

Well posedness of an angiogenesis related integrodifferential diffusion model

Ana Carpio

Departamento de Matematica Aplicada, Universidad Complutense, 28040 Madrid, Spain

Gema Duro¹

Departamento de Análisis Económico: Economía Cuantitativa, Universidad Autónoma de Madrid, 28049 Madrid, Spain

Abstract

We prove existence and uniqueness of nonnegative solutions for a nonlocal in time integrodifferential diffusion system related to angiogenesis descriptions. Fundamental solutions of appropriately chosen parabolic operators with bounded coefficients allow us to generate sequences of approximate solutions. Comparison principles and integral equations provide uniform bounds ensuring some convergence properties for iterative schemes and providing stability bounds. Uniqueness follows from chained integral inequalities.

Keywords: Integrodifferential, diffusion, nonlocal, fundamental solutions.

1. Introduction

Models for angiogenesis (blood vessel generation) have a complex mathematical structure, involving integral terms, transport operators, degenerate diffusion and eventually measure valued coefficients. In [5], for instance, the following model for the dynamics of the density of blood vessels is used to describe tumor

Email addresses: carpio@mat.ucm.es (Ana Carpio), gema.duro@uam.es (Gema Duro)

¹Corresponding author

induced angiogenesis:

$$\begin{aligned} \frac{\partial}{\partial t} p(t, \mathbf{x}, \mathbf{v}) &= \alpha(c(t, \mathbf{x})) \rho(\mathbf{v}) p(t, \mathbf{x}, \mathbf{v}) - \gamma p(t, \mathbf{x}, \mathbf{v}) \int_0^t ds \tilde{p}(s, \mathbf{x}) \\ &\quad - \mathbf{v} \cdot \nabla_{\mathbf{x}} p(t, \mathbf{x}, \mathbf{v}) + k \nabla_{\mathbf{v}} \cdot (\mathbf{v} p(t, \mathbf{x}, \mathbf{v})) \\ &\quad - \nabla_{\mathbf{v}} \cdot [\mathbf{F}(c(t, \mathbf{x})) p(t, \mathbf{x}, \mathbf{v})] + \sigma \Delta_{\mathbf{v}} p(t, \mathbf{x}, \mathbf{v}), \end{aligned} \quad (1)$$

$$\alpha(c) = \alpha_1 \frac{\frac{c}{c_R}}{1 + \frac{c}{c_R}} \geq 0, \quad \mathbf{F}(c) = \frac{d_1}{(1 + \gamma_1 c)^{q_1}} \nabla c, \quad (2)$$

$$\frac{\partial}{\partial t} c(t, \mathbf{x}) = d \Delta_{\mathbf{x}} c(t, \mathbf{x}) - \eta c(t, \mathbf{x}) |\mathbf{j}(t, \mathbf{x})|, \quad (3)$$

$$\mathbf{j}(t, \mathbf{x}) = \int_{\mathbb{R}^2} \mathbf{v}' p(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}', \quad \tilde{p}(t, \mathbf{x}) = \int_{\mathbb{R}^2} d\mathbf{v}' p(t, \mathbf{x}, \mathbf{v}'), \quad (4)$$

when $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $t \in [0, \infty)$. This type of models is inspired in previous work on self-organized phenomena and retinal angiogenesis [6, 7]. Here, p represents the density (in space and velocity) of blood vessels growing in response to the distribution of the tumor angiogenic factor c . Parameters σ , γ , k , α_1 , c_R , d_1 , γ_1 , d and η are positive constants. $\rho(\mathbf{v})$ is a gaussian profile centered at a point \mathbf{v}_0 . The original model contains a Dirac mass $\delta_{\mathbf{v}_0}$ instead of a gaussian. Regularizations of the form $\rho_\varepsilon(\mathbf{v}) = \frac{1}{(\pi\varepsilon)^{N/2}} e^{-\frac{|\mathbf{v}-\mathbf{v}_0|^2}{\varepsilon}}$ are used for numerical purposes. The functions ρ_ε tend to $\delta_{\mathbf{v}_0}$ as ε tends to 0. $\rho(\mathbf{v})$ stands for any of them. From the theoretical point of view, we may seek to construct solutions for delta valued coefficients as limits of solutions for these gaussians approximations. However, proving compactness of such sequences of solutions, even in simpler models like the ones we consider here, is yet an open problem, as we will discuss in Section 5.

Establishing well posedness of these regularized problems is a challenging task due to the combination of degenerate diffusion, nonlinear transport terms and integrodifferential sources. Here, we will use a simpler diffusion model as a basis to develop strategies to handle some of the technical difficulties:

$$\begin{aligned} \frac{\partial}{\partial t} p(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} p(t, \mathbf{x}, \mathbf{v}) &= \alpha(c(t, \mathbf{x})) \rho(\mathbf{v}) p(t, \mathbf{x}, \mathbf{v}) \\ &\quad - \gamma p(t, \mathbf{x}, \mathbf{v}) \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p(s, \mathbf{x}, \mathbf{v}'), \end{aligned} \quad (5)$$

$$p(0, \mathbf{x}, \mathbf{v}) = p_0(\mathbf{x}, \mathbf{v}), \quad (6)$$

$$\frac{\partial}{\partial t} c(t, \mathbf{x}) - d \Delta_{\mathbf{x}} c(t, \mathbf{x}) = -\eta c(t, \mathbf{x}) j(t, \mathbf{x}), \quad (7)$$

$$c(0, \mathbf{x}) = c_0(\mathbf{x}), \quad (8)$$

set in the whole space $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$, for $t \in [0, \infty)$. Here,

$$j(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}'| p(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}'. \quad (9)$$

The variable coefficient $\alpha(c)$ is still defined by (2). There are two reasons to replace $|\mathbf{j}|$ with j . From the modeling point of view, the euclidean norm of \mathbf{j} might vanish under certain symmetry conditions, whereas in practice the concentration of tumor angiogenic factor c would still decrease due to cell consumption. From the mathematical point of view, $|\mathbf{j}(p)| = (j_1(p)^2 + j_2(p)^2)^{1/2}$ may bring about lipschitzianity problems near zero. This might cause uniqueness problems when $\mathbf{j}(p)$ approaches zero, which happens at infinity when \mathbf{x} is allowed to vary in an unbounded domain. Our existence proof holds for both choices, $|\mathbf{j}|$ and j . However, we can only guarantee uniqueness in the latter case.

Notice that we have also included a viscosity term $\Delta_{\mathbf{v}}p$. Adding a vanishing viscosity term $\delta\Delta_{\mathbf{v}}p$, δ small, to problems with degenerate diffusion in that variable is a standard numerical strategy to devise numerical schemes ensuring positivity of solutions and avoiding sign related artifacts. Our results extend to problems with asymmetric diffusion $\sigma_1\Delta_{\mathbf{x}}p + \sigma_2\Delta_{\mathbf{v}}p$. We have set $\sigma_1 = \sigma_2 = \sigma$ for simplicity.

The unknown p represents a density of blood vessels. Positivity is therefore a key property of the solutions. For positive p the sign of the source term in (5) cannot be controlled. A possibility to generate approximate solutions with a controlled sign is to freeze the integral coefficient and to include the linearized integral source in a linearized diffusion operator. The paper is organized as follows. Section 2 recalls and establishes results on existence of fundamental solutions, solutions for initial value problems, properties of such solutions and comparison principles, mostly for a linearized version of (5)-(6) in which the dependence on c is ignored. Section 3 constructs the unique nonnegative solution p of the nonlinear problem when $\alpha = 0$ as limit of solutions of an iterative scheme. Existence of nonnegative solutions for the scheme follows using fundamental solutions. Heat equations provide upper solutions yielding uniform L^q bounds when $\alpha = 0$. Energy inequalities produce the uniform bounds on derivatives required for compactness. The limiting function is the unique solution sought for. Once the strategy to handle the integral term is clear, section 4 considers the full coupled problem (5)-(8). These results pave the way for the study of more realistic problems in which the heat operator is replaced by a Fokker-Planck operator with degenerate diffusion and transport terms, for which the theory of fundamental solutions is more involved [8]. Our iterative schemes may be used for the numerical approximation of the solutions. The additional properties we establish on the solutions and the iterates may be exploited to render formal derivations of these models rigorous.

2. Linear problem

The key underlying linearized problems are:

$$Lp = \frac{\partial}{\partial t}p(t, \mathbf{x}, \mathbf{v}) - \sigma\Delta_{\mathbf{x}\mathbf{v}}p(t, \mathbf{x}, \mathbf{v}) + a(t, \mathbf{x}, \mathbf{v})p(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v}), \quad (10)$$

$$p(0, \mathbf{x}, \mathbf{v}) = p_0(\mathbf{x}, \mathbf{v}), \quad (11)$$

when $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t \in [0, \infty)$, with $a \in L^\infty([0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$, $\sigma \in \mathbb{R}^+$, $f \in L^\infty(0, \infty; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $p_0 \in L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, as well as:

$$\frac{\partial}{\partial t} c(t, \mathbf{x}) - d\Delta_{\mathbf{x}} c(t, \mathbf{x}) + a(t, \mathbf{x})c(t, \mathbf{x}) = f(t, \mathbf{x}), \quad (12)$$

$$c(0, \mathbf{x}) = c_0(\mathbf{x}), \quad (13)$$

when $\mathbf{x} \in \mathbb{R}^2$, $t \in [0, \infty)$, with $a \in L^\infty([0, \infty) \times \mathbb{R}^2)$, $d \in \mathbb{R}^+$, $f \in L^\infty(0, \infty; L^q(\mathbb{R}^2))$ and $c_0 \in L^q(\mathbb{R}^2)$, $1 \leq q \leq \infty$.

Solutions may be constructed using fundamental solutions of the parabolic operator, whose properties depend on the smoothness of the coefficient a . We recall below the known theory of classical and weak fundamental solutions and discuss additional bounds for solutions of (10)-(11) and (12)-(13).

2.1. Existence using classical fundamental solutions

Existence of a classical smooth solution of (10)-(11) follows from the theory of fundamental solutions for parabolic equations with smooth bounded coefficients [10]. Let us assume that $a(t, \mathbf{x}, \mathbf{v})$ is a continuous function satisfying

$$|a(t, \mathbf{x}, \mathbf{v}) - a(t, \mathbf{x}^0, \mathbf{v}^0)| \leq A|\mathbf{x} - \mathbf{x}^0, \mathbf{v} - \mathbf{v}^0|^\beta, \quad 0 < \beta < 1, \quad (14)$$

for some $A > 0$. A fundamental solution of $Lu = 0$ is a function $\Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}')$ defined for $\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}' \in \mathbb{R}^2$ and $t, \tau \in \mathbb{R}^+$, $t > \tau$, which verifies:

- (i) for fixed $(\tau, \mathbf{x}', \mathbf{v}')$, the equation $L\Gamma = 0$ holds for all $\mathbf{x}, \mathbf{v}, t > \tau$,
- (ii) for every continuous function ψ

$$\lim_{t \rightarrow \tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') \psi(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' = \psi(\mathbf{x}, \mathbf{v}).$$

According to Theorems 10 and 11 in Chapter 1 of reference [10], there exists a fundamental solution for our operator L under hypothesis (14) on a , which satisfies the following bounds:

$$0 < \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') \leq C(T)(t - \tau)^{-n/2} e^{-\sigma^* \frac{(|\mathbf{x} - \mathbf{x}'|^2 + |\mathbf{v} - \mathbf{v}'|^2)}{4(t - \tau)}}, \quad (15)$$

$$\left| \frac{\partial \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}')}{\partial z_i} \right| \leq C(T)(t - \tau)^{-(n+1)/2} e^{-\sigma^* \frac{(|\mathbf{x} - \mathbf{x}'|^2 + |\mathbf{v} - \mathbf{v}'|^2)}{4(t - \tau)}}, \quad (16)$$

for $t \in [0, T]$, where $z_i = x_i$ or v_i , $i = 1, 2$, for $\sigma^* < \sigma$ and $C(T) > 0$ (see also reference [15], pp. 124-125). The constants $C(T)$ appearing in estimates (15)-(16) depend on the parabolicity constant σ , the number of independent space and velocity variables n , the final time T , the maximum modulus of the coefficients $M_0 = \max_{t \in [0, T], \mathbf{x} \in \mathbb{R}^2} |a(t, \mathbf{x})|$ and the Hölder constants A, β in inequality (14), see reference [15]. Given continuous functions $f(t, \mathbf{x}, \mathbf{v})$ and $p_0(\mathbf{x}, \mathbf{v})$ defined on $[0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}^2$, respectively, and satisfying:

$$|f(t, \mathbf{x}, \mathbf{v})| \leq C_f e^{h(|\mathbf{x}|^2 + |\mathbf{v}|^2)}, \quad |p_0(\mathbf{x}, \mathbf{v})| \leq C_p e^{h(|\mathbf{x}|^2 + |\mathbf{v}|^2)},$$

for $C_f, C_p > 0$ and $h < \frac{\sigma}{4T}$, the function

$$\begin{aligned} p(t, \mathbf{x}, \mathbf{v}) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Gamma(t, \mathbf{x}, \mathbf{v}; 0, \mathbf{x}', \mathbf{v}') p_0(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' \\ &+ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') f(\tau, \mathbf{x}', \mathbf{v}') d\tau d\mathbf{x}' d\mathbf{v}' \end{aligned} \quad (17)$$

is a solution of problem (10)-(11), as shown in reference [10]. The positivity of the fundamental solution implies positivity of solutions for $f \geq 0$ and $p_0 \geq 0$. The integral expression (17), together with the bound (15), implies:

$$\|p(t)\|_\infty \leq C_\infty \left(\|p_0\|_\infty + \int_0^t \|f(s)\|_\infty ds \right) \quad (18)$$

$$\|p(t)\|_1 \leq C_1 \left(\|p_0\|_1 + \int_0^t \|f(s)\|_1 ds \right), \quad (19)$$

where C_∞, C_1 depend on σ, n, T, M_0, A and β .

