NUCLEAR EMBEDDINGS OF BESOV SPACES INTO ZYGMUND SPACES

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Dedicated to the memory of Professor Jaak Peetre

Abstract. Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. We prove that the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is nuclear if $a < -1$ and $1 \le p, q \le \infty$, while if $-1 < a < 0$, $2 < p < \infty$ and $p \le q \le \infty$ the embedding I_d fails to be nuclear. Furthermore, if $a = -1$, the embedding $I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega)$ is not nuclear.

1. Introduction

The research on nuclearity of embeddings between function spaces started with the study of embeddings between certain Sobolev spaces of Hilbert type in the paper by Maurin [14, p. 366] and the books by Yosida [26, p.279] and Maurin [15, p. 336]. Outside the framework of Hilbert spaces, results on nuclearity of Sobolev embeddings were obtained by Pietsch and Triebel [21] and Pietsch [18] (see also [22, p. 354]). Recent contributions are due to Edmunds, Gurka and Lang [7], Triebel [25] and Cobos, Domínguez and Kühn [3].

Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For $1 < p < \infty, 1 \le q \le \infty$ consider the Besov space $B^d_{p,q}(\Omega)$ and the Lebesgue space $L_p(\Omega)$. The embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\Omega)$ is compact. In fact, its approximation numbers $a_n(T)$ behave as n^{-1} (see [8, Theorem 3.3.4]). However, it follows from a recent result by Triebel [25, Theorem, p. 3039] that I_d is not nuclear for $1 < p, q < \infty$. Indeed, the result of Triebel implies that the embedding $B^d_{p,\min(p,q)}(\Omega) \hookrightarrow B^0_{p,\max(p,2)}(\Omega)$ is not nuclear. Since we have the factorization

$$B^d_{p,\min(p,q)}(\Omega) \hookrightarrow B^d_{p,q}(\Omega) \xrightarrow{I_d} L_p(\Omega) = F^0_{p,2}(\Omega) \hookrightarrow B^0_{p,\max(p,2)}(\Omega)\,,$$

it follows that I_d cannot be nuclear. If $q = 1$ or ∞ we can proceed similarly but using now [3, Theorem 4.5] with the same result that I_d is not nuclear.

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Starting from the non-nuclear embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\Omega)$ there are two natural ways to achieve nuclearity: One could either use a smaller source space or a larger target space. The first option was investigated by Cobos, Domínguez and Kühn [3], where we added some logarithmic smoothness to the Besov space $B^d_{p,q}(\Omega)$. Here we study the second way.

If we take any $r \in (1,p)$ and replace $L_p(\Omega)$ by the bigger space $L_r(\Omega)$ then $I_d : B^d_{p,q}(\Omega) \longrightarrow L_r(\Omega)$ can be factorized as follows:

$$B^d_{p,q}(\Omega) \hookrightarrow B^0_{r,\min(2,r)}(\Omega) \hookrightarrow F^0_{r,2}(\Omega) = L_r(\Omega).$$

According to [25, Theorem], the embedding $B^d_{p,q}(\Omega) \hookrightarrow B^0_{r,\min(2,r)}(\Omega)$ is nuclear. Therefore, $I_d : B^d_{p,q}(\Omega) \longrightarrow L_r(\Omega)$ is also nuclear. Consequently, the problem is to find a target space X, bigger than $L_p(\Omega)$ and smaller than $L_r(\Omega)$ for any $r \in (1,p)$ such that $I_d : B^d_{p,q}(\Omega) \longrightarrow X$ is still nuclear. Since the Zygmund space $L_p(\log L)_a(\Omega)$ has the property that $L_p(\Omega) \hookrightarrow L_p(\log L)_a(\Omega) \hookrightarrow L_r(\Omega)$ for any $a < 0$ and $r < p$, this leads us naturally to study whether or not the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is nuclear for $a < 0$. This is the aim of the present paper.

Note that the spaces $B^d_{p,q}(\Omega)$ and $L_p(\log L)_a(\Omega)$ belong to different scales. Indeed, the Zygmund spaces $L_p(\log L)_a(\Omega)$ are related to Triebel-Lizorkin spaces $F^s_{p,q}(\Omega)$ because $F^0_{p,q}(\Omega) = L_p(\Omega)$ and $L_p(\log L)_a(\Omega)$ can be obtained by extrapolation of Lebesgue spaces (see [8, Theorem 1 of Section 2.6.2] or [4, Corollary 3.1]). The fact that these scales of spaces are different produces several difficulties in the research and makes it more interesting. For example, it is not possible to reduce the problem to sequence space considerations by using wavelet bases because the sequence spaces associated to $B^d_{p,q}(\Omega)$ and $L_p(\log L)_a(\Omega)$ have different structure (see [23, Section 1.7 and Chapter 3] and [24, pp. 13-17]). To overcome this obstruction we will use the inclusion relations between Lebesgue spaces and Besov spaces with smoothness 0, the representation of $L_p(\log L)_a(\Omega)$ as interpolation space generated by a couple of Lebesgue spaces by using a logarithmic perturbation of the real method, and the characterization of $L_p(\log L)_a(\Omega)$ by extrapolation of Lebesgue spaces.

