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ON COMPACTNESS THEOREMS FOR LOGARITHMIC INTERPOLATION METHODS

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Abstract. Let (A_0, A_1) be a Banach couple, (B_0, B_1) a quasi-Banach couple, $0 < q \leq \infty$ and T a linear operator. We prove that if $T : A_0 \to B_0$ is bounded and $T : A_1 \to B_1$ is compact, then the interpolated operator by the logarithmic method $T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}}$ is compact too. This result allows the extension of some limit variants of Krasnosel'skiĭ's compact interpolation theorem.

1. Introduction. In 1960, Krasnosel'skiĭ [20] gave a reinforced version of the Riesz-Thorin theorem involving compactness. He proved that if T is a linear operator such that $T : L_{p_0} \to L_{q_0}$ compactly and $T : L_{p_1} \to L_{q_1}$ boundedly with $1 \leq p_0, p_1, q_1 \leq \infty$, $1 \leq q_0 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then $T : L_p \to L_q$ is also compact. This result promoted the study of compact operators between abstract interpolation spaces. The first results were due to Lions and Peetre [21] and to Persson [23] (see also [2, 24] and the references given there). In 1992, it was proven by Cwikel [15] and Cobos, Kühn and Schonbek [12] that if (A_0, A_1), (B_0, B_1) are Banach couples and T is a linear operator such that $T : A_j \to B_j$ is bounded, for $j = 0, 1$, and one of the restrictions is compact, then the interpolated operator by the real method $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also compact. In 1998, Cobos and Persson proved in [13] that the previous result is still valid for quasi-Banach couples. As a particular application of this result, they gave an extension of Krasnosel'skiĭ's theorem to Lorentz spaces with no restrictions on parameters q_j, that is to say, $0 < q_0 \neq q_1 \leq \infty$.

The logarithmic perturbations $(A_0, A_1)_{\theta,q,\mathbb{A}}$ of the real method have attracted considerable attention in the last years (see [18, 19, 14, 3]). When $\theta = 0$ and $\theta = 1$, these

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spaces are related to the limiting interpolation spaces [5, 10, 11]. Applying the logarithmic methods to the couple (L_r, L_∞) one can get generalized Lorentz-Zygmund spaces $L_{p,q,\mathbb{A}}$ (see [16, 22]).

Edmunds and Opic established in [17] the following limit version of Krasnosel'skiĭ's theorem: let (R,μ) and (S,ν) be finite measure spaces, $1 < p_0 < p_1 \leq \infty$, $1 < q_0 < q_1 \leq \infty$, $1 \leq q < \infty$ and $\alpha + 1/q > 0$. If T is a linear operator such that $T : L_{p_0}(R) \to L_{q_0}(S)$ compactly and $T : L_{p_1}(R) \to L_{q_1}(S)$ boundedly then $T : L_{p_0,q,\alpha+\frac{1}{\min(p_0,q)}}(R) \to L_{q_0,q,\alpha+\frac{1}{\max(q_0,q)}}(S)$ is also compact.

Later Cobos, Fernández Cabrera and Martínez [6] and Cobos and Segurado [14] obtained abstract versions of this result. They work with logarithmic interpolation methods with limit values of θ applied to Banach couples and $1 \leq q \leq \infty$. In particular, it is shown in [14] that the result of Edmunds and Opic also holds when the spaces are defined over any σ-finite measure spaces.

The first objective of this paper is to extend the abstract results for $0 < q \leq \infty$ and a quasi-Banach target couple. Then, as a consequence, we prove an extended version of the limit Krasnosel'skiĭ type result for $0 < q_0 < q_1 \leq \infty$ and $0 < q < \infty$.

The organization of the paper is as follows. In Section 2 we review the definition and some properties of limit logarithmic interpolation spaces. In Section 3 we prove the abstract compactness theorem for logarithmic spaces. As the proof is quite technical, we settle several auxiliary lemmas in advance. Finally, in Section 4 we derive the Krasnosel'skiĭ's type result.

2. Logarithmic interpolation spaces. Let $\bar{A} = (A_0, A_1)$ be a *quasi-Banach couple*, that is to say, two quasi-Banach spaces A_j, $j = 0, 1$, which are continuously embedded in some Hausdorff topological vector space. We put $c_{A_j} \geq 1$ for the constants in the quasi-triangle inequality, $j = 0, 1$. Let $t > 0$, the Peetre's *K- and J- functionals* are defined by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : \ a = a_0 + a_1, \ a_j \in A_j, \ j = 0,1\}$$

where $a \in A_0 + A_1$ and

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \ a \in A_0 \cap A_1.$$

Observe that $K(1,\cdot)$ is the quasi-norm of $A_0 + A_1$ and $J(1,\cdot)$ the quasi-norm of $A_0 \cap A_1$. In both cases, the quasi-triangular inequality holds with constant $c = \max\{c_{A_0}, c_{A_1}\}$. When $c_{A_0} = c_{A_1} = 1$ we say that $\bar{A} = (A_0, A_1)$ is a Banach couple.

For a quasi-Banach couple $\bar{A} = (A_0, A_1)$, the *Gagliardo completion* $A_j^\sim$ of A_j is formed of all those $a \in A_0 + A_1$ such that

$$\|a\|_{A_j^\sim} := \sup\left\{t^{-j}K(t,a) : t > 0\right\} < \infty,$$

(see [1, 2, 4]). Clearly $A_j \hookrightarrow A_j^\sim$, where $\hookrightarrow$ means continuous embedding. Note that

$$K(t,a;A_0^\sim,A_1^\sim) \leq K(t,a;A_0,A_1) \leq \max\{c_{A_0}, c_{A_1}\}K(t,a;A_0^\sim,A_1^\sim), \tag{1}$$

for $t > 0$ and $a \in A_0 + A_1$. Indeed, for any decomposition $a = a_0 + a_1$, with $a_j \in A_j \hookrightarrow A_j^\sim$, we have that

$$K(t,a;A_0^\sim,A_1^\sim)\le \|a_0\|_{A_0^\sim}+t\|a_1\|_{A_1^\sim}\le \|a_0\|_{A_0}+t\|a_1\|_{A_1}.$$

Hence $K(t,a;A_0^\sim,A_1^\sim)\le K(t,a;A_0,A_1)$. On the other hand, if $a=b_0+b_1$ with $b_j\in A_j^\sim\hookrightarrow A_0+A_1$, then

$$\begin{aligned}K(t,a;A_0,A_1)&\le \max\{c_{A_0},c_{A_1}\}\left(K(t,b_0;A_0,A_1)+K(t,b_1;A_0,A_1)\right)\\&\le \max\{c_{A_0},c_{A_1}\}\left(\|b_0\|_{A_0^\sim}+t\|b_1\|_{A_1^\sim}\right).\end{aligned}$$

Thus $K(t,a;A_0,A_1)\le \max\{c_{A_0},c_{A_1}\}K(t,a;A_0^\sim,A_1^\sim)$. In particular, if $\bar{A}=(A_0,A_1)$ is a Banach couple, we get an equality in (1) as it can be seen in [1, Theorem V.1.5].

