ_m) be a sequence of positive numbers and let (W_m) be a sequence of quasi-Banach spaces with the same constant in the quasi-triangle inequality for all W_m. We put

$$\ell_q(\lambda_m W_m) = \{w = (w_m) : \ w_m \in W_m \text{ and } (\lambda_m \|w_m\|_{W_m})_{m\in\mathbb{Z}} \in \ell_q\}.$$

We endow $\ell_q(\lambda_m W_m)$ with the quasi-norm $\|w\|_{\ell_q(\lambda_m W_m)} = \left\|(\lambda_m \|w_m\|_{W_m})\right\|_{\ell_q}$. The space $\Gamma(\lambda_m W_m)$ is defined similarly.

Given a quasi-Banach couple $\bar{E} = (E_0, E_1)$, let $W_m = (E_0 + E_1, K\,(2^m, \cdot))$. We denote by ι the linear operator assigning to any $w \in E_0 + E_1$, the sequence $\iota w = (..., w, w, w, ...)$ with all co-ordinates equal to w. For $j = 0, 1$, it is easy to check that $\iota : E_j \to \ell_\infty(2^{-mj} W_m)$ is bounded with norm less than or equal to 1.

The following result is related with [12, Lemma 3.3] but now we allow that $T : A_j \times B_j \longrightarrow E_j$ might not be compact.

Lemma 3.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples and let $T : \bar{A} \times \bar{B} \to \bar{E}$. Fix $j \in \{0, 1\}$ and put $\beta_j = \beta\,(T : A_j \times B_j \to E_j)$. Assume that there are quasi-Banach spaces X, Y and bounded linear operators $R_n \in \mathcal{L}\,(X, A_j)$, $S_n \in \mathcal{L}\,(Y, B_j)$ such that $\|R_n\|_{X,A_j} \le 1$, $\|S_n\|_{Y,B_j} \le 1$ and $\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y, E_0+E_1} = 0$. Then the following holds.*

a) If $\beta_j = 0$, then there is a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj} W_m)} = 0.$$

b) If $\beta_j > 0$, then there is a constant C independent of T and a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj} W_m)} \le C\beta_j.$$

Proof. Since $\sup_{n\in\mathbb{N}} \|\iota T(R_n, S_n)\|_{X\times Y, \ell_\infty(2^{-mj} W_m)} \le \|T\|_j < \infty$, there exists a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj} W_m)} = \lambda \ge 0.$$

Let $(x_{n'}) \subseteq U_X$, $(y_{n'}) \subseteq U_Y$ so that

$$\|\iota T(R_{n'} x_{n'}, S_{n'} y_{n'})\|_{\ell_\infty(2^{-mj} W_m)} \to \lambda \text{ as } n' \to \infty.$$

Take any $\sigma > \beta_j$. There exists a finite set $\{z_1, ..., z_s\} \subseteq E_j$ such that

$$T(U_{A_j}, U_{B_j}) \subseteq \bigcup_{k=1}^{s} \{z_k + \sigma U_{E_j}\}.$$

Passing to another subsequence if necessary that we continue denoting by (n'), we may find $k \in [1, s]$ such that

$$T(R_{n'}x_{n'}, S_{n'}y_{n'}) \in z_k + \sigma U_{E_j} \text{ for all } n'. \tag{3.1}$$

Now we estimate the quasi-norm of $\iota(z_k)$ in $\ell_\infty(2^{-mj}W_m)$. Take any $m \in \mathbb{Z}$. Using that $\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y, E_0+E_1} = 0$, we can find n' belonging to the subsequence and sufficiently large so that

$$2^{-jm} \max(1, 2^m) \|T(R_{n'}, S_{n'})\|_{X\times Y, E_0+E_1} \leq \sigma.$$

Whence,

$$\begin{aligned}
2^{-jm} K(2^m, z_k) &\leq c_{\bar{E}}\Big(2^{-jm} K(2^m, z_k - T(R_{n'}x_{n'}, S_{n'}y_{n'})) \\
&\qquad + 2^{-mj} K(2^m, T(R_{n'}x_{n'}, S_{n'}y_{n'}))\Big) \\
&\leq c_{\bar{E}}\Big(\|z_k - T(R_{n'}x_{n'}, S_{n'}y_{n'})\|_{E_j} \\
&\qquad + 2^{-jm} \max(1, 2^m) \|T(R_{n'}, S_{n'})\|_{X\times Y, E_0+E_1}\Big) \\
&\leq 2c_{\bar{E}}\sigma.
\end{aligned}$$

This yields that $\|\iota z_k\|_{\ell_\infty(2^{-mj}W_m)} \leq 2c_{\bar{E}}\sigma$. Consequently, using that $\|\iota\|_{E_j, \ell_\infty(2^{-mj}W_m)} \leq 1$ and (3.1), we obtain with $C = 2c_{\bar{E}}(1 + 2c_{\bar{E}})$ that

$$\begin{aligned}
&\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj}W_m)} \\
&\quad \leq c_{\bar{E}}\Big(\|\iota T(R_{n'}x_{n'}, S_{n'}y_{n'}) - \iota z_k\|_{\ell_\infty(2^{-mj}W_m)} + \|\iota z_k\|_{\ell_\infty(2^{-mj}W_m)}\Big) \\
&\quad \leq C\sigma/2.
\end{aligned}$$

If $\beta_j = 0$, it follows that $\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj}W_m)} = 0$. If $\beta_j > 0$, then taking $\sigma = 2\beta_j$ we conclude that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})\|_{X\times Y, \ell_\infty(2^{-mj}W_m)} \leq C\beta_j.$$