The organization of the paper is as follows. In Section 2 we recall some basic properties of nuclear operators. In Section 3 we review the definitions of the function spaces that we need. Finally, in Section 4, we study nuclearity of the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ for $a < 0$. We show that I_d is nuclear if $a < -1$ and that this result is almost optimal in the sense that if $-1 < a < 0, 2 < p < \infty$ and $p \leq q \leq \infty$ then I_d is not nuclear. Furthermore, we show that, corresponding to the choice $a = -1$ and $p = q = \infty$, the embedding $I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega)$ also fails to be nuclear.

2. Nuclear operators

Let E, F be Banach spaces and let E' be the dual space of E. We write $\mathcal{L}(E,F)$ for the space of all bounded linear operators from E to F. According to Grothendieck [10], an operator $T \in \mathcal{L}(E,F)$ is said to be *nuclear* if T can be represented as

$$Tx = \sum_{k=1}^{\infty} f_k(x) y_k \quad \text{with} \quad \sum_{k=1}^{\infty} \|f_k\|_{E'} \|y_k\|_F < \infty \,,$$

where $(f_k) \subseteq E'$ and $(y_k) \subseteq F$. The collection $\mathcal{N}(E,F)$ of all nuclear operators from E to F is a Banach space endowed with the norm

$$\nu(T) = \nu(T : E \longrightarrow F) = \inf \Big\{ \sum_{k=1}^{\infty} \|f_k\|_{E'} \|y_k\|_F : Tx = \sum_{k=1}^{\infty} f_k(x) y_k \Big\} .$$

Note that if $T \in \mathcal{N}(E,F)$ then T is the limit of a sequence of finite rank operators, so T is compact. Note also that $\|T : E \longrightarrow F\| \leq \nu(T : E \longrightarrow F)$.

The collection of all nuclear operators $\mathcal{N} = \bigcup_{E,F} \mathcal{N}(E,F)$ is a Banach operator ideal in the sense of [19, 13, 6]. Hence, if $R \in \mathcal{L}(E_0, E)$, $T \in \mathcal{N}(E,F)$ and $S \in \mathcal{L}(F, F_0)$, then the composite operator STR is nuclear and

$$\nu(STR : E_0 \longrightarrow F_0) \leq \|S : F \longrightarrow F_0\| \nu(T : E \longrightarrow F) \|R : E_0 \longrightarrow E\| .$$

If $n \in \mathbb{N}$, E is an n-dimensional space and $id : E \longrightarrow E$ is the identity operator, then $\nu(id : E \longrightarrow E) = n$ (see [12, p. 18] or [20, pp. 65-66]).

Let E be a complex Banach space and let $T \in \mathcal{L}(E,E)$ be a compact operator. We denote by $(\lambda_k(T))$ the sequence of eigenvalues of T, counted according to their algebraic multiplicity and ordered by decreasing modulus. If T has only a finite number of eigenvalues, then we complete the sequence with 0. If $T \in \mathcal{N}(E,E)$ then it was shown by Grothendieck [10] that

$$\Big(\sum_{k=1}^{\infty} |\lambda_k(T)|^2 \Big)^{1/2} \leq \nu(T : E \longrightarrow E)$$

(see also [20, p. 160] or [13, p. 105]). This result can be improved if E is a Hilbert space H with the effect that

$$\sum_{k=1}^{\infty} |\lambda_k(T)| \leq \nu(T : H \longrightarrow H)$$

(see [20, 3.8.3]).

Let E be an n-dimensional Banach space and let $T \in \mathcal{L}(E,E)$. Clearly we can find a finite nuclear representation $Tx = \sum_{k=1}^{m} f_k(x) x_k$ of T with $(f_k)_{k=1}^{m} \subseteq E'$ and

$(x_k)_{k=1}^m \subseteq E$. The value $\sum_{k=1}^m f_k(x_k)$ does not depend of the particular representation of T (see [12, pp. 13-15] or [20, Lemma 4.2.2]). Thus, the *trace* of T is defined by

$$\text{trace } T = \sum_{k=1}^{m} f_k(x_k)\,.$$

According to [12, 1.10.(ii)], we have that

$$|\text{trace } T| \leq n\|T : E \longrightarrow E\|\,. \tag{2.1}$$

The following property follows from results of Grothendieck [10, I.5.1]. For completeness we include a proof.

