Christoffel transformation for a matrix of bi-variate measures

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Abstract. We consider the sequences of matrix bi-orthogonal polynomials with respect to the bilinear forms $\langle\cdot,\cdot\rangle_{\hat{R}}$ and $\langle\cdot,\cdot\rangle_{\hat{L}}$

$$\begin{aligned}\langle P(z_1), Q(z_2)\rangle_{\hat{R}} &= \int_{\mathbb{T}\times\mathbb{T}} P(z_1)^{\dagger} L(z_1) d\mu(z_1,z_2) Q(z_2),\\ \langle P(z_1), Q(z_2)\rangle_{\hat{L}} &= \int_{\mathbb{T}\times\mathbb{T}} P(z_1) L(z_1) d\mu(z_1,z_2) Q(z_2)^{\dagger},\end{aligned} \qquad P,Q \in \mathbb{L}^{p\times p}[z]$$

where $\mu(z_1,z_2)$ is a matrix of bi-variate measures supported on $\mathbb{T}\times\mathbb{T}$, with $\mathbb{T}$ the unit circle, $L^{p\times p}[z]$ is the set of matrix Laurent polynomials of size $p\times p$ and $L(z)$ is a special polynomial in $L^{p\times p}[z]$. A connection formula between the sequences of matrix Laurent bi-orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\hat{R}}$, (resp. $\langle\cdot,\cdot\rangle_{\hat{L}}$) and the sequence of matrix Laurent bi-orthogonal polynomials with respect to $d\mu(z_1,z_2)$ is given.

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1. Introduction

Perturbations of a linear functional supported on an infinite subset of the real line (see for example [10, 20, 24, 25, 34, 50, 48]) and the unit circle (see for example [16, 17, 30, 31, 35] and the references therein) have been extensively studied in the literature, in particular when these linear functionals are positive definite since they have an integral representation [19, 46]. More precisely, there are three perturbations that have historically highlighted, the so called Christoffel [20], Geronimus [34] and Uvarov [48, 49] transformations. Later on, in [51] Zhedanov stressed the importance of the first two to show that every spectral linear transformation of a linear functional supported on the real line can be written as finite superposition of Christoffel and Geronimus transformations.

The Christoffel and Geronimus transformations have been studied from different points of view. For example in [10] M. Bueno and F. Marcellán analyzed the relation between the semi-infinite matrices associated with the multiplication operator by x (Jacobi matrices) of the original and the perturbed (Christoffel or Geronimus) linear functional finding an important relation between them by using LU and UL factorization, respectively, as well as QR factorization (see [11, 12]). On the other hand, in [20, 34, 49, 50] the relation between the original and perturbed monic orthogonal polynomials is given in terms of a determinantal formula. In [50] the author combined these two methods in such a way a relation between perturbation of linear functionals, factorization of Jacobi matrices and relation between the corresponding orthogonal polynomials is deduced. Returning to the Jacobi matrix, recall that since its tridiagonal shape, it plays a crucial role in the study of the zeros of orthogonal polynomials taking into account they are the eigenvalues of their leading principal submatrices. However, the situation for linear functionals supported on the unit circle is rather more complicated that in the real line because the semi-infinite matrix associated with the multiplication operator by z in terms of a basis of orthogonal polynomials is not a band matrix but an irreducible Hessenberg one. This is a consequence of the fact that the multiplication operator by z is not symmetric for the bilinear form associated with the linear functional. The above problem was solved by Cantero, Moral and Velázquez in [15], where they obtain a new orthonormal basis $(\mathcal{X}_n(z))_{n\in\mathbb{N}}$ (the Laurent orthogonal polynomial basis or CMV basis) as a result of the Gram-Schmidt orthonormal process applied to the basis $\{1, z, z^{-1}, \dots\}$ of the linear space of Laurent polynomials. This new basis satisfies a five term recurrence relation, or, equivalently, there exists a unitary semi-infinite five diagonal band matrix $\mathfrak{C}$ such that $z\mathcal{X}(z) = \mathfrak{C}\mathcal{X}(z)$, where $\mathcal{X}(z) = (\mathcal{X}_0(z), \mathcal{X}_1(z) \cdots)^\top$. $\mathfrak{C}$ is known in the literature as CMV matrix. As an application, in [5] the theory of orthogonal Laurent polynomials on the unit circle and the theory of Toda-like integrable systems using the Gauss–Borel factorization of a CMV moment matrix, constructed from a complex quasi-definite measure supported on the unit circle, is studied.

Recently, Cantero, Marcellán, Moral and Velázquez [14] presented an approach to the Darboux transformations for CMV matrices. In particular, for the Christoffel transformation they show that given a Hermitian polynomial $L(z)$, a linear functional μ (supported on the unit circle) and the perturbed one $\hat{\mu} = L(z)\mu$, if $L(\mathfrak{C})$ has Cholesky factorization $L(\mathfrak{C}) = AA^\dagger$, then $L(\hat{\mathfrak{C}}) = A^\dagger A$, where $\mathfrak{C}$ and $\hat{\mathfrak{C}}$ are the CMV matrices associated with μ and $\hat{\mu}$, respectively.

On the other hand, in [38] the authors deal with a measure supported on the unit circle multiplied by a non-negative trigonometric polynomial $g(\theta)$. Using the fact that for $g(\theta)$ there exists a positive integer number $m \in \mathbb{N}$ and a polynomial $G_{2m}(z)$ of degree $2m$ such that $g(\theta) = z^{-m}G_{2m}(z)$, they give a determinantal expression for the perturbed monic orthogonal polynomial of degree n multiplied by $G_{2m}(z)$. In this expression, the original orthogonal polynomials, from degree n until degree $n + m$, their corresponding reversed polynomials (see Eq. (1)) as well as the zeros of the polynomial $G_{2m}(z)$, are involved.

Concerning the theory of matrix orthogonal polynomials and their applications (see for example [13, 37, 47]), there is an exhaustive bibliography focused on matrix bilinear forms as well as the existence of the corresponding sequences of matrix bi-orthogonal polynomials, both in the real line [43, 45, 47] and in the unit circle [22, 23, 26, 33, 44, 47]. In [4, 27, 28] the authors studied sequences of matrix orthogonal polynomials which are eigenfunctions of a second order linear matrix differential operator (right-hand and left-hand side) with polynomial matrices as coefficients. Moreover in [18], M. Castro and F. Grünbaum showed that there exist sequences of matrix orthogonal polynomials satisfying a first order linear matrix differential equation with polynomial matrices as coefficients, a situation that does not appear in the scalar case. Concerning spectral transformations, in [24, 25] the authors show that all multiple Geronimus transformations of a measure supported on the real line yield a simple Geronimus transformation for a matrix of measures. More recently, in [1, 2, 3] the authors have done a complete study of spectral transformations for matrix sesquilinear forms supported on an infinite set of the real line, and the corresponding connection formulas (Christoffel type formulas) between the bi-orthogonal sequences of original and perturbed bilinear forms. These connection formulas are given in terms of quasi-determinants [32]. Here, we will use similar techniques to find formulas for the Christoffel transformations for matrix Laurent polynomials. Finally, the connection between orthogonal polynomials with respect to measures supported on lemniscates and harmonic algebraic curves, respectively, and matrix orthogonal polynomials with respect to a matrix of measures supported on the unit and the real line has been pointed out in [39] and [40], respectively.

Returning to the scalar case, if we take $v \in (L[x,y])'$ (the algebraic dual of the set of Laurent polynomials of two variables), then we can consider the following bilinear form $\langle p(z_1), q(z_2)\rangle := \left\langle v, p(z_1) \otimes \overline{q(z_2)}\right\rangle$, where $\otimes$ is the tensor product. In particular, if v is associated with a bi-variate complex measure $d\mu(z_1, z_2)$, then

$$\langle p(z_1), q(z_2)\rangle = \left\langle v, p(z_1) \otimes \overline{q(z_2)}\right\rangle = \int\int p(z_1)\overline{q(z_2)}\, d\mu(z_1, z_2).$$

Taking into account the above discussion, if μ is now a matrix of bi-variate complex measures supported on $\mathbb{T} \times \mathbb{T}$

$$d\mu(z_1, z_2) := \begin{pmatrix} du_{1,1}(z_1, z_2) & \cdots & du_{1,p}(z_1, z_2) \\ \vdots & & \vdots \\ du_{p,1}(z_1, z_2) & \cdots & du_{p,p}(z_1, z_2) \end{pmatrix},$$

then we can define the following matrix bilinear forms $\langle \cdot, \cdot\rangle_R$ and $\langle \cdot, \cdot\rangle_L$ from their entries (see also (5) and (6))

$$(\langle P(z_1), Q(z_2)\rangle_R)_{i,j} := \sum_{m,l=1}^{p} \int_{\mathbb{T}\times\mathbb{T}} (\overline{P(z_1)})_{l,i}(Q(z_2))_{m,j} du_{l,m}(z_1, z_2),$$

$$(\langle P(z_1), Q(z_2)\rangle_L)_{i,j} := \sum_{m,l=1}^{p} \int_{\mathbb{T}\times\mathbb{T}} (P(z_1))_{i,l}(\overline{Q(z_2)})_{j,m} du_{l,m}(z_1, z_2),$$

with $P(z), Q(z) \in \mathbb{L}^{p\times p}[z]$.
In this contribution we focus our attention on the study of matrix Christoffel transformations for a matrix of measures supported on the bi-circle, i. e. given a matrix of measures $d\mu(z_1, z_2)$ supported on $\mathbb{T}\times\mathbb{T}$ and a matrix prepared Laurent polynomial $L(z)$ (see Definition 1), we are interested in dealing with the following bilinear forms

$$\begin{aligned} \langle P(z_1), Q(z_2)\rangle_{\hat{R}} &= \int_{\mathbb{T}\times\mathbb{T}} P(z_1)^{\dagger} L(z_1) d\mu(z_1, z_2) Q(z_2), \\ \langle P(z_1), Q(z_2)\rangle_{\hat{L}} &= \int_{\mathbb{T}\times\mathbb{T}} P(z_1) L(z_1) d\mu(z_1, z_2) Q(z_2)^{\dagger}. \end{aligned} \qquad P, Q \in \mathbb{L}^{p\times p}[z],$$

Here, † means the conjugate transpose of a matrix.

The structure of the manuscript is as follows. In Section 2 the basic background about matrix polynomials and matrix Laurent polynomials is presented. Section 3 deals with matrix bi-orthogonal Laurent polynomials and its relation with the Gauss-Borel factorization. In Section 4, the Christoffel transformation of a matrix of bi-variate measures is considered. Connection formulas for their corresponding sequences of matrix bi-orthogonal Laurent polynomials as well as for the matrix kernel polynomials are obtained.