□

Given $n \in \mathbb{N}$, if $x = (x_k)_{k=-n}^{n} \in \mathbb{R}^{2n+1}$ we write $\tilde{x} = \sum_{k=-n}^{n} x_k e_k$, where $e_k = \left(\delta_m^k\right)_{m\in\mathbb{Z}}$ and δ_m^k is the Kronecker delta. If Γ is a quasi-Banach sequence lattice and $\|\cdot\|_\Gamma$ is a p-norm, then the functional $\|x\|_{\tilde{\Gamma}} = \|\tilde{x}\|_\Gamma$ defines a p-norm on $\mathbb{R}^{2n+1}$. It is not hard to check that $\|\cdot\|_{\tilde{\Gamma}}$ is equivalent to $\|x\|_p = \left(\sum_{k=-n}^{n} |x_k|^p\right)^{1/p}$ on $\mathbb{R}^{2n+1}$ and that $U_{\left(\mathbb{R}^{2n+1},\ \|\cdot\|_{\tilde{\Gamma}}\right)}$ is compact in $\left(\mathbb{R}^{2n+1},\ \|\cdot\|_{\tilde{\Gamma}}\right)$. This yields that for any quasi-Banach sequence lattice Γ and for any $\varepsilon > 0$, there exists an ε-net for $U_{\left(\mathbb{R}^{2n+1},\ \|\cdot\|_{\tilde{\Gamma}}\right)}$. That is to say, there is a finite set $\{v_1, ..., v_s\} \subseteq \mathbb{R}^{2n+1}$ such that for any $x \in U_{\left(\mathbb{R}^{2n+1},\ \|\cdot\|_{\tilde{\Gamma}}\right)}$ we have

$$\min_{1\leq k\leq s} \|x - v_k\|_{\tilde{\Gamma}} \leq \varepsilon. \tag{3.2}$$

This remark will be useful in the proof of the next theorem, which is the main result of the paper.

Theorem 3.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be p-normed quasi-Banach couples $(0 < p \leq 1)$, let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \leq 1)$ and let $\Gamma_0, \Gamma_1, \Gamma_2$ be quasi-Banach sequence lattices. We assume that Γ_0, Γ_1 satisfy (2.1) and (2.2) and that Γ_2 satisfies (2.1) with parameter r. Suppose also that the sequence spaces satisfy the condition (2.5) on convolutions. Let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and put $\beta_j = \beta\,(T : A_j \times B_j \longrightarrow E_j)$, $j = 0, 1$. Then*

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}\right) \leq \begin{cases} 0 & \text{if} \quad \beta_j = 0 \text{ for } j = 0 \text{ or } 1, \\ C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) & \text{otherwise.} \end{cases} \tag{3.3}$$

Here C is a constant independent of T.

Proof. *Step 1.* Since $\bar{A}$ and $\bar{B}$ are p-normed, the spaces $F_m = (A_0 \cap A_1, J\,(2^m, \cdot; A_0, A_1))$ and $G_m = (B_0 \cap B_1, J\,(2^m, \cdot; B_0, B_1))$ are also p-normed for each $m \in \mathbb{Z}$. Consider the couples

$$\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m}F_m)),\ \bar{G}_p = (\ell_p(G_m), \ell_p(2^{-m}G_m)).$$

According to [12, Lemma 2.4], we have with equivalence of quasi-norms

$$(\ell_p(F_m), \ell_p(2^{-m}F_m))_{\Gamma_0} = \Gamma_0(F_m),\ (\ell_p(G_m), \ell_p(2^{-m}G_m))_{\Gamma_1} = \Gamma_1(G_m). \tag{3.4}$$

Let $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ be the linear operator assigning to any sequence (u_m) its sum in A_0+A_1. Realizing $\bar{A}_{\Gamma_0}$ by means of the J-functional, the map $\pi : \Gamma_0(F_m) \to \bar{A}_{\Gamma_0}$ is bounded and for any $a \in \bar{A}_{\Gamma_0}$ with $\|a\|_{\bar{A}_{\Gamma_0;J}} < 1$ there is $(u_m) \in \Gamma_0(F_m)$ with $\|(u_m)\|_{\Gamma_0(F_m)} < 1$ such that $\pi(u_m) = a$. Moreover, $\pi : \ell_p(2^{-mj}F_m) \to A_j$ is bounded with norm less than or equal 1 for $j = 0, 1$. Similar properties hold for $\pi : \Gamma_1(G_m) \to \bar{B}_{\Gamma_1}$ and $\pi : \ell_p(2^{-mj}G_m) \to B_j$.

As for the r-normed couple (E_0, E_1), put $W_m = (E_0+E_1, K(2^m, \cdot; E_0, E_1))$, consider the couple $\overline{W}_\infty = (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ and the linear operator $\iota w = (..., w, w, w, ...)$ introduced before Lemma 3.2. If we realize $\bar{E}_{\Gamma_2}$ by means of the K-functional, then $\iota : \bar{E}_{\Gamma_2} \to \Gamma_2(W_m)$ is bounded with $\|\iota w\|_{\Gamma_2(W_m)} = \|w\|_{\bar{E}_{\Gamma_2;K}}$. Moreover, if we consider $\iota : E_j \to \ell_\infty(2^{-mj}W_m)$

then its norm is less than or equal to 1 for $j = 0, 1$, and the following interpolation formula holds $(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{\Gamma_2} = \Gamma_2(W_m)$ (see [12, Lemma 2.4]).