Lemma 2.1. *If E is a finite-dimensional Banach space and $T \in \mathcal{L}(E, E)$, then*

$$|\textit{trace } T| \leq \nu(T : E \longrightarrow E)\,.$$

Proof. Let $\dim E = n$, $\varepsilon > 0$ and $\delta > 0$. We can choose a nuclear representation of T, $Tx = \sum_{k=1}^{\infty} f_k(x)x_k$, with $(f_k) \subseteq E', (x_k) \subseteq E$ and $\sum_{k=1}^{\infty} \|f_k\|_{E'}\|x_k\|_E \leq (1+\varepsilon)\nu(T)$. Now select $N \in \mathbb{N}$ such that $\sum_{k>N} \|f_k\|_{E'}\|x_k\|_E \leq \delta$ and consider the operators R_N and S_N defined by $R_N x = \sum_{k=1}^{N} f_k(x)x_k$ and $S_N x = \sum_{k=N+1}^{\infty} f_k(x)x_k$. Then

$$\begin{aligned}|\text{trace } R_N| &= |\sum_{k=1}^{N} f_k(x_k)| \leq \sum_{k=1}^{N} \|f_k\|_{E'}\|x_k\|_E \\ &\leq \sum_{k=1}^{\infty} \|f_k\|_{E'}\|x_k\|_E \leq (1+\varepsilon)\nu(T)\,.\end{aligned}$$

On the other hand, by (2.1),

$$\begin{aligned}|\text{trace } S_N| &\leq n\|S_N : E \longrightarrow E\| \leq n\,\nu(S_N : E \longrightarrow E) \\ &\leq n \sum_{k=N+1}^{\infty} \|f_k\|_{E'}\|x_k\|_E \leq n\delta\,.\end{aligned}$$

Since the trace is linear, using the triangle inequality we get

$$|\text{trace } T| \leq |\text{trace } R_N| + |\text{trace } S_N| \leq (1+\varepsilon)\nu(T : E \longrightarrow E) + \delta n\,.$$

Finally, letting $\delta \to 0$ and $\varepsilon \to 0$, we obtain the desired result. □

3. Function spaces

Let $d \in \mathbb{N}$. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions on $\mathbb{R}^d$. We write $\mathfrak{F}$ for the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ and $\mathfrak{F}^{-1}$ for the inverse Fourier transform.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ with supp $\varphi_0 \subset \{x \in \mathbb{R}^d : |x| \leq 2\}$ and $\varphi_0(x) = 1$ if $|x| \leq 1$. For $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$ put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^d$, and for any $f \in \mathcal{S}'(\mathbb{R}^d)$ and $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the entire analytic functions $\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)$ make sense pointwise in $\mathbb{R}^d$.

Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Then the Besov space $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ having a finite norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} (2^{js} \|\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)\|_{L_p(\mathbb{R}^d)})^q \right)^{1/q}$$

with the usual modification if $q = \infty$.

For s and q as before and $1 \leq p < \infty$, the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$ is formed by all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d)} = \left\| \left(\sum_{j=0}^{\infty} (2^{js} |\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)(\cdot)|)^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}$$

is finite. If $1 < p < \infty$, we put $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$. The choice $s = 0$ produces the well-known Lebesgue spaces $H^0_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.

Details of these two well-known scales of spaces can be found in the monographs [16, 22, 23, 24]. It turns out that

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d)$$

where $\hookrightarrow$ means continuous embedding.

Next we recall briefly the characterization of Besov spaces in terms of wavelets. We refer to [23, Section 1.7 and Chapter 3] and [24, pp. 13-17] for full details, see also [25].

Put $L_0 = 1$ and $L_j = 2^d - 1$ if $j \in \mathbb{N}$. In what follows we assume that $j \in \mathbb{N}_0$, $1 \leq l \leq L_j$ and $m \in \mathbb{Z}^d$. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the space $b^s_{p,q}$ consists of all scalar sequences $\lambda = (\lambda_{jlm})$ having a finite norm

$$\|\lambda\|_{b^s_{p,q}} = \Big(\sum_j 2^{j(s-d/p)q} \big(\sum_{l,m} |\lambda_{jlm}|^p \big)^{q/p} \Big)^{1/q}.$$

Given $r \in \mathbb{N}$, for $1 \leq l \leq 2^d - 1$ take real compactly supported functions $\psi_0, \psi^l \in \mathcal{C}^r(\mathbb{R}^d)$ (i.e. having continuous bounded derivatives up to order r) satisfying the moment conditions

$$\int_{\mathbb{R}^d} x^\alpha \psi^l(x) dx = 0 \text{ for all } \alpha \in \mathbb{N}_0 \text{ with } |\alpha| \leq r\,,$$

and such that the system

$$\big\{ 2^{jd/2} \psi_{jlm} : j \in \mathbb{N}_0,\, 1 \leq l \leq L_j,\, m \in \mathbb{Z}^d \big\}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$, where the functions ψ_{jlm} are defined by

$$\psi_{jlm}(x) = \begin{cases} \psi_0(x-m) & \text{if} \quad j=0,\, l=1,\, m\in\mathbb{Z}^d \\ \psi^l(2^{j-1}x-m) & \text{if} \quad j\in\mathbb{N},\, 1\le l\le 2^d-1,\, m\in\mathbb{Z}^d\,. \end{cases} \tag{3.1}$$

Let $s\in\mathbb{R}$ and $1\le p,q\le\infty$. Then there is a number $r(s,p)>0$ such that if $\{\psi_{jlm}\}$ is a system of functions as above with $r>r(s,p)$, then the following holds:

A distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if and only if it can be represented as $f=\sum_{jlm}\lambda_{jlm}\psi_{jlm}$ with $\lambda(f):=(\lambda_{jlm})\in b^s_{p,q}$, where the series converges unconditionally in $\mathcal{S}'(\mathbb{R}^d)$ and the coefficients are determined by

$$\lambda_{jlm}=\lambda_{jlm}(f)=2^{jd}\langle f,\psi_{jlm}\rangle = 2^{jd}\int_{\mathbb{R}^d} f(x)\psi_{jlm}(x)dx\,.$$

Moreover,

$$\|\lambda(f)\|_{b^s_{p,q}} \text{ defines an equivalent norm to } \|f\|_{B^s_{p,q}(\mathbb{R}^d)}\,. \tag{3.2}$$

The spaces $F^s_{p,q}(\mathbb{R}^d)$ can be also characterized in terms of wavelets, but the nature of the corresponding sequence spaces is different from that of the spaces $b^s_{p,q}$ (see [23, 24]).

Subsequently, Ω stands for a bounded Lipschitz domain in $\mathbb{R}^d$ (see [23, pp. 63-64]). As usual the space $B^s_{p,q}(\Omega)$ is defined by restriction of $B^s_{p,q}(\mathbb{R}^d)$ to Ω. The norm in $B^s_{p,q}(\Omega)$ is given by

$$\|f\|_{B^s_{p,q}(\Omega)} = \inf\left\{\|g\|_{B^s_{p,q}(\mathbb{R}^d)} : g\in B^s_{p,q}(\mathbb{R}^d),\, g_{|\Omega}=f\right\}.$$

The spaces $F^s_{p,q}(\Omega)$ are defined similarly.

Let (Λ,μ) be a σ-finite measure space, let $1\le p\le\infty$ and $a\in\mathbb{R}$. The Zygmund space $L_p(\log L)_a(\Lambda)$ is formed by all (equivalence classes of) μ-measurable functions f on Λ for which

$$\|f\|_{L_p(\log L)_a(\Lambda)} = \Big(\int_0^{\mu(\Lambda)} [(1+|\log t|)^a f^*(t)]^p dt\Big)^{1/p} < \infty$$

(the integral should be replaced by the supremum if $p=\infty$). Here f^* is the non-increasing rearrangement of f

$$f^*(t)=\inf\{s>0 : \mu\{x\in\Lambda : |f(x)|>s\}\le t\}\,.$$

See [1, 8] for properties of Zygmund spaces. Note that if $a=0$ we get the Lebesgue spaces $L_p(\Lambda)$. Clearly, if $a<0$ then

$$L_p(\log L)_{-a}(\Lambda)\hookrightarrow L_p(\Lambda)\hookrightarrow L_p(\log L)_a(\Lambda)\,.$$

When working with Zygmund spaces, two different descriptions will be very useful, either as extrapolation spaces (see [8, Section 2.6.2] or [4, Section 3.3]) or as interpolation spaces generated by the *logarithmic interpolation method* $(A_0,A_1)_{\theta,\gamma,q}$. Next we recall the definition of this interpolation method.

Let A_0, A_1 be Banach spaces continuously embedded in a Hausdorff topological vector space. For $0 < \theta < 1, \gamma \in \mathbb{R}$ and $1 \le q \le \infty$, the space $(A_0, A_1)_{\theta,\gamma,q}$ consists of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,\gamma,q}} = \left(\int_0^\infty \big(t^{-\theta} (1 + |\log t|)^{-\gamma} K(t,a) \big)^q \frac{dt}{t} \right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here $K(t,a)$ is the K-functional of Peetre,

$$K(t,a) = \inf\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j \} .$$

We refer to [11, 17, 9, 5] for details of these spaces. For $\gamma = 0$ we get the classical real interpolation space $(A_0, A_1)_{\theta,q}$ (see [2, 22, 1]).

In what follows we use the symbols $\lesssim$ and $\sim$ with the usual meaning: If X and Y are quantities depending on certain parameters, we write $X \lesssim Y$ if there is a constant c independent of the parameters such that $X \le cY$. We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

4. Embeddings and nuclearity

Next we establish the result announced in the Introduction on nuclearity of embeddings.

Theorem 4.1. *Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that $1 \le p, q \le \infty$ and $a \in \mathbb{R}$. If $a < -1$, then the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is nuclear.*

Proof. Our arguments are based on the wavelet representation of $B^d_{p,q}(\mathbb{R}^d)$ and the existence of a (bounded linear) extension operator $\mathrm{ext} : B^d_{p,q}(\Omega) \to B^d_{p,q}(\mathbb{R}^d)$, see [25] for full details. The embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ can be factorized as

$$I_d = \mathrm{restr} \circ \mathrm{id} \circ \mathrm{ext} ,$$

where $\mathrm{id} : B^d_{p,q}(\mathbb{R}^d) \longrightarrow L_p(\log L)_a(\mathbb{R}^d)$ denotes the formal identity and $\mathrm{restr} : L_p(\log L)_a(\mathbb{R}^d) \longrightarrow L_p(\log L)_a(\Omega)$ is the restriction operator.