2. Preliminaries

First of all we will fix some notation. Let $\mathbb{C}$ and $\mathbb{Z}$ be the set of complex and integer numbers, respectively, and denote by $\mathbb{C}^{p\times p}$ the linear space of $p\times p$ matrices with complex entries. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ will denote the unit circle. For an arbitrary finite or infinite matrix A, $A^{\top}$ is the transpose of the matrix A, and $A^{\dagger} = \bar{A}^{\top}$. When $A = (\alpha_{i,j})_{i,j=0}$ is a (finite or semi-infinite) block square matrix with $\alpha_{i,j} \in \mathbb{C}^{p\times p}$, $A_{[n]} := (\alpha_{i,j})_{i,j=0}^{n-1}$ means the principal leading $p\times p$ block sub-matrix of A of order n. Given matrices $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}^{m\times p}$ and $D \in \mathbb{C}^{p\times p}$, we denote the last quasi-determinant (or Schur complement) of the block matrix $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, as

$$\Theta_* \begin{bmatrix} A & B \\ C & D \end{bmatrix} := D - CA^{-1}B.$$

This is a very instrumental approach, as we do not want to go into details to quasi-determinants, however we stress that the theory goes beyond this introduction (see [32]). For a deeper discussion of the use of quasi-determinants within orthogonal polynomials see the recent paper [8].

The product AB of two semi-infinite matrices A and B is said to be admissible if any matrix entry $(AB)_{i,j} = \sum_k A_{i,k}B_{k,j}$ involves only a finite number of non-null terms. As in the finite case, the product of semi-infinite matrices satisfies the distributive law $A(B + C) = AB + BC$ when the products AB and BC are admissible. Besides, if AB is admissible, then $(AB)^{\dagger}$ is also admissible and $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. However, the associative law can fail even if all the involved matrix products are admissible [14].

Proposition 1 ([14, 21]). *Let A, B and C be semi-infinite matrices. The associative property $(AB)C = A(BC)$ is valid in any of the following cases*

i) *A and B are lower Hessenberg type.*
ii) *B and C are upper Hessenberg type.*
iii) *A is lower Hessenberg type and B is upper Hessenberg type.*

Corollary 1 ([21])**.** *If A is either a lower or upper triangular block matrix such that the blocks of its main diagonal are nonsingular matrices, then A has a unique inverse.*

Remark 1. *In this manuscript we always deal with Hessenberg block matrices or matrices that can be factorized in terms of them. Thus, when we need to use the associative law of the product, and the hypothesis of Proposition* 1 *will be satisfied, we will forget the associativity parenthesis.*

Recall that for any matrices $A_k \in \mathbb{C}^{p\times p}$, $k = 0, \dots, n$, with A_n non-sigular, the matrix $P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$ is said to be a matrix polynomial of degree n. In particular, if $A_n = I_p$, the identity $p \times p$ matrix, then the polynomial is said to be monic. The set of matrix polynomials with coefficients in $\mathbb{C}^{p\times p}$ will be denoted by $\mathbb{C}^{p\times p}[z]$. $y_0 \in \mathbb{C}$ is said to be a zero of $P(z)$ if $\det[P(y_0)] = 0$. Clearly, from the above definition, $P(z)$ has at most np zeros. If $\deg(P) = n$, then the reversed polynomial of $P(z)$ is defined as

$$(P(z))^* := z^n (P(1/\bar{z}))^\dagger. \tag{1}$$

Definition 1. *Given a family of matrices $(A_k)_{k=m}^n$ with $m, n \in \mathbb{Z}$ and $m \leqslant n$, the matrix $L(z) = \sum_{k=m}^n A_k z^k$ is said to be a Laurent matrix polynomial. The set of matrix Laurent polynomials will be denoted by $\mathbb{L}^{p\times p}[z]$. In particular, if $L(z) \in \mathbb{L}^{p\times p}[z]$ has the form*

$$L(z) = \sum_{k=-d}^{d} A_k z^k, \quad A_{-d}, A_d \neq 0_{p\times p},$$

with $A_k^\dagger = A_{-k}$, $k = 0, \dots, d$, then it is said to be a matrix prepared Laurent polynomial of "degree" d.

In the context of orthogonal polynomials, the definition of prepared Laurent polynomial was originated in [9], where they received the name of nice polynomials. In [9] one can find a study of perturbations of complex multivariate measures by multiplication with multivariate Laurent polynomials in the algebraic torus, and the corresponding Christoffel formulas were deduced. In [7] a similar study for scalar Laurent polynomial type perturbations of bi-variate linear functionals was given. Here the name of prepared polynomial appears at first time. We stress that in other frameworks as signal preprocessing or control theory, the prepared Laurent polynomials are known as parahermitian (or Para-Hermitian) polynomials [29, 42].

Given the basis $\eta(z) := (I_p, zI_p, z^{-1}I_p, z^2I_p, z^{-2}I_p \cdots)^\top$ in the bi-module of matrix Laurent polynomials $\mathbb{L}^{p\times p}[z]$, the semi-infinite matrix

$$T := \left(\begin{array}{cc|cc|cc|c}
0 & I_p & 0 & 0 & & & \\
0 & 0 & 0 & I_p & & & \\
\hline
I_p & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & I_p & \\
\hline
0 & 0 & I_p & 0 & 0 & 0 & \ddots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}\right) \tag{2}$$

represents the shift operator $z^n \mapsto z^{n+1}$ with respect to the basis $\eta(z)$, i.e. $T\eta(z) = z\eta(z)$ where T is a unitary matrix $TT^\dagger = T^\dagger T = I$. Here I denote the semi-infinite identity matrix [1].

Since we need some basic tools of the spectral theory of matrix polynomials, we will define the concept of Canonical Jordan Chain (this generalizes the concept of Jordan chain for matrix polynomials of degree 1 [36, 41]). For this aim, we translate here some of the constructions presented in [1, 2].

Given $W(z) \in \mathbb{C}^{p\times p}[z]$, a monic matrix polynomial of degree N, let $y_1, \dots, y_q$, be its zeros and let $\alpha_1, \dots, \alpha_q$ be their corresponding multiplicities. Since $W(z)$ is a monic polynomial, $\sum_{i=1}^q \alpha_i = Np$. If for the zero y_i there exists a nonzero vector $r_0^{(i)}$ (resp. $l_0^{(i)}$) such that

$$W(y_i)r_0^{(i)} = 0_p, \qquad (\text{resp.} \quad l_0^{(i)}W(y_i) = 0_p^\top), \quad \text{where} \quad 0_p = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{1\times p},$$

then $r_0^{(i)}$ (resp. $l_0^{(i)}$) is said to be a right (resp. left) eigenvector of $W(z)$ associated with y_i. A sequence of vectors $\{r_0^{(i)}, r_1^{(i)} \dots, r_{m_i-1}^{(i)}\}$ (resp. $\{l_0^{(i)}, l_1^{(i)} \dots, l_{m_i-1}^{(i)}\}$) is said to be a right (resp. left) Jordan chain of length m_i associated with y_i if $r_0^{(i)}$ (resp.$l_0^{(i)}$) is a right (resp. left) eigenvector of $W(z)$ corresponding to y_i and

$$\sum_{k=0}^{j} \frac{1}{k!} W^{(k)}(y_i) r_{j-k}^{(i)} = 0_p, \quad (\text{resp.} \quad \sum_{k=0}^{j} \frac{1}{k!} l_{j-k}^{(i)} W^{(k)}(y_i) = 0_p^\top), \quad j = 0, \dots, m_i - 1.$$

A right (resp. left) Jordan chain $\{r_0^{(i)}, r_1^{(i)} \dots, r_{k_i-1}^{(i)}\}$ (resp. $\{l_0^{(i)}, l_1^{(i)} \dots, l_{\kappa_i-1}^{(i)}\}$) is said to be of maximal length k_i if there does not an other one with length κ_i+1. The maximal length of a Jordan chain corresponding to the zero y_i is called the multiplicity of right (resp. left) eigenvector $r_0^{(i)}$ (resp. $l_0^{(i)}$) and is denoted by $m(r_0^{(i)})$ (resp. $m(l_0^{(i)})$). In the sequel we will only deal with Jordan chains of maximal length. In general we can assume that $m(r_0^{(i)}) = m(l_0^{(i)})$ taking into account the following proposition (see [36]).

[1]Notice that for the matrix polynomial basis $\{I_p, zI_p, z^2I_p, \dots\}$ the multiplication operator by z is not unitary.

Proposition 2. *Given a zero y_i of $W(z)$, there exists a right eigenvector $r_0^{(i)}$ associated with y_i and multiplicity $\kappa^{(i)}$ if and only if there exists a left eigenvector $l_0^{(i)}$ associated with y_i and multiplicity $\kappa^{(i)}$.*

Thus, given a basis $\{r_{1,0}^{(i)},\dots,r_{s_i,0}^{(i)}\}$ of the linear subspace $\operatorname{Ker}(W(y_i))$ and $\{l_{1,0}^{(i)},\dots,l_{s_i,0}^{(i)}\}$ a basis of the linear subspace $\operatorname{CoKer}(W(y_i))$ with

$$\dim(\operatorname{Ker}(W(y_i))) = \dim(\operatorname{CoKer}(W(y_i))) = s_i,$$

a right (resp. left) canonical Jordan chain associated with the zero y_i is defined as a system of right (resp. left) Jordan chains with maximal length

$$r_{j,0}^{(i)}, r_{j,1}^{(i)}, \cdots, r_{j,\kappa_j^{(i)}-1,}^{(i)} \quad (\text{resp.} \quad l_{j,0}^{(i)}, l_{j,1}^{(i)}, \cdots, l_{j,\kappa_j^{(i)}-1,}^{(i)}) \quad j=1,\dots,s_i.$$

The number $m(y_i, W(z)) := \sum_{j=1}^{s_i} \kappa_j^{(i)}$ is said to be the Jordan multiplicity of y_i. The following result, that is a direct consequence of Proposition 1.13 of [36] (see also [41]), will be a main tool in the sequel.

Proposition 3. *Let y_i be a zero of $W(z)$ with multiplicity α_i. If*

$$r_{j,0}^{(i)}, r_{j,1}^{(i)}, \cdots, r_{j,\kappa_j^{(i)}-1}^{(i)} \quad (\textit{resp.} \quad l_{j,0}^{(i)}, l_{j,1}^{(i)}, \cdots, l_{j,\kappa_j^{(i)}-1}^{(i)}), \quad j=1,\dots s_i, \tag{3}$$

is a right (resp. left) canonical Jordan chain corresponding to y_i, then $m(y_i, W(z)) = \alpha_i$.