Bounds on classical fundamental solutions are studied more in detail in reference [15], that admits $0 < \beta \leq 1$ and includes classical time derivatives and second order derivatives in space:

$$|\partial_t^{m_0} \partial_{\mathbf{x}}^{m_x} \partial_{\mathbf{v}}^{m_v} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}')| \leq \frac{C(T) e^{-\sigma^* \frac{(|\mathbf{x}-\mathbf{x}'|^2 + |\mathbf{v}-\mathbf{v}'|^2)}{(t-\tau)}}}{(t-\tau)^{(n+2m_0+|m_x|+|m_v|)/2}} \quad (20)$$

for $0 < t - \tau \leq T$, $2m_0 + |m_x| + |m_v| \leq 2$. The constants σ^* and C depend on σ, n, M_0, A, β and T (see [15], Theorem 1.1).

Existence results for measurable, bounded or integrable data follow by a regularization procedure. We smooth the data using convenient mollifiers [4, 12]: $p_0 = \rho_\varepsilon * p_0$ and $f = \psi_\varepsilon * f$ in adequate variables. These C^∞ families converge to p_0 and f in L^q , $1 \leq q < \infty$, as ε tends to zero, and are bounded when $p_0, f \in L^q$. When $q = \infty$, we have weak* convergence. The corresponding classical sequences of solutions, and their derivatives, are bounded in L^q . Passing to the limit in the linear equation, the limit is a weak solution. Passing to the limit in the integral equation, it verifies a similar integral equation. The limit solution inherits nonnegativity for positive data. It also inherits the L^q bounds (18)-(19).

2.2. Existence using weak fundamental solutions

When the coefficient a is only measurable and bounded, existence of weak fundamental solutions has been established. A measurable function of the form $\Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}')$ is a weak fundamental solution of the initial value problem for (10)-(11) if the function

$$P_{t,\tau} \psi(\mathbf{x}, \mathbf{v}) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') \psi(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' \quad (21)$$

satisfies:

$$\frac{\partial}{\partial t} P_{t,\tau} \psi(\mathbf{x}, \mathbf{v}) = [\sigma \Delta_{\mathbf{x}, \mathbf{v}} - a(t, \mathbf{x}, \mathbf{v})] P_{t,\tau} \psi(\mathbf{x}, \mathbf{v}) \quad (22)$$

$$\lim_{t \rightarrow \tau} P_{t,\tau} \psi(\mathbf{x}, \mathbf{v}) = \psi(\mathbf{x}, \mathbf{v}) \quad (23)$$

for any continuous function ψ with compact support, $\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}' \in \mathbb{R}^2 \times \mathbb{R}^2$, $t, \tau \in [0, \infty)$, $t > \tau$. The weak solution of the initial value problem (22)-(23) is unique since σ is constant [11]. A weak solution $P_{t,\tau} \psi(\mathbf{x}, \mathbf{v})$ of (22)-(23) satisfies

$$\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} P_{s,0} \psi(\mathbf{x}, \mathbf{v}) \left[\frac{\partial}{\partial t} + \sigma \Delta_{\mathbf{x}, \mathbf{v}} - a(s, \mathbf{x}, \mathbf{v}) \right] \phi(t, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} ds + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \psi(\mathbf{x}, \mathbf{v}) \phi(0, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} = 0,$$

for any $\phi \in C_c^\infty([0, T] \times \mathbb{R}^2 \times \mathbb{R}^2)$.

Under hypothesis on the second order operator that are satisfied for constant positive σ and measurable, bounded a , a fundamental solution Γ satisfying the bounds [1, 2, 13, 11]:

$$\frac{C_1 e^{-C_1(t-\tau)}}{(t-\tau)^{n/2}} e^{-\gamma_1 \frac{(|\mathbf{x}-\mathbf{x}'|^2 + |\mathbf{v}-\mathbf{v}'|^2)}{t-\tau}} \leq \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') \quad (24)$$

$$\Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') \leq \frac{C_2 e^{C_2(t-\tau)}}{(t-\tau)^{n/2}} e^{-\gamma_2 \frac{(|\mathbf{x}-\mathbf{x}'|^2 + |\mathbf{v}-\mathbf{v}'|^2)}{t-\tau}}, \quad (25)$$

exists for $t, \tau \in [0, \infty)$ such that $\tau < t$ and $\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}' \in \mathbb{R}^2$. The dimension $n = 4$ in our particular case. The constants $C_1, C_2, \gamma_1, \gamma_2$ depend on σ and $\|a\|_{L^\infty}$.

The existence and regularity of weak fundamental solutions for general parabolic problems of the form $u_t = \nabla \cdot (\sigma \nabla u) + b \cdot \nabla u - au$ with measurable coefficients has been studied in a series of papers. β -Hölder continuity, $\beta \in (0, 1]$, is discussed in [2, 18, 11]. When b is not continuous, differentiability cannot be expected. Depending on the regularity of the coefficients [13, 11], the equivalent of initial value problem (22)-(23) for general parabolic operators should be understood in a merely weak sense.

In our case, σ is constant, $b = 0$, and a is bounded. Therefore, both the fundamental solution Γ and $P_{t,\tau} \psi$ are smoother. Once its existence is guaranteed, the fundamental solution can be seen as a solution of a heat equation with a source $-ap = -a\Gamma$, bounded in terms of a heat kernel. It admits the integral expression

$$\begin{aligned} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') &= G(t - \tau, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') \\ &- \int_\tau^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} G(t - s, \mathbf{x} - \boldsymbol{\xi}, \mathbf{v} - \boldsymbol{\nu}) a(s, \boldsymbol{\xi}, \boldsymbol{\nu}) \Gamma(s, \boldsymbol{\xi}, \boldsymbol{\nu}; \tau, \mathbf{x}', \mathbf{v}') d\boldsymbol{\xi} d\boldsymbol{\nu} ds, \end{aligned} \quad (26)$$

where G is the heat kernel for diffusivity σ . In fact, Γ can be constructed as the solution of integral equation (26) using an iterative scheme that yields bound (15), but with coefficients depending on $\|a\|_\infty$, n , σ and T , for $t \in [0, T]$. This way of reasoning is standard in kinetic models, see [20, 8] and references therein.

The derivatives of Γ satisfy a similar integral expression. For $z_i = x_i$ or $z_i = v_i$, $i = 1, 2$:

$$\begin{aligned} \frac{\partial}{\partial z_i} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') &= \frac{\partial}{\partial z_i} G(t - \tau, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') \\ - \int_{\tau}^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\partial}{\partial z_i} G(t - s, \mathbf{x} - \boldsymbol{\xi}, \mathbf{v} - \boldsymbol{\nu}) a(s, \boldsymbol{\xi}, \boldsymbol{\nu}) \Gamma(s, \boldsymbol{\xi}, \boldsymbol{\nu}; \tau, \mathbf{x}', \mathbf{v}') d\boldsymbol{\xi} d\boldsymbol{\nu} ds. \end{aligned} \quad (27)$$

The coefficient a being bounded, inequality (25) allows us to obtain estimates of the form (16), like in the classical case, but with coefficients depending on $\|a\|_{\infty}$, n , σ and T , for $t \in [0, T]$.

We summarize these observations for the problems we consider here in the following Lemmas.

Lemma 2.1. *When a is a bounded function, the initial value problem (10)-(11) has a fundamental solution Γ satisfying estimates (24)-(25) and (16) with parameters depending on the norm $\|a\|_{\infty}$, the dimension n , the diffusivity σ and $T > 0$, for $t \in [0, T]$. An analogous result holds for (12)-(13).*

Proposition 2.2. *For any $a \in L^{\infty}([0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$, $p_0 \in L^{\infty} \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and $f \in L^{\infty}(0, T; L^{\infty} \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, there exists a unique solution $p \in C([0, T]; L^{\infty} \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ of the initial value problem (10)-(11) satisfying the weak formulation:*

$$\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} p \left[\frac{\partial \phi}{\partial t} + \sigma \Delta_{\mathbf{xv}} \phi - a \phi \right] d\mathbf{x} d\mathbf{v} ds + \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} f \phi d\mathbf{x} d\mathbf{v} ds + \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_0 \phi(0) d\mathbf{x} d\mathbf{v} = 0, \quad (28)$$

for $\phi \in C_c^{\infty}([0, T] \times \mathbb{R}^2 \times \mathbb{R}^2)$. This solution admits the integral expression (17) in terms of the fundamental solution Γ given by (26). It is positive for positive f and p_0 and satisfies estimates (18)-(19) with constants depending on σ , T , $\|a\|_{\infty}$ and the dimension. The solution has the same regularity as solutions of heat equations with L^q data and sources, $1 \leq q \leq \infty$.

Proof. We can either exploit (21) and the semigroup theory [14], or construct the solution as limit of classical solutions as in reference [8]. Following [8], let us consider regularized coefficients $a_k = a * \phi_k$ where ϕ_k is a positive C^{∞} mollifying family (see reference [4], p. 108). We then have $\|a_k\|_{\infty} \leq \|a\|_{\infty}$ and $a_k \rightharpoonup a$ in $L_{t\mathbf{xv}}^{\infty}$ weak* when $k \rightarrow \infty$ (see reference [4], p. 126). Let us first assume that $p_0, f \in C_c^{\infty}$ are smooth functions with compact support. Let p_k be the corresponding classical solutions of (10)-(11) with coefficient a_k , satisfying the integral equations (17) in terms of fundamental solutions Γ_k . Thanks to estimates (24)-(25) and the fact that $\|a_k\|_{\infty}$ is uniformly bounded, p_k are bounded in $L^2(0, T; L_{\mathbf{xv}}^2)$. Then, the energy inequality (see [9], also Lemma 2.9 below) provides a uniform bound on $\frac{\partial p_k}{\partial z_i}$ in $L^2(0, T; L_{\mathbf{xv}}^2)$ for $z_i = v_i$ or x_i , $i = 1, 2$. Therefore, p_k is bounded in $L^2(0, T; H_{\mathbf{xv}}^1)$. Using equation (10), the time derivative $\frac{\partial p_k}{\partial t}$ is bounded in $L^{\infty}(0, T; H_{\mathbf{xv}}^{-1})$ (see [4], proposition 9.20). Since the injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact for any smooth bounded

Ω (see [4], theorem 9.16), classical compactness results (see Theorem 12.1 in [12], Corollary 4 in [17]) imply that p_k is compact in $L^2(0, T; L^2_{\mathbf{xv}}(\Omega))$, for any bounded Ω . We can extract subsequences converging strongly and pointwise to p in any Ω . Choosing $\Omega = B(0, M)$, for integers M tending to infinity, a diagonal extraction procedure allows us to obtain a subsequence $p_{k'}$ converging pointwise to a limit p in $[0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$, strongly in $L^2(0, T; L^2_{loc})$, and weakly in $L^2(0, T; L^2_{\mathbf{xv}})$. For $\phi \in C_c^\infty([0, T] \times \mathbb{R}^2 \times \mathbb{R}^2)$, the weak formulation of problem (10)-(11) with coefficient a_k reads:

$$\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_k \left[\frac{\partial \phi}{\partial t} + \sigma \Delta_{\mathbf{xv}} \phi - a_k \phi \right] d\mathbf{x} d\mathbf{v} ds + \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} f \phi d\mathbf{x} d\mathbf{v} ds + \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_0 \phi(0) d\mathbf{x} d\mathbf{v} = 0. \quad (29)$$

The support of ϕ being a compact set contained in $[0, T] \times \Omega$, Ω bounded, we may select a subsequence converging to p in $L^2(0, T; L^2(\Omega))$, and therefore in $L^1(0, T; L^1(\Omega))$. Taking limits, p is a weak solution of (10)-(11) with coefficient a .

Now, let us pass to the limit in the integral expressions (17) for $p_{k'}$ and $\Gamma_{k'}$. Thanks to inequality (25), the fundamental solutions $\Gamma_{k'}$ are bounded in $L^r_{t\mathbf{xv}}([0, T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ for any $r \in (1, n/(n-2))$, $n = 4$. Therefore, a subsequence converges weakly in $L^r_{t\mathbf{xv}}$ to a limit Γ . Taking limits in the integral expressions, we see that the solution p of (10)-(11) with coefficient a verifies (17). Setting $f = 0$, Γ fulfills the definition of fundamental solution for a .