The diagram which illustrates the situation is

$$\begin{array}{ccccccc} \ell_p(F_m) \times \ell_p(G_m) & \xrightarrow{(\pi,\pi)} & A_0 \times B_0 & \xrightarrow{T} & E_0 & \xrightarrow{\iota} & \ell_\infty(W_m) \\ \ell_p(2^{-m}F_m) \times \ell_p(2^{-m}G_m) & \xrightarrow{(\pi,\pi)} & A_1 \times B_1 & \xrightarrow{T} & E_1 & \xrightarrow{\iota} & \ell_\infty(2^{-m}W_m) \\ \hline \Gamma_0(F_m) \times \Gamma_1(G_m) & \xrightarrow{(\pi,\pi)} & \overline{A}_{\Gamma_0} \times \overline{B}_{\Gamma_1} & \xrightarrow{T} & \overline{E}_{\Gamma_2} & \xrightarrow{\iota} & \Gamma_2(W_m). \end{array}$$

Put $\widehat{T} = \iota T(\pi, \pi)$. Then $\widehat{T} : \overline{F}_p \times \overline{G}_p \to \overline{W}_\infty$.

According to (v) and (vi) and properties of π and ι, we get

$$\begin{aligned} \beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) &\le 2c_{\bar{E}}c_{\Gamma_2}\beta\left(\iota T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \Gamma_2(W_m)\right) \\ &\le 2c_{\bar{E}}c_{\Gamma_2}\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right). \end{aligned} \tag{3.5}$$

It is easier to estimate $\beta(\widehat{T})$ than $\beta(T)$ because on the couples $\overline{F}_p$, $\overline{G}_p$, $\overline{W}_\infty$ we can use the following families of projections: For $n \in \mathbb{N}$, let

$$\begin{aligned} R_n(u_m) &= (..., 0, 0, u_{-n}, u_{-n+1}, ..., u_{n-1}, u_n, 0, 0, ...), \\ R_n^+(u_m) &= (..., 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, ...), \\ R_n^-(u_m) &= (..., u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, ...). \end{aligned}$$

It is clear that the identity operator I on $\ell_p(F_m) + \ell_p(2^{-m}F_m)$ can be decomposed as $I = R_n + R_n^+ + R_n^-$, $n \in \mathbb{N}$. These projections are bounded from $\ell_p(2^{-mj}F_m)$ into $\ell_p(2^{-mj}F_m)$ with norm less than or equal to 1 for $j = 0, 1$, and the same happens on $\Gamma_0(F_m)$. Moreover, the restrictions $R_n : \ell_p(F_m) + \ell_p(2^{-m}F_m) \to \ell_p(F_m) \cap \ell_p(2^{-m}F_m)$, $R_n^+ : \ell_p(F_m) \to \ell_p(2^{-m}F_m)$ and $R_n^- : \ell_p(2^{-m}F_m) \to \ell_p(F_m)$ are bounded with

$$\begin{aligned} &\|R_n\|_{\ell_p(F_m)+\ell_p(2^{-m}F_m), \ell_p(F_m)\cap\ell_p(2^{-m}F_m)} \le c_{\bar{A}}2^{1/p}2^n, \\ &\left\|R_n^+\right\|_{\ell_p(F_m), \ell_p(2^{-m}F_m)} = 2^{-(n+1)} = \left\|R_n^-\right\|_{\ell_p(2^{-m}F_m), \ell_p(F_m)}. \end{aligned} \tag{3.6}$$

Let S_n, S_n^+, S_n^- and P_n, P_n^+, P_n^- similar sequences of projections defined on the couples $\overline{G}_p$, $\overline{W}_\infty$, respectively. They satisfy the corresponding version of (3.6).

Having in mind (3.5), in order to prove (3.3) it suffices to show that if $\beta_j > 0$ for $j = 0$ and $j = 1$, then there is a constant C independent of T such that for any $\varepsilon > 0$ we have

$$\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \le C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) + \varepsilon,$$

and if $\beta_j = 0$ for $j = 0$ or $j = 1$, then

$$\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) = 0.$$

With this aim, for $n \in \mathbb{N}$ we decompose $\widehat{T}$ as

$$\begin{aligned}
\widehat{T} &= P_{3n}\widehat{T} + P_{3n}^{+}\widehat{T} + P_{3n}^{-}\widehat{T} = P_{3n}\widehat{T}(R_{4n}, S_{4n}) \\
&+ P_{3n}\widehat{T}(R_{4n}, S_{4n}^{+}) + P_{3n}\widehat{T}(R_{4n}, S_{4n}^{-}) + P_{3n}\widehat{T}(R_{4n}^{+}, S_{4n} + S_{4n}^{+}) \\
&+ P_{3n}\widehat{T}(R_{4n}^{-}, S_{4n} + S_{4n}^{-}) + P_{3n}\widehat{T}(R_{4n}^{+}, S_{4n}^{-}) \\
&+ P_{3n}\widehat{T}(R_{4n}^{-}, S_{4n}^{+}) + P_{3n}^{+}\widehat{T} + P_{3n}^{-}\widehat{T}. \qquad (3.7)
\end{aligned}$$

Step 2. Now we proceed to give a direct estimate for the measure of non-compactness of the operator $P_{3n}\widehat{T}(R_{4n}, S_{4n})$. First note that we have by (v) that

$$\begin{aligned}
&\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \\
&\quad \le \beta\left(T(\pi R_{4n}, \pi S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \bar{E}_{\Gamma_2}\right) \\
&\quad \le c\beta\left(T(\pi R_{4n}, \pi S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \bar{E}_{\Gamma_2;J}\right)
\end{aligned}$$

where the last target space is provided with the J-quasi-norm.