Definition 2. *Given a right (left) canonical Jordan chain as in* (3) *corresponding to y_i, for each $j = 1,\dots,s_i$, we define the following right (resp. left) root vector polynomials*

$$r_j^{(i)}(z) = \sum_{t=0}^{\kappa_j^{(i)}-1} (z-y_i)^t r_{j,t}^{(i)} \quad (\textit{resp.} \quad l_j^{(i)}(z) = \sum_{t=0}^{\kappa_j^{(i)}-1} (z-y_i)^t l_{j,t}^{(i)}). \tag{4}$$

Proposition 4. *Given the monic matrix polynomial $W(z)$, the right (resp. left) root vector polynomials introduced above (see* (4)*) satisfy*

$$\left.\frac{\mathrm{d}^t}{\mathrm{d}\, z^t}\right|_{z=y_i} (W(z) r_j^{(i)}(z)) = 0_p \quad (\textit{resp.} \quad \left.\frac{\mathrm{d}^t}{\mathrm{d}\, z^t}\right|_{z=y_i} (l_j^{(i)}(z) W(z) = 0_p^\top)),$$

where $t = 0,\dots,\kappa_j^{(i)}-1, \quad j = 1\dots,s_i$.

Definition 3. *Let y_i be a zero of $W(z)$ and let $r_j^{(i)}(z)$ and $l_j^{(i)}(z)$, $1 \le j \le s_i$, be its associated right and left root vector polynomial defined as above. Given a matrix function $f(z)$ which is smooth in its domain of definition, we consider its matrix spectral jets*

$$\begin{aligned}
J_f^{r,(j)}(y_i) &:= \left[f(y_i) r_j^{(i)}(y_i), \dots, \frac{(f(z) r_j^{(i)}(z))_{y_i}^{(\kappa_j^{(i)}-1)}}{(\kappa_j^{(i)}-1)!} \right] \in \mathbb{C}^{p\times\kappa_j^{(i)}},\\
J_f^{r}(y_i) &:= \left[J_f^{r,(1)}(y_i) \dots, J_f^{r,(s_i)}(y_i) \right] \in \mathbb{C}^{p\times\alpha_i},\\
J_f^{r} &:= \left[J_f^{r}(y_1), \dots, J_f^{r}(y_q) \right] \in \mathbb{C}^{p\times Np},
\end{aligned}$$

$$J_f^{l,(j)}(y_i) := \begin{bmatrix} l_j^{(i)}(y_i)f(y_i) \\ \vdots \\ \frac{(l_j^{(i)}(z)f(z))_{y_i}^{(\kappa_j^{(i)}-1)}}{(\kappa_j^{(i)}-1)!} \end{bmatrix} \in \mathbb{C}^{\kappa_j^{(i)}\times p}, \quad J_f^{l}(y_i) := \begin{bmatrix} J_f^{l,(1)}(y_i) \\ \vdots \\ J_f^{l,(s_i)}(y_i) \end{bmatrix} \in \mathbb{C}^{\alpha_i\times p},$$

$$J_f^{l} := \begin{bmatrix} J_f^{l}(y_1) \\ \vdots \\ J_f^{l}(y_q) \end{bmatrix} \in \mathbb{C}^{Np\times p}.$$

In particular, if $f(z)$ and $g(z)$ are matrix polynomials with $f(z) = \sum_{k=0}^{m} B_j z^j$, then

$$J_f^{r,(j)}(y_i) = \sum_{k=0}^{m} B_j J_{z^k}^{r,(j)}(y_i), \quad J_f^{l,(j)}(y_i) = \sum_{k=0}^{m} J_{z^k}^{l,(j)}(y_i) B_j,$$

and

$$J_{f+g}^{r,(j)}(y_i) = J_f^{r,(j)}(y_i) + J_g^{r,(j)}(y_i), \quad J_{f+g}^{l,(j)}(y_i) = J_f^{l,(j)}(y_i) + J_g^{r,(j)}(y_i).$$

3. Matrix Orthogonal Laurent polynomials

In this section, for the reader's commodity, we recall the material we will need in the sequel from [6]. Given a bi-variate matrix of measures (not necessarily Hermitian) $d\mu(z_1, z_2)$ supported on $\mathbb{T}\times\mathbb{T}$, we can define the bilinear forms on the bi-module of matrix Laurent polynomials,

$$\langle\cdot,\cdot\rangle_L, \langle\cdot,\cdot\rangle_R : \mathbb{L}^{p\times p}[z]\times\mathbb{L}^{p\times p}[z] \longrightarrow \mathbb{C}^{p\times p}$$

as follows

$$\langle f(z_1), g(z_2)\rangle_R = \int_{\mathbb{T}\times\mathbb{T}} f(z_1)^{\dagger} d\mu(z_1, z_2) g(z_2), \tag{5}$$

$$\langle f(z_1), g(z_2)\rangle_L = \int_{\mathbb{T}\times\mathbb{T}} f(z_1) d\mu(z_1, z_2) g(z_2)^{\dagger}. \tag{6}$$

They are related by

$$\left\langle f^{\dagger}(z_1), g^{\dagger}(z_2)\right\rangle_L = \langle f(z_1), g(z_2)\rangle_R. \tag{7}$$

In [6] the authors constructed a new sequence of matrix bi-orthogonal Laurent polynomials for (6) from the basis

$$\epsilon(z) = (I_p, z^{-1}I_p, zI_p, z^{-2}I_p, z^2 I_p \cdots)^{\top}$$

that constitutes an analog of the one analyzed in [15] (see also [14]) for the scalar case. In a similar way, under certain hypotheses (see below), we can construct sequences of matrix bi-orthogonal Laurent polynomials $(\mathcal{X}_n^{@[1]}(z_1), \mathcal{X}_n^{@[2]}(z_2))_{n\in\mathbb{N}}$, $@ = R, L$, for (6) from the canonical basis,

$$\eta(z) := (I_p, zI_p, z^{-1}I_p, z^2 I_p, z^{-2}I_p \cdots)^{\top}. \tag{8}$$

Such bi-orthogonal Laurent polynomials must satisfy, for $i = 1, 2$, and $@ = R, L$,

$$\mathcal{X}_n^{@[i]}(z) \in \begin{cases} \mathrm{Span}\{z^{-k}I_p \cdots z^k I_p\} & (\text{coefficient of } z^{-k}I_P = I_p), \quad n = 2k, \\ \mathrm{Span}\{z^{-k}I_p \cdots z^{k+1}I_p\} & (\text{coefficient of } z^{k+1}I_p = I_p), \quad n = 2k+1, \end{cases} \tag{9}$$

and

$$\left\langle \chi_m^{R[1]}(z_1), \chi_n^{R[2]}(z_2) \right\rangle_R = \delta_{n,m} D_n^R, \quad \left\langle \chi_m^{L[1]}(z_1), \chi_n^{L[2]}(z_2) \right\rangle_L = \delta_{m,n} D_n^L, \tag{10}$$

with D_n^R, D_n^L nonsingular matrices. The sequence $(\chi_n^{@[i]})_{n\in\mathbb{N}}$, $i = 1,2$, and $@ = R, L$, is said to be a zig-zag basis. One can characterize these zig-zag bases, but first we give the following definition

Definition 4. *The semi-infinite matrices M_L and M_R*

$$M_R = \int_{\mathbb{T}\times\mathbb{T}} (\eta(z_1)^\top)^\dagger d\mu(z_1,z_2)\, \eta(z_2)^\top = \left\langle \eta(z_1)^\top, \eta(z_2)^\top \right\rangle_R,$$
$$M_L = \int_{\mathbb{T}\times\mathbb{T}} \eta(z_1) d\mu(z_1,z_2)\, \eta(z_2)^\dagger = \langle \eta(z_1), \eta(z_2) \rangle_L,$$

are said to be the Gram moment matrices with respect to the basis $\eta(z)$.

For our purposes, we assume that the bilinear form $\langle\cdot,\cdot\rangle_R$ (resp. $\langle\cdot,\cdot\rangle_L$) is quasi-definite, this is equivalently to $\det(M_R)_{[n]} \neq 0$, (resp. $\det(M_L)_{[n]} \neq 0$) for every $n \in \mathbb{N}$. Under this assumption, there exists a block Gauss-Borel factorization for the matrix M_R (resp M_L) [6], i.e.

$$M_R = S_1^{-1} D^R S_2^{-1}, \qquad M_L = Z_1^{-1} D^L Z_2^{-1},$$

where S_1, Z_1 and S_2, Z_2 are lower and upper triangular block matrices, respectively, with blocks I_p in their main diagonal, and D^R, D^L are non-singular diagonal block matrices. Let us define the block vectors

$$\chi^{R[i]}(z) := \begin{pmatrix} \chi_0^{R[i]}(z) & \chi_1^{R[i]}(z) & \cdots \end{pmatrix}, \quad \chi^{[i]}(z) = \begin{pmatrix} \chi_0^{L[i]}(z) \\ \chi_1^{L[i]}(z) \\ \vdots \end{pmatrix}, \quad i = 1,2,$$

as

$$\chi^{R[1]}(z) := \eta(z)^\top S_1^\dagger, \qquad \chi^{R[2]}(z) := \eta(z)^\top S_2, \tag{11}$$
$$\chi^{L[1]}(z) := Z_1 \eta(z), \qquad \chi^{L[2]}(z) := Z_2^\dagger \eta(z).$$

We will see that they satisfy (9) and (10).