Once identity (17) is established for solutions of (10)-(11) with bounded coefficient a and C_c^∞ data, we extend it to L^q data by density. When $1 \leq q < \infty$, there are sequences $p_{k,0} \in C_c^\infty$ and $f_k \in C_c^\infty$ converging to p_0 in $L^q_{\mathbf{xv}}$ and f in $L^1_t L^q_{\mathbf{xv}}$ (see [4], corollary 4.23). Identity (17) and inequality (25) yield a uniform bound on $\|p_k(t)\|_r$ for $t \in [0, T]$, $r \geq q$. Therefore, we can extract a subsequence converging weakly to a limit p . Taking limits in the weak formulation and the integral equation for p_k , we prove (17) for a weak solution p of the initial value problem (10)-(11) with data p_0 and f .

Using identity (17), the positivity of the solution follows from the positivity of the data. The upper bound (25) on the fundamental solution yields estimates (18)-(19) with constants depending on $\sigma, n, T, \|a\|_\infty$.

To prove uniqueness, assume we have two solutions $p_1, p_2 \in L^\infty(0, T; L^q_{\mathbf{xv}})$ of (10)-(11) with initial and source data in L^q and coefficient a bounded. The difference $\bar{p} = p_1 - p_2$ is a solution of a heat equation with $\bar{p}(0) = 0$ and source $-a\bar{p}$. The integral expression for solutions of heat equations yields:

$$\bar{p}(t) = - \int_0^t G(t-s) * a(s) \bar{p} ds \Rightarrow \|\bar{p}(t)\|_q \leq \|a\|_\infty \int_0^t \|\bar{p}(s)\|_q ds.$$

This Gronwall inequality implies $\|\bar{p}(t)\|_q = 0$ for any $t \in [0, T]$.

Similar results hold suppressing the dependence on the variable \mathbf{v} . We state the result for data in $L^q_{\mathbf{x}}$ spaces since we do not assume $c_0 \in L^\infty_{\mathbf{x}} \cap L^1_{\mathbf{x}}$.

Proposition 2.3. *For any $a \in L^\infty([0, \infty) \times \mathbb{R}^2)$, $c_0 \in L^q(\mathbb{R}^2)$, and $f \in L^\infty(0, T; L^q(\mathbb{R}^2))$, $1 \leq q \leq \infty$, there exists a unique solution $c \in C([0, T]; L^\infty(\mathbb{R}^2))$*

of the initial value problem (12)-(13). This solution admits the integral expression (17) in terms of the fundamental solution Γ (suppressing the dependence on \mathbf{v}). The solution of (12)-(13) is positive for positive f and p_0 and satisfies estimates (18)-(19) with constants depending on σ , T , $\|a\|_\infty$ and the dimension.

Proof. The proof is similar to that of Proposition 2.2, except for the uniform bounds on p_k and $\frac{\partial p_k}{\partial z_i}$.

Let us first assume that $q = \infty$. Thanks to estimates (24)-(25) and the fact that $\|a_k\|_\infty$ is uniformly bounded, p_k are bounded in $L^\infty(0, T; L^\infty_{\mathbf{xv}})$. Moreover, differentiating (17) and using the estimates on the derivatives of Γ_k referred to in Lemma 2.1, we see that $\frac{\partial p_k}{\partial z_i}$ are uniformly bounded in $L^{r_1}(0, T; L^\infty_{\mathbf{xv}})$ for $z_i = v_i$ or x_i , $i = 1, 2$, and $r_1 \in [1, 2)$. Therefore, p_k is uniformly bounded in $L^{r_1}(0, T; W^{1, r_2}_{\mathbf{xv}}(\Omega))$ for any $r_2 \in [1, \infty)$ and any Ω bounded. Using equation (10), the time derivative $\frac{\partial p_k}{\partial t}$ is bounded in $L^{r_1}(0, T; W^{-1, r'_2}_{\mathbf{xv}})$ when $r_2 < \infty$ (see [4], proposition 9.20). Since the injection of $W^{1, r_2}(\Omega)$ in $L^{r_2}(\Omega)$ is compact for any smooth bounded Ω ([4], theorem 9.16), classical compactness results [3, 12, 17] imply that p_k is compact in $L^{r_1}(0, T; L^{r_2}_{\mathbf{xv}}(\Omega))$. As in Proposition 2.2, a diagonal extraction procedure allows us to obtain a subsequence $p_{k'}$ converging pointwise to a limit p in $[0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$, strongly in $L^{r_1}(0, T; L^{r_2}_{loc})$, and weak* in $L^\infty(0, T; L^\infty_{\mathbf{xv}})$. In particular, we have strong convergence in $L^1(0, T; L^1_{loc})$ which allows us to pass to the limit in the weak formulation of the equations (29) and establish that p satisfies (28). When $q < \infty$, a similar proof works.

Identity (17) when $q < \infty$ is proven as in Proposition 2.2. When $q = \infty$, we first have to extend it to solutions with coefficient a and differentiable bounded data, with bounded derivatives. The proof proceeds as the proof in Proposition 2.2 for C^∞_c data since classical solutions for these data satisfy the integral equation, and bounds in $L^\infty(0, T; W^{1, \infty}_{\mathbf{xv}})$ imply bounds in $L^2(0, T; H^1(\Omega))$, Ω bounded. This allows to pass to the limit in the weak formulations of the initial values problem with coefficient a_k . Also, the associated fundamental solutions Γ_k satisfy the Dunford-Pettis criterion for weak compactness in L^1 (see reference [4], theorem 4.30), which allows us taking limits in the integral equation. Once (17) is established for solutions with coefficient a and differentiable bounded data, with bounded derivatives, L^∞ data are handled approximating p_0 and f by mollified sequences $p_{k,0}$ and f_k tending to p_0 and f in L^∞ weak*.

2.3. Comparison principle and heat estimates

This section recalls comparison principles and basic $L^r - L^q$ estimates for solutions of diffusion problems.

Lemma 2.4. *Let $p^{(1)}$, $p^{(2)}$ be the solutions of the initial value problem (10)-(11) with bounded coefficient a and data $f^{(1)}$, $p_0^{(1)}$ and $f^{(2)}$, $p_0^{(2)}$, respectively, constructed in Proposition 2.2. Assume that:*

$$f^{(1)} \leq f^{(2)}, \quad p_0^{(1)} \leq p_0^{(2)}. \quad (30)$$

Then, the corresponding solutions $p^{(1)}$ and $p^{(2)}$ preserve the ordering:

$$p^{(1)} \leq p^{(2)}. \quad (31)$$

In particular, any solution p is nonnegative if $p_0 \geq 0$ and $f \geq 0$. The same positivity and comparison principles hold for (12)-(13).

Proof. It follows from the positivity of the fundamental solutions and the integral expression (17) for the solutions of the initial value problem (10)-(11). Similarly for (12)-(13).

Lemma 2.5. *When the bounded coefficient $a \geq 0$, any positive solution p of the initial value problem (10)-(11) with L^q data is bounded from above by a solution of a heat equation with the same initial and source data. Moreover, the following estimates hold for any $q \in [1, \infty]$:*

$$\|p\|_q \leq \|p_0\|_q + t \max_{s \in [0, t]} \|f(s)\|_q, \quad (32)$$

$$\|p\|_r \leq C_1 t^{-(\frac{1}{q} - \frac{1}{r})\frac{n}{2}} \|p_0\|_q + C_2 t^{-(\frac{1}{q} - \frac{1}{r})\frac{n}{2} + 1} \max_{s \in [0, t]} \|f(s)\|_q, \quad (33)$$

provided $r \geq q$, $(\frac{1}{q} - \frac{1}{r})\frac{n}{2} < 1$, n being the dimension. Analogous estimates hold for solutions of (12)-(13) adapting the dimension.

Proof. Notice that p is the solution of the heat equation with source $g = f - ap \leq f$. Let u be the solution of:

$$\frac{\partial}{\partial t} u(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{x}\mathbf{v}} u(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v}), \quad u(0, \mathbf{x}, \mathbf{v}) = p_0(\mathbf{x}, \mathbf{v}). \quad (34)$$

The solution of problem (34) admits integral expressions in terms of the heat kernel $G(t, \mathbf{x}, \mathbf{v})$. It is then straightforward that:

$$\begin{aligned} p(t) &= G(t) * p_0 + \int_0^t G(t - \tau) * [f(\tau) - a(\tau)p(\tau)] d\tau \\ &\leq u(t) = G(t) * p_0 + \int_0^t G(t - \tau) * f(\tau) d\tau, \end{aligned} \quad (35)$$

where $*$ denotes convolution in the \mathbf{x}, \mathbf{v} variables. Setting $f = 0$, the well known $L^r - L^q$ estimates for heat operators $\|u\|_q = \|G(t) * p_0\|_q$ follow [9]:

$$\|u\|_q \leq \|G(t)\|_1 \|p_0\|_q \leq \|p_0\|_q, \quad (36)$$

$$\|u\|_r \leq \|G(t)\|_{q'} \|p_0\|_q \leq C_q t^{-(\frac{1}{q} - \frac{1}{r})\frac{n}{2}} \|p_0\|_q, \quad 1/r = 1/q + 1/q' - 1, \quad (37)$$

for $r \geq q$. The parameters n stands for the space-velocity dimension. When $f \neq 0$ we find estimates (32)-(33) for u . They extend to p since $p \leq u$. Analogous arguments work for (12)-(13), with n representing only the spatial dimension.

2.4. Velocity decay

Integral expressions for solutions of the initial value problem (10)-(11) yield additional information on their velocity decay, depending on the initial and source data.

Lemma 2.6. *For the solution constructed in Proposition 2.2, let us assume that $|\mathbf{v}|^\beta f \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $|\mathbf{v}|^\beta p_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, for $\beta > 0$. Then, the solution p of (10)-(11) satisfies $|\mathbf{v}|^\beta p \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$.*

Moreover, if $|\mathbf{v}|^\beta p_0 \in L_x^q(\mathbb{R}^2; L_v^1(\mathbb{R}^2))$ and $|\mathbf{v}|^\beta f \in L^\infty(0, T; L_x^q(\mathbb{R}^2; L_v^1(\mathbb{R}^2)))$, for $1 \leq q \leq \infty$, $\beta = 0, 1, 2$, then, the solution p of the problem (10)-(11) satisfies $|\mathbf{v}|^\beta p \in L^\infty(0, T; L_x^q(\mathbb{R}^2; L_v^1(\mathbb{R}^2)))$ for $\beta = 0, 1, 2$.

Proof. Using the integral expression (17) for p , taking absolute values, multiplying by $|\mathbf{v}|^\beta$ and integrating we obtain:

$$\begin{aligned} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}|^\beta |p(t, \mathbf{x}, \mathbf{v})| d\mathbf{v} d\mathbf{x} &\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}|^\beta \Gamma(t, \mathbf{x}, \mathbf{v}; 0, \mathbf{x}', \mathbf{v}') |p_0(\mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' d\mathbf{x} d\mathbf{v} + \\ &\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}|^\beta \Gamma(t, \mathbf{x}, \mathbf{v}; s, \mathbf{x}', \mathbf{v}') |f(s, \mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' ds d\mathbf{x} d\mathbf{v}, \end{aligned} \quad (38)$$

for any $t \in [0, T]$. Thanks to estimate (25), the first integral is bounded from above in terms of a heat kernel G :

$$\begin{aligned} C_1(T) t^{\frac{\beta}{2}} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} \frac{|\mathbf{v} - \mathbf{v}'|^\beta}{t^{\beta/2}} G(t, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') |p_0(\mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' d\mathbf{x} d\mathbf{v} \\ + C_1(T) \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} G(t, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') |\mathbf{v}'|^\beta |p_0(\mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' d\mathbf{x} d\mathbf{v} \\ \leq C_2(T) \|p_0\|_{L_{\mathbf{x}\mathbf{v}}^1} + C_3(T) \| |\mathbf{v}|^\beta p_0 \|_{L_{\mathbf{x}\mathbf{v}}^1}. \end{aligned} \quad (39)$$

Proceeding in a similar way, the integral involving f is bounded by $\hat{C}_2(T) \|f\|_{L_t^\infty L_{\mathbf{x}\mathbf{v}}^1} + \hat{C}_3(T) \| |\mathbf{v}|^\beta f \|_{L_t^\infty L_{\mathbf{x}\mathbf{v}}^1}$.

For the last part, integrating (17) with respect to \mathbf{v} , using (25), and

$$\int_{\mathbb{R}^2} d\mathbf{v} (t-\tau)^{-4/2} e^{-\gamma_2 \frac{(|\mathbf{x}-\mathbf{x}'|^2 + |\mathbf{v}-\mathbf{v}'|^2)}{t-\tau}} = C_1 (t-\tau)^{-2/2} e^{-\gamma_2 \frac{|\mathbf{x}-\mathbf{x}'|^2}{t-\tau}} = K(t-s, \mathbf{x}-\mathbf{x}'),$$

we see that:

$$\int_{\mathbb{R}^2} p d\mathbf{v} \leq C_2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} K(t, \mathbf{x}-\mathbf{x}') p_0(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' + C_2 \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} K(t-s, \mathbf{x}-\mathbf{x}') f(s, \mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' ds. \quad (40)$$

This yields the L_x^q estimate on $\int_{\mathbb{R}^2} p d\mathbf{v}$.