Let $\ell(t)=1+|\log t|$, $\ell\ell(t)=1+(\log(1+|\log t|))$ and for $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$

$$\ell^{\mathbb{A}}(t)=\ell^{(\alpha_0,\alpha_\infty)}(t)=\begin{cases}\ell^{\alpha_0}(t) & \text{if } 0<t\le 1,\\ \ell^{\alpha_\infty}(t) & \text{if } 1<t<\infty,\end{cases}$$

and define $\ell\ell^{\mathbb{A}}(t)$ similarly.

Given $0\le\theta\le 1$, $0<q\le\infty$, $\mathbb{A}\in\mathbb{R}^2$ and a quasi-Banach couple $\bar{A}=(A_0,A_1)$, the *logarithmic interpolation space* $(A_0,A_1)_{\theta,q,\mathbb{A}}$ consists of all $a\in A_0+A_1$ such that

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}}=\|\left(K(2^m,a)2^{-m\theta}\ell^{\mathbb{A}}(2^m)\right)_{m\in\mathbb{Z}}\|_{\ell_q}<\infty.$$

Since this definition requires the weighted sequence space $\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m))$, we also use the notation $(A_0,A_1)_{\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m))}$. It is not difficult to check that the quasi-norm of $(A_0,A_1)_{\theta,q,\mathbb{A}}$ is equivalent to the continuous quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}}\sim\begin{cases}\left(\int_0^\infty\left[t^{-\theta}\ell^{\mathbb{A}}(t)K(t,a)\right]^q\frac{dt}{t}\right)^{1/q} & \text{if } 0<q<\infty,\\ \sup\{t^{-\theta}\ell^{\mathbb{A}}(t)K(t,a):t>0\} & \text{if } q=\infty.\end{cases}$$

See [18, 19] for more details on $(A_0,A_1)_{\theta,q,\mathbb{A}}$.

We are interested in the limiting interpolation spaces that appear when $\theta=0$ and $\theta=1$. Note that $K(t,a;A_0,A_1)=tK(t^{-1},a;A_1,A_0)$ and therefore

$$(A_0,A_1)_{\theta,q,(\alpha_0,\alpha_\infty)}=(A_1,A_0)_{1-\theta,q,(\alpha_\infty,\alpha_0)}\tag{2}$$

with equal quasi-norms. In particular, $(A_0,A_1)_{0,q,(\alpha_0,\alpha_\infty)}=(A_1,A_0)_{1,q,(\alpha_\infty,\alpha_0)}$. Subsequently we focus on the case $\theta=1$.

Under the assumptions

$$\begin{cases}\alpha_0+\frac{1}{q}<0 & \text{if } 0<q<\infty,\\ \alpha_0\le 0 & \text{if } q=\infty,\end{cases}\tag{3}$$

we have that $A_0\cap A_1\hookrightarrow(A_0,A_1)_{1,q,\mathbb{A}}\hookrightarrow A_0+A_1$, for any quasi-Banach couple $\bar{A}=(A_0,A_1)$, otherwise $(A_0,A_1)_{1,q,\mathbb{A}}=\{0\}$ (see [19, Theorem 2.2]).

When $\bar{A}=(A_0,A_1)$ is a Banach couple, it will be useful to represent the space $(A_0,A_1)_{1,q,\mathbb{A}}$ by means of the J-functional.

Let $\bar{A}=(A_0,A_1)$ be a Banach couple, $0<q\le\infty$, $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$. Assume that

$$\begin{cases}\alpha_\infty>0, \text{ or } \alpha_\infty=0 \text{ and } \beta_\infty\ge 0 & \text{if } 0<q\le 1,\\ \alpha_\infty-\frac{1}{q'}>0, \text{ or } \alpha_\infty=\frac{1}{q'} \text{ and } \beta_\infty-\frac{1}{q'}>0 & \text{if } 1<q\le\infty,\end{cases}\tag{4}$$

where $1/q+1/q'=1$. The space $(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}=(A_0,A_1)^J_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))}$ is formed of all those $a\in A_0+A_1$ for which there exists $(u_m)\subseteq A_0\cap A_1$ such that

$$a=\sum_{m=-\infty}^{\infty}u_m \text{ (convergence in } A_0+A_1)$$

and

$$\| \left(J(2^m,u_m)2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)\right)_{m\in\mathbb{Z}}\|_{\ell_q}<\infty.$$

We set

$$\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}}=\inf\Big\{\| \left(J(2^m,u_m)2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)\right)\|_{\ell_q}:a=\sum_{m=-\infty}^{\infty}u_m\Big\}.$$

If $\mathbb{B}=(0,0)$, we simply write $(A_0,A_1)^J_{1,q,\mathbb{A}}$. It is proven in [3, Section 2] that under the assumptions in (4), $A_0\cap A_1\hookrightarrow(A_0,A_1)_{1,q,\mathbb{A},\mathbb{B}}\hookrightarrow A_0+A_1$ for every Banach couple $\bar{A}=(A_0,A_1)$. If $1\le q\le\infty$ there exists an equivalent continuous representation for the J-spaces (see [14, Definition 3.1]).