Therefore, for any $f \in B^d_{p,q}(\Omega)$, the following wavelet representation holds

$$\mathrm{ext}(f) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \sum_{m \in \mathbb{Z}^d} \lambda_{jlm} \psi_{jlm} \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^d)) \tag{4.1}$$

with $\lambda_{jlm} = \lambda_{jlm}(\mathrm{ext}(f)) = 2^{jd} \langle \mathrm{ext}(f), \psi_{jlm} \rangle$ and

$$\|f\|_{B^d_{p,q}(\Omega)} \sim \|\mathrm{ext}(f)\|_{B^d_{p,q}(\mathbb{R}^d)} \sim \Big(\sum_{j=0}^{\infty} \big(2^{jd/p'} \big(\sum_{l=1}^{L_j} \sum_{m \in \mathbb{Z}^d} |\lambda_{jlm}|^p \big)^{1/p} \big)^q \Big)^{1/q} , \tag{4.2}$$

where $1/p + 1/p' = 1$. The coefficients define bounded linear functionals on $B^d_{p,q}(\Omega)$,

$$F_{j\ell m}(f) := \lambda_{jlm}(\mathrm{ext}(f))\,.$$

Indeed, according to (4.1) and (4.2), we have for all $f \in B^d_{p,q}(\Omega)$ the estimate

$$|F_{jlm}(f)| \lesssim 2^{-jd/p'} \|\mathrm{ext}(f)\|_{B^d_{p,q}(\mathbb{R}^d)} \sim 2^{-jd/p'} \|f\|_{B^d_{p,q}(\Omega)}\,,$$

whence

$$\|F_{jlm}\|_{(B^d_{p,q}(\Omega))'} \lesssim 2^{-jd/p'}\,. \tag{4.3}$$

This implies for the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ the expansion

$$I_d(f) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \sum_{m \in Z_j} F_{jlm}(f) \psi_{jlm|\Omega}\,, \tag{4.4}$$

where the index sets Z_j are defined by

$$Z_j := \{m \in \mathbb{Z}^d : \mathrm{supp}\ (\psi_{jlm}) \cap \Omega \neq \varnothing\} \quad \text{for} \quad j \in \mathbb{N}_0\,.$$

We show now that (4.4) is a nuclear representation of the operator I_d. Since the functions ψ_0 and $\psi^l, 1 \le l \le 2^d - 1$, are compactly supported, there is $r > 0$ such that

$$\mathrm{supp}\ (\psi_0) \cup \bigcup_{l=1}^{2^d-1} \mathrm{supp}\ (\psi^l) \subseteq [-r, r]^d\,.$$

By the definition of ψ_{jlm} in (3.1), and since Ω is a bounded domain with non-empty interior, one can easily check that

$$M_j := \mathrm{card}\ Z_j \sim 2^{jd} \qquad \text{and} \qquad \mathrm{V}_{j\ell m} := \mathrm{vol}\ (\mathrm{supp}\ (\psi_{jlm})) \lesssim 2^{-jd}\,. \tag{4.5}$$

Moreover, by the boundedness of ψ_0 and ψ^l, we have $\|\psi_{jlm}\|_{L_\infty(\mathbb{R}^d)} \lesssim 1$. This yields

$$\begin{aligned}
\|\psi_{jlm|\Omega}\|_{L_p(\log L)_a(\Omega)} &\le \|\psi_{jlm}\|_{L_p(\log L)_a(\mathbb{R}^d)} \\
&= \Big(\int_0^{\mathrm{V}_{j\ell m}} [(1 + |\log t|)^a \psi^*_{jlm}(t)]^p dt \Big)^{1/p} \\
&\lesssim \Big(\int_0^{\mathrm{V}_{j\ell m}} (1 + |\log t|)^{ap} dt \Big)^{1/p} \\
&\lesssim 2^{-jd/p} (1 + j)^a\,,
\end{aligned}$$

where we have used the fact that the function $(1 + |\log t|)^a$ is increasing in $(0, 1]$.

Consequently, collecting all these estimates and taking into account that $L_j \le 2^d$, we derive the desired result

$$\begin{aligned}\nu(I_d) &\le \sum_{j=0}^{\infty}\sum_{l=1}^{L_j}\sum_{m\in Z_j} \|F_{jlm}\|_{(B^d_{p,q}(\Omega))'}\|\psi_{jlm|\Omega}\|_{L_p(\log L)_a(\Omega)} \\ &\lesssim \sum_{j=0}^{\infty} 2^d\cdot 2^{jd}\cdot 2^{-jd/p'}\cdot 2^{-jd/p}(1+j)^a \sim \sum_{j=0}^{\infty}(1+j)^a < \infty\,,\end{aligned}$$

because $a < -1$. □

Since the parameter q plays no role in Theorem 4.1, we can derive the following consequence for Triebel-Lizorkin spaces.