Proposition 5. *The sequences $(\chi_n^{@[1]}(z_1), \chi_n^{@[2]}(z_2))_{n\in N}$ are bi-orthogonal with respect to $\langle\cdot,\cdot\rangle_@$, $@ = L, R$. Moreover*

$$\left\langle \chi_n^{@[1]}(z_1), \chi_n^{@[2]}(z_2) \right\rangle_@ = D_n^@.$$

Proof. The proof of the above statement is equivalent to prove that

$$\left\langle \chi^{R[1]}(z_1), \chi^{R[2]}(z_2) \right\rangle_R = D^R, \quad \left\langle \chi^{L[1]}(z_1), \chi^{L[2]}(z_2) \right\rangle_L = D^L.$$

But, from (11), we get

$$\begin{aligned} \left\langle \chi^{R[1]}(z_1), \chi^{R[2]}(z_2) \right\rangle_R &= \left\langle \eta(z_1)^\top S_1^\dagger, \eta(z_2)^\top S_2 \right\rangle_R \\ &= \int_{\mathbb{T}\times\mathbb{T}} S_1 \overline{\eta(z_1)} d\mu(z_1,z_2)\, \eta(z_2)^\top S_2 \\ &= S_1 \left\langle \eta(z_1)^\top, \eta(z_2)^\top \right\rangle S_2 = D^R. \end{aligned}$$

For $(\chi_n^{L[1]}(z_1), \chi_n^{L[2]}(z_2))_{n\in N}$ the proof is similar. □

Remark 2. *The bi-orthogonality of* $(\mathcal{X}_n^{@[1]}(z_1),\mathcal{X}_n^{@[2]}(z_2))_{n\in N}$ *is equivalent to*

$$\left\langle \mathcal{X}_{2k}^{@[1]}(z_1), z_2^j I_p\right\rangle_@ = 0_{p\times p} \quad for \quad -k+1\leq j\leq k, \quad and \quad \left\langle \mathcal{X}_{2k}^{@[1]}(z_1), z_2^{-k} I_p\right\rangle_@ = D_{2k}^@,$$
$$\left\langle z_1^j I_p, \mathcal{X}_{2k}^{@[2]}(z_2)\right\rangle_@ = 0_{p\times p} \quad for \quad -k+1\leq j\leq k, \quad and \quad \left\langle z_1^{-k} I_p, \mathcal{X}_{2k}^{@[2]}(z_2)\right\rangle_@ = D_{2k}^@,$$

and

$$\left\langle \mathcal{X}_{2k+1}^{@[1]}(z_1), z_2^j I_p\right\rangle_@ = 0_{p\times p} \quad for \quad -k\leq j\leq k, \quad and \quad \left\langle \mathcal{X}_{2k+1}^{@[1]}(z_1), z_2^{k+1} I_p\right\rangle_@ = D_{2k+1}^@,$$
$$\left\langle z_1^j I_p, \mathcal{X}_{2k+1}^{@[2]}(z_2)\right\rangle_@ = 0_{p\times p} \quad for \quad -k\leq j\leq k, \quad and \quad \left\langle z_1^{k+1} I_p, \mathcal{X}_{2k+1}^{@[2]}(z_2)\right\rangle_@ = D_{2k+1}^@,$$

where $D_{2k}^@$, $D_{2k+1}^@$ *are nonsingular matrices. Here* $@ = R, L$.

Definition 5. *The sequences of matrix Laurent polynomials* $(\varphi_n^{L[1]}(z_1),\varphi_n^{L[2]}(z_2))_{n\in\mathbb{N}}$, *and* $(\varphi_n^{R[1]}(z_1),\varphi_n^{R[2]}(z_2))_{n\in\mathbb{N}}$ *defined as follows*

$$\varphi_n^{R[1]}(z_1) := \mathcal{X}_n^{R[1]}(z_1), \qquad \varphi_n^{R[2]}(z_2) := \mathcal{X}_n^{R[2]}(z_2)(D_n^R)^{-1},$$
$$\varphi_n^{L[1]}(z_1) := (D_n^L)^{-1}\mathcal{X}_n^{L[1]}(z_1), \qquad \varphi_n^{L[2]}(z_2) := \mathcal{X}_n^{L[2]}(z_2).$$

are said to be the CMV bases. Notice that the above sequences are bi-orthonormal.

Definition 6. *In the same way as for orthogonal matrix polynomials, given the bi-orthogonal sequences* $(\mathcal{X}_n^{L[1]}(z_1),\mathcal{X}_n^{L[2]}(z_2))_{n\in\mathbb{N}}$, $(\mathcal{X}_n^{R[1]}(z_1),\mathcal{X}_n^{R[2]}(z_2))_{n\in\mathbb{N}}$, *we define the Kernel Laurent polynomials*

$$\mathcal{K}_n^R(z_1,z_2) := \sum_{j=0}^n \mathcal{X}_j^{R[2]}(z_2)(D_j^R)^{-1}(\mathcal{X}_j^{R[1]}(z_1))^\dagger = \sum_{j=0}^n \varphi_j^{R[2]}(z_2)(\varphi_j^{R[1]}(z_1))^\dagger,$$
$$\mathcal{K}_n^L(z_1,z_2) := \sum_{j=0}^n (\mathcal{X}_j^{L[2]}(z_2))^\dagger(D_j^L)^{-1}\mathcal{X}_j^{L[1]}(z_1) = \sum_{j=0}^n (\varphi_j^{L[2]}(z_2))^\dagger\varphi_j^{L[1]}(z_1).$$

Proposition 6. *The Kernel Laurent polynomials satisfy a reproducing property, i.e. for every matrix Laurent polynomial*

$$Q(z) \in \operatorname{Span}\{z^{-k}I_p \cdots z^t I_p\} \quad with \quad \begin{cases} t = k, & if \quad n = 2k, \\ t = k+1, & if \quad n = 2k+1, \end{cases} \tag{12}$$

we have

$$\left\langle (\mathcal{K}_n^R(z_1,y))^\dagger, Q(z_2)\right\rangle_R = Q(y), \quad \left\langle Q(z_1), \mathcal{K}_n^R(y,z_2)\right\rangle_R = Q(y)^\dagger,$$

$$\left\langle \mathcal{K}_n^L(z_1,z), Q(z_2)\right\rangle_L = Q(z)^\dagger, \quad \left\langle Q(z_1), (\mathcal{K}_n^L(z,z_2))^\dagger\right\rangle_L = Q(z).$$

Proof. Let $Q(z)$ be a matrix Laurent polynomial as in (12), then there exist $n+1$ matrix coefficients $(\alpha_l)_{l=0}^n$ (some can be the null matrix) such that

$$Q(z) = \sum_{l=0}^n \varphi_l^{R[2]}(z)\alpha_l.$$

From here

$$\left\langle (\mathcal{K}_n^R(z_1,y))^\dagger, Q(z_2)\right\rangle_R = \sum_{j=0}^{n} \varphi_j^{R[2]}(y)\int_{\mathbb{T}\times\mathbb{T}} (\varphi_j^{R[1]}(z_1))^\dagger d\mu(z_1,z_2)\sum_{l=0}^{n}\varphi_l^{R[2]}(z_2)\alpha_l$$
$$= \sum_{j=0}^{m} \varphi_j^{R[2]}(y)\int_{\mathbb{T}\times\mathbb{T}} (\varphi_j^{R[1]}(z_1))^\dagger d\mu(z_1,z_2)\varphi_j^{R[2]}(z_2)\alpha_j = Q(y).$$

The other identities follow in a similar way. □

4. Christoffel transformation

In this section we give the main result of the paper, the Christoffel formulas for perturbations associated with the multiplication by a prepared Laurent polynomial. This constitutes a matrix extension, as we did in [1, 2, 3] in the real case, for the Laurent polynomial perturbations analyzed in the scalar case in [7].
Let $L(z)$ be a matrix prepared Laurent polynomial of degree d (see Definition 1)

$$L(z) := \sum_{j=-d}^{d} \beta_j z^j, \quad \beta_j^\dagger = \beta_{-j}, \quad j = 0, \cdots, d.$$

For simplicity, we will assume that $\beta_d = I_p$. Let $W(z) := z^d L(z)$. Notice that $W(z)$ is a monic polynomial in $\mathbb{C}^{p\times p}[z]$ of degree $N := 2d$. As a consequence, $W(z)$ has Np zeros (counting multiplicities). If $y_1, \ldots, y_q$ are their zeros and $\alpha_1, \ldots, \alpha_q$ the corresponding multiplicities, then $\sum_{j=0}^{q} \alpha_j = Np$. Notice also that from the definition of $W(z)$, $\det(W(0)) \neq 0$.

Next, we will deal with a new matrix of measures

$$d\hat{\mu}(z_1,z_2) = L(z_1)d\mu(z_1,z_2),$$

as well as with the corresponding perturbed bilinear forms

$$\langle f(z_1), g(z_2)\rangle_{\hat{R}} = \int_{\mathbb{T}\times\mathbb{T}} f(z_1)^\dagger L(z_1)d\mu(z_1,z_2)g(z_2),$$
$$\langle f(z_1), g(z_2)\rangle_{\hat{L}} = \int_{\mathbb{T}\times\mathbb{T}} f(z_1)L(z_1)d\mu(z_1,z_2)g(z_2)^\dagger. \tag{13}$$

Observe that for $d\hat{\mu}(z_1,z_2)$, the property (7) is still preserved because $(L(z))^\dagger = L(z)$ on $\mathbb{T}$. Let $\hat{M}_R := \langle \eta(z_1)^\top, \eta(z_2)^\top\rangle_{\hat{R}}$ and $\hat{M}_L := \langle \eta(z_1), \eta(z_2)\rangle_{\hat{L}}$ be the block semi-infinite Gram matrices corresponding to $\langle\cdot,\cdot\rangle_{\hat{R}}$ and $\langle\cdot,\cdot\rangle_{\hat{L}}$, respectively. Let us assume that they have a Gauss-Borel factorization [6]

$$\hat{M}_R = \hat{S}_1^{-1}\hat{D}^R\hat{S}_2^{-1}, \qquad \hat{M}_L = \hat{Z}_1^{-1}\hat{D}^L\hat{Z}_2^{-1},$$

where $\hat{S}_1, \hat{Z}_1$ and $\hat{S}_2, \hat{Z}_2$ are lower and upper triangular block matrices, respectively, with I_p in their main diagonal entries and $\hat{D}^R, \hat{D}^L$ are non-singular diagonal block matrices. With this in mind, we can define the zig-zag basis $(\hat{\chi}_n^{@[i]}(z))_{n\in\mathbb{N}}$, $i = 1,2$, $@ = R, L$, from the following block vectors

$$\hat{\chi}^{L[1]}(z) = \hat{Z}_1\eta(z), \qquad \hat{\chi}^{L[2]}(z) = \hat{Z}_2^{\dagger}\eta(z), \tag{14}$$

$$\hat{\chi}^{R[1]}(z) = \eta(z)^{\top}\hat{S}_1^{\dagger}, \qquad \hat{\chi}^{R[2]}(z) = \eta(z)^{\top}\hat{S}_2. \tag{15}$$

A straightforward consequence of the above definition is

Corollary 2. *The following relations hold*

$$\hat{\chi}^{R[1]}(z) = \hat{\chi}^{R[2]}(z)\hat{S}_2^{-1}\hat{S}_1^{\dagger} = \chi^{R[1]}(z)S_1^{-\dagger}\hat{S}_1^{\dagger} = \chi^{R[2]}(z)S_2^{-1}\hat{S}_1^{\dagger},$$
$$\hat{\chi}^{L[1]}(z) = \hat{Z}_1\hat{Z}_2^{-\dagger}\hat{\chi}^{L[2]}(z) = \hat{Z}_1Z_1^{-1}\chi^{L[1]}(z) = \hat{Z}_1Z_2^{-\dagger}\chi^{L[2]}(z).$$