Let $\bar{A}=(A_0,A_1)$ be a Banach couple. If $1\le q\le\infty$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)$ satisfies (3), then [14, Theorem 3.5 and Theorem 3.6] state that

$$(A_0,A_1)_{1,q,\mathbb{A}}=\begin{cases}(A_0,A_1)^J_{1,q,\mathbb{A}+1} & \text{if } \alpha_\infty+1/q>0,\\ (A_0,A_1)^J_{1,q,\mathbb{A}+1,(0,1)} & \text{if } \alpha_\infty+1/q=0,\end{cases}\tag{5}$$

with equivalent norms. Here $\mathbb{A}+\lambda=(\alpha_0+\lambda,\alpha_\infty+\lambda)$, for any $\lambda\in\mathbb{R}$. If $0<q<1$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)$ satisfies (3), then [3, Theorem 3.2] shows that

$$(A_0,A_1)_{1,q,\mathbb{A}}=\begin{cases}(A_0^{\sim},A_1^{\sim})^J_{1,q,\mathbb{A}+1/q} & \text{if } \alpha_\infty+1/q>0,\\ (A_0^{\sim},A_1^{\sim})^J_{1,q,\mathbb{A}+1/q,(0,1/q)} & \text{if } \alpha_\infty+1/q=0,\end{cases}\tag{6}$$

with equivalent quasi-norms. In general, when $\alpha_\infty+1/q<0$ and $0<q\le\infty$, or $\alpha_\infty=0$ and $q=\infty$, the K-space $(A_0,A_1)_{1,q,\mathbb{A}}$ does not admit a J-representation (see [14, Proposition 3.4] and [3, Example 2.1]). In this case, the following result is useful. For a given quasi-Banach couple $\bar{A}=(A_0,A_1)$, $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $0<q\le\infty$ satisfying

$$\begin{cases}\alpha_0+1/q<0 \text{ and } \alpha_\infty+1/q<0 & \text{if } 0<q<\infty,\\ \alpha_0\le 0 \text{ and } \alpha_\infty\le 0 & \text{if } q=\infty,\end{cases}$$

we have that for any $\alpha>-1/q$

$$(A_0,A_1)_{1,q,\mathbb{A}}=(A_0+A_1,A_1)_{1,q,(\alpha_0,\alpha)},\tag{7}$$

with equivalent quasi-norms. This result was proven in [14, Corollary 2.5] for Banach couples and $1\le q\le\infty$, but the proof remains valid for quasi-Banach couples and $0<q\le\infty$ just taking into account the constant in the quasi-triangle inequality.

3. Compactness theorem. In what follows, if X and Y are quantities depending on certain parameters, we write $X\lesssim Y$ if $X\le CY$ with a constant C independent of all the parameters. We put $X\sim Y$ if $X\lesssim Y$ and $Y\lesssim X$.

Let A be a quasi-Banach space. For $M>0$, we put $MU_A=\{a\in A:\|a\|_A\le M\}$ and just U_A when $M=1$. If B is another quasi-Banach space, let $\mathcal{L}(A,B)$ denote the set of bounded linear operators from A to B and $\mathcal{K}(A,B)$ the set of linear compact operators from A to B. If $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ are two quasi-Banach couples, we put $T\in\mathcal{L}(\bar{A},\bar{B})$ if $T\in\mathcal{L}(A_0+A_1,B_0+B_1)$ and the restrictions $T:A_j\to B_j$ are also bounded with quasi-norm $\|T\|_j$, for $j=0,1$. If $A_0=A_1=A$ or $B_0=B_1=B$, then we simply write $T\in\mathcal{L}(A,\bar{B})$ or $T\in\mathcal{L}(\bar{A},B)$. For $\lambda\in\mathbb{R}$, we set $\lambda^+=\max\{0,\lambda\}$.

Let $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ be quasi-Banach couples, $0<q\le\infty$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ satisfying (3). If $T\in\mathcal{L}(\bar{A},\bar{B})$, then $T\in\mathcal{L}\left(\bar{A}_{1,q,\mathbb{A}};\bar{B}_{1,q,\mathbb{A}}\right)$ and the following norm estimate holds

$$\|T\|_{\bar{A}_{1,q,\mathbb{A}};\bar{B}_{1,q,\mathbb{A}}}\lesssim\begin{cases}\|T\|_1\left(1+\left(\log\frac{\|T\|_0}{\|T\|_1}\right)^+\right)^{\alpha_\infty^+-\alpha_0} & \text{if } \|T\|_j\neq 0, j=0,1;\\ \|T\|_1 & \text{if } \|T\|_j=0, j=0 \text{ or } j=1.\end{cases}\tag{8}$$

This result was proven in [7, Theorem 2.2] for Banach couples and $1\le q\le\infty$. The proof remains true in our hypothesis.

Our goal in this section is to prove the compactness of the interpolated operator $T:(A_0,A_1)_{1,q,\mathbb{A}}\to(B_0,B_1)_{1,q,\mathbb{A}}$, for $\bar{A}$ a Banach couple and $\bar{B}$ a quasi-Banach couple, under the assumptions that $T:A_1\to B_1$ is compact and $T:A_0\to B_0$ is bounded. For this purpose we establish first a simplified version of this result and some auxiliary lemmas.

Lemma 3.1. *Let $\bar{A}=(A_0,A_1)$ be a quasi-Banach couple and let B be a quasi-Banach space. Take $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $0<q\le\infty$ satisfying (3).*

1. *If $T\in\mathcal{L}(B,\bar{A})$ with $T:B\to A_1$ compact, then $T:B\to(A_0,A_1)_{1,q,\mathbb{A}}$ is compact.*
2. *If $T\in\mathcal{L}(\bar{A},B)$ with $T:A_1\to B$ compact, then $T:(A_0,A_1)_{1,q,\mathbb{A}}\to B$ is compact.*

Proof. For the first case, the proof given in [14, Lemma 4.1,(a)] is still valid. However, for the second case, [14, Lemma 4.2,(b)] uses Hahn-Banach theorem and we have to proceed differently. It is clear that for any $m\in\mathbb{Z}$

$$\sup\left\{\frac{K(2^m,a)}{\|a\|_{\bar{A}_{1,q,\mathbb{A}}}}:a\in\bar{A}_{1,q,\mathbb{A}},a\neq 0\right\}\le 2^m\ell^{-\mathbb{A}}(2^m).\tag{9}$$