Theorem 4.2. *Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that $1 \le p < \infty, 1 \le q \le \infty$ and $a \in \mathbb{R}$. If $a < -1$ then the embedding $I_d : F^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is nuclear.*

Proof. Since $F^d_{p,q}(\Omega) \hookrightarrow B^d_{p,\max(p,q)}(\Omega)$, nuclearity of $I_d : F^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ follows from Theorem 4.1 and the ideal property of $\mathcal{N}$. □

The remaining of this section is devoted to show that Theorem 4.1 is almost optimal. This is done by means of two negative results. First we introduce some notation based on the representation (4.1).

For $N \in \mathbb{N}$ let

$$\Lambda_N = \{(j,l,m) : 0 \le j \le N,\ 1 \le l \le L_j,\ m \in Z_j\} \tag{4.6}$$

be endowed with the measure

$$\mu_N = \sum_{j=0}^{N}\frac{1}{L_jM_j}\sum_{l=1}^{L_j}\sum_{m\in Z_j}\delta_{(j,l,m)}$$

where

$$\delta_{(j,l,m)}\{(x,y,z)\} = \begin{cases}1 & \text{if} \quad j=x,\ l=y \text{ and } m=z\\ 0 & \text{otherwise}\,.\end{cases}$$

Then (Λ_N, μ_N) is a completely atomic measure space, with finite measure $\mu_N(\Lambda_N) = N+1$.

Subsequently we will also work with Lebesgue and Zygmund spaces defined on Λ_N. As usual, we identify a function g on Λ_N with the sequence of values $(\xi_{jlm})_{(j,l,m)\in\Lambda_N}$ that g takes.

Next we establish an auxiliary result. First note that, since $L_p(\Omega) = F^0_{p,2}(\Omega)$ (see [23, 24]) and any function $f \in L_p(\Omega)$ can be written in the form

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \sum_{m \in Z_j} \lambda_{jlm} \psi_{jlm|\Omega}\,,$$

the following operator is well-defined for every $N \in \mathbb{N}$

$$(4.7) \qquad P_N f = (\lambda_{jlm})_{(j,l,m)\in\Lambda_N} \quad \text{for } f = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \sum_{m \in Z_j} \lambda_{jlm} \psi_{jlm|\Omega} \in L_p(\Omega)\,.$$

Lemma 4.3. *Let $d, N \in \mathbb{N}$, let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Consider the measure space (Λ_N, μ_N) introduced in (4.6) and the operator P_N defined in (4.7). If $2 < p < \infty$ and $a < 0$ then $P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\Lambda_N)$ is bounded with*

$$\|P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\Lambda_N)\| \lesssim N^{-a}\,.$$

Proof. Take $2 \le p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $1/p = (1-\theta)/p_0 + \theta/p_1$. By (3.2), the following diagram holds

$$L_{p_j}(\Omega) = F^0_{p_j,2}(\Omega) \hookrightarrow B^0_{p_j,p_j}(\Omega) \xrightarrow{P_N} L_{p_j}(\Lambda)\,,\; j = 0, 1.$$

According to [17, Proposition 6.2] we have that

$$L_p(\log L)_a(\Omega) = (L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,-a,p} \quad \text{and}$$
$$L_p(\log L)_a(\Lambda_N) = (L_{p_0}(\Lambda_N), L_{p_1}(\Lambda_N))_{\theta,-a,p}\,.$$

Then it follows from the interpolation theorem for the logarithmic interpolation method (see [11, Theorem 2.1]) that $P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\log L)_a(\Lambda_N)$ is bounded with

$$\|P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\log L)_a(\Lambda_N)\| \lesssim 1\,.$$

Therefore, to establish the lemma it suffices to prove that the embedding J_N from $L_p(\log L)_a(\Lambda_N)$ into $L_p(\Lambda_N)$ satisfies that

$$(4.8) \qquad \|J_N : L_p(\log L)_a(\Lambda_N) \longrightarrow L_p(\Lambda_N)\| \lesssim N^{-a}.$$

With this aim, note that the smallest measure of an atom in Λ_N is $\tau_N = (2^d - 1)^{-1} M_N^{-1}$, which behaves as 2^{-Nd} according to (4.5). Moreover, without loss of generality, we may assume that N is sufficiently large so that

$$(4.9) \qquad (1 + \log(N+1))^{-ap} \le (1 + |\log \tau_N|)^{-ap} \sim N^{-ap}\,.$$

Let $g \in L_p(\log L)_a(\Lambda_N)$. The non-increasing rearrangement g^* of g takes only a finite number of values, say $g^* = \sum_{k=1}^{R} \alpha_k \chi_{(b_{k-1},b_k)}$ with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_R \ge 0$ and

$0 = b_0 < \tau_N = b_1 < b_2 < \cdots < b_R = N+1$. Hence

$$\|g\|^p_{L_p(\Lambda_N)} = \int_0^{N+1} g^*(t)^p dt = \sum_{k=1}^R \int_{b_{k-1}}^{b_k} \alpha_k^p dt\,.$$