Proposition 7. *If $L(z) = \sum_{j=-d}^{d}\beta_j z^j$ is a matrix prepared Laurent polynomial, then*

$$\eta(z)^{\top}\left[L(T^{\dagger})\right]^{\dagger} = L(z)\eta(z)^{\top},$$
$$L(T)\eta(z) = \eta(z)L(z), \tag{16}$$

where the semi-infinite matrix T and the block vector $\eta(z)$ were defined in (2) *and* (8)*, respectively.*

Proof. Using the fact that $T^{\top} = T^{\dagger}$ and

$$(I_p, zI_p, z^{-1}I_p, z^2I_p, z^{-2}I_p\cdots)T^{\dagger} = (zI_p, z^2I_p, I_p, z^3I_p, z^{-1}I_p\cdots) = z\eta(z)^{\top},$$
$$(I_p, zI_p, z^{-1}I_p, z^2I_p, z^{-2}I_p\cdots)T = (z^{-1}I_p, I_p, z^{-2}I_p, zI_p, z^3I_p\cdots) = z^{-1}\eta(z)^{\top},$$

we have that

$$\eta(z)^{\top}[L(T^{\dagger})]^{\dagger} = \eta(z)^{\top}\left[\sum_{j=-d}^{d}\beta_j(T^{\dagger})^j\right]^{\dagger} = \eta(z)^{\top}\left[\sum_{j=-d}^{d}T^j\beta_{-j}\right] = \eta(z)^{\top}L(z).$$

Thus the result is proved. The equation (16) is obtained in a similar way. □

Proposition 8. *The perturbed block moment matrices $\hat{M}_R$ and $\hat{M}_L$ satisfy*

$$\hat{M}_R = L(T^{\dagger})M_R, \qquad \hat{M}_L = L(T)M_L.$$

Proof. From (13) and Proposition 7 we get

$$\hat{M}_R = \int_{\mathbb{T}\times\mathbb{T}}\left(L(z_1)\eta(z_1)^{\top}\right)^{\dagger}d\mu(z_1,z_2)\eta(z_2)^{\top} = L(T^{\dagger})M_R,$$

$$\hat{M}_L = L(T)\int_{\mathbb{T}\times\mathbb{T}}\eta(z_1)d\mu(z_1,z_2)\eta(z_2)^{\dagger} = L(T)M_L.$$

□

Proposition 9. *Let us introduce the following semi-infinite block matrices*

$$\omega := (D^R)^{-\dagger}S_2^{\dagger}\hat{S}_2^{-\dagger}(\hat{D}^R)^{\dagger}, \qquad \tilde{\omega} := \hat{D}^L\hat{Z}_2^{-1}Z_2(D^L)^{-1}.$$

Then

$$L(z)\hat{\chi}^{R[1]}(z) = \chi^{R[1]}(z)\omega, \quad \hat{\chi}^{L[1]}(z)L(z) = \tilde{\omega}\chi^{L[1]}(z). \tag{17}$$

ω and $\tilde{\omega}$ are called connection matrices.

Proof. From Proposition 8 we have that $\hat{M}_R = L(T^\dagger)M_R$. Thus

$$\begin{aligned} \hat{S}_1^{-1}\hat{D}^R\hat{S}_2^{-1} &= L(T^\dagger)S_1^{-1}D^RS_2^{-1}, \\ \hat{S}_2^{-\dagger}(\hat{D}^R)^\dagger\hat{S}_1^{-\dagger} &= S_2^{-\dagger}(D^R)^\dagger S_1^{-\dagger}[L(T^\dagger)]^\dagger, \\ S_1^\dagger(D^R)^{-\dagger}S_2^\dagger\hat{S}_2^{-\dagger}(\hat{D}^R)^\dagger &= [L(T^\dagger)]^\dagger\hat{S}_1^\dagger, \\ \mathcal{X}^{R[1]}(z)(D^R)^{-\dagger}S_2^\dagger\hat{S}_2^{-\dagger}(\hat{D}^R)^\dagger &= L(z)\hat{\mathcal{X}}^{R[1]}(z). \end{aligned}$$

Notice that from Proposition 1, we have ensured the associativity of the product for the above semi-infinite matrices (see also [14]), besides the matrix $(D^R)^{-\dagger}S_2^\dagger\hat{S}_2^{-\dagger}(\hat{D}^R)^\dagger$ is a lower triangular block matrix. The other relation is obtained in a similar way. □

From (14), (15) and the definition of prepared polynomial of degree d, for $@ = R, L$, we get

$$L(z)\hat{\mathcal{X}}_n^{@[1]}(z) \in \begin{cases} \text{Span}\{z^{-(k+d)}I_p \cdots z^{k+d}I_p\} & (\text{coefficient of } z^{-(k+d)}I_P = I_p), \ n = 2k, \\ \text{Span}\{z^{-(k-d)}I_p \cdots z^{k+d+1}I_p\} & (\text{coefficient of } z^{k+d+1}I_p = I_p), \ n = 2k+1. \end{cases}$$

With this in mind and Proposition 9

$$L(z)\hat{\mathcal{X}}_n^{@[1]}(z) \in \text{Span}\{\mathcal{X}_0^{@[1]}(z)\cdots\mathcal{X}_{n+N}^{@[1]}\}, \ (\text{with coefficient of } \mathcal{X}_{n+N}^{@[1]} = I_p), \ N = 2d. \tag{18}$$

With this in mind, we have the following result

Proposition 10. *The connection matrices ω and $\tilde{\omega}$ have the following form*

$$\omega := \begin{pmatrix} \omega_{0,0} & 0 & 0 & \cdots & \cdots \\ \omega_{0,1} & \omega_{1,1} & 0 & \cdots & \cdots \\ \vdots & \vdots & \omega_{2,2} & 0 & \cdots \\ \omega_{0,N-1} & \vdots & \vdots & \ddots & \ddots \\ I_p & \omega_{1,N} & & & \\ 0 & I_p & & & \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}, \quad \tilde{\omega} := \begin{pmatrix} \tilde{\omega}_{0,0} & \tilde{\omega}_{0,1} & \cdots & \tilde{\omega}_{0,N-1} & I_p & 0 & \cdots \\ 0 & \tilde{\omega}_{1,1} & \cdots & \tilde{\omega}_{1,N-1} & \tilde{\omega}_{1,N} & I_p & \ddots \\ & \ddots & \tilde{\omega}_{2,2} & \cdots & \tilde{\omega}_{2,N} & \tilde{\omega}_{2,N+1} & I_p & \ddots \\ & & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The above is equivalent to the following connection formulas

$$L(z)\hat{\mathcal{X}}_n^{R[1]}(z) = \mathcal{X}_{n+N}^{R[1]}(z) + \sum_{k=n}^{n+N-1}\mathcal{X}_k^{R[1]}(z)\omega_{n,k}, \quad N = 2d, \tag{19}$$

$$\hat{\mathcal{X}}_n^{L[1]}(z)L(z) = \mathcal{X}_{n+N}^{L[1]}(z) + \sum_{k=n}^{n+N-1}\tilde{\omega}_{n,k}\mathcal{X}_k^{L[1]}(z).$$

Proof. From Proposition 9 and (18) we know that

$$L(z)\hat{\mathcal{X}}_n^{R[1]}(z) = \mathcal{X}_{n+N}^{R[1]}(z) + \sum_{k=0}^{n+N-1}\mathcal{X}_k^{R[1]}(z)\omega_{n,k},$$

and, since for $0 \leq k \leq n-1$,

$$\left\langle L(z_1)\hat{\mathcal{X}}_n^{R[1]}(z_1), \mathcal{X}_k^{R[1]}(z_2)\right\rangle_R = \left\langle \hat{\mathcal{X}}_n^{R[1]}(z_1), \mathcal{X}_k^{R[1]}(z_2)\right\rangle_{\hat{R}} = 0_{p\times p},$$

then $\omega_{n,k} = 0_{p\times p}$ for $k = 0,\dots,n-1$. The proof for $\tilde{\omega}$ follows in a similar way. □

The following theorem gives formulas that relate the first families of bi-orthogonal Laurent polynomials $(\mathcal{X}_n^{@[1]}(z_1))_{n\in\mathbb{N}}$ and $(\hat{\mathcal{X}}_n^{@[1]}(z_1))_{n\in\mathbb{N}}$, which are generated by the bi-variate matrices of measures $d\mu(z_1,z_2)$, and $L(z_1)d\mu(z_1,z_2)$, respectively.

Theorem 1. *The sequences $(\mathcal{X}_n^{@[1]}(z))_{n\in\mathbb{N}}$ and $(\hat{\mathcal{X}}_n^{@[1]}(z))_{n\in\mathbb{N}}$, $@ = L,R$, are related as follows (see Definition 3)*

$$L(z)\hat{\mathcal{X}}_n^{R[1]}(z) = \Theta_* \left[\begin{array}{ccc|c} J^l_{z^d\mathcal{X}_n^{R[1]}(z)} & \cdots & J^l_{z^d\mathcal{X}_{n+N-1}^{R[1]}(z)} & J^l_{z^d\mathcal{X}_{n+N}^{R[1]}(z)} \\ \hline z^d\mathcal{X}_n^{R[1]}(z) & \cdots & z^d\mathcal{X}_{n+N-1}^{R[1]}(z) & z^d\mathcal{X}_{n+N}^{R[1]}(z) \end{array}\right], \tag{20}$$

$$\hat{\mathcal{X}}_n^{L[1]}(z)L(z) = \Theta_* \left[\begin{array}{c|c} J^r_{z^d\mathcal{X}_n^{L[1]}(z)} & \mathcal{X}_n^{L[1]}(z) \\ \vdots & \vdots \\ J^r_{z^d\mathcal{X}_{n+N-1}^{L[1]}(z)} & \mathcal{X}_{n+N-1}^{L[1]}(z) \\ \hline J^r_{z^d\mathcal{X}_{n+N}^{L[1]}(z)} & \mathcal{X}_{n+N}^{L[1]}(z) \end{array}\right]. \tag{21}$$

Proof. We will prove (20). The proof of (21) is similar. Let $W(z) := z^d L(z)$. Then $W(z)$ is a monic matrix polynomial of degree $N := 2d$. Let $y_1,\dots,y_q$, be its zeros and let $\alpha_1,\dots,\alpha_q$, be their corresponding multiplicities. Notice that $\sum_{k=1}^q \alpha_k = Np$. From Proposition 10 we have

$$W(z)\hat{\mathcal{X}}_n^{R[1]}(z) = z^d\mathcal{X}_{n+N}^{R[1]}(z) + \sum_{k=n}^{n+N-1} z^d\mathcal{X}_k^{R[1]}(z)\omega_{n,k}.$$