Consider on $\mathbb{R}^{8n+1}$ the quasi-norms

$$\|x\|_{\tilde{\Gamma}_j} = \| \sum_{k=-4n}^{4n} x_k e_k \|_{\Gamma_j} = \|(\dots,0,0,x_{-4n},\dots,x_{4n},0,0,\dots)\|_{\Gamma_j}\,,\ j = 0,1,$$

for $x = (x_k)_{k=-4n}^{4n}$. Let $\eta = \left(\max_{j=0,1}\left\|\sum_{k=-4n}^{4n} e_k/\left\|e_k\right\|_{\Gamma_j}\right\|_{\Gamma_j}\right)^{-1}$. By (3.2), there exists a finite η-net for $U_{(\mathbb{R}^{8n+1},\|\cdot\|_{\tilde{\Gamma}_0})}$. That is, there is a finite set $\Lambda_0 = \{\lambda^1, \dots, \lambda^s\} \subseteq U_{(\mathbb{R}^{8n+1},\|\cdot\|_{\tilde{\Gamma}_0})}$ such that for any $x \in \mathbb{R}^{8n+1}$ with $\|x\|_{\tilde{\Gamma}_0} \le 1$ we can find $\lambda^d \in \Lambda_0$ with $\left\|x - \lambda^d\right\|_{\tilde{\Gamma}_0} \le \eta$. Similarly, let $\Lambda_1 = \{\mu^1, \dots, \mu^t\} \subseteq U_{(\mathbb{R}^{8n+1},\|\cdot\|_{\tilde{\Gamma}_1})}$ be an η-net for $U_{(\mathbb{R}^{8n+1},\|\cdot\|_{\tilde{\Gamma}_1})}$. We can associate to each $\lambda^d = (\lambda_k^d)_{k=-4n}^{4n}$ the positive numbers

$$\varphi_k^j = \varphi_{k,\lambda^d}^j = \left(\frac{\eta}{\|e_k\|_{\Gamma_0}} + |\lambda_k^d|\right) 2^{-kj},\ j = 0,1.$$

In a parallel way, we associate to each $\mu^z = (\mu_k^z)_{k=-4n}^{4n} \in \Lambda_1$ the positive numbers

$$\psi_k^j = \psi_{k,\mu^z}^j = \left(\frac{\eta}{\|e_k\|_{\Gamma_1}} + |\mu_k^z|\right) 2^{-kj},\ j = 0,1.$$

Let $\sigma_0 > \beta_0$, $\sigma_1 > \beta_1$ and choose $N \in \mathbb{Z}$ such that $2^N \le \sigma_1/\sigma_0 < 2^{N+1}$. There are finite sets

$$\Delta_0 = \{h_l : l = 1, ..., L_0\} \subseteq E_0,\ \Delta_1 = \{f_y : y = 1, ..., L_1\} \subseteq E_1$$

such that

$$T(U_{A_0}, U_{B_0}) \subseteq \bigcup_{l=1}^{L_0} \{h_l + \sigma_0 U_{E_0}\},\ T(U_{A_1}, U_{B_1}) \subseteq \bigcup_{y=1}^{L_1} \{f_y + \sigma_1 U_{E_1}\}.$$

Take any $\lambda^d \in \Lambda_0$, $\mu^z \in \Lambda_1$, $h_l \in \Delta_0$ and $f_y \in \Delta_1$. For any $-4n \le k, s \le 4n$, take an element $g_{k,s} = g_{k,s,\lambda^d,\mu^z,h_l,f_y}$ belonging to

$$\left(\varphi_k^0 \psi_s^0 \{h_l + \sigma_0 U_{E_0}\}\right) \cap \left(\varphi_k^1 \psi_s^1 \{f_y + \sigma_1 U_{E_1}\}\right) \tag{3.8}$$

provided the intersection is non-empty. Put $g_{k,s} = 0$ if (3.8) is empty. Let

$$\bar{g}_{k,s} = \begin{cases} g_{k,s} & \text{if} \quad k \in [-4n, 4n] \text{ and } s \in [-4n, 4n], \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{Z}$, put $\xi_m = \sum_{k=-\infty}^{\infty} \bar{g}_{k,m+N-k}$. This series is convergent, with $\xi_m \in E_0 \cap E_1$ and $\xi_m = 0$ if $m \notin [-8n - N, 8n - N]$. Put $\xi = \sum_{m=-\infty}^{\infty} \xi_m$. Then $\xi \in E_0 \cap E_1 \subseteq \bar{E}_{\Gamma_2;J}$. Let Υ be the collection of all elements ξ as constructed above. The set Υ is finite because $\Lambda_0, \Lambda_1, \Delta_0$ and Δ_1 are finite. Next we show that there is a constant L independent of T such that Υ is an $L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0)$-net for $T(U_{\Gamma_0(F_m)}, U_{\Gamma_1(G_m)})$ in $\bar{E}_{\Gamma_2}$.