Given $\varepsilon>0$, we fix $m<0$ such that $2^m\ell^{-\mathbb{A}}(2^m)\le\varepsilon/(4c_B\|T\|_{A_0,B})$. Using (9), we get that for any $a\in U_{\bar{A}_{1,q,\mathbb{A}}}$ there exists $a_j\in A_j$, $j=0,1$ such that $a=a_0+a_1$ and

$$\|a_0\|_{A_0}+2^m\|a_1\|_{A_1}\le 2K(2^m,a)\le 2^{m+1}\ell^{-\mathbb{A}}(2^m)\le\varepsilon/(2c_B\|T\|_{A_0,B}).$$

Let $M=2^{-m}\varepsilon/(2c_B\|T\|_{A_0,B})$. By compactness of the operator $T:A_1\to B$, there exists $\{b_1,...,b_k\}\subset B$ such that $\min\{\|Tx-b_j\|_B:1\le j\le k\}\le\varepsilon/(2c_B)$, for every $x\in MU_{A_1}$. Consequently, for each $a\in U_{\bar{A}_{1,q,\mathbb{A}}}$ we can take $j\in\{1,...,k\}$ such that $\|Ta_1-b_j\|_B\le\varepsilon/(2c_B)$ and

$$\|Ta-b_j\|_B\le c_B\left(\|Ta_0\|_B+\|Ta_1-b_j\|_B\right)\le\varepsilon.$$

Therefore, $T:(A_0,A_1)_{1,q,\mathbb{A}}\to B$ is compact. ∎

Lemma 3.2. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple. Let $\bar{B}=(B_0,B_1)$ be a quasi-Banach couple and $T\in\mathcal{L}(\bar{A},\bar{B})$. If $T:A_1\to B_1$ is compact, then $T:A_1^\sim\to B_1^\sim$ is also compact.*

Proof. Let $\varepsilon > 0$ and $a \in U_{A_1^\sim} = \{a \in A_0 + A_1 : \sup_{t>0} K(t,a)/t \le 1\}$. For every $n \in \mathbb{N}$ there exists $a_{0n} \in A_0$ and $a_{1n} \in A_1$, satisfying that $a = a_{0n} + a_{1n}$ and $\|a_{0n}\|_{A_0} + 1/n\|a_{1n}\|_{A_1} \le 2K(1/n, a) \le 2/n$. Note that $\lim_{n\to\infty} Ta_{1n} = Ta$ in $B_0 + B_1$, since $\lim_{n\to\infty} a_{1n} = a$ in $A_0 + A_1$. Moreover, the sequence (a_{1n}) is contained in $2U_{A_1}$ and the operator T is compact from A_1 to B_1, therefore there exists a subsequence $(Ta_{1n'})$ that is convergent in B_1. Using compatibility, we obtain that $Ta_{1n'} \xrightarrow{n'\to\infty} Ta$ in B_1 and then we can find $n_0' \in \mathbb{N}$ such that $\|Ta_{1n_0'} - Ta\|_{B_1} \le \varepsilon/(2c_{B_1})$.

Again by compactness of $T : A_1 \to B_1$, there exists $\{b_1, ..., b_k\} \subset B_1$ such that $\min\{\|Tx - b_j\|_{B_1} : 1 \le j \le k\} \le \varepsilon/(2c_{B_1})$, for every $x \in 2U_{A_1}$. Hence, we can take $j \in \{1, ..., k\}$ such that $\|Ta_{1n_0'} - b_j\|_{B_1} \le \varepsilon/(2c_{B_1})$ and

$$\|Ta - b_j\|_{B_1} \le c_{B_1}\left(\|Ta - Ta_{1n_0'}\|_{B_1} + \|Ta_{1n_0'} - b_j\|_{B_1}\right) \le c_{B_1}\left(\varepsilon/(2c_{B_1}) + \varepsilon/(2c_{B_1})\right) = \varepsilon.$$

Thus $T : A_1^\sim \to B_1$ is compact. Since $B_1 \hookrightarrow B_1^\sim$, it follows that $T : A_1^\sim \to B_1^\sim$ is also compact. ■

The previous lemma for Banach couples and compactness on the restriction $T : A_0 \to B_0$ was given in [6, Theorem 2.2]. The formulation of the next two lemmas correspond to [9, Lemma 2.3 and Corollary 2.2] in the Banach case. The proofs can be found in [8, Lemma 3.2 and Lemma 3.3] for quasi-Banach spaces and bilinear operators.

LEMMA 3.3. *Let A, B, Z be quasi-Banach spaces, D a dense subspace of A and $T \in \mathcal{K}(A, B)$. Let $(S_n)_{n\in\mathbb{N}} \subset \mathcal{L}(B, Z)$ such that $M := \sup\{\|S_n\|_{B,Z} : n \ge 1\} < \infty$. If $\lim_{n\to\infty} \|S_nTu\|_Z = 0$ for all $u \in D$ then $\lim_{n\to\infty} \|S_nT\|_{A,Z} = 0$.*

LEMMA 3.4. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let A, B be intermediate spaces with respect to $\bar{A}$ and $\bar{B}$, respectively. Assume that $T \in \mathcal{L}(A_0 + A_1, B_0 + B_1) \cap \mathcal{K}(A, B)$. Let X be a quasi-Banach space and $(R_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X, A)$ such that $M := \sup\{\|R_n\|_{X,A} : n \ge 1\} < \infty$ and $\lim_{n\to\infty} \|TR_n\|_{X,B_0+B_1} = 0$. Then $\lim_{n\to\infty} \|TR_n\|_{X,B} = 0$.*

Let (λ_m) be a sequence of positive numbers and (W_m) a sequence of quasi-Banach spaces with the same constant $c \ge 1$ in the quasi-triangle inequality. For any $0 < q \le \infty$, we put

$$\ell_q(\lambda_m W_m) = \{w = (w_m)_{m\in\mathbb{Z}} : w_m \in W_m \text{ and } (\lambda_m \|w_m\|_{W_m}) \in \ell_q\}.$$

The quasi-norm in $\ell_q(\lambda_m W_m)$ is given by $\|w\|_{\ell_q(\lambda_m W_m)} = \|(\lambda_m \|w_m\|_{W_m})_{m\in\mathbb{Z}}\|_{\ell_q}$.