Since $\hat{\mathcal{X}}_n^{R[1]}(z)$ is a matrix Laurent polynomial, then it is an analytic function in $\mathbb{C}\setminus\{0\}$ and, in addition, 0 is not a zero of $W(z)$. Let $l_j^{(i)}(z)$, $j = 1,\dots,s_i$, be the left root vector polynomials of degree $\kappa_j^{(i)} - 1$ associated with the zero y_i (see Definition 2). Then from Proposition 4, for $t = 0,\cdots,\kappa_j^{(i)} - 1$, we have

$$\begin{aligned} 0_p^\top = \frac{d^t}{dz^t}\left(l_j^{(i)}(z)W(z)\hat{\mathcal{X}}_n^{R[1]}(z)\right)\Big|_{z=y_i} &= \frac{d^t}{dz^t}\left(l_j^{(i)}(z)z^d\mathcal{X}_{n+N}^{R[1]}(z)\right)\Big|_{z=y_i} \\ &+ \sum_{k=n}^{n+N-1}\frac{d^t}{dz^t}\left(l_j^{(i)}(z)z^d\mathcal{X}_k^{R[1]}(z)\right)\Big|_{z=y_i}\omega_{n,k}. \end{aligned}$$

Using Definition 3, we have the following relation between the spectral jets

$$-J^{l,(j)}_{z^d\mathcal{X}_{n+N}^{R[1]}(z)}(y_i) = \left(J^{l,(j)}_{z^d\mathcal{X}_n^{R[1]}(z)}(y_i) \quad \cdots \quad J^{l,(j)}_{z^d\mathcal{X}_{n+N-1}^{R[1]}(z)}(y_i)\right)\begin{pmatrix}\omega_{n,n}\\ \vdots \\ \omega_{n,n+N-1}\end{pmatrix}.$$

Proceeding in such a way for each zero y_i, $i = 1, \cdots, q$, we get

$$-\begin{pmatrix} J^l_{z^d \mathcal{X}^{R[1]}_{n+N}(z)}(y_1) \\ \vdots \\ J^l_{z^d \mathcal{X}^{R[1]}_{n+N}}(y_q) \end{pmatrix} = \begin{pmatrix} J^l_{z^d \mathcal{X}^{R[1]}_{n}(z)}(y_1) & \cdots & J^l_{z^d \mathcal{X}^{R[1]}_{n+N-1}(z)}(y_1) \\ \vdots & & \vdots \\ J^l_{z^d \mathcal{X}^{R[1]}_{n}(z)}(y_q) & \cdots & J^l_{z^d \mathcal{X}^{R[1]}_{n+N-1}(z)}(y_q) \end{pmatrix}_{Np\times Np} \begin{pmatrix} \omega_{n,n} \\ \vdots \\ \omega_{n,n+N-1} \end{pmatrix}$$

$$-J^l_{z^d \mathcal{X}^{R[1]}_{n+N}(z)} = \begin{pmatrix} J^l_{z^d \mathcal{X}^{R[1]}_{n}(z)} & \cdots & J^l_{z^d \mathcal{X}^{R[1]}_{n+N-1}(z)} \end{pmatrix}_{Np\times Np} \begin{pmatrix} \omega_{n,n} \\ \vdots \\ \omega_{n,n+N-1} \end{pmatrix},$$

and since

$$L(z)\hat{\mathcal{X}}^{R[1]}_{n+N}(z) = \mathcal{X}^{R[1]}_{n+N}(z) + \begin{pmatrix} \mathcal{X}^{R[1]}_{n}(z) \cdots \mathcal{X}^{R[1]}_{n+N-1}(z) \end{pmatrix} \begin{pmatrix} \omega_{n,n} \\ \vdots \\ \omega_{n,n+N-1}, \end{pmatrix},$$

then the result follows. □

As in the above case, since $(\hat{\mathcal{X}}^{L[2]}_n(z))_{n\in\mathbb{N}}$ and $(\hat{\mathcal{X}}^{R[2]}_n(z))_{n\in\mathbb{N}}$ are bases of $\mathbb{L}^{p\times p}[z]$, then there exist matrix coefficients $(b_{n,k})^n_{k=0}$ and $(\tilde{b}_{n,k})^n_{k=0}$ such that

$$\mathcal{X}^{R[2]}_n(z) = \sum_{k=0}^{n} \hat{\mathcal{X}}^{R[2]}_k(z) b_{n,k}, \qquad \mathcal{X}^{R[2]}(z) = \hat{\mathcal{X}}^{R[2]}(z)B,$$

$$\mathcal{X}^{L[2]}_n(z) = \sum_{k=0}^{n} \tilde{b}_{n,k} \hat{\mathcal{X}}^{L[2]}_k(z), \qquad \mathcal{X}^{L[2]}(z) = \tilde{B}\hat{\mathcal{X}}^{L[2]}(z),$$

where B is the connection matrix between $(\mathcal{X}^{R[2]}_n(z))_{n\in\mathbb{N}}$ and $(\hat{\mathcal{X}}^{R[2]}_n(z))_{n\in\mathbb{N}}$ and $\tilde{B}$ is the connection matrix between $(\mathcal{X}^{L[2]}_n(z))_{n\in\mathbb{N}}$ and $(\tilde{\mathcal{X}}^{L[2]}_n(z))_{n\in\mathbb{N}}$.

Proposition 11. *Given the connection matrices ω and $\tilde{\omega}$ (see* (17)*), the following relations hold*

$$B = \hat{S}_2^{-1} S_2, \qquad \tilde{B} = Z_2^{\dagger} \hat{Z}_2^{-\dagger}, \tag{22}$$

$$\omega = (D^R)^{-\dagger} B^{\dagger} (\hat{D}^R)^{\dagger}, \qquad \tilde{\omega} = \hat{D}^L \tilde{B}^{\dagger} (D^L)^{-1}. \tag{23}$$

Proof. To prove (22), let B' be the upper triangular block matrix, with I_p in its main diagonal, defined as $B' = \hat{S}_2^{-1} S_2$. Then

$$\hat{S}_2 B' = S_2 \longrightarrow \eta^{\top}(z)\hat{S}_2 B' = \eta^{\top}(z) S_2 \longrightarrow \hat{\mathcal{X}}^{R[2]}(z)B' = \mathcal{X}^{R[2]}(z).$$

Since the representation in this basis is unique, we get $B' = B$. For $\tilde{B}$ the result follows in a similar way. (23) is a straightforward consequence of Proposition 9 and (22). □

Proposition 12. *Let $\hat{\mathcal{K}}^{@}_n(z_1, z_2)$, $@ = L, R$, be the perturbed Laurent Kernel polynomials associated with $(\hat{\mathcal{X}}^{@[1]}_n(z_1), \hat{\mathcal{X}}^{@[2]}_n(z_2))_{n\in\mathbb{N}}$. Then the perturbed and original*

Laurent kernel polynomials are related by the following connection formulas

$$\mathcal{K}^R_n(z_1,z_2)=\hat{\mathcal{K}}^R_n(z_1,z_2)(L(z_1))^\dagger - \left(\hat{\mathcal{X}}^{R[2]}_{n-N+1}(z_2)(\hat{D}^R_{n-N+1})^{-1}\cdots\hat{\mathcal{X}}^{R[2]}_{n}(z_2)(\hat{D}^R_{n})^{-1}\right)\omega^\dagger_{[n,N]}\begin{pmatrix}(\mathcal{X}^{R[1]}_{n+1}(z_1))^\dagger\\ \vdots\\ (\mathcal{X}^{R[1]}_{n+N}(z_1))^\dagger\end{pmatrix}, \quad (24)$$

$$\mathcal{K}^L_n(z_1,z_2)=\hat{\mathcal{K}}^L_n(z_1,z_2)L(z_1) - \left((\hat{\mathcal{X}}^{L[2]}_{n-N+1}(z_2))^\dagger(\hat{D}^L_{n-N+1})^{-1}\cdots(\hat{\mathcal{X}}^{L[2]}_{n}(z_2))^\dagger(\hat{D}^L_{n})^{-1}\right)\tilde{\omega}_{[n,N]}\begin{pmatrix}\mathcal{X}^{[1]L}_{n+1}(z_1)\\ \vdots\\ \mathcal{X}^{[1]L}_{n+N}(z_1)\end{pmatrix}, \quad (25)$$

where

$$\omega_{[n,N]}:=\begin{pmatrix}I_p & \omega_{n-N+2,n+1} & \cdots & \cdots & \omega_{n,n+1}\\ & I_p & \ddots & & \vdots\\ & & \ddots & \ddots & \vdots\\ & & & \ddots & \omega_{n,n+N-1}\\ & & & & I_p\end{pmatrix},\quad \tilde{\omega}_{[n,N]}:=\begin{pmatrix}I_p & & & & \\ \tilde{\omega}_{n-N+2,n+1} & I_p & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \tilde{\omega}_{n,n+1} & \cdots & \cdots & \tilde{\omega}_{n,n+N-1} & I_p\end{pmatrix},$$

are truncations of the connection matrices ω and $\tilde{\omega}$, respectively.