Given any $u = (u_m) \in U_{\Gamma_0(F_m)}$, $v = (v_m) \in U_{\Gamma_1}(G_m)$, there exists $\lambda^d = (\lambda_k) \in \Lambda_0$, $\mu^z = (\mu_k) \in \Lambda_1$ such that for $k = -4n, ..., 4n$ we have

$$\begin{aligned} |J(2^k, u_k) - \lambda_k| \, \|e_k\|_{\Gamma_0} &\le \|(J(2^m, u_m) - \lambda_m)\|_{\tilde{\Gamma}_0} \le \eta, \\ |J(2^k, v_k) - \mu_k| \, \|e_k\|_{\Gamma_1} &\le \|(J(2^m, v_m) - \mu_m)\|_{\tilde{\Gamma}_1} \le \eta. \end{aligned}$$

Hence,

$$J(2^k, u_k) \le \frac{\eta}{\|e_k\|_{\Gamma_0}} + |\lambda_k|\,,\ J(2^k, v_k) \le \frac{\eta}{\|e_k\|_{\Gamma_1}} + |\mu_k|\,.$$

This yields that

$$\|u_k\|_{A_j} \le \varphi_k^j,\ \|v_k\|_{B_j} \le \psi_k^j,\ j = 0, 1,\ -4n \le k \le 4n.$$

Therefore,

$$u_k \in \varphi_k^0 U_{A_0} \cap \varphi_k^1 U_{A_1},\ v_s \in \psi_s^0 U_{B_0} \cap \psi_s^1 U_{B_1},\ -4n \le k, s \le 4n.$$

We can find $h_l \in \Delta_0$, $f_y \in \Delta_1$ such that

$$\begin{aligned} \left\|T(u_k, v_s) - \varphi_k^0 \psi_s^0 h_l\right\|_{E_0} &\le \varphi_k^0 \psi_s^0 \sigma_0, \\ \left\|T(u_k, v_s) - \varphi_k^1 \psi_s^1 f_y\right\|_{E_1} &\le \varphi_k^1 \psi_s^1 \sigma_1, \end{aligned} \tag{3.9}$$

and so the intersection (3.8) is non-empty. Let $\xi \in \Upsilon$ the vector associated to λ^d, μ^z, h_l and f_y. Put

$$\bar{u}_m = \begin{cases} u_m & \text{if} \quad m \in [-4n, 4n], \\ 0 & \text{if} \quad m \notin [-4n, 4n], \end{cases}$$

define $\bar{v}_m$, $\bar{\varphi}^j_m$, $\bar{\psi}^j_m$ similarly, and write

$$T_k(u, v) = \sum_{m=-\infty}^{\infty} T(\bar{u}_m, \bar{v}_{k+N-m}) \in E_0 \cap E_1, \ k \in \mathbb{Z}.$$

We have

$$T(\pi R_{4n} u, \pi S_{4n} v) = \sum_{k=-\infty}^{\infty} T_k(u, v).$$

Since $\bar{E}$ is r-normed, using (3.9) we get

$$\begin{aligned} J\left(2^k, T_k(u,v) - \xi_k\right) &= J\left(2^k, \sum_{m=-\infty}^{\infty} (T(\bar{u}_m, \bar{v}_{k+N-m}) - \bar{g}_{m,k+N-m})\right) \\ &\leq \left(\sum_{m=-\infty}^{\infty} J\left(2^k, T(\bar{u}_m, \bar{v}_{k+N-m}) - \bar{g}_{m,k+N-m}\right)^r\right)^{1/r} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \max\left(2\bar{\varphi}^0_m \bar{\psi}^0_{k+N-m}\sigma_0, 22^k \bar{\varphi}^1_m \bar{\psi}^1_{k+N-m}\sigma_1\right)^r\right)^{1/r} \\ &\leq 4\sigma_0 \left(\sum_{m=-\infty}^{\infty} \left(\bar{\varphi}^0_m \bar{\psi}^0_{k+N-m}\right)^r\right)^{1/r} \end{aligned}$$

where in the last inequality we have used that $\bar{\varphi}^1_m = 2^{-m}\bar{\varphi}^0_m, \bar{\psi}^1_{k+N-m} = 2^{-k-N+m}\bar{\psi}^0_{k+N-m}$ and that $2^{-N}\sigma_1 < 2\sigma_0$. Consequently, by condition (2.5) on convolutions and definition of f_{Γ_1}, we obtain

$$\begin{aligned} \|T(\pi R_{4n} u, \pi S_{4n} v) - \xi\|_{\bar{E}_{\Gamma_2;J}} &\leq \left\|\left(J(2^k, T_k(u,v) - \xi_k)\right)\right\|_{\Gamma_2} \\ &\leq 4\sigma_0 \left\|\left(\sum_{m=-\infty}^{\infty} \left(\bar{\varphi}^0_m \bar{\psi}^0_{k+N-m}\right)^r\right)^{1/r}\right\|_{\Gamma_2} \\ &\leq 4M\sigma_0 \left\|(\bar{\varphi}^0_m)\right\|_{\Gamma_0} \left\|(\bar{\psi}^0_{m+N})\right\|_{\Gamma_1} \\ &\leq 4M\sigma_0 \|\tau_N\|_{\Gamma_1,\Gamma_1} c_{\Gamma_0} \left(1 + \eta \left\|\sum_{k=-4n}^{4n} \frac{e_k}{\|e_k\|_{\Gamma_0}}\right\|_{\Gamma_0}\right) \\ &\qquad \times c_{\Gamma_1} \left(1 + \eta \left\|\sum_{k=-4n}^{4n} \frac{e_k}{\|e_k\|_{\Gamma_1}}\right\|_{\Gamma_1}\right) \\ &\leq L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0) \end{aligned}$$

where $L = c16Mc_{\Gamma_0}c_{\Gamma_1}$.