For $\int_{\mathbb{R}^2} |\mathbf{v}| p d\mathbf{v}$, we multiply identity (17) by $|\mathbf{v}|$, integrate with respect to \mathbf{v} , replace in the right hand side $|\mathbf{v}|$ with $|\mathbf{v} - \mathbf{v}'| + |\mathbf{v}'|$ and use estimate (25). We then notice that

$$\int_{\mathbb{R}^2} d\mathbf{v} \frac{|\mathbf{v} - \mathbf{v}'|}{(t-\tau)^2} e^{-\gamma_2 \frac{(|\mathbf{x}-\mathbf{x}'|^2 + |\mathbf{v}-\mathbf{v}'|^2)}{t-\tau}} = \frac{\hat{C}_1}{t-\tau} e^{-\gamma_2 \frac{|\mathbf{x}-\mathbf{x}'|^2}{t-\tau}} = \tilde{K}(t-s, \mathbf{x}-\mathbf{x}').$$

We finally find:

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{v}| p d\mathbf{v} &\leq \hat{C}_2 \left[K(t) * \int_{\mathbb{R}^2} |\mathbf{v}'| p_0 d\mathbf{v}' + \int_0^t K(t-s) * \left[\int_{\mathbb{R}^2} |\mathbf{v}'| f(s) d\mathbf{v}' \right] ds \right. \\ &\quad \left. + \tilde{K}(t) * \int_{\mathbb{R}^2} p_0 d\mathbf{v}' + \int_0^t \tilde{K}(t-s) * \left[\int_{\mathbb{R}^2} f(s) d\mathbf{v}' \right] ds \right]. \end{aligned} \quad (41)$$

This yields the $L_{\mathbf{x}}^q$ estimate on $\int_{\mathbb{R}^2} |\mathbf{v}| p d\mathbf{v}$. The proof for $\int_{\mathbb{R}^2} |\mathbf{v}|^2 p d\mathbf{v}$ is analogous.

Lemma 2.7. *For the solution constructed in Proposition 2.2, let us assume that $|\mathbf{v}|^\beta f \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $|\mathbf{v}|^\beta p_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, for $\beta \geq 0$. Then, the velocity derivatives of the solution p of (10)-(11) satisfy $|\mathbf{v}|^\beta \frac{\partial p}{\partial v_i} \in L^1(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $i = 1, 2$.*

Proof. Differentiating identity (17), we find an expression for the derivatives of the solution:

$$\begin{aligned} \frac{\partial p_k}{\partial z_i}(t, \mathbf{x}, \mathbf{v}) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\partial \Gamma_k}{\partial z_i}(t, \mathbf{x}, \mathbf{v}; 0, \mathbf{x}', \mathbf{v}') p_0(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}' \\ &+ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\partial \Gamma_k}{\partial z_i}(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') f(\tau, \mathbf{x}', \mathbf{v}') d\tau d\mathbf{x}' d\mathbf{v}', \end{aligned} \quad (42)$$

with $z_i = v_i$ or $z_i = x_i$. When $\beta = 0$, $\frac{\partial p}{\partial v_i}(t)$ is an integrable function for $t > 0$ thanks to Lemma 2.1 and estimate (16). When $\beta > 0$, we argue as in the proof of Lemma 2.6 using estimate (16) to obtain, for any $t \in [0, T]$:

$$\begin{aligned} \|\mathbf{v}|\beta \frac{\partial p}{\partial v_i}(t)\|_{L_{\mathbf{x}\mathbf{v}}^1} &\leq t^{-1/2} C_2(T) [\|p_0\|_{L_{\mathbf{x}\mathbf{v}}^1} + \|\mathbf{v}|\beta p_0\|_{L_{\mathbf{x}\mathbf{v}}^1}] \\ &+ C_3(T) T^{1/2} [\|f\|_{L_t^\infty L_{\mathbf{x}\mathbf{v}}^1} + \|\mathbf{v}|\beta f\|_{L_t^\infty L_{\mathbf{x}\mathbf{v}}^1}]. \end{aligned}$$

Lemma 2.8. *For the solution p constructed in Proposition 2.2 when $a = a(t, \mathbf{x}) \in L^\infty((0, T) \times \mathbb{R}^2)$, $p_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and $f \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, the function $\tilde{p} = \int_{\mathbb{R}^2} p d\mathbf{v}$ satisfies*

$$\frac{\partial}{\partial t} \tilde{p}(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} \tilde{p}(t, \mathbf{x}) + a(t, \mathbf{x}) \tilde{p}(t, \mathbf{x}) = \tilde{f}(t, \mathbf{x}), \quad (43)$$

$$\tilde{p}_k(0, \mathbf{x}) = \tilde{p}_0(\mathbf{x}), \quad (44)$$

with source $\tilde{f}(t, \mathbf{x}) = \int_{\mathbb{R}^2} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$ and initial datum $\tilde{p}_0(t, \mathbf{x}) = \int_{\mathbb{R}^2} p_0(\mathbf{x}, \mathbf{v}) d\mathbf{v}$.

Proof. To justify this, we may use the integral expression (17) and integrate with respect to \mathbf{v} . Since the coefficients do not depend on \mathbf{v} , the fundamental solution is invariant by translations in \mathbf{v} and depends on $\mathbf{v} - \mathbf{v}'$. Thanks to (26)

$$\begin{aligned} \int_{\mathbb{R}^2} \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') d\mathbf{v} &= \int_{\mathbb{R}^2} G(t - \tau, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') d\mathbf{v} \\ - \int_\tau^t \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} G(t - s, \mathbf{x} - \boldsymbol{\xi}, \mathbf{v} - \boldsymbol{\nu}) d\mathbf{v} \right] a(s, \boldsymbol{\xi}) \left[\int_{\mathbb{R}^2} \Gamma(s, \boldsymbol{\xi}, \boldsymbol{\nu}; \tau, \mathbf{x}', \mathbf{v}') d\boldsymbol{\nu} \right] d\boldsymbol{\xi} ds, \end{aligned}$$

where G is the heat kernel for diffusivity σ in the variables \mathbf{x}, \mathbf{v} . Notice that $\int G(t - \tau, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') d\mathbf{v} = K(t - \tau, \mathbf{x} - \mathbf{x}')$ is the heat kernel for diffusivity σ in the variable \mathbf{x} . Therefore, $\tilde{\Gamma} = \int \Gamma(t, \mathbf{x}, \mathbf{v}; \tau, \mathbf{x}', \mathbf{v}') d\mathbf{v}$ is the fundamental solution for the operator $\frac{\partial}{\partial t} \tilde{p}(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} \tilde{p}(t, \mathbf{x}) + a(t, \mathbf{x}) \tilde{p}(t, \mathbf{x})$. Integrating (17) with respect to \mathbf{v} , we conclude that \tilde{p} is a solution of (43)-(44).

Alternatively, we may first consider smooth solutions approximating a by a mollified sequence a_k (as in the proof of Proposition 2.2) and p_0, f by C_c^∞ data with compact support $p_{k,0}$ and f_k . We may then integrate equations (10)-(11), noticing that $\int_{\mathbb{R}^2} \Delta_{\mathbf{v}} p_k d\mathbf{v} = \lim_{R \rightarrow 0} \int_{|\mathbf{v}|=R} \nabla_{\mathbf{v}} p_k \cdot \mathbf{n} dS = 0$, since the derivatives of p_k are integrable functions by Lemma 2.7. Letting $k \rightarrow \infty$, we obtain the equation for \tilde{p} as limit of the problems for \tilde{p}_k .

Lemma 2.9. *Any solution p of the initial value problem (10)-(11) with bounded coefficient $a \geq 0$, initial datum $u_0 \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ and source $f \in L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$ satisfies the energy inequality:*

$$\|p(t)\|_2^2 + 2\sigma \int_0^t \|\nabla_{\mathbf{xv}} p(s)\|_2^2 ds \leq \|p_0\|_2^2 + 2 \int_0^t f(s) p(s) ds. \quad (45)$$

An analogous inequality holds for solutions \tilde{p} of (43)-(44) and c of (12)-(13).

Proof. Let us first assume that $p_0, f \in C_c^\infty$ and a is replaced by a smooth mollified sequence $a_k \geq 0$ converging to a in L^∞ weak*. By Section 2.1, $p_k \in C([0, T], L^2(\mathbb{R}^2 \times \mathbb{R}^2))$ and $\nabla_{\mathbf{xv}} p_k \in L^2((0, T) \times \mathbb{R}^2 \times \mathbb{R}^2)$. Using the integral expression (35) and the differential equation, $\Delta_{\mathbf{xv}} p_k$ and $\frac{\partial p_k}{\partial t}$ belong to $L^2((0, T) \times \mathbb{R}^2 \times \mathbb{R}^2)$. The equation holds in L^2 . Multiplying the equation by p_k , integrating over $(0, T) \times \mathbb{R}^2 \times \mathbb{R}^2$ and integrating by parts, we find:

$$\begin{aligned} \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_k(t)^2 d\mathbf{x} d\mathbf{v} + 2 \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} [\sigma |\nabla_{\mathbf{xv}} p_k|^2 + a_k |p_k|^2] d\mathbf{x} d\mathbf{v} ds = \\ 2 \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} f p_k d\mathbf{x} d\mathbf{v} ds + \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_0^2 d\mathbf{x} d\mathbf{v}. \end{aligned} \quad (46)$$

Lemma 2.1 provides a uniform bound on p_k in $L^2(0, T; L_{\mathbf{xv}}^2)$. The energy inequality (46) extends this uniform bound to $L^2(0, T; H_{\mathbf{xv}}^1)$. Arguing as in Proposition 2.2, the solutions p_k of the regularized problems tend to the solution of the problem with coefficient a in $L^2(0, T; H_{\mathbf{xv}}^1)$ weak. Since the limit of their norms is bounded from below by the norms of the weak limits, taking limits in identity (46) and neglecting a positive term, we get inequality (45) for coefficient a and smooth data of compact support.

Now, take C_c^∞ sequences $p_{k,0}, f_k$ converging to p_0, f in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ and $L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$, respectively. Let p_k be the corresponding solutions of problem (10)-(11). Inequality (45) yields uniform $L^2(0, T; H_{\mathbf{xv}}^1)$ estimates, implying weak convergence of a subsequence to a limit p in $L^2(0, T; H_{\mathbf{xv}}^1)$. Taking limits in the weak formulations for p_k , it follows that p is a weak solution with data p_0 and f . Taking limits in the energy identities for p_k , we get the energy inequality (45) for p .

Lemma 2.10. *For the solution p constructed in Proposition 2.2 when $a = a(t, \mathbf{x}) \in L^\infty((0, T) \times \mathbb{R}^2)$, $(1 + |\mathbf{v}|^2)p_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and $(1 + |\mathbf{v}|^2)f \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, the function $m = \int_{\mathbb{R}^2} |\mathbf{v}|^2 p d\mathbf{v}$ satisfies*

$$\frac{\partial}{\partial t} m(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} m(t, \mathbf{x}) + (a(t, \mathbf{x}) - 4\sigma) m(t, \mathbf{x}) = \hat{f}(t, \mathbf{x}), \quad (47)$$

with integrable source $\hat{f}(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}|^2 f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$ and initial datum $\hat{p}_0(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}|^2 p_0(\mathbf{x}, \mathbf{v}) d\mathbf{v}$.

Proof. We argue first for smooth solutions corresponding to smooth a , p_0 , f . Multiplying (10) by $|\mathbf{v}|^2$ and integrating with respect to \mathbf{v} we obtain (47). Indeed, integrating by parts over balls of radius R in velocity and letting $R \rightarrow \infty$ we find:

$$\sigma \int_{\mathbb{R}^2 \times \mathbb{R}^2} v_i^2 \frac{\partial^2}{\partial^2 v_i} p_k d\mathbf{x} d\mathbf{v} = -2\sigma \int_{\mathbb{R}^2 \times \mathbb{R}^2} v_i \frac{\partial}{\partial v_i} p_k d\mathbf{x} d\mathbf{v} = 2\sigma \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_k d\mathbf{x} d\mathbf{v}.$$

The boundary integrals $\int_{\mathbb{R}^2} \int_{|\mathbf{v}|=R} v_i^2 \frac{\partial}{\partial v_i} p_k n_i d\mathbf{x} dS_{\mathbf{v}}$ and $\int_{\mathbb{R}^2} \int_{|\mathbf{v}|=R} v_i p_k n_i d\mathbf{x} dS_{\mathbf{v}}$ tend to zero as R tends to infinity as a consequence of Lemmas 2.6 and 2.7, which ensure $(1 + |\mathbf{v}|^2)p \in C(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $(1 + |\mathbf{v}|^2) \frac{\partial p}{\partial v} \in L^1(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$. The result extends to non smooth data and coefficients by employing approximating sequences and taking limits, as in Proposition 2.2.