Proof. Given the connection matrix ω (see (17)), consider its block sub-matrix of order $n+1$

$$\omega_{[n+1]}=\begin{pmatrix}\omega_{0,0} & & & & \\ \vdots & \ddots & & & \\ \omega_{0,N-1} & & \ddots & & \\ I_p & \ddots & & \ddots & \\ & \ddots & \ddots & & \ddots & \\ & & I_p & \omega_{n-N+1,n} & \cdots & \omega_{n,n}\end{pmatrix}$$

and define the following expression in two variables

$$\mathcal{L}(z_1,z_2):=\left(\hat{\mathcal{X}}^{R[2]}_0(z_2)(\hat{D}^R_0)^{-1}\quad\cdots\quad\hat{\mathcal{X}}^{R[2]}_n(z_2)(\hat{D}^R_n)^{-1}\right)\omega^\dagger_{[n+1]}\begin{pmatrix}(\mathcal{X}^{R[1]}_0(z_1))^\dagger\\ \vdots\\ (\mathcal{X}^{R[1]}_n(z_1))^\dagger\end{pmatrix}.$$

On one hand, using the fact that ω is a lower triangular block matrix and (23), we have that $\omega^\dagger_{[n+1]}=\hat{D}^R_{[n+1]}B_{[n+1]}(D^R)^{-1}_{[n+1]}$. Thus

$$\begin{aligned}\left(\hat{\mathcal{X}}^{R[2]}_0(z_2)(\hat{D}^R_0)^{-1}\quad\cdots\quad\hat{\mathcal{X}}^{R[2]}_n(z_2)(\hat{D}^R_n)^{-1}\right)\omega^\dagger_{[n+1]}&=\left(\hat{\mathcal{X}}^{R[2]}_0(z_2)\quad\cdots\quad\hat{\mathcal{X}}^{R[2]}_n(z_2)\right)B_{[n+1]}D^R_{[n+1]}\\&=\left(\mathcal{X}^{R[2]}_0(z_2)(D^R_0)^{-1}\quad\cdots\quad\mathcal{X}^{R[2]}_n(z_2)(D^R_n)^{-1}\right),\end{aligned}$$

and this implies that $\mathcal{L}(z_1,z_2)=\mathcal{K}^R_n(z_1,z_2)$. On the other hand, notice that for $n\geq N$, the relation (19) yields

$$(\hat{\mathcal{X}}^{R[1]}_{n-N+m}(z_1))^\dagger (L(z_1))^\dagger = (\mathcal{X}^{R[1]}_{n+m}(z_1))^\dagger + \sum_{k=n-N+m}^{n+m-1} \omega^\dagger_{n-N+m,k} (\mathcal{X}^{R[1]}_k(z_1))^\dagger, \quad m = 1, \cdots, N.$$

This expression is equivalent to

$$\begin{pmatrix} \omega^\dagger_{n-N+m,n-N+m} & \cdots & \omega^\dagger_{n-N+m,n} \end{pmatrix} \begin{pmatrix} (\mathcal{X}^{R[1]}_{n-N+m}(z_1))^\dagger \\ \vdots \\ (\mathcal{X}^{R[1]}_n(z_1))^\dagger \end{pmatrix} =$$

$$(\hat{\mathcal{X}}^{R[1]}_{n-N+m}(z_1))^\dagger (L(z_1))^\dagger - \begin{pmatrix} \omega^\dagger_{n-N+m,n+1} & \cdots & \omega^\dagger_{n-N+m,n+m-1} & I_p \end{pmatrix} \begin{pmatrix} (\mathcal{X}^{R[1]}_{n+1}(z_1))^\dagger \\ \vdots \\ (\mathcal{X}^{R[1]}_{n+m}(z_1))^\dagger \end{pmatrix}.$$

With this in mind

$$\omega^\dagger_{[n+1]} \begin{pmatrix} (\mathcal{X}^{R[1]}_0(z_1))^\dagger \\ \vdots \\ (\mathcal{X}^{R[1]}_n(z_1))^\dagger \end{pmatrix} = \begin{pmatrix} (\hat{\mathcal{X}}^{R[1]}_0(z_1))^\dagger \\ \vdots \\ (\hat{\mathcal{X}}^{R[1]}_n(z_1))^\dagger \end{pmatrix} (L(z_1))^\dagger - \begin{pmatrix} 0_{(n-N)p\times p} \\ R_N(z_1) \end{pmatrix},$$

where

$$R_N(z_1) = \omega^\dagger_{[n,N]} \begin{pmatrix} (\mathcal{X}^{R[1]}_{n+1}(z_1))^\dagger \\ \vdots \\ (\mathcal{X}^{R[1]}_{n+N}(z_1))^\dagger \end{pmatrix}.$$

Thus

$$\mathcal{L}(z_1, z_2) = \hat{\mathcal{K}}^R_n(z_1, z_2)(L(z_1))^\dagger - \omega^\dagger_{[n,N]} \begin{pmatrix} (\mathcal{X}^{R[1]}_{n+1}(z_1))^\dagger \\ \vdots \\ (\mathcal{X}^{R[1]}_{n+N}(z_1))^\dagger \end{pmatrix}$$

and we get (24). To prove (25) we compute

$$\begin{pmatrix} (\hat{\mathcal{X}}^{L[2]}_0(z_2))^\dagger (\hat{D}^L_0)^{-1} & \cdots & (\hat{\mathcal{X}}^{L[2]}_n(z_2))^\dagger (\hat{D}^L_n)^{-1} \end{pmatrix} \tilde{\omega}_{[n+1]} \begin{pmatrix} \mathcal{X}^{[1]L}_0(z_1) \\ \vdots \\ \mathcal{X}^{[1]L}_n(z_1) \end{pmatrix}$$

in two different ways and we proceed as above. □

The following theorem gives formulas that relate the second families of bi-orthogonal Laurent polynomials $(\mathcal{X}^{@[2]}_n(z_2))_{n\in\mathbb{N}}$ and $(\hat{\mathcal{X}}^{@[2]}_n(z_2))_{n\in\mathbb{N}}$, which are generated by the bi-variate matrices of measures $d\mu(z_1, z_2)$, and $L(z_1)d\mu(z_1, z_2)$, respectively.

Theorem 2. *The sequences* $(\mathcal{X}_n^{@[2]}(z))$ *and* $(\hat{\mathcal{X}}_n^{@[2]}(z))$, $@ = L, R$, *are related as follows*

$$(\hat{D}_n^R)^{-\dagger}(\hat{\mathcal{X}}_n^{R[2]}(z_2))^\dagger = \Theta_* \left[\begin{array}{ccc|c} J^l_{z_1^d\mathcal{X}^{R[2]}_{n+1}(z_1)} & \cdots & J^l_{z_1^d\mathcal{X}^{R[2]}_{n+N}(z_1)} & J^l_{z_1^d(\mathcal{K}_n^R(z_1,z_2))^\dagger} \\ \hline 0_{p\times p} & \cdots 0_{p\times p} & I_p & 0_{p\times p}\end{array}\right],$$

$$(\hat{\mathcal{X}}_n^{L[2]}(z_2))^\dagger(\hat{D}_n^L)^{-1} = \Theta_* \left[\begin{array}{c|c} J^r_{z_1^d\mathcal{X}^{L[2]}_{n+1}(z_1)} & 0_{p\times p} \\ \vdots & \vdots \\ J^r_{z_1^d\mathcal{X}^{L[2]}_{n+N}(z_1)} & \begin{matrix}0_{p\times p}\\ I_p\end{matrix} \\ \hline J^r_{z_1^d\mathcal{K}_n^L(z_1,z_2)} & 0_{p\times p}\end{array}\right].$$

Proof. From Proposition 12

$$z_1^d(\mathcal{K}_n^R(z_1,z_2))^\dagger = W(z_1)(\tilde{\mathcal{K}}_n^R(z_1,z_2))^\dagger \\ - \left(z_1^d\mathcal{X}^{R[1]}_{n+1}(z_1)\cdots z_1^d\mathcal{X}^{R[1]}_{n+N}(z_1)\right)\omega_{[n,N]}\begin{pmatrix}(\hat{D}^R_{n-N+1})^{-\dagger}(\hat{\mathcal{X}}^{R[2]}_{n-N+1}(z_2))^\dagger \\ \vdots \\ (\hat{D}^R_n)^{-\dagger}(\hat{\mathcal{X}}^{R[2]}_n(z_2))^\dagger\end{pmatrix}.$$

Taking into account that for a fixed $z_2 \neq 0$, $(\tilde{\mathcal{K}}_n^R(z_1,z_2))^\dagger$ is a matrix Laurent polynomial in the variable z_1, then it is analytic in $\mathbb{C}\setminus\{0\}$ and since 0 is not a zero of $W(z_1)$, we get $J^{l,(j)}_{W(z_1)(\tilde{\mathcal{K}}_n^R(z_1,z_2))^\dagger}(y_i) = 0_p$, for $1 \leq j \leq s_i$, $i = 1,\cdots,q$, where the spectral jets $J^{l,(j)}_{\cdot}(y_i)$ act on the variable z_1. From here

$$J^{l,(j)}_{z_1^d(\mathcal{K}_n^R(z_1,z_2))^\dagger}(y_i) = -\left(J^{l,(j)}_{z_1^d\mathcal{X}^{R[1]}_{n+1}(z_1)}(y_i) \quad\cdots\quad J^{l,(j)}_{z_1^d\mathcal{X}^{R[1]}_{n+N}(z_1)}(y_i)\right)\omega_{[n,N]}\begin{pmatrix}(\hat{D}^R_{n-N+1})^{-\dagger}(\hat{\mathcal{X}}^{R[2]}_{n-N+1}(z_2))^\dagger \\ \vdots \\ (\hat{D}^R_n)^{-\dagger}(\hat{\mathcal{X}}^{R[2]}_n(z_2))^\dagger\end{pmatrix}.$$

Summarizing all these equations in a matrix form and taking into account that

$$\begin{pmatrix}0 & \cdots & 0 & I_p\end{pmatrix}\omega_{[n,N]} = \begin{pmatrix}0 & \cdots & 0 & I_p\end{pmatrix},$$

the statement follows. □

Remark 3. *If* $d\nu(z)$ *is a univariate matrix of measures supported on* $\mathbb{T}$ *and we define the following matrix of bi-variate measures*

$$d\mu(z_1,z_2) := d\nu(z_1)\delta_{z_1}(z_2), \quad \operatorname{supp}\mu(z_1,z_2) = \mathbb{T}\times\mathbb{T},$$

where $\delta_a(z)$ *is de Dirac Delta function supported in* $z = a$, *then the sequences*

$$(\mathcal{X}_n^{R[1]}(z_1), \mathcal{X}_n^{R[2]}(z_2))_{n\in\mathbb{N}} \quad and \quad (\mathcal{X}_n^{L[1]}(z_1), \mathcal{X}_n^{L[2]}(z_2))_{n\in\mathbb{N}}$$

are bi-orthogonal with respect to (5) *and* (6), *respectively, if and only if*

$$(\mathcal{X}_n^{R[1]}(z), \mathcal{X}_n^{R[2]}(z))_{n\in\mathbb{N}} \quad and \quad (\mathcal{X}_n^{L[1]}(z), \mathcal{X}_n^{L[2]}(z))_{n\in\mathbb{N}}$$

are bi-orthogonal with respect to

$$\langle f(z), g(z)\rangle_r = \int_{\mathbb{T}} f(z)^\dagger d\nu(z) g(z), \quad and \quad \langle f(z), g(z)\rangle_l = \int_{\mathbb{T}} f(z) d\nu(z) g(z)^\dagger,$$

respectively. Thus, the results obtained here can be also used for univariate matrix of measures supported on $\mathbb{T}$.