We claim that

$$\int_{b_{k-1}}^{b_k} \alpha_k^p dt \lesssim N^{-ap} \int_{b_{k-1}}^{b_k} (1+|\log t|)^{ap} \alpha_k^p dt\,,\ k = 1,\ldots,R\,.$$

Indeed, since the function $(1+|\log t|)^{ap}$ is increasing in $(0,1]$ and decreasing in $[1,N+1]$, we have for the integral with $k=1$ that

$$\begin{aligned}\int_0^{\tau_N} \alpha_1^p dt &= 2(\tau_N/2)\alpha_1^p = 2\int_{\tau_N/2}^{\tau_N} (1+|\log t|)^{-ap}(1+|\log t|)^{ap}\alpha_1^p dt\\ &\le 2(1-\log(\tau_N/2))^{-ap}\int_0^{\tau_N}(1+|\log t|)^{ap}\alpha_1^p dt\\ &\lesssim N^{-ap}\int_0^{\tau_N}(1+|\log t|)^{ap}\alpha_1^p dt\,.\end{aligned}$$

For $k>1$, we get

$$\begin{aligned}\int_{b_{k-1}}^{b_k} \alpha_k^p dt &= \int_{b_{k-1}}^{b_k} (1+|\log t|)^{-ap}(1+|\log t|)^{ap}\alpha_k^p dt\\ &\le \sup_{b_{k-1}<s<b_k}(1+|\log s|)^{-ap}\int_{b_{k-1}}^{b_k}(1+|\log t|)^{ap}\alpha_k^p dt\\ &\le \max\{(1+|\log \tau_N|)^{-ap},(1+\log(N+1))^{-ap}\}\int_{b_{k-1}}^{b_k}(1+|\log t|)^{ap}\alpha_k^p dt\\ &\lesssim N^{-ap}\int_{b_{k-1}}^{b_k}(1+|\log t|)^{ap}\alpha_k^p dt\end{aligned}$$

by (4.9).

Therefore, we derive that

$$\begin{aligned}\|g\|^p_{L_p(\Lambda_N)} &= \sum_{k=1}^R \int_{b_{k-1}}^{b_k}\alpha_k^p dt \lesssim N^{-ap}\sum_{k=1}^R\int_{b_{k-1}}^{b_k}(1+|\log t|)^{ap}\alpha_k^p dt\\ &= N^{-ap}\|g\|^p_{L_p(\log L)_a(\Lambda_N)}\,.\end{aligned}$$

This establishes (4.8) and completes the proof. □

Now we are ready for establishing the negative results.

Theorem 4.4. *Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that $2 < p < \infty, p \le q \le \infty$ and $-1 < a < 0$. Then the embedding $I_d : B^d_{p,q}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is not nuclear.*

Proof. By the ideal property of nuclear operators and the diagram

$$B^d_{p,p}(\Omega) \hookrightarrow B^d_{p,q}(\Omega) \xrightarrow{I_d} L_p(\log L)_a(\Omega)\,,$$

it suffices to show that $I_d : B^d_{p,p}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is not nuclear. With this aim, take any $N \in \mathbb{N}$, consider the finite dimensional space $L_p(\Lambda_N)$ and the factorization

$$L_p(\Lambda_N) \xrightarrow{A_N} B^d_{p,p}(\Omega) \xrightarrow{I_d} L_p(\log L)_a(\Omega) \xrightarrow{P_N} L_p(\Lambda_N)\,.$$

Here

$$A_N(\lambda_{jlm}) = \sum_{j=0}^{N} 2^{-jd} \sum_{l=1}^{L_j} \sum_{m\in Z_j} \lambda_{jlm}\psi_{jlm|\Omega}$$

and P_N is the operator defined in (4.7). By (3.2), we have

$$\begin{aligned}\|A_N(\lambda_{jlm})\|_{B^d_{p,p}(\Omega)} &\sim \Big(\sum_{j=0}^{N} 2^{jdp/p'-jdp} \sum_{l=1}^{L_j} \sum_{m\in Z_j} |\lambda_{jlm}|^p\Big)^{1/p} \\ &\sim \|(\lambda_{jlm})\|_{L_p(\Lambda_N)}\,,\end{aligned}$$

so

$$\|A_N : L_p(\Lambda_N) \longrightarrow B^d_{p,p}(\Omega)\| \lesssim 1\,. \tag{4.10}$$

Furthermore, according to Lemma 4.3, we get

$$\|P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\Lambda_N)\| \lesssim N^{-a}\,.$$

Let $D_N := P_N I_d A_N : L_p(\Lambda_N) \longrightarrow L_p(\Lambda_N)$. The operator D_N is diagonal. In fact,

$$D_N(\lambda_{jlm}) = (2^{-jd}\lambda_{jlm})_{(j,l,m)\in\Lambda_N}\,.$$

Then for the trace of D_N, having in mind (4.5), we obtain

$$\text{trace } D_N = \sum_{j=0}^{N} 2^{-jd} \sum_{l=1}^{L_j} \sum_{m\in Z_j} 1 \sim N\,.$$