Now we establish the analogous results to [14, Lemma 4.2].

LEMMA 3.5. *Let $(W_m)_{m\in\mathbb{N}}$ be a sequence of quasi-Banach spaces with constant $c \ge 1$ in the quasi-triangle inequality. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (3). Then*

$$(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}} \hookrightarrow \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)W_m).$$

Proof. Let $x = (x_m) \in (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}$. Given any decomposition $x = y + z$ with $y = (y_m) \in \ell_\infty(W_m)$ and $z = (z_m) \in \ell_\infty(2^{-m}(W_m))$, we have

$$\|x_k\|_{W_k} \le c\left(\|y_k\|_{W_k} + \|z_k\|_{W_k}\right) \le c\left(\|y\|_{\ell_\infty(W_m)} + 2^k \|z\|_{\ell_\infty(2^{-m}W_m)}\right), k \in \mathbb{Z}.$$

Then $\|x_k\|_{W_k} \le cK(2^k, x; \ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ for every $k \in \mathbb{Z}$, which yields that $\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)W_m)} \le c\|x\|_{(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}}$. ■

For a sequence of Banach spaces we also have the following result.

LEMMA 3.6. *Let $(W_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces. Let $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $0<q\le 1$ satisfying (3).*

1. *If $\alpha_\infty+1/q>0$, then*
$$\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)W_m)\hookrightarrow(\ell_1(W_m),\ell_1(2^{-m}W_m))_{1,q,\mathbb{A}}.$$
2. *If $\alpha_\infty+1/q=0$, then*
$$\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)\ell\ell^{(0,1/q)}(2^m)W_m)\hookrightarrow(\ell_1(W_m),\ell_1(2^{-m}W_m))_{1,q,\mathbb{A}}.$$

Proof.

1. Let $x=(x_m)\in\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)W_m)$ and let δ_m^k the Kronecker delta. We set $u_k=(\delta_m^k x_k)_{m\in\mathbb{Z}}\in\ell_1(W_m)\cap\ell_1(2^{-m}W_m)\hookrightarrow\ell_1(W_m)^\sim\cap\ell_1(2^{-m}W_m)^\sim$. Using (6), we now derive that
$$\begin{aligned}&\|x\|_{(\ell_1(W_m),\ell_1(2^{-m}W_m))_{1,q,\mathbb{A}}}\sim\|x\|_{(\ell_1(W_m)^\sim,\ell_1(2^{-m}W_m)^\sim)^J_{1,q,\mathbb{A}+1/q}}\\&\le\left(\sum_{k=-\infty}^{\infty}[2^{-k}\ell^{\mathbb{A}+1/q}(2^k)J(2^k,u_k;\ell_1(W_m)^\sim,\ell_1(2^{-m}W_m)^\sim)]^q\right)^{1/q}\\&\le\left(\sum_{k=-\infty}^{\infty}[2^{-k}\ell^{\mathbb{A}+1/q}(2^k)J(2^k,u_k;\ell_1(W_m),\ell_1(2^{-m}W_m))]^q\right)^{1/q}\\&=\|x\|_{\ell_q(2^{-k}\ell^{\mathbb{A}+1/q}(2^k))}.\end{aligned}$$
2. This case can be handled as the previous one but using the appropriate equality of (6).

■

We now prove the main result of this section.

THEOREM 3.7. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple. Let $\bar{B}=(B_0,B_1)$ be a quasi-Banach couple and $T\in\mathcal{L}(\bar{A},\bar{B})$ such that $T:A_1\to B_1$ is compact. For any $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $0<q\le\infty$ satisfying (3), we have that*
$$T:(A_0,A_1)_{1,q,\mathbb{A}}\to(B_0,B_1)_{1,q,\mathbb{A}}$$
is also compact.

Proof. Step 1. Let $0<q\le 1$ and assume that $\alpha_\infty+1/q\ge 0$. For $m\in\mathbb{Z}$, let
$$G_m=(A_0^\sim\cap A_1^\sim,J(2^m,\cdot;A_0^\sim,A_1^\sim))\text{ and}$$
$$F_m=(B_0^\sim+B_1^\sim,K(2^m,\cdot;B_0^\sim,B_1^\sim)).$$
We define $\mu_m=2^{-m}\ell^{\mathbb{A}}(2^m)$ and
$$\lambda_m=\begin{cases}2^{-m}\ell^{\mathbb{A}+1/q}(2^m) & \text{if }\alpha_\infty+1/q>0,\\ 2^{-m}\ell^{\mathbb{A}+1/q}(2^m)\ell\ell^{(0,1/q)}(2^m) & \text{if }\alpha_\infty+1/q=0.\end{cases}$$
By (1) and (6), we have that
$$(A_0^\sim,A_1^\sim)_{\ell_q(\mu_m)}=(A_0,A_1)_{\ell_q(\mu_m)}=(A_0^\sim,A_1^\sim)^J_{\ell_q(\lambda_m)}$$

with equivalent quasi-norms.

Consider the operators $\pi(u)=\sum_m u_m$ and $jb=(...,b,b,b,...)$. Observe that

$$\pi:\ell_q(\lambda_m G_m)\to (A_0^{\sim},A_1^{\sim})_{\ell_q(\mu_m)}$$

is a metric surjection if we consider on $(A_0^{\sim},A_1^{\sim})_{\ell_q(\mu_m)}$ the J-quasi-norm. Moreover, restrictions $\pi:\ell_1(2^{mj}G_m)\to A_j^{\sim}$, $j=0,1$, are bounded operators with norm ≤ 1. On the other hand,

$$j:(B_0^{\sim},B_1^{\sim})_{1,q,\mathbb{A}}\to \ell_q(\mu_m F_m)$$

is a metric injection and restrictions $j:B_j^{\sim}\to\ell_\infty(2^{-mj}F_m)$, $j=0,1$, are bounded with quasi-norm ≤ 1. Applying Lemma 3.5 and Lemma 3.6 we obtain the following diagram that illustrates the situation