It follows that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0).$$

If $\beta_0 = 0$ or $\beta_1 = 0$, then (2.3) implies that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) = 0.$$

Otherwise, the choice $\sigma_j = (1+\varepsilon)\beta_j$ with $\varepsilon > 0$ yields that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L(1+\varepsilon)\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

Letting $\varepsilon \to 0$ we conclude that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

Step 3. Now we show that each one of the other six operators involving P_{3n} in the decomposition (3.7) has norm which tends to 0 as $n \to \infty$. To establish it we will use the norm estimate given by Theorem 2.1 and also the fact that $T : (A_0 + A_1) \times (B_0 + B_1) \to E_0 + E_1$ is bounded. Hence, $T : A_i \times B_j \to E_0 + E_1$ is also bounded for $i = 0, 1$, $j = 0, 1$.

Consider, for example, $P_{3n}\widehat{T}(R^+_{4n}, S_{4n}+S^+_{4n})$. The following commutative diagram holds:

$$\begin{array}{ccc}
\ell_p(F_m) \times \ell_p(G_m) & \xrightarrow{P_{3n}\widehat{T}(R^+_{4n}, S_{4n} + S^+_{4n})} & \ell_\infty(W_m) \\
\Big\downarrow (R^+_{4n}, S_{4n} + S^+_{4n}) & & \Big\uparrow P_{3n} \\
\ell_p(2^{-m}F_m) \times \ell_p(G_m) & \xrightarrow{\widehat{T}} & \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m).
\end{array}$$

Moreover, by (3.6), we know that

$$\begin{aligned}
&\left\|R^+_{4n}\right\|_{\ell_p(F_m),\ell_p(2^{-m}F_m)} \leq 2^{-4n}, \ \left\|S_{4n} + S^+_{4n}\right\|_{\ell_p(G_m),\ell_p(G_m)} \leq 1 \\
&\text{and } \ \|P_{3n}\|_{\ell_\infty(W_m)+\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \leq c_{\bar{E}} 2^{3n}.
\end{aligned}$$

Hence,

$$\left\|P_{3n}\widehat{T}(R^+_{4n}, S_{4n} + S^+_{4n})\right\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)} \leq c_{\bar{E}} 2^{-n} \|T\|_{A_1\times B_0, E_0+E_1} \xrightarrow{n\to\infty} 0.$$

On the other hand,

$$\left\|P_{3n}\widehat{T}(R^+_{4n}, S_{4n} + S^+_{4n})\right\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \leq \|T\|_1 = \|T\|_{A_1\times B_1, E_1}.$$

Using the interpolation formulae (3.4), the corresponding formula for $\Gamma_2(W_m)$ and Theorem 2.1, we conclude that

$$
\begin{aligned}
&\beta\left(P_{3n}\widehat{T}(R_{4n}^+, S_{4n}+S_{4n}^+) : \Gamma_0(F_m)\times\Gamma_1(G_m)\to\Gamma_2(W_m)\right)\\
&\quad\le \left\|P_{3n}\widehat{T}(R_{4n}^+, S_{4n}+S_{4n}^+)\right\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)}\\
&\quad\le C2^{-n}f_{\Gamma_1}(2^n)\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Operators $P_{3n}\widehat{T}(R_{4n}, S_{4n}^+)$, $P_{3n}\widehat{T}(R_{4n}, S_{4n}^-)$, $P_{3n}\widehat{T}(R_{4n}^-, S_{4n}+S_{4n}^-)$, $P_{3n}\widehat{T}(R_{4n}^+, S_{4n}^-)$, $P_{3n}\widehat{T}(R_{4n}^-, S_{4n}^+)$ can be treated similarly.

Step 4. Next we work with the other two operators $P_{3n}^+\widehat{T}$, $P_{3n}^-\widehat{T}$ in the decomposition (3.7). It is convenient to split them as follows

$$
\begin{aligned}
P_{3n}^+\widehat{T}+P_{3n}^-\widehat{T} &= P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-)+P_{3n}^-\widehat{T}(R_n+R_n^+, S_n+S_n^+)\\
&+ P_{3n}^+\widehat{T}(R_n+R_n^-, S_n^+)+P_{3n}^-\widehat{T}(R_n+R_n^+, S_n^-)\\
&+ P_{3n}^+\widehat{T}(R_n^+, I)+P_{3n}^-\widehat{T}(R_n^-, I).
\end{aligned}
$$

Factorization

$$
\begin{array}{ccc}
\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) & \xrightarrow{\ P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-)\ } & \ell_\infty(2^{-m}W_m)\\
\Big\downarrow (R_n+R_n^-, S_n+S_n^-) & & \Big\uparrow P_{3n}^+\\
\ell_p(F_m)\times\ell_p(G_m) & \xrightarrow{\qquad\widehat{T}\qquad} & \ell_\infty(W_m)
\end{array}
$$

and the fact that $\|P_{3n}^+\|_{\ell_\infty(W_m),\ell_\infty(2^{-m}W_m)}\le 2^{-3n}$, $\|R_n+R_n^-\|_{\ell_p(2^{-m}F_m),\ell_p(F_m)}\le 2^n$ and $\|S_n+S_n^-\|_{\ell_p(2^{-m}G_m),\ell_p(G_m)}\le 2^n$ yields that

$$
\begin{aligned}
&\left\|P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-)\right\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}\\
&\qquad\le 2^{2n}2^{-3n}\,\|T\|_0\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Since

$$
\left\|P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-)\right\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)}\le\|T\|_0\,,
$$

it follows from Theorem 2.1 and properties of f_{Γ_1} that

$$
\begin{aligned}
&\beta\left(P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-) : \Gamma_0(F_m)\times\Gamma_1(G_m)\to\Gamma_2(W_m)\right)\\
&\quad\le\left\|P_{3n}^+\widehat{T}(R_n+R_n^-, S_n+S_n^-)\right\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)}\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

With the operator $P_{3n}^{-}\widehat{T}(R_n + R_n^{+}, S_n + S_n^{+})$ we can proceed in a similar way.