3. Integrodifferential problem for the density

For $k \geq 2$, we consider the iterative scheme:

$$\frac{\partial}{\partial t} p_k(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{x}\mathbf{v}} p_k(t, \mathbf{x}, \mathbf{v}) + a_{k-1}(t, \mathbf{x}) p_k(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v}), \quad (48)$$

$$p_k(0, \mathbf{x}, \mathbf{v}) = p_0(\mathbf{x}, \mathbf{v}), \quad (49)$$

for $(t, \mathbf{x}, \mathbf{v}) \in [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$, with $\sigma \in \mathbb{R}^+$, $f \in L^\infty(0, \infty; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $p_0 \in L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, $f \geq 0$ and $p_0 \geq 0$. The coefficient a_{k-1} is defined as:

$$a_{k-1}(t, \mathbf{x}) = a(p_{k-1}) = \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p_{k-1}(s, \mathbf{x}, \mathbf{v}') = \int_0^t ds \tilde{p}_{k-1}(s, \mathbf{x}), \quad (50)$$

for $k \geq 2$.

We set p_1 equal to the solution of the heat equation obtained when $a_0 = 0$. Thanks to the integral expression (35), $p_1 \in L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $p_1 \in L^\infty(0, T; L^\infty_{\mathbf{x}} \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2)))$. Additionally, estimate (33) holds.

An induction procedure guarantees the existence of iterates p_k satisfying $p_k \in L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ and $p_k \in L^\infty(0, T; L^\infty_{\mathbf{x}} \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2)))$. Indeed, assuming that a_{k-1} is measurable and bounded, a unique positive solution p_k exists in view of Proposition 2.2. Then, we must check that a_k is a bounded function, and that we can construct p_{k+1} . By Proposition 2.2 and Lemma 2.6, the integral expression (17) in terms of fundamental solutions satisfying (25) yields the $L^1 - L^\infty$ bounds (18)-(19) on p_k and L^∞ bounds on a_k with constants depending on σ , T , $\|a_{k-1}\|_\infty$. Therefore, a_k is a bounded function. By induction, we can construct the sequence p_k for all k and a_k is a bounded function for all k .

Equations (48)-(49) and their fundamental solutions in dimension $n = 4$ ensure $L^q_{\mathbf{x}\mathbf{v}}$ regularity for p_k , thanks to Proposition 2.2 and Lemma 2.1. Integrating in velocity and time, we deduce $L^q_{\mathbf{x}}$ regularity for \tilde{p}_k and a_k , either

exploiting the integral equations for p_k as in Lemma 2.6, or applying Proposition 2.2 to the differential equations (43)-(44) for \tilde{p}_k established in Lemma 2.8.

We use this iterative scheme to establish the following existence result:

Theorem 3.1. *There exists a nonnegative solution p of the system:*

$$\frac{\partial}{\partial t} p(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} p(t, \mathbf{x}, \mathbf{v}) + \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p(s, \mathbf{x}, \mathbf{v}') p(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v}), \quad (51)$$

$$p(0, \mathbf{x}, \mathbf{v}) = p_0(\mathbf{x}, \mathbf{v}), \quad (52)$$

satisfying

$$p \in L^2(0, T; H^1(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)), \\ p \in L^\infty(0, T; L^\infty \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2))),$$

if $f \in L^\infty(0, T; L^\infty \cap L^1 \cap H^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $p_0 \in L^\infty(0, T; L^\infty \cap L^1 \cap H^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $f \in L^\infty(0, T; L^\infty \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2)))$, $p_0 \in L^\infty \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2))$, $f \geq 0, p_0 \geq 0$ and $\sigma \in \mathbb{R}^+$. This solution is unique and its norms are bounded in terms of the norms of the data.

We detail the steps of the proof below. After establishing a priori bounds of p_k , a_k , we will pass to the limit in (48), obtaining a solution of (51)-(52). We will show that this solution is unique and study its regularity. Let us collect first the relevant a priori bounds.

3.1. A priori bounds

Fundamental solutions provide existence, positivity and basic regularity. However, $\|a_{k-1}\|_\infty$ affects the estimates in a way difficult to control. Uniform L^q bounds on p_k follow from comparison principles. Since the fundamental solution of (48) is positive and $p_0, f \geq 0$, we have $p_k, a_k \geq 0$ for all k . By Lemmas 2.4 and 2.5, p_k is bounded from above by the solution of the heat equation with the same data, that is, p_1 . For $k \geq 2$, we have:

$$0 \leq p_k(t) \leq p_1(t), \quad t \in [0, T]. \quad (53)$$

Since the data are integrable and bounded, p_1 satisfies the $L^r - L^q$ estimates (33). This yields uniform bounds for p_k and a_k . Indeed, combining (53) and (33), we get:

$$\|p_k(t)\|_{L^q_{\mathbf{xv}}} \leq \|p_1(t)\|_{L^q_{\mathbf{xv}}} \leq C(T, \|p_0\|_{L^q_{\mathbf{xv}}}, \|f\|_{L^r_t L^q_{\mathbf{xv}}}), \quad (54)$$

for $t \in [0, T]$, when $k \geq 2$, $T > 0$, and $1 \leq q \leq \infty$. Integrating (53) with respect to velocity and time, and applying Lemma 2.6 to p_1 , we find:

$$0 \leq a_{k-1}(t) \leq a_1(t) \leq \|p_1\|_{L^1(0, t; L^\infty(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2)))} \leq C(T, \|\tilde{p}_0\|_{L^\infty}, \|\tilde{f}\|_{L^r_t L^\infty_{\mathbf{x}}}), \quad (55)$$

for $\mathbf{x} \in \mathbb{R}^2$, $t \in [0, T]$, when $k \geq 2$, $T > 0$.

Once we have obtained uniform bounds on p_k and a_k , the heat operator provides uniform bounds on the derivatives of p_k . The starting function for the iteration p_1 is the solution of a problem for a heat equation with $L^1 \cap L^\infty$ data. Using its integral expression in terms of heat kernels,

$$\nabla_{\mathbf{xv}} p_1(t) = G(t) * \nabla_{\mathbf{xv}} p_0 + \int_0^t \nabla_{\mathbf{xv}} G(t - \tau) * f(\tau) d\tau, \quad (56)$$

its derivatives are bounded in terms of the L^q norms of the initial and source data:

$$\|\nabla_{\mathbf{xv}} p_1(t)\|_q \leq \|\nabla_{\mathbf{xv}} p_0\|_q + 2t^{1/2} \max_{s \in [0, t]} \|f(s)\|_q, \quad t \in [0, T]. \quad (57)$$

We have only assumed that $p_0 \in H^1$, H^1 being the usual Sobolev space. Therefore, we set $q = 2$. For any $k \geq 2$, p_k is a solution of a heat equation with a source term $a_{k-1} p_k$, which we have bounded in $L_t^\infty(L_{\mathbf{xv}}^q)$, $1 \leq q \leq \infty$:

$$\frac{\partial}{\partial t} p_k(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} p_k(t, \mathbf{x}, \mathbf{v}) = -a_{k-1}(t, \mathbf{x}) p_k(t, \mathbf{x}, \mathbf{v}) + f(t, \mathbf{x}, \mathbf{v}). \quad (58)$$

This yields $L_{\mathbf{xv}}^q$ bounds on derivatives of p_k . Inequality (57) with $q = 2$ implies:

$$\begin{aligned} \|\nabla_{\mathbf{xv}} p_k(t)\|_2 &\leq \|\nabla_{\mathbf{xv}} p_0\|_2 + 2t^{1/2} \max_{s \in [0, t]} (\|a_{k-1}(s)\|_\infty \|p_k(s)\|_2 + \|f(s)\|_2) \\ &\leq \|\nabla_{\mathbf{xv}} p_0\|_2 + 2t^{1/2} (C(p_0, f, T) + \|f\|_{L^\infty(0, T; L_{\mathbf{xv}}^2)}) \end{aligned} \quad (59)$$

in $[0, T]$, thanks to estimates (54) and (55).

As a consequence, we obtain a uniform bound on $\|p_k(t)\|_{L^2(0, T; H^1(\mathbb{R}^2 \times \mathbb{R}^2))}$ (which might also have been derived from energy inequalities). Uniform bounds on $\|\frac{\partial}{\partial t} p_k(t)\|_{L^2(0, T; H^{-1}(\mathbb{R}^2 \times \mathbb{R}^2))}$ follow then from (48). Notice that the injection $H^1(\Omega) \subset L^2(\Omega)$ is compact for any bounded set Ω [4]. Arguing as in the proof of Proposition 2.2, the compactness results in references [12, 17] (see Theorem 12.1 in [12], Corollary 4 in [17]) imply the existence of a subsequence $p_{k'}$ tending to a limit p strongly on compact sets, that is, in $L^2(0, T; L_{loc}^2(\mathbb{R}^2 \times \mathbb{R}^2))$, and almost everywhere [4]. It also converges weakly in the reflexive Banach spaces in which we have uniform bounds.

3.2. Convergence to a solution

Thanks to the pointwise convergence obtained in the previous step we may pass to the limit in inequality (53) to obtain:

$$0 \leq p \leq p_1(t) \quad \Rightarrow \quad \|p(t)\|_q \leq \|p_1(t)\|_q \leq C(T, \|p_0\|_{L_{\mathbf{xv}}^q}, \|f\|_{L_t^\infty L_{\mathbf{xv}}^q}), \quad (60)$$

for $t \in [0, T]$. The uniform bound (53) also shows that $|p_{k'}|$ is uniformly bounded from above by a function p_1 belonging to $L^r(0, T; L_{\mathbf{xv}}^q)$ for any $1 \leq r, q < \infty$. Lebesgue's dominated convergence theorem implies that $p_{k'}$ converges to p in $L^r(0, T; L_{\mathbf{xv}}^q)$ strongly for any $1 \leq r, q < \infty$.

Recall that $a_{k'-1}(t, \mathbf{x}) = \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p_{k'-1}(s, \mathbf{x}, \mathbf{v}')$. By inequality (53), the integrand satisfies $0 \leq p_{k'-1} \leq p_1$. On the other hand, p_1 is integrable in $[0, t] \times$

$\mathbb{R}^2 \times \mathbb{R}^2$. By Lebesgue's dominated convergence theorem, pointwise convergence implies:

$$a_{k'-1}(t, \mathbf{x}) = \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p_{k'-1}(s, \mathbf{x}, \mathbf{v}') \longrightarrow a(t, \mathbf{x}) = \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p(s, \mathbf{x}, \mathbf{v}'),$$

as k tends to infinity, for any $t \in [0, T]$ and $\mathbf{x} \in \mathbb{R}^2$ fixed. Let us now pass to the limit in the nonlinear term $a_{k'-1}p_{k'}$. It tends to ap almost everywhere. Thanks to estimates (53) and (55), $0 \leq a_{k'-1}p_{k'} \leq a_1p_1$. The upper bound a_1p_1 is integrable in $[0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$ because a_1 is bounded. Lebesgue's dominated convergence theorem yields convergence in L^1 and in the sense of distributions.

As specified above, due to the uniform bounds on $\|p_{k'}(t)\|_{L^2(0, T; H_{\mathbf{x}\mathbf{v}}^1)}$ and $\|\frac{\partial}{\partial t} p_{k'}(t)\|_{L^2(0, T; H_{\mathbf{x}\mathbf{v}}^{-1})}$, $p_{k'}$ tends to p weakly in $L^2(0, T; H_{\mathbf{x}\mathbf{v}}^1)$ and $\frac{\partial}{\partial t} p_{k'}$ tends to $\frac{\partial}{\partial t} p$ in $L^2(0, T; H_{\mathbf{x}\mathbf{v}}^{-1})$ weakly.

Let us write down the weak formulation of the initial value problem (48)-(49). For any $\phi(t, \mathbf{x}, \mathbf{v}) \in C_c^\infty([0, T] \times \mathbb{R}^2 \times \mathbb{R}^2)$,

$$\begin{aligned} & - \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_0(\mathbf{x}, \mathbf{v}) \phi(0, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} - \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(s, \mathbf{x}, \mathbf{v}) \phi(s, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} ds \\ & = \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[\frac{\partial}{\partial t} + \sigma \Delta_{\mathbf{x}\mathbf{v}} - a_{k'-1}(s, \mathbf{x}) \right] \phi(s, \mathbf{x}, \mathbf{v}) p_{k'}(s, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} ds. \end{aligned}$$

Letting $k' \rightarrow \infty$ we find that p is a solution of (51)-(52) in the sense of distributions and in $L^2(0, T; H_{\mathbf{x}\mathbf{v}}^{-1})$.

3.3. Uniqueness result

Let us consider first the integrated problem (43)-(44) for \tilde{p} introduced in Lemma 2.8:

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{p}(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} \tilde{p}(t, \mathbf{x}) + a(t, \mathbf{x}) \tilde{p}(t, \mathbf{x}) &= \tilde{f}(t, \mathbf{x}), \\ \tilde{p}(0, \mathbf{x}) &= \tilde{p}_0(\mathbf{x}), \end{aligned}$$

where $a(t, \mathbf{x}) = \int_0^t ds \tilde{p}(s, \mathbf{x}) \geq 0$.