Consequently, if we assume that $I_d : B^d_{p,p}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)$ is nuclear, then it follows from Lemma 2.1 that

$$\begin{aligned} N &\lesssim |\text{trace } D_N| \le \nu(D_N : L_p(\Lambda_N) \longrightarrow L_p(\Lambda_N)) \\ &\le \|A_N : L_p(\Lambda_N) \longrightarrow B^d_{p,p}(\Omega)\|\,\nu(I_d : B^d_{p,p}(\Omega) \longrightarrow L_p(\log L)_a(\Omega)) \\ &\qquad \times \|P_N : L_p(\log L)_a(\Omega) \longrightarrow L_p(\Lambda_N)\| \\ &\lesssim \nu(I_d : B^d_{p,p}(\Omega) \longrightarrow L_p(\log L)_a(\Omega))\, N^{-a} \end{aligned}$$

which is a contradiction because $-1 < a < 0$. □

We close the paper with a result concerning the limit case $a = -1$.

Theorem 4.5. *Let $d \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then the embedding $I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega)$ is not nuclear.*

Proof. This time the argument relies on the extrapolation description of the Zygmund space $L_\infty(\log L)_{-1}(\Omega)$ (see [8, Theorem 2.6.2/1]). Namely

$$\|f\|_{L_\infty(\log L)_{-1}(\Omega)} \sim \sup_{r\geq 1} r^{-1}\|f\|_{L_r(\Omega)} \, . \tag{4.11}$$

Take any $N \in \mathbb{N}$ and consider the factorization

$$L_\infty(\Lambda_N) \xrightarrow{A_N} B^d_{\infty,\infty}(\Omega) \xrightarrow{I_d} L_\infty(\log L)_{-1}(\Omega) \xrightarrow{S_N} L_\infty(\Lambda_N)$$

where again

$$A_N(\lambda_{jlm}) = \sum_{j=0}^{N} 2^{-jd} \sum_{l=1}^{L_j} \sum_{m\in Z_j} \lambda_{jlm}\psi_{jlm|\Omega}$$

and now

$$S_N f = \Big(\frac{\lambda_{jlm}}{j+1}\Big)_{(j,l,m)\in\Lambda_N} \quad \text{for } f = \sum_{j=0}^{\infty}\sum_{l=1}^{L_j}\sum_{m\in Z_j} \lambda_{jlm}\psi_{jlm|\Omega} \, .$$

Proceeding as in (4.10) we have that $\|A_N : L_\infty(\Lambda_N) \longrightarrow B^d_{\infty,\infty}(\Omega)\| \lesssim 1$. In order to estimate the norm of S_N, take any $f \in L_\infty(\log L)_{-1}(\Omega)$ and any $r \geq 1$. Using Hölder's inequality and (4.5), we have

$$\begin{aligned} |\lambda_{jlm}| &\leq 2^{jd}\int_\Omega |f\psi_{jlm}|dx \\ &\leq 2^{jd}\|f\|_{L_r(\Omega)}\|\psi_{jlm}\|_{L_{r'}(\Omega)} \\ &\lesssim 2^{jd/r}\|f\|_{L_r(\Omega)} \\ &= 2^{jd/r} r\big(r^{-1}\|f\|_{L_r(\Omega)}\big) \, . \end{aligned}$$

The choice $r = jd$ and (4.11) yield that

$$|\lambda_{jlm}| \lesssim j\|f\|_{L_\infty(\log L)_{-1}(\Omega)} \, .$$

Hence

$$\|S_N : L_\infty(\log L)_{-1} \longrightarrow L_\infty(\Lambda_N)\| \lesssim 1 \, .$$

Put $D_N = S_N I_d A_N$. Then D_N is the diagonal operator in $L_\infty(\Lambda_N)$ defined by $D_N(\lambda_{jlm}) = (\frac{2^{-jd}}{j+1}\lambda_{jlm})$. Therefore

$$\text{trace } D_N = \sum_{j=0}^{N} \frac{1}{2^{jd}(j+1)} \sum_{l=1}^{L_j}\sum_{m\in Z_j} 1 \sim \sum_{j=0}^{N} \frac{1}{j+1} \sim \log N \, .$$

If we assume that $I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega)$ is nuclear then, according to Lemma 2.1, we obtain

$$\begin{aligned}\log N \sim |\text{trace } D_N| &\leq \nu(D_N : L_\infty(\Lambda_N) \longrightarrow L_\infty(\Lambda_N))\\ &\leq \|A_N : L_\infty(\Lambda_N) \longrightarrow B^d_{\infty,\infty}(\Omega)\| \, \nu(I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega))\\ &\qquad \times \|S_N : L_\infty(\log L)_{-1}(\Omega) \longrightarrow L_\infty(\Lambda_N)\|\\ &\lesssim \nu(I_d : B^d_{\infty,\infty}(\Omega) \longrightarrow L_\infty(\log L)_{-1}(\Omega))\end{aligned}$$

which is a contradiction.

□

Note that the argument of Theorem 4.5 for the case $a = -1$ is based on the fact that $p = q = \infty$.

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