For the four remaining operators we shall need Lemmata 3.1 and 3.2. In applications of Lemma 3.1, as dense subspace of $\ell_p(2^{-mj}F_m)$ (respectively, $\ell_p(2^{-mj}G_m)$) for $j = 0, 1$, we take the subspace of all sequences having only a finite number of co-ordinates different from 0. Besides, if $S : \bar{F}_p \times \bar{G}_p \to \bar{W}_\infty$, we put

$$\|S\|_j = \|S\|_{\ell_p(2^{-mj}F_m)\times\ell_p(2^{-mj}G_m),\ell_\infty(2^{-mj}W_m)}\ ,\ j = 0, 1\,.$$

Consider $P_{3n}^{+}\widehat{T}(R_n + R_n^{-}, S_n^{+})$. Factorization

$$\begin{array}{ccc}
\ell_p(F_m) \times \ell_p(G_m) & \xrightarrow{\quad T(\pi(R_n + R_n^{-}), \pi S_n^{+}) \quad} & E_0 + E_1 \\
\Big\downarrow {\scriptstyle (R_n + R_n^{-}, S_n^{+})} & & \Big\uparrow {\scriptstyle T} \\
\ell_p(F_m) \times \ell_p(2^{-m}G_m) & \xrightarrow{\quad (\pi, \pi) \quad} & A_0 \times B_1
\end{array}$$

shows that

$$\left\|T(\pi(R_n + R_n^{-}), \pi S_n^{+})\right\|_{\ell_p(F_m)\times\ell_p(G_m),E_0+E_1} \leq 2^{-n}\, \|T\|_{A_0\times B_1,E_0+E_1} \xrightarrow{n\to\infty} 0.$$

Since

$$\|P_{3n}^{+}\widehat{T}(R_n + R_n^{-}, S_n^{+})\|_0 \leq \|\widehat{T}(R_n + R_n^{-}, S_n^{+})\|_0,$$

it follows from Lemma 3.2 that there are a constant C_1 independent of T, a subsequence (n') and $N_1 \in \mathbb{N}$ such that for any $n' \geq N_1$ we have

$$\|P_{3n'}^{+}\widehat{T}(R_{n'} + R_{n'}^{-}, S_{n'}^{+})\|_0 \leq C_1\beta_0 \tag{3.10}$$

provided that $\beta_0 > 0$. If $\beta_0 = 0$, we obtain that

$$\lim_{n'\to\infty} \|P_{3n'}^{+}\widehat{T}(R_{n'} + R_{n'}^{-}, S_{n'}^{+})\|_0 = 0.$$

On the other hand, if $u \in \ell_p(2^{-m}F_m)$ and $v \in \ell_p(2^{-m}G_m)$ are sequence with only a finite number of co-ordinates different from 0, we have

$$\|P_{3n}^{+}\widehat{T}(u, v)\|_{\ell_\infty(2^{-m}W_m)} \leq 2^{-3n}\|\widehat{T}(u, v)\|_{\ell_\infty(W_m)} \to 0 \ \text{ as } \ n \to \infty.$$

Moreover,

$$\|P_{3n}^{+}\widehat{T}(R_n + R_n^{-}, S_n^{+})\|_1 \leq \|P_{3n}^{+}\widehat{T}\|_1.$$

Hence, applying Lemma 3.1, we obtain that there is a constant C_2 independent of T and $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$

$$\|P_{3n}^{+}\widehat{T}(R_n + R_n^{-}, S_n^{+})\|_1 \leq C_2\beta_1 \tag{3.11}$$

provided that $\beta_1 > 0$. If $\beta_1 = 0$, then we get that

$$\lim_{n\to\infty} \|P^+_{3n}\widehat{T}(R_n + R^-_n, S^+_n)\|_1 = 0.$$

Put $L = \max\{C_1, C_2\}$ and take any n' from the subsequence with $n' \geq \max\{N_1, N_2\}$. If $\beta_j > 0$ for $j = 0, 1$, it follows from Theorem 2.1 and estimates (3.10), (3.11) that

$$\begin{aligned}
\beta &\left(P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \\
&\leq \|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)} \\
&\leq C\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_0 f_{\Gamma_1}\left(\frac{\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_1}{\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_0}\right) \\
&\leq CL\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).
\end{aligned}$$

If $\beta_j = 0$ for $j = 0$ or $j = 1$, then we obtain

$$\beta\left(P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \xrightarrow{n'\to\infty} 0.$$

Proceeding similarly, an analogous conclusion holds for each one of the operators $P^-_{3n}\widehat{T}(R_n + R^+_n, S^-_n)$, $P^+_{3n}\widehat{T}(R^+_n, I)$ and $P^-_{3n}\widehat{T}(R^-_n, I)$.

Step 5. Having in mind (3.5), (3.7) and collecting the estimates in the previous steps, if $\beta_j > 0$ for $j = 0, 1$, then we conclude that there is a constant $C > 0$ independent of T such that for any $\varepsilon > 0$ we can decompose the operator by (3.7) with $n = n'$ belonging to the subsequence appeared in Step 4 and being sufficiently large, with the result that

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) \leq C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) + \varepsilon.$$

Consequently,

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) \leq C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

If $\beta_j = 0$ for $j = 0$ or $j = 1$, then we derive that

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) = 0.$$

This finishes the proof. □

For the case of the real method with a function parameter (Example 2.3), we obtain the following result.