Let us assume that we have two nonnegative solutions $\tilde{p}^{(1)}$ and $\tilde{p}^{(2)}$ belonging to $L^\infty(0, T; L_{\mathbf{x}}^\infty \cap L_{\mathbf{x}}^1)$. Set $\bar{p} = \tilde{p}^{(1)} - \tilde{p}^{(2)}$ and $\bar{a} = a^{(1)} - a^{(2)}$, with $a^{(i)}(t, \mathbf{x}) = \int_0^t ds \tilde{p}^{(i)}(s, \mathbf{x}) \geq 0$, $i = 1, 2$. Subtracting the equations for $\tilde{p}^{(1)}$ and $\tilde{p}^{(2)}$ we find:

$$\frac{\partial}{\partial t} \bar{p} - \sigma \Delta_{\mathbf{x}} \bar{p} + a^{(1)} \bar{p} = -\tilde{p}^{(2)}(a^{(1)} - a^{(2)}) = -\tilde{p}^{(2)} \int_0^t (\tilde{p}^{(1)} - \tilde{p}^{(2)})(s) ds, \quad (61)$$

$$\bar{p}(0) = 0. \quad (62)$$

Observing that $a^{(1)} \geq 0$, the energy inequality in Lemma 2.9 yields:

$$\frac{1}{2} \|\bar{p}(t)\|_2^2 + \sigma \int_0^t \|\nabla_{\mathbf{x}} \bar{p}(s)\|_2^2 ds \leq - \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{x} \tilde{p}^{(2)}(s, \mathbf{x}) \bar{a}(s, \mathbf{x}) \bar{p}(s, \mathbf{x}). \quad (63)$$

Set $M_{\hat{T}} = \max_{s \in [0, \hat{T}]} \|\bar{p}(s)\|_2$, $\hat{T} \leq T$, and $|\tilde{p}^{(2)}(t, \mathbf{x})| \leq M$ for $0 \leq t \leq T$ and $\mathbf{x} \in \mathbb{R}^2$. Thanks to Jensen's inequality for convex functions the following inequalities hold:

$$\begin{aligned} \left| \int_0^s \bar{p}(s', \mathbf{x}) ds' \right|^2 &= |\bar{a}(s, \mathbf{x})|^2 \leq s \int_0^s ds' |\bar{p}(s', \mathbf{x})|^2 \\ \Rightarrow \|\bar{a}(s)\|_2^2 &\leq s \int_0^s ds' \|\bar{p}(s')\|_2^2 \leq s^2 M_s^2. \end{aligned} \quad (64)$$

Inserting (64) in (63), we obtain:

$$\frac{1}{2} \|\bar{p}(t)\|_2^2 \leq M(M_{\hat{T}})^2 \frac{\hat{T}^2}{2}, \quad t \in [0, \hat{T}], \quad (65)$$

which implies:

$$(1 - M\hat{T}^2)M_{\hat{T}}^2 \leq 0.$$

If $\hat{T} < \frac{1}{\sqrt{M}}$, this implies $M_{\hat{T}} = 0$ and $\bar{p} = 0$ in $[0, \hat{T}]$. The procedure can be repeated at time \hat{T} to get $\bar{p} = 0$ in $[\hat{T}, 2\hat{T}]$. Iteratively, we find $\bar{p} = 0$ up to time T , thus $a^{(1)} = a^{(2)} = a$. Then, $p^{(1)}$ and $p^{(2)}$ are solutions of the same linear equations, with the same initial and source data, and the same coefficient a . Therefore, they are equal (as a consequence of either Proposition 2.2 or Lemma 2.9) and the constructed solution is unique.

3.4. Regularity of the solutions

We have constructed a solution of:

$$\frac{\partial}{\partial t} p - \sigma \Delta_{\mathbf{xv}} p = -a p + f, \quad p(0) = p_0 \quad (66)$$

where $a = \int_0^t ds \int_{\mathbb{R}^2} d\mathbf{v}' p(s, \mathbf{x}, \mathbf{v}')$. This solution satisfies:

$$0 \leq p \leq p_1, \quad 0 \leq a \leq a_1,$$

using first (60) and then integrating in velocity and time. In view of the L^∞ bound on the coefficient a , the term $a p$ belongs to $L_t^\infty(L_{\mathbf{xv}}^q)$ for any $1 \leq q \leq \infty$. The regularity of solutions of heat equations implies that the derivatives of p with respect to any variable remain in $L_{\mathbf{xv}}^q$. However, the $L_{\mathbf{xv}}^q$ norms become singular as $t \rightarrow 0$ unless we assume regularity of the derivatives of the initial data.

Assuming $\nabla_{\mathbf{xv}} p_0 \in L_{\mathbf{xv}}^q$, the integral reformulation of the heat equation (66) in terms of its heat kernel G

$$\nabla_{\mathbf{xv}} p(t) = G(t) * \nabla_{\mathbf{xv}} p_0 + \int_0^t \nabla_{\mathbf{xv}} G(t - \tau) * [f(\tau) - a(\tau)p(\tau)] d\tau,$$

yields

$$\|\nabla_{\mathbf{xv}} p(t)\|_q \leq \|\nabla_{\mathbf{xv}} p_0\|_q + 2t^{1/2} \max_{s \in [0, t]} \|f(s) - a(s)p(s)\|_q, \quad t \in [0, T]. \quad (67)$$

Otherwise, the alternative expression

$$\nabla_{\mathbf{xv}}p(t) = \nabla_{\mathbf{xv}}G(t) * p_0 + \int_0^t \nabla_{\mathbf{xv}}G(t - \tau) * [f(\tau) - a(\tau)p(\tau)]d\tau,$$

only implies

$$\|\nabla_{\mathbf{xv}}p(t)\|_q \leq t^{-1/2}\|p_0\|_q + 2t^{1/2}\max_{s \in [0, t]} \|f(s) - a(s)p(s)\|_q, \quad t \in [0, T].$$

Once estimates on the first order derivatives are available, second order derivatives can be estimated in a similar way splitting the derivatives between the heat kernel and the source, provided the derivatives of f also belong to $L^q_{\mathbf{xv}}$, and the derivatives of a are bounded functions. The regularity of the time derivatives follows using the differential equation.

4. Coupling with the diffusion equation

Let us consider now the full problem (5)-(8) coupling the density p to the variable c . The equation for the density includes now a linear source in p :

$$\frac{\partial}{\partial t}p(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}}p(t, \mathbf{x}, \mathbf{v}) + \gamma a(t, \mathbf{x})p(t, \mathbf{x}, \mathbf{v}) = \alpha(c(t, \mathbf{x}))\rho(\mathbf{v})p(t, \mathbf{x}, \mathbf{v}).$$

$\rho(\mathbf{v})$ is a smooth, bounded and integrable positive function. This equation is coupled with a diffusion equation for c :

$$\frac{\partial}{\partial t}c(t, \mathbf{x}) - d\Delta_{\mathbf{x}}c(t, \mathbf{x}) = -\eta c(t, \mathbf{x})j(t, \mathbf{x}).$$

Let us recall that:

$$\alpha(c) = \alpha_1 \frac{\frac{c}{c_R}}{1 + \frac{c}{c_R}}, \quad j(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}'| p(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}', \quad a(t, \mathbf{x}) = \int_0^t \int_{\mathbb{R}^2} p(s, \mathbf{x}, \mathbf{v}') d\mathbf{v}' ds.$$

The function c is expected to decay at infinity, except for a finite interval of x_2 for which it tends to a constant k_∞ as x_1 grows. For any $t > 0$ and $x_2 \in [a, b] \subset \mathbb{R}$,

$$c(t, x_1, x_2) \rightarrow k_\infty \quad \text{as } x_1 \rightarrow \infty. \quad (68)$$

That interval represents the location of a confined distant source. We impose the same behavior on $c(0) = c_0 \geq 0$. Writing $c = c_\infty + \hat{c}$ where c_∞ is a solution of the heat equation with the same initial datum, \hat{c} is a solution of:

$$\frac{\partial}{\partial t}\hat{c}(t, \mathbf{x}) = d\Delta_{\mathbf{x}}\hat{c}(t, \mathbf{x}) - \eta\hat{c}(t, \mathbf{x})j(t, \mathbf{x}) - \eta c_\infty(t, \mathbf{x})j(t, \mathbf{x}), \quad (69)$$

vanishing at infinity with initial datum $\hat{c}(0) = 0$.

The following existence and uniqueness result holds:

Theorem 4.1. *There exists a unique nonnegative solution (p, c) of (5)-(8) satisfying:*

$$\begin{aligned} p &\in L^2(0, T; H^1(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)), \\ p, |\mathbf{v}|^2 p &\in L^\infty(0, T; L^\infty \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2))), \\ \hat{c} = c - c_\infty &\in L^2(0, T; H^1(\mathbb{R}^2)) \cap L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^2)), \end{aligned}$$

when $p_0 \in L^\infty \cap L^1 \cap H^1(\mathbb{R}^2 \times \mathbb{R}^2)$, $p_0, |\mathbf{v}|^2 p_0 \in L^\infty \cap L^1_{\mathbf{x}}(\mathbb{R}^2; L^1_{\mathbf{v}}(\mathbb{R}^2))$, $p_0 \geq 0$ and $c_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, $c_0 \geq 0$. The norms of this solution are bounded in terms of the norms of the data.

The existence proof relies on an iterative scheme. After showing that the scheme is well defined, we obtain uniform a priori bounds on p_k , a_k , j_k and c_k . A solution of (5)-(8) follows passing to the limit. This solution inherits the bounds on the iterates in terms of the norms of the data, which implies stability of the solution. Uniqueness follows from integral inequalities. We detail the proofs in the next four subsections.

4.1. Iterative scheme

For $k \geq 2$ we consider the iterative scheme:

$$\begin{aligned} \frac{\partial}{\partial t} p_k(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} p_k(t, \mathbf{x}, \mathbf{v}) + \gamma a_{k-1}(t, \mathbf{x}) p_k(t, \mathbf{x}, \mathbf{v}) & \quad (70) \\ & = \alpha(c_{k-1}(t, \mathbf{x})) \rho(\mathbf{v}) p_k(t, \mathbf{x}, \mathbf{v}), \end{aligned}$$

$$\frac{\partial}{\partial t} c_{k-1}(t, \mathbf{x}) = d \Delta_{\mathbf{x}} c_{k-1}(t, \mathbf{x}) - \eta c_{k-1}(t, \mathbf{x}) j_{k-1}(t, \mathbf{x}). \quad (71)$$

We initialize the iteration setting $p_1 = 0$. c_1 is the solution of (71) with initial datum c_0 . Let us show that the iterative scheme is well defined. This follows using the fundamental solutions of the corresponding linear problems with bounded coefficients, the integral expressions for their solutions and the upper uniform bounds for the fundamental solutions involving constants depending on the L^∞ norm of the coefficients, as we argue by induction.

Let us assume that $a_{k-1} = \int_0^t ds \int d\mathbf{v}' p_{k-1}(s, \mathbf{x}, \mathbf{v}') \geq 0$ and $c_{k-1} \geq 0$ are bounded. By Proposition 2.2 there exists a unique positive solution p_k of the initial value problem for (70), which admits an integral expression in terms of the fundamental solution introduced in Lemma 2.1. This implies that a_k, j_k are bounded functions and $a_k, j_k \geq 0$. Indeed, the fundamental solution is positive and bounded from above by (25). This yields the L^1, L^∞ bounds (18)-(19) on p_k and, by Lemma 2.6, L^∞ bounds on a_k, j_k with constants depending on $\sigma, T, \gamma, \|a_{k-1}\|_\infty, \|\rho\|_\infty$ and α_1 .

For any bounded j_k , we construct a positive solution c_k of (71) using the corresponding fundamental solution, thanks to Lemma 2.1 and Proposition 2.3. This yields the L^∞ bound (18) on c_k with constants depending on d, T, η , and $\|j_k\|_\infty$. Moreover, c_k are positive bounded functions.

Therefore, we may repeat the procedure and construct p_{k+1}, c_{k+1} enjoying the same properties. The iterative sequence is well defined.

4.2. Uniform estimates

Uniform estimates with respect to k are a consequence of the positivity of the solutions and adequate comparison principles.

To obtain uniform L^q estimates on p_k we resort to the comparison principle in Lemma 2.5. Since $\gamma a_{k-1} p_k \geq 0$, the functions p_k are bounded from above by the solution of the heat equation with the same initial datum and source $\alpha(c_{k-1})\rho p_k$:

$$\frac{\partial}{\partial t} u_k(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} u_k(t, \mathbf{x}, \mathbf{v}) = \alpha(c_{k-1}(t, \mathbf{x})) \rho(\mathbf{v}) p_k(t, \mathbf{x}, \mathbf{v}). \quad (72)$$

Using the integral expression (35) for u_k and $p_k \leq u_k$, we get the inequality:

$$\|p_k\|_{L_{\mathbf{xv}}^q} \leq \|u_k\|_{L_{\mathbf{xv}}^q} \leq \|p_0\|_{L_{\mathbf{xv}}^q} + \alpha_1 \|\rho\|_{L_{\mathbf{v}}^\infty} \int_0^t \|p_k(s)\|_{L_{\mathbf{xv}}^q} ds. \quad (73)$$

Applying Gronwall's lemma, we find:

$$\|p_k(t)\|_{L_{\mathbf{xv}}^q} \leq \|p_0\|_{L_{\mathbf{xv}}^q} e^{t\alpha_1 \|\rho\|_\infty}, \quad t \in [0, T], \quad 1 \leq q \leq \infty. \quad (74)$$

Applying again Lemmas 2.4 and 2.5, $p_k \leq u_k \leq \mathcal{P}$, where \mathcal{P} is a solution of

$$\frac{\partial}{\partial t} \mathcal{P}(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} \mathcal{P}(t, \mathbf{x}, \mathbf{v}) = \alpha_1 \|\rho\|_\infty \mathcal{P}(t, \mathbf{x}, \mathbf{v}), \quad (75)$$

with the same initial datum. Changing variables, it comes out that $\mathcal{P}(t) = e^{\alpha_1 \|\rho\|_\infty t} (G(t) *_{\mathbf{xv}} p_0)$, where $G(t)$ is the heat kernel for diffusivity σ .