$$
\begin{array}{ccccccc}
\ell_1(G_m) & \xrightarrow{\pi} & A_0^{\sim} & \xrightarrow{T} & B_0^{\sim} & \xrightarrow{j} & \ell_\infty(F_m)\\
\ell_1(2^{-m}G_m) & \xrightarrow{\pi} & A_1^{\sim} & \xrightarrow{T} & B_1^{\sim} & \xrightarrow{j} & \ell_\infty(2^{-m}F_m)\\
\hline\hline
\bar{\ell}_1(G_m)_{1,q,\mathbb{A}} & \xrightarrow{\pi} & (A_0^{\sim},A_1^{\sim})_{1,q,\mathbb{A}} & \xrightarrow{T} & (B_0^{\sim},B_1^{\sim})_{1,q,\mathbb{A}} & \xrightarrow{j} & \bar{\ell}_\infty(F_m)_{1,q,\mathbb{A}},\\
\uparrow & & & & & & \downarrow\\
\ell_q(\lambda_m G_m) & & & & & & \ell_q(\mu_m F_m)
\end{array}
$$

where

$$\bar{\ell}_1(G_m)_{1,q,\mathbb{A}}:=(\ell_1(G_m),\ell_1(2^{-m}G_m))_{1,q,\mathbb{A}} \text{ and}$$
$$\bar{\ell}_\infty(F_m)_{1,q,\mathbb{A}}:=(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m))_{1,q,\mathbb{A}}.$$

Let $\hat{T}=jT\pi$. Properties of π and j yield that compactness of $T:(A_0,A_1)_{1,q,\mathbb{A}}\to (B_0,B_1)_{1,q,\mathbb{A}}$ is equivalent to compactness of $\hat{T}:\ell_q(\lambda_m G_m)\to\ell_q(\mu_m F_m)$. Observe that applying Lemma 3.2, $T:A_1^{\sim}\to B_1^{\sim}$ is compact and so $\hat{T}:\ell_1(2^{-m}G_m)\to\ell_\infty(2^{-m}F_m)$ is also compact. We shall check the compactness of $\hat{T}$ with the help of the following projections. For $n\in\mathbb{N}$ we define

$$
\begin{aligned}
Q_n(u_m)&=(...,0,0,u_{-n},...,u_n,0,0,...),\\
Q_n^+(u_m)&=(...,0,0,u_{n+1},u_{n+2},...),\\
Q_n^-(u_m)&=(...,u_{-n-2},u_{-n-1},0,0,...).
\end{aligned}
$$

The identity operator on $\ell_1(G_m)+\ell_1(2^{-m}G_m)$ can be written as $I=Q_n+Q_n^++Q_n^-$. These projections have the following properties:

$$\|Q_n\|_{E,E}=\|Q_n^+\|_{E,E}=\|Q_n^-\|_{E,E}=1 \text{ for } E=\ell_1(G_m),\ell_1(2^{-m}G_m),\ell_q(\lambda_m G_m), \tag{10}$$

$$\|Q_n\|_{\ell_1(2^{-m}G_m),\ell_1(G_m)}=\|Q_n\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)}=2^n, n\ge 1, \tag{11}$$

$$\|Q_n^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)}=2^{-(n+1)}, n\ge 1, \tag{12}$$

$$\|Q_n^-\|_{\ell_1(2^{-m}G_m),\ell_1(G_m)}=2^{-(n+1)}, n\ge 1. \tag{13}$$

On the couple $(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m))$ we can define similar projections P_n,P_n^+,P_n^- satisfying analogous properties.

We have

$$\hat{T}=\hat{T}Q_n+\hat{T}Q_n^-+\hat{T}Q_n^+=\hat{T}Q_n+\hat{T}Q_n^-+P_n\hat{T}Q_n^++P_n^-\hat{T}Q_n^++P_n^+\hat{T}Q_n^+.$$

Next we show that $\hat{T}Q_n$, $P_n\hat{T}Q_n^+$, and $P_n^-\hat{T}Q_n^+$ are compact from $\ell_q(\lambda_m G_m)$ to $\ell_q(\mu_m F_m)$ and that the quasi-norms of the other two operators converge to 0.

Using (11) and Lemma 3.6, we have the factorization

$$\begin{tikzcd}[column sep=small]
 & & \ell_1(G_m) \arrow[r, "\hat{T}"] & \ell_\infty(F_m) \\
\ell_q(\lambda_m G_m) \arrow[r, hook] & \ell_1(G_m)+\ell_1(2^{-m}G_m) \arrow[ur, "Q_n"] \arrow[dr, "Q_n"'] & & \\
 & & \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m),
\end{tikzcd}$$

which allows to apply Lemma 3.1 to obtain the compactness of

$$\hat{T}Q_n : \ell_q(\lambda_m G_m) \to (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{1,q,\mathbb{A}}.$$

Now from Lemma 3.5, we conclude that $\hat{T}Q_n : \ell_q(\lambda_m G_m) \to \ell_q(\mu_m F_m)$ is compact.

Considering (10), (12), the analogous properties to (10) and (11) for the operator P_n and Lemma 3.5, we have the factorization

$$\begin{tikzcd}[column sep=small]
\ell_1(G_m) \arrow[dr, "Q_n^+"] & & & & \\
 & \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m) \arrow[r, "P_n"] & \ell_\infty(F_m)\cap\ell_\infty(2^{-m}F_m) \arrow[r, hook] & \ell_q(\mu_m F_m). \\
\ell_1(2^{-m}G_m) \arrow[ur, "Q_n^+"'] & & & &
\end{tikzcd}$$

Thus, by Lemma 3.1 and Lemma 3.6, the operator $P_n\hat{T}Q_n^+ : \ell_q(\lambda_m G_m) \to \ell_q(\mu_m F_m)$ is compact.