Theorem 3.4. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple ($0 < r \leq 1$). Suppose that ρ_0, ρ_1, ρ_2 are function parameters such that for some constant L we have*

$$\rho_0(t)\rho_1(s) \leq L\rho_2(ts), \ \ t, s > 0. \tag{3.12}$$

Let $0 < q_0, q_1 \leq \infty$ *and write*

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r, \\ \frac{1}{\max(q_0,q_1)} & \text{if} \quad q_0 < r \text{ or } q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ *and* $\beta_j = \beta\left(T : A_j \times B_j \longrightarrow E_j\right)$, $j = 0, 1$, *then we have:*

a) $\beta\left(T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1,q_1} \longrightarrow (E_0, E_1)_{\rho_2,q}\right) = 0$, *if* $\beta_0 = 0$ *or* $\beta_1 = 0$.

b) $\beta\left(T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1,q_1} \longrightarrow (E_0, E_1)_{\rho_2,q}\right) \leq C\beta_0 s_{\rho_1}(\beta_1/\beta_0)$ *if* $\beta_0 > 0$ *and* $\beta_1 > 0$.

Here C *is a constant independent of* T.

Proof. Proceeding as in [12, Theorem 4.8], using (3.12) and Young's inequality, one can check that assumptions of Theorem 3.3 hold. Having in mind that we can replace $f_{\ell_q(1/\rho_1(2^m))}$ by s_{ρ_1}, the result follows from (3.3). □

For the case of the real method, that is, when $\rho_0(t) = \rho_1(t) = \rho_2(t) = t^\theta$ with $0 < \theta < 1$, we get the following result.

Theorem 3.5. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *be quasi-Banach couples and let* $\bar{E} = (E_0, E_1)$ *be an* r*-normed quasi-Banach couple* ($0 < r \leq 1$)*. Let* $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ *and put*

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r, \\ \frac{1}{\max(q_0,q_1)} & \text{if} \quad q_0 < r \text{ or } q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ *and* $\beta_j = \beta\left(T : A_j \times B_j \longrightarrow E_j\right)$, $j = 0, 1$, *then we have that*

$$\beta\left(T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q}\right) \leq C\beta_0^{1-\theta}\beta_1^{\theta}.$$

Here C *is a constant independent of* T.

Theorems 3.3, 3.4 and 3.5 refine the compactness result for bilinear operators established in [12, Theorems 4.7, 4.8 and 4.9].

When $\bar{A}, \bar{B}, \bar{E}$ are Banach couples, so $r = 1$, and $1 \leq q_0, q_1, q \leq \infty$ with $1/q = 1/q_0 + 1/q_1 - 1$, Theorem 3.5 includes [37, Theorem 3.2] and shows that the estimate for the measure of non-compactness holds in any of the cases $q_0 = \infty, q_1 = \infty, q = \infty$ or $q = 1$, cases which have not been studied in [37].

Remark 3.6. Observe that the assumptions on the quasi-Banach sequence lattices, function parameters and scalar parameters in Theorems 3.3, 3.4 and 3.5 are the same as in the corresponding interpolation theorems for bounded bilinear operators (see, for example, [12, Theorem 3.1], [27, Corollary 3.2], [36, Théorème 4.1] and [47, 1.19.5]).

Remark 3.7. In the assumptions of Theorem 3.3, if $\beta_j > 0$ for $j = 0, 1$, then it also holds

$$\beta(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}) \leq C\beta_0 f_{\Gamma_0}(\beta_1/\beta_0)\,. \tag{3.13}$$

This follows by applying Theorem 3.3 to the operator

$$\widetilde{T} : (B_0 + B_1) \times (A_0 + A_1) \longrightarrow (E_0 + E_1)\,, \quad \widetilde{T}(b, a) = T(a, b)\,,$$

exchanging the roles of $\bar{A}$ and $\bar{B}$, and of Γ_0 and Γ_1.

Estimates (3.3) and (3.13) are not comparable as we show next with an example.

Let $\bar{A}, \bar{B}, \bar{E}, r, q_0, q_1, q$ as in the statement of Theorem 3.4. Assume that $0 < \theta < 1, -\infty < \alpha_2 < 0 < \alpha_0, \alpha_1 < \infty$ and put $\rho_k(t) = t^\theta(1 + |\log t|)^{-\alpha_k}$ for $k = 0, 1, 2$. Then (3.12) is satisfied. Since $s_{\rho_k}(t) = t^\theta(1 + |\log t|)^{|\alpha_k|}$, Theorem 3.4 (or Theorem 3.3) yields

$$\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) \leq C\beta_0^{1-\theta}\beta_1^\theta(1 + |\log(\beta_1/\beta_0)|)^{\alpha_1} \tag{3.14}$$

while it follows from (3.13) that

$$\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) \leq C\beta_0^{1-\theta}\beta_1^\theta(1 + |\log(\beta_1/\beta_0)|)^{\alpha_0}\,. \tag{3.15}$$

Therefore, if $\alpha_1 < \alpha_0$ then it is clear that (3.14) is a better estimate than (3.15), while if $\alpha_0 < \alpha_1$ then (3.15) is better than (3.14).

In the special case when we have equality in (3.12), i.e. $\rho_0(t)\rho_1(s) = L\rho_2(ts), t, s > 0$, then we have that $s_{\rho_0} = s_{\rho_1}$ and the estimates coincide. This is the case in the assumptions of Theorem 3.5.

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