Now, Lemma 2.8 yields the following equation for $\tilde{p}_k(s, \mathbf{x}) = \int_{\mathbb{R}^2} p_k(s, \mathbf{x}, \mathbf{v}) d\mathbf{v}$:

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{p}_k(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} \tilde{p}_k(t, \mathbf{x}) &= \alpha(c_{k-1}(t, \mathbf{x})) \int_{\mathbb{R}^2} d\mathbf{v} \rho(\mathbf{v}) p_k(t, \mathbf{v}, \mathbf{x}) - \gamma a_{k-1}(t, \mathbf{x}) \tilde{p}_k(t, \mathbf{x}) \\ &\leq \alpha_1 \|\rho\|_\infty \tilde{p}_k(t, \mathbf{x}), \end{aligned} \quad (76)$$

where $a_{k-1}(t, \mathbf{x}) = \int_0^t \tilde{p}_{k-1}(s, \mathbf{x}) ds$ and $\tilde{p}_k(0, \mathbf{x}) = \tilde{p}_0(\mathbf{x})$. By Lemma 2.4, \tilde{p}_k is bounded from above by the solution of a heat equation with source $\alpha_1 \|\rho\|_\infty \tilde{p}_k(t, \mathbf{x})$. Repeating the Gronwall argument used to estimate $\|p_k\|_{L^q_{\mathbf{xv}}}$ we find that

$$\|\tilde{p}_k(t)\|_{L_{\mathbf{x}}^q} \leq \|\tilde{p}_0\|_{L_{\mathbf{x}}^q L_{\mathbf{v}}^1} e^{t\alpha_1 \|\rho\|_\infty}, \quad t \in [0, T], \quad 1 \leq q \leq \infty. \quad (77)$$

Therefore, a_k is uniformly bounded in $L^\infty(0, T; L_{\mathbf{x}}^q(\mathbb{R}^2))$ for any $q \in [1, \infty]$ and any $T > 0$. Notice that estimate (77) would also follow directly taking into account that $p_k \leq \mathcal{P}$, with \mathcal{P} defined in (75). Integrating with respect to \mathbf{v} , we find $0 \leq \tilde{p}_k \leq \tilde{\mathcal{P}}$ and, consequently, estimate (77). Integrating in time, we find $0 \leq a_k \leq \int_0^t \tilde{\mathcal{P}}(s, \mathbf{x}) ds$.

Uniform bounds on the derivatives of p_k are obtained observing that the functions p_k solve heat equations with uniformly bounded sources in $L^\infty(0, T; L_{\mathbf{xv}}^q)$ for all $1 \leq q \leq \infty$:

$$\frac{\partial}{\partial t} p_k(t, \mathbf{x}, \mathbf{v}) - \sigma \Delta_{\mathbf{xv}} p_k(t, \mathbf{x}, \mathbf{v}) = (\alpha(c_{k-1}(t, \mathbf{x})) \rho(\mathbf{v}) - \gamma a_{k-1}(t, \mathbf{x})) p_k(t, \mathbf{x}, \mathbf{v}), \quad (78)$$

as in (56)-(59). As discussed in subsection 3.4, the derivatives of p_k with respect to any variable belong to $L^q_{\mathbf{xv}}$ for all $1 \leq q \leq \infty$ and all $t > 0$. When $p_0 \in H^1(\mathbb{R}^2 \times \mathbb{R}^2)$, the partial derivatives $\frac{\partial p_k}{\partial z_i} \in L^\infty(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$, for $z_i = x_i$ and $z = v_i$, $i = 1, 2$, thanks to inequality (67). A uniform bound for p_k in $L^2(0, T, H^1_{\mathbf{xv}})$ follows. Equation (70) yields then a uniform bound on the time derivatives $\|\frac{\partial}{\partial t} p_k(t)\|_{L^2(0, T; H^{-1}_{\mathbf{xv}})}$.

Now, we need uniform estimates on j_k . Since we know that the L^∞ norms of the coefficients $\alpha(c_{k-1})\rho - \gamma a_{k-1}$ in equation (70) are uniformly bounded, we may resort to Lemma 2.6 to obtain direct uniform estimates on $\|j_k\|_{L^q_t L^q_x} = \|j(p_k)\|_{L^q_t L^q_x}$ in terms of $\|\tilde{p}_0\|_{L^q_t L^q_x}$ and $\|j(p_0)\|_{L^q_t L^q_x}$. Alternatively, we can resort to uniform bounds on $m_k = \int |\mathbf{v}|^2 p_k d\mathbf{v}$ obtained from differential inequalities provided by Lemma 2.10:

$$\begin{aligned} \frac{\partial}{\partial t} m_k(t, \mathbf{x}) - \sigma \Delta_{\mathbf{x}} m_k(t, \mathbf{x}) &= \alpha(c_{k-1}(t, \mathbf{x})) \int_{\mathbb{R}^2} d\mathbf{v} \rho(\mathbf{v}) |\mathbf{v}|^2 p_k(t, \mathbf{x}, \mathbf{v}) \\ &+ (4\sigma - \gamma a_{k-1}(t, \mathbf{x})) m_k(t, \mathbf{x}) \leq (\alpha_1 \|\rho\|_\infty + 4\sigma) m_k(t, \mathbf{x}). \end{aligned} \quad (79)$$

In view of inequality (79), Lemma 2.4 and 2.5 imply that $m_k \geq 0$ is bounded from above by the solution \mathcal{M} of a heat equation with source $(\alpha_1 \|\rho\|_\infty + 2\sigma) m_k(t, \mathbf{x})$. Repeating the Gronwall argument:

$$\|m_k(t)\|_{L^q_x} \leq \| |\mathbf{v}|^2 p_0 \|_{L^q_x L^1_v} e^{t(\alpha_1 \|\rho\|_\infty + 2\sigma)}, \quad t \in [0, T], \quad 1 \leq q \leq \infty. \quad (80)$$

In fact, $0 \leq m_k \leq \mathcal{M} = e^{(\alpha_1 \|\rho\|_\infty + 2\sigma)t} (G(t) *_{\mathbf{x}} m_0)$, where $G(t)$ is the heat kernel for diffusivity σ .

Set $\mathbf{v} = (v_1, v_2)$. Notice that, for any $R > 0$:

$$j_k = \int_{\mathbb{R}^2} |\mathbf{v}| p_k d\mathbf{v} \leq \left[R \int_{|\mathbf{v}| \leq R} p_k d\mathbf{v} + \int_{|\mathbf{v}| > R} \frac{|\mathbf{v}|^2}{R} p_k d\mathbf{v} \right] \leq (R \tilde{p}_k + \frac{1}{R} m_k). \quad (81)$$

Therefore, j_k is uniformly bounded in $L^\infty(0, T; L^q_x(\mathbb{R}^2 \times \mathbb{R}^2))$ for any $q \in [1, \infty]$ and $T > 0$.

Let us now obtain uniform estimates on c_k . The source term in (71) being negative, c_k is uniformly bounded from above by the solution c_∞ of the heat equation with the same initial datum c_0 by Lemma 2.5. Thus, $\|c_k(t)\|_\infty \leq \|c_\infty\|_\infty \leq \|c_0\|_\infty$ for all $t \in [0, T]$. Writing down the equation satisfied by $\hat{c}_k = c - c_\infty$,

$$\frac{\partial}{\partial t} \hat{c}_k(t, \mathbf{x}) - d \Delta_{\mathbf{x}} \hat{c}_k(t, \mathbf{x}) = -\eta c_k(t, \mathbf{x}) j_k(t, \mathbf{x}), \quad \hat{c}_k(0, \mathbf{x}) = 0, \quad (82)$$

we see that the source term is uniformly bounded in $L^\infty(0, T; L^q_x(\mathbb{R}^2))$ for any $q \in [1, \infty]$. Then, $\hat{c}_k, \nabla_{\mathbf{x}} \hat{c}_k \in L^\infty(0, T; L^2_x(\mathbb{R}^2))$ thanks to Lemma 2.5 and inequality (67). A uniform bound for \hat{c}_k in $L^2(0, T; H^1_x)$ follows. Equation (82) yields then a uniform estimate on the time derivatives $\|\frac{\partial}{\partial t} \hat{c}_k(t)\|_{L^2(0, T; H^{-1}_x)}$.

4.3. Passage to the limit

The densities p_k are uniformly bounded in $L^2(0, T; H_{\mathbf{xv}}^1)$ and their time derivatives are bounded in $L^2(0, T; H_{\mathbf{xv}}^{-1})$. The modified variables \hat{c}_k are bounded in $L^2(0, T; H_{\mathbf{x}}^1)$ and their time derivatives are bounded in $L^2(0, T; H_{\mathbf{x}}^{-1})$. Since the injection $H^1(\Omega) \subset L^2(\Omega)$ is compact for any bounded Ω , the classical compactness results in [12, 17] imply compactness of p_k and \hat{c}_k in $L^2(0, T; L_{loc}^2)$, that is, over bounded sets. Arguing as in the proof of Proposition 2.2, we extract subsequences $p_{k'}$, $\hat{c}_{k'}$ tending pointwise and strongly to limits p and \hat{c} in $L^2(0, T; L_{\mathbf{xv}}^2(\Omega))$ and $L^2(0, T; L_{\mathbf{x}}^2(\omega))$, respectively, for bounded sets Ω and ω . Weak convergence implies that $p \in L^2(0, T; H_{\mathbf{xv}}^1)$ and $\hat{c} \in L^2(0, T; H_{\mathbf{x}}^1)$. Weak convergences also imply that the limits p and \hat{c} inherit the bounds established on the converging sequences.

The subsequences $p_{k'}$ and $\hat{c}_{k'}$ converge to p and \hat{c} pointwise almost everywhere. We know that $0 \leq p_{k'} \leq \mathcal{P}$, where \mathcal{P} is defined in (75) and satisfies $\mathcal{P} \in L^r(0, T; L_{\mathbf{xv}}^q)$, and any $1 \leq r, q < \infty$. Lebesgue's dominated convergence theorem implies that $p_{k'}$ converges to p in $L^r(0, T; L_{\mathbf{xv}}^q)$. The passage to the limit in $a_{k'-1}$ and $a_{k'-1}p_{k'}$ proceeds as in Section 3.2.

Combining continuity of $\alpha(x)$ and pointwise convergence of $c_{k'}$ to $\hat{c} + c_\infty = c$, we obtain pointwise convergence of $\alpha(c_{k'-1})$ to $\alpha(c)$. The positive term $\alpha(c_{k'-1})p_{k'}$ converges almost everywhere to $\alpha(c)p$ and is bounded by $\alpha_1\mathcal{P}$. Therefore, it converges strongly in any $L^r(0, T; L_{\mathbf{xv}}^q)$.

Recall that $j_{k'}(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}| p_{k'}(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$. The integrand satisfies $|\mathbf{v}| p_{k'} \leq |\mathbf{v}| \mathcal{P}$, which is integrable over $[0, t] \times \mathbb{R}^2 \times \mathbb{R}^2$. By Lebesgue's dominated convergence theorem,

$$j_{k'}(t, \mathbf{x}) \longrightarrow j(t, \mathbf{x}) = \int_{\mathbb{R}^2} |\mathbf{v}| p(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$

as k tends to infinity, for any $t \in [0, T]$ and $\mathbf{x} \in \mathbb{R}^2$ fixed. Let us now pass to the limit in the nonlinear term $c_{k'}j_{k'}$. It tends to cj almost everywhere. Using (81), we find $|c_{k'}j_{k'}| \leq (R\tilde{\mathcal{P}} + \frac{1}{R}\mathcal{M})$, which is integrable in $[0, T] \times \mathbb{R}^2$. Lebesgue's dominated convergence theorem yields convergence in L^1 and in the sense of distributions.

Passing to the limit in weak versions of the equations, we find that (p, c) is a solution of (5)-(8) in the sense of distributions and in $L^2(0, T; H_{\mathbf{xv}}^{-1})$.