For $P_n^-\hat{T}Q_n^+$, we first use (10) and (12) to get the next diagram

$$\begin{tikzcd}
\ell_1(G_m) \arrow[dr, "Q_n^+"] & & \\
 & \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m). \\
\ell_1(2^{-m}G_m) \arrow[ur, "Q_n^+"'] & &
\end{tikzcd}$$

Again from Lemma 3.1 and Lemma 3.6, we infer the compactness of $TQ_n^+ : \ell_q(\lambda_m G_m) \to \ell_\infty(2^{-m}F_m)$. Now using the analogous property to (13) for the operator P_n^-, we have the factorization

$$\begin{tikzcd}
 & & \ell_\infty(F_m) \\
\ell_q(\lambda_m G_m) \arrow[r, "TQ_n^+"] & \ell_\infty(2^{-m}F_m) \arrow[ur, "P_n^-"] \arrow[dr, "P_n^-"'] & \\
 & & \ell_\infty(2^{-m}F_m).
\end{tikzcd}$$

Applying again Lemma 3.1 and Lemma 3.5, we deduce that $P_n^-\hat{T}Q_n^+ : \ell_q(\lambda_m G_m) \to \ell_q(\mu_m F_m)$ is compact.

We shall now prove that $\|\hat{T}Q_n^-\|_{\ell_q(\lambda_m G_m),\ell_q(\mu_m F_m)} \xrightarrow{n\to\infty} 0$. Using (13) we get that

$$\|\hat{T}Q_n^-\|_{\ell_1(2^{-m}G_m),\ell_\infty(F_m)+\ell_\infty(2^{-m}F_m)} \leq 2^{-(n+1)}\|\hat{T}\|_{\ell_1(G_m),\ell_\infty(F_m)+\ell_\infty(2^{-m}F_m)} \xrightarrow{n\to\infty} 0.$$

Then Lemma 3.4 implies that $\|\hat{T}Q_n^-\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \xrightarrow{n\to\infty} 0$. Note also that

$$\|\hat{T}Q_n^-\|_{\ell_1(G_m),\ell_\infty(F_m)} \le \|\hat{T}\|_{\ell_1(G_m),\ell_\infty(F_m)}, \text{ for every } n \in \mathbb{N}.$$

Thus, using (8), Lemma 3.5 and Lemma 3.6, we conclude that

$$\lim_{n\to\infty} \|\hat{T}Q_n^-\|_{\ell_q(\lambda_m G_m),\ell_q(\mu_m F_m)} \lesssim \lim_{n\to\infty} \|\hat{T}Q_n^-\|_{\bar{\ell}_1(G_m)_{1,q,\mathbb{A}};\bar{\ell}_\infty(F_m)_{1,q,\mathbb{A}}}$$
$$\lesssim \lim_{n\to\infty} \|\hat{T}Q_n^-\|_1 \left(1 + \left(\log \frac{\|\hat{T}Q_n^-\|_0}{\|\hat{T}Q_n^-\|_1}\right)^+\right)^{\alpha_\infty^+ - \alpha_0} = 0.$$

Now we show that $\lim_{n\to\infty} \|P_n^+\hat{T}Q_n^+\|_{\ell_q(\lambda_m G_m),\ell_q(\mu_m F_m)} = 0$. We define

$$D = \{u = (u_m)_{m=-\infty}^{\infty} : u_m \in G_m \text{ with a finite number of no-null coordinates}\}.$$

Since D is dense in $\ell_1(2^{-m}G_m)$ and for any $u \in D$,

$$\|P_n^+\hat{T}u\|_{\ell_\infty(2^{-m}F_m)} \le 2^{-(n+1)}\|\hat{T}\|_{\ell_1(G_m),\ell_\infty(F_m)}\|u\|_{\ell_1(G_m)} \xrightarrow{n\to\infty} 0,$$

by Lemma 3.3 we get that

$$\lim_{n\to\infty} \|P_n^+\hat{T}Q_n^+\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \le \lim_{n\to\infty} \|P_n^+\hat{T}\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} = 0.$$

Then, proceeding as in the previous case we infer that

$$\lim_{n\to\infty} \|P_n^+\hat{T}Q_n^+\|_{\ell_q(\lambda_m G_m),\ell_q(\lambda_m F_m)} = 0.$$

Step 2. Let $0 < q \le 1$ and suppose now that $\alpha_\infty + 1/q < 0$. Take $\alpha > -1/q$. By (7), we get that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,(\alpha_0,\alpha)}$ and $(B_0, B_1)_{1,q,\mathbb{A}} = (B_0 + B_1, B_1)_{1,q,(\alpha_0,\alpha)}$. Applying the previous case we prove the compactness of

$$T : (A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,(\alpha_0,\alpha)} \to (B_0 + B_1, B_1)_{1,q,(\alpha_0,\alpha)} = (B_0, B_1)_{1,q,\mathbb{A}}$$

Step 3. Assume now that $1 < q \le \infty$. In this case we can proceed as when $0 < q \le 1$ but defining

$$\lambda_m = \begin{cases} 2^{-m}\ell^{\mathbb{A}+1}(2^m) & \text{if } \alpha_\infty + 1/q > 0, \\ 2^{-m}\ell^{\mathbb{A}+1}(2^m)\ell\ell^{(0,1)}(2^m) & \text{if } \alpha_\infty + 1/q = 0 \text{ and } 1 < q < \infty, \end{cases}$$

and using (5) instead of (6) and [14, Lemma 4.2] instead of Lemma 3.1.

This completes the proof. ■

The corresponding result for the $0, q, \mathbb{A}-$method is a consequence of (2) and reads as follows.

Corollary 3.8. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\bar{B} = (B_0, B_1)$ be a quasi-Banach couple and $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that $T : A_0 \to B_0$ is compact. For any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ such that*

$$\begin{cases} \alpha_\infty + 1/q < 0 & \textit{if } q < \infty, \\ \alpha_\infty \le 0 & \textit{if } q = \infty, \end{cases}$$

we have that $T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}$ is also compact.

4. Applications to Lorentz-Zygmund spaces. Let (R,μ) be a $\sigma-$finite measure space. For f a $\mu-$measurable function on R, let f^* be the *non-increasing rearrangement* of f defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{x \in R : |f(x)| > s\}) \le t\}.$$

Let $0 < p, q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. The *generalized Lorentz-Zygmund space* $L_{p,q,\mathbb{A}}(R,\mu)$ is formed of all the (classes of) $\mu-$measurable functions f on R having a finite quasi-norm

$$\|f\|_{p,q,\mathbb{A}} = \left(\int_0^{\mu(R)} \left[t^{1/p} \ell^{\mathbb{A}}(t) f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}.$$

See [22, 16].

Now we are going to extend the result given in [14, Corollary 4.5] to the case $0 < q < \infty$ and $0 < q_0 < q_1 \le \infty$.

THEOREM 4.1. *Let (R,μ) and (S,ν) be $\sigma-$finite measure spaces. Take $1 < p_0 < p_1 \le \infty$, $0 < q_0 < q_1 \le \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that*

$$T : L_{p_0}(R) \to L_{q_0}(S) \text{ is compact and}$$
$$T : L_{p_1}(R) \to L_{q_1}(S) \text{ is bounded.}$$

Then $T : L_{p_0,q,\mathbb{A}+\frac{1}{\min(p_0,q)}}(R) \to L_{q_0,q,\mathbb{A}+\frac{1}{\max(q_0,q)}}(S)$ is also compact.

Proof. By Corollary 3.8,

$$T : (L_{p_0}(R), L_{p_1}(R))_{0,q,\mathbb{A}} \to (L_{q_0}(S), L_{q_1}(S))_{0,q,\mathbb{A}}$$

is compact.

On the other hand, according to [2, Theorem 5.2.1] for any $r < q_0$ we have

$$\begin{aligned} L_{p_0}(R) &= (L_1(R), L_\infty(R))_{1-1/p_0,p_0}, \\ L_{p_1}(R) &= (L_1(R), L_\infty(R))_{1-1/p_1,p_1}, \\ L_{q_0}(S) &= (L_r(S), L_\infty(S))_{1-r/q_0,q_0}, \\ L_{q_1}(S) &= (L_r(S), L_\infty(S))_{1-r/q_1,q_1}. \end{aligned}$$

It follows from [18, Theorem 4.7 and Theorem 5.9]

$$(L_1(R), L_\infty(R))_{1-1/p_0,q,\mathbb{A}+\frac{1}{\min(p_0,q)}} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q,\mathbb{A}} \text{ and}$$
$$(L_{q_0}(S), L_{q_1}(S))_{0,q,\mathbb{A}} \hookrightarrow (L_r(S), L_\infty(S))_{1-r/q_0,q,\mathbb{A}+\frac{1}{\max(q,q_0)}}.$$

Besides by [18, Corollary 8.4] we have

$$\begin{aligned} L_{p_0,q,\mathbb{A}+\frac{1}{\min(p_0,q)}} &= (L_1(R), L_\infty(R))_{1-1/p_0,q,\mathbb{A}+\frac{1}{\min(p_0,q)}}, \\ L_{q_0,q,\mathbb{A}+\frac{1}{\max(q_0,q)}} &= (L_r(S), L_\infty(S))_{1-r/q_0,q,\mathbb{A}+\frac{1}{\max(q,q_0)}}. \end{aligned}$$

Consequently, the operator

$$T : L_{p_0,q,\mathbb{A}+\frac{1}{\min(p_0,q)}} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q,\mathbb{A}} \to (L_{q_0}(S), L_{q_1}(S))_{0,q,\mathbb{A}} \hookrightarrow L_{q_0,q,\mathbb{A}+\frac{1}{\max(q_0,q)}}$$

is compact. ∎

Next we consider the case of compactness on the second restriction.

Corollary 4.2. *Let (R,μ) and (S,ν) be $\sigma-$finite measure spaces. Take $1 \le p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Let T be a linear operator such that*

$$T : L_{p_0}(R) \to L_{q_0}(S) \text{ is bounded and}$$
$$T : L_{p_1}(R) \to L_{q_1}(S) \text{ is compact.}$$

Then $T : L_{p_1,q,\mathbb{A}+\frac{1}{\min(p_1,q)}}(R) \to L_{q_1,q,\mathbb{A}+\frac{1}{\max(q_1,q)}}(S)$ is also compact.

Proof. By Theorem 3.7 and (2),

$$T : (L_{p_1}(R), L_{p_0}(R))_{0,q,(\alpha_\infty,\alpha_0)} \to (L_{q_1}(S), L_{q_0}(S))_{0,q,(\alpha_\infty,\alpha_0)}$$

is compact.

Using [2, Theorem 5.2.1 and Theorem 3.4.1/(a)], for any $r < q_0$ we get

$$L_{p_0}(R) = (L_\infty(R), L_1(R))_{1/p_0,p_0} \text{ if } p_0 > 1,$$
$$L_{p_1}(R) = (L_\infty(R), L_1(R))_{1/p_1,p_1},$$
$$L_{q_0}(S) = (L_\infty(S), L_r(S))_{r/q_0,q_0},$$
$$L_{q_1}(S) = (L_\infty(S), L_r(S))_{r/q_1,q_1}.$$

It follows from [18, Theorem 4.7 and Theorem 5.9] that

$$(L_\infty(R), L_1(R))_{1/p_1,q,(\alpha_\infty,\alpha_0)+1/\min(p_1,q)} \hookrightarrow (L_{p_1}(R), L_{p_0}(R))_{0,q,(\alpha_\infty,\alpha_0)} \text{ and}$$
$$(L_{q_1}(S), L_{q_0}(S))_{0,q,(\alpha_\infty,\alpha_0)} \hookrightarrow (L_\infty(S), L_r(S))_{r/q_1,q,(\alpha_\infty,\alpha_0)+1/\max(q,q_1)}.$$

If $p_0 = 1$, these inclusions also follow from [18, Theorem 4.7 and Theorem 5.9]. Furthermore, according to [18, Corollary 8.4] and (2) we have

$$L_{p_1,q,\mathbb{A}+1/\min(p_1,q)} = (L_\infty(R), L_1(R))_{1/p_1,q,(\alpha_\infty,\alpha_0)+1/\min(p_1,q)},$$
$$L_{q_1,q,\mathbb{A}+1/\max(q_1,q)} = (L_\infty(S), L_r(S))_{r/q_1,q,(\alpha_\infty,\alpha_0)+1/\max(q,q_1)}.$$

Consequently, the operator

$$T : L_{p_1,q,\mathbb{A}+\frac{1}{\min(p_1,q)}} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{1,q,\mathbb{A}} \to (L_{q_0}(S), L_{q_1}(S))_{1,q,\mathbb{A}} \hookrightarrow L_{q_1,q,\mathbb{A}+\frac{1}{\max(q_1,q)}}$$

is compact. ■

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