4.4. Uniqueness

Uniqueness follows subtracting the equations for two possible sets of solutions p_1, p_2, c_1, c_2 . Let us set $\bar{p} = p_1 - p_2$ and $\bar{c} = c_1 - c_2$. These differences satisfy the equations:

$$\frac{\partial}{\partial t} \bar{p} - \sigma \Delta_{\mathbf{xv}} \bar{p} + [\gamma a(p_1) - \alpha(c_1)\rho] \bar{p} = [-\gamma a(\bar{p}) + (\alpha(c_1) - \alpha(c_2))\rho] p_2, \quad (83)$$

$$\frac{\partial}{\partial t} \bar{c} - d \Delta_{\mathbf{x}} \bar{c} + \eta j(p_1) \bar{c} = -\eta j(\bar{p}) c_2, \quad (84)$$

with $\bar{p}(0) = 0$ and $\bar{c}(0) = 0$. The mean value theorem applied to the definition $\alpha(c)$ yields:

$$|\alpha(c_1) - \alpha(c_2)| = \alpha_1 \left| \frac{c_1}{c_R + c_1} - \frac{c_2}{c_R + c_2} \right| = \frac{\alpha_1 c_R}{(c_R + \xi)^2} |c_1 - c_2| \leq \frac{\alpha_1}{c_R} |c_1 - c_2|, \quad (85)$$

where $\xi \in [c_1, c_2]$. Since c_1 and c_2 are nonnegative, $\xi \geq 0$.

Let Γ_p and Γ_c denote the fundamental solutions provided by Lemma 2.1 associated to the parabolic operators $\frac{\partial}{\partial t} \bar{p} - \sigma \Delta_{\mathbf{x}\mathbf{v}} \bar{p} + [\gamma a(p_1) - \alpha(c_1) \rho] \bar{p}$, and $\frac{\partial}{\partial t} \bar{c} - d \Delta_{\mathbf{x}} \bar{c} + \eta j(p_1) \bar{c}$, respectively. By the regularity properties of the solution (c_1, p_1) , their coefficients are bounded functions. Particularizing the integral expressions (17) provided by Propositions 2.2 and 2.3 for p and c , and using the upper bounds (25) on the fundamental solutions Γ_p and Γ_c , we find:

$$\begin{aligned} \|\bar{p}(t)\|_{L^1_{\mathbf{x}\mathbf{v}}} &\leq C_p \left[\gamma \|p_2\|_{L_t^\infty L_x^\infty L_v^1} \int_0^t ds \int_0^s d\tau \|\bar{p}(\tau)\|_{L^1_{\mathbf{x}\mathbf{v}}} \right. \\ &\quad \left. + \frac{\alpha_1 \|\rho\|_\infty}{c_R} \|p_2\|_{L_t^\infty L_x^\infty L_v^1} \int_0^t ds \|\bar{c}(s)\|_{L^1_{\mathbf{x}}} \right], \end{aligned} \quad (86)$$

$$\|\bar{c}(t)\|_{L^1_{\mathbf{x}}} \leq C_c \eta \|c_2\|_{L_t^\infty L_x^\infty} \int_0^t ds \|j(\bar{p})(s)\|_{L^1_{\mathbf{x}}}, \quad (87)$$

thanks to (85). The constants C_p and C_c depend on σ , the dimension, T , and the L^∞ norm of the coefficients.

Now, notice that $\|j(\bar{p})\|_{L^1_{\mathbf{x}}} \leq \|\mathbf{v}|\bar{p}\|_{L^1_{\mathbf{x}\mathbf{v}}}$. This latter norm can be estimated as done in Lemma 2.6. Using again the integral expression (17) for \bar{p} , multiplying by $|\mathbf{v}|$, taking absolute values, and integrating we obtain:

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}| |\bar{p}(t, \mathbf{x}, \mathbf{v})| d\mathbf{v} d\mathbf{x} \leq \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}| |\Gamma_p(t, \mathbf{x}, \mathbf{v}; s, \mathbf{x}', \mathbf{v}')| |f(s, \mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' ds d\mathbf{x} d\mathbf{v} = I, \quad (88)$$

with $f = [-\gamma a(\bar{p}) + (\alpha(c_1) - \alpha(c_2)) \rho] p_2$. Thanks to estimate (25),

$$\Gamma_p(t, \mathbf{x}, \mathbf{v}; s, \mathbf{x}', \mathbf{v}') \leq C_p G(t - s, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}'),$$

for a heat kernel G . Then, for $t \in [0, T]$, we have $I \leq I_1 + I_2$, where

$$I_1 = C_p \int_0^t (t - s)^{\frac{1}{2}} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} G(t - s, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') \frac{|\mathbf{v} - \mathbf{v}'|}{(t - s)^{1/2}} |f(s, \mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' ds d\mathbf{x} d\mathbf{v},$$

$$I_2 = C_p \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} G(t - s, \mathbf{x} - \mathbf{x}', \mathbf{v} - \mathbf{v}') |\mathbf{v}'| |f(s, \mathbf{x}', \mathbf{v}')| d\mathbf{x}' d\mathbf{v}' ds d\mathbf{x} d\mathbf{v}.$$

Using the property of convolutions $\|a * b\|_1 \leq \|a\|_1 \|b\|_1$ and computing the norms involving heat kernels, we find:

$$I_1 \leq M_1 \int_0^t \|f(s)\|_{L^1_{\mathbf{x}\mathbf{v}}} ds, \quad I_2 \leq M_2 \int_0^t \|\mathbf{v} |f(s)\|_{L^1_{\mathbf{x}\mathbf{v}}} ds.$$

The norm $\|f\|_{L^1_{\mathbf{xv}}}$ has already been estimated in terms of the right hand side in inequality (86) and $\|\mathbf{v}|f|\|_{L^1_{\mathbf{xv}}}$ is similarly bounded taking into account the weight $|\mathbf{v}|$. Inserting this information in inequality (88), we get:

$$\begin{aligned} \|\mathbf{v}|\bar{p}(t)\|_{L^1_{\mathbf{xv}}} &\leq T \left[\gamma [M_1 \|p_2\|_{L_t^\infty L_x^\infty L_v^1} + M_2 \|\mathbf{v}|p_2|\|_{L_t^\infty L_x^\infty L_v^1}] \int_0^t ds \|\bar{p}(s)\|_{L^1_{\mathbf{xv}}} \right. \\ &\quad \left. + \frac{\alpha_1 \|\rho\|_\infty}{c_R} [M_1 \|p_2\|_{L_t^\infty L_x^\infty L_v^1} + M_2 \|\mathbf{v}|p_2|\|_{L_t^\infty L_x^\infty L_v^1}] \int_0^t ds \|\bar{c}(s)\|_{L^1_x} \right]. \end{aligned} \quad (89)$$

Notice that $\int_0^t ds \int_0^s d\tau \|\bar{p}(\tau)\|_{L^1_{\mathbf{xv}}} \leq T \int_0^t ds \|\bar{p}(s)\|_{L^1_{\mathbf{xv}}}$.

Combining inequalities (86), (87) and (89), we find:

$$\|\bar{p}(t)\|_{L^1_{\mathbf{xv}}} \leq A \int_0^t ds \|\bar{p}(s)\|_{L^1_{\mathbf{xv}}} + B \int_0^t ds \|\mathbf{v}|\bar{p}(s)\|_{L^1_{\mathbf{xv}}}, \quad (90)$$

$$\|\mathbf{v}|\bar{p}(t)\|_{L^1_{\mathbf{xv}}} \leq \hat{A} \int_0^t ds \|\bar{p}(s)\|_{L^1_{\mathbf{xv}}} + \hat{B} \int_0^t ds \|\mathbf{v}|\bar{p}(s)\|_{L^1_{\mathbf{xv}}}. \quad (91)$$

Setting $U(t) = \max\{\|\bar{p}(t)\|_{L^1_{\mathbf{xv}}}, \|\mathbf{v}|\bar{p}(t)\|_{L^1_{\mathbf{xv}}}\}$, we deduce from (90)-(91) that $U(t)$ satisfies a Gronwall inequality of the form:

$$U(t) \leq D \int_0^t U(s) ds$$

for $t \in [0, T]$, with $D > 0$. Therefore, Gronwall's lemma implies $U = 0$ and $p_1 = p_2$ in $[0, T]$ for any $T > 0$. Then, estimate (87) implies $c_1 = c_2$.

Remark 4.2. The hypotheses $p_0 \in H^1_{\mathbf{xv}}$ is required to establish that the iterates p_k satisfy $\nabla_{\mathbf{xv}} p_k \in L^2((0, T) \times \mathbb{R}^2 \times \mathbb{R}^2)$ through integral equations for heat equations. This bound still holds true when $p_0 \in L^2_{\mathbf{xv}}$ resorting to energy inequalities as in Lemma 2.9 instead. Therefore, $H^1_{\mathbf{xv}}$ regularity is not needed to establish the existence and uniqueness of a solution.

Remark 4.3. We have seen in the proof that the hypotheses on $|\mathbf{v}|^2 p_0$ can be replaced by hypotheses on $|\mathbf{v}| p_0$ exploiting the integral equations according to Lemma 2.6 instead of the differential inequalities provided by Lemma 2.10. Also, the uniqueness proof only needs information on $|\mathbf{v}| p$. However, arguments based on differential inequalities are more likely to apply when trying to extend these results to bounded sets in space $\Omega \subset \mathbb{R}^2$. Thus, it is worth keeping in mind both procedures.

Remark 4.4. The same existence result holds replacing j by $|\mathbf{j}|$, with essentially the same proof.

5. Discussion and future work

Models for angiogenesis display mathematical structures of increasing complexity, which require the introduction of adequate strategies for their analysis

and numerical simulation. We have considered here a simplified model, including regularizations that are frequent in numerical approximations: replacement of Dirac measures by gaussians and inclusion of viscosity in degenerate directions. We have shown that nonnegative solutions of these regularized models may be constructed as limits of solutions of an iterative scheme, obtaining stability bounds in terms of the norms of the data. Uniqueness conditions are also established. Whether regularized problems approximating measure valued coefficients with gaussians can be shown to effectively converge to the original measure valued problem even in our simpler framework is an open issue.

The main ingredient missing in the model considered here is the transport operator in the equation for the blood vessel density. This operator describes blood vessel extension in response to the chemotactic force created by the concentration of tumor angiogenic factor. In principle, more realistic models including such transport operators might be handled implementing a similar iterative procedure relying on fundamental solutions of Fokker-Planck operators for the blood vessel density, instead of fundamental solutions of a diffusion operator.

Acknowledgements. This work has been supported by MICINN and MINECO grants no. FIS2011-28838-C02-02 and No. MTM2014-56948-C2-1-P.

References

- [1] D.G. Aronson, Bounds for the fundamental solution of a parabolic equations, Bull. of the AMS 73, 890-896, 1967.
- [2] D.G. Aronson, Nonnegative solutions of linear parabolic equations, Ann. Sci. Norm. Sup. Pisa 22, 607-694, 1968.
- [3] J.P. Aubin, Un théorème de compacité, C. R. Acad. Sci. 256, 5042-5044, 1963.
- [4] H. Brézis, Functional analysis, Sobolev spaces and partial differential equations, Springer, 2011.
- [5] LL Bonilla, V Capasso, M. Alvaro, M. Carretero, Hybrid modelling of tumor induced angiogenesis, Phys. Rev. E 90, 062716, 2014.
- [6] V. Capasso, D. Morale, G. Facchetti, Randomness in self-organized phenomena. A case study: Retinal angiogenesis, BioSystems 112, 292-297, 2013.
- [7] V. Capasso, D. Morale, G. Facchetti, The role of stochasticity in a model of retinal angiogenesis, IMA J Appl Math 77 (6), 729-747 (2012).
- [8] A. Carpio, Long time behavior of solutions of the Vlasov-Poisson-Fokker-Planck equation, Math. Meth. Appl. Sc., 21, 985-1014, 1998.
- [9] T. Cazenave, A. Haraux, An introduction to semilinear evolution equations, Oxford University Press, 1998.

- [10] A. Friedman, Partial differential equations of parabolic type, Dover, 2008.
- [11] S. Kusuoka, Hölder continuity and bounds for fundamental solutions to nondivergence form parabolic equations, *Analysis & PDE*, 8, 1-32, 2015.
- [12] J.L. Lions, Quelques méthodes pour les problèmes aux limites nonlinéaires, Gauthier-Villards, 1969.
- [13] J.R. Norris, D.W. Stroock, Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients, *Proc. London Math. Soc. (3)* 62, 373-402, 1991.
- [14] A. Pazy, Semigroups of linear operators and applications to partial differential equations, *Applied Mathematical Sciences* 44, Springer-Verlag 1983.
- [15] F.O. Porper, S.D. Eidel'man, Two-sided estimates of fundamental solutions of second order parabolic equations, and some applications, *Russian Math. Surveys* 39(3), 119-178, 1984.
- [16] M.H. Protter, H.F. Weinberger, Maximum principles in differential equations, Springer, 1999.
- [17] J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura ed Applicata (IV)* CXLVI, 65-96, 1987.
- [18] D.W. Stroock, Diffusion semigroups corresponding to uniformly elliptic divergence form operators, In *Séminaire de Probabilités, XXII*, vol. 1321 of *Lecture Notes in Math.*, 316347, Springer, 1988.
- [19] D.W. Stroock and S. R. Srinivasa Varadhan, Multidimensional diffusion processes, vol. 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.
- [20] H.D. Victory, B.P. O'Dwyer, On classical solutions of Vlasov-Poisson-Fokker-Planck systems, *Ind. Univ. Math. Math. J.*, 3 (1), 105-155, 1990.