In the remainder of the paper we will analyze two examples. We consider a sequence of monic Bernstein-Szegő polynomials $\{p_n(z)\}_{n\geq 0}$ (see [46]) defined by

$$p_n(z) = z^n - \frac{z^{n-1}}{2}, n \geq 1,$$

as well as the Laurent polynomials defined by $x_{2k}(z) = z^{-k}(p_{2k}(z))^*$ (see (1)) and $x_{2k+1}(z) = z^{-k}p_{2k+1}(z)$, or equivalently,

$$x_{2k}(z) = z^{-k}(1 - (1/2)z), \qquad x_{2k+1}(z) = z^k(z - 1/2) \quad k = 0, 1, \cdots . \tag{26}$$

If we define $\mathcal{X}_n(z) := x_n(z)I_2$, then $(\mathcal{X}_n(z))_{n\in\mathbb{N}}$ is the sequence of matrix orthogonal Laurent polynomials with respect to the bilinear forms [46]

$$\langle f, g\rangle_R = \frac{1}{2\pi}\int_0^{2\pi} f(z)^\dagger \frac{d\theta}{|h(e^{i\theta})|^2} I_2 g(z), \quad z = e^{i\theta}, \quad \text{where} \quad h(z) = \frac{2z-1}{\sqrt{3}}.$$

Let us define the bilinear form

$$\langle f, g\rangle_{\hat{R}} = \frac{1}{2\pi}\int_0^{2\pi} f(z)^\dagger \frac{L(z)\, d\theta}{|h(e^{i\theta})|^2} I_2 g(z), \quad z = e^{i\theta}, \tag{27}$$

where $L(z)$ is a matrix prepared Laurent polynomial. We are going to represent the sequences of bi-orthogonal with respect to (27), for two different Laurent polynomials $L(z)$.

Example 1 (Diagonal example). *Let* $L(z)$ *be the Laurent polynomial of "degree"* 1

$$L(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^{-1} + 2\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z.$$

It is not difficult to check that $L(z) = U\tilde{L}(z)U^\dagger$, *where*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad \tilde{L}(z) = \begin{pmatrix} 1/z + z & 0 \\ 0 & 1/z + 4 + z \end{pmatrix}.$$

Let $((\hat{\mathcal{X}}_n^{R[1]}(z), \hat{\mathcal{X}}_n^{R[2]}(z))_{n\in\mathbb{N}})$ *be the sequences of matrix orthogonal Laurent polynomials associated with* $L(z)\frac{d\theta}{|h(e^{i\theta})|^2}$. *Taking into account this fact, we have that for every* $n \in \mathbb{N}$, $\hat{\mathcal{X}}_n^{R[1]}(z) = \hat{\mathcal{X}}_n^{R[2]}(z) := \hat{\mathcal{X}}_n^{R}(z)$ *and they can be written as* $\hat{\mathcal{X}}_n^{R}(z) = U^\dagger \hat{X}_n(z) U$, *where* $(\hat{X}_n(z))_{n\in\mathbb{N}}$ *is the sequence of matrix orthogonal Laurent polynomials associated with* $\tilde{L}(z)\frac{d\theta}{|h(e^{i\theta})|^2}$. *Using Theorem* 1, *we are going to find the representation for* $(\hat{X}_n(z))_{n\in\mathbb{N}}$.

Let $\tilde{W}(z)$ *be the matrix polynomial*

$$\tilde{W}(z) := z\tilde{L}(z) = \begin{pmatrix} 1 + z^2 & 0 \\ 0 & 1 + 4z + z^2 \end{pmatrix}.$$

Observe that since $\det(\tilde{W}(z)) = (1+z^2)(1+4z+z^2)$*, then* $\tilde{W}(z)$ *has four simple zeros* $(y_i)_{i=1}^4$*. Moreover, the right (resp. left) eigenvectors will be any nonzero solution of the equation* $\tilde{W}(y_i)r_{i,0} = 0_p$ *(resp.* $l_{i,0}\tilde{W}(y_i) = 0_p^\top$*).*

Zeros	*Right eigenvector*	*Left eigenvector*
$y_1 = i$	$r_{1,0} = (1,0)^\top$	$l_{1,0} = (1,0)$
$y_1 = -i$	$r_{2,0} = (1,0)^\top$	$l_{2,0} = (1,0)$
$y_3 = -2+\sqrt{3}$	$r_{3,0} = (0,1)^\top$	$l_{3,0} = (0,1)$
$y_4 = -2-\sqrt{3}$	$r_{4,0} = (0,1)^\top$	$l_{4,0} = (0,1)$

.

Using Theorem 1*, we deduce that the sequences of matrix orthogonal Laurent polynomials* $(\hat{X}_n(z))_{n\in\mathbb{N}}$ *are given by*

$\tilde{L}(z)\hat{X}_{2k}(z) =$

$$\Theta_* \left|\begin{array}{cccc|cc} \left(\frac{1}{2}+i\right)i^{-k} & 0 & \left(-1-\frac{i}{2}\right)i^{k} & 0 & \left(-1+\frac{i}{2}\right)i^{-k} & 0 \\ \left(\frac{1}{2}-i\right)(-i)^{-k} & 0 & \left(-1+\frac{i}{2}\right)(-i)^{k} & 0 & \left(-1-\frac{i}{2}\right)(-i)^{-k} & 0 \\ 0 & -\frac{1}{2}\left(\sqrt{3}-4\right)\left(\sqrt{3}-2\right)^{1-k} & 0 & \left(\sqrt{3}-\frac{5}{2}\right)\left(\sqrt{3}-2\right)^{k+1} & 0 & \frac{1}{2}\left(\sqrt{3}-4\right)\left(\sqrt{3}-2\right)^{-k} \\ 0 & \frac{1}{2}\left(4+\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{1-k} & 0 & \left(-\frac{5}{2}-\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{k+1} & 0 & -\frac{1}{2}\left(4+\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{-k} \\ \hline \multicolumn{2}{c}{z^{-k+1}(1-(1/2)z)I_2} & \multicolumn{2}{c|}{z^{k+1}(z-1/2)I_2} & \multicolumn{2}{c}{z^{-k}(1-(1/2)z)I_2} \end{array}\right|,$$

$\tilde{L}(z)\hat{X}_{2k+1}(z) =$

$$\Theta_* \left|\begin{array}{cccc|cc} \left(-1-\frac{i}{2}\right)i^{k} & 0 & \left(1-\frac{i}{2}\right)i^{-k} & 0 & \left(-\frac{1}{2}+i\right)i^{k} & 0 \\ \left(-1+\frac{i}{2}\right)(-i)^{k} & 0 & \left(1+\frac{i}{2}\right)(-i)^{-k} & 0 & \left(-\frac{1}{2}-i\right)(-i)^{k} & 0 \\ 0 & \left(\sqrt{3}-\frac{5}{2}\right)\left(\sqrt{3}-2\right)^{k+1} & 0 & -\frac{1}{2}\left(\sqrt{3}-4\right)\left(\sqrt{3}-2\right)^{-k} & 0 & -\left(\sqrt{3}-\frac{5}{2}\right)\left(\sqrt{3}-2\right)^{k+2} \\ 0 & \left(-\frac{5}{2}-\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{k+1} & 0 & \frac{1}{2}\left(4+\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{-k} & 0 & \left(\frac{5}{2}+\sqrt{3}\right)\left(-2-\sqrt{3}\right)^{k+2} \\ \hline \multicolumn{2}{c}{z^{k+1}(z-1/2)I_2} & \multicolumn{2}{c|}{z^{-k}(1-1/2z)I_2} & \multicolumn{2}{c}{z^{k+2}(z-1/2)} \end{array}\right|.$$

Example 2 (Non-diagonal example). *Let* $L(z)$ *be a Laurent polynomial of "degree"* 2

$$L(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^{-2} - \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^{-1} - \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^2.$$

Here, $L(z)$ *can be written* $L(z) = U(z)\tilde{L}(z)U^{-1}(z)$*, where*

$$U^{-1}(z) = \begin{pmatrix} z & -z \\ 1 & 1 \end{pmatrix}, \quad \tilde{L}(z) = \begin{pmatrix} z^2 - \sqrt{2} + \frac{1}{z^2} & 0 \\ 0 & z^2 + \sqrt{2} + \frac{1}{z^2} \end{pmatrix},$$

and $U^{-1}(z) = U^\dagger(z)$ *on* $\mathbb{T}$*. Observe that this is not a diagonal case, because the matrix* $U(z)$ *depends on the variable* z*.*
Define the polynomial $W(z)$ *as follows*

$$W(z) := z^2 L(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z - \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^4.$$

Since $\det(W(z)) = z^8+1$, *then* $W(z)$ *has eight simple zeros* $(y_i)_{i=1}^{8}$. *Moreover, the right (resp. left) eigenvectors will be any nonzero solution of the equation* $W(y_i)r_{i,0} = 0_p$ *(resp.* $l_{i,0}W(y_i) = 0_p^{\top}$*).*

Zeros	*Right eigenvector*	*Left eigenvector*
$y_1 = i^{1/4}$	$r_{1,0} = ((1-i)i^{3/4}, \sqrt{2})^{\top}$	$l_{1,0} = (-(1+i)i^{5/4}, \sqrt{2})$
$y_2 = -i^{1/4}$	$r_{2,0} = (-(1-i)i^{3/4}, \sqrt{2})^{\top}$	$l_{2,0} = ((1+i)i^{5/4}, \sqrt{2})$
$y_3 = i^{3/4}$	$r_{3,0} = (-(1+i)i^{1/4}, \sqrt{2})^{\top}$	$l_{3,0} = (i^{5/4}, 1)$
$y_4 = -i^{3/4}$	$r_{4,0} = ((1+i)i^{1/4}, \sqrt{2})^{\top}$	$l_{4,0} = (-(1-i)i^{7/4}, \sqrt{2})$
$y_5 = i^{5/4}$	$r_{5,0} = (-(1+i)i^{3/4}, \sqrt{2})^{\top}$	$l_{5,0} = (i^{3/4}, 1)$
$y_6 = -i^{5/4}$	$r_{6,0} = ((1+i)i^{3/4}, \sqrt{2})^{\top}$	$l_{6,0} = (-i^{3/4}, 1)$
$y_7 = i^{7/4}$	$r_{7,0} = (-(1-i)i^{1/4}, \sqrt{2})^{\top}$	$l_{7,0} = (-(1-i)i^{3/4}, \sqrt{2})$
$y_8 = -i^{7/4}$	$r_{8,0} = ((1-i)i^{1/4}, \sqrt{2})^{\top}$	$l_{8,0} = ((1-i)i^{3/4}, \sqrt{2})$

.

Let $((\hat{\mathcal{X}}_n^{R[1]}(z), \hat{\mathcal{X}}_n^{R[2]}(z))_{n\in\mathbb{N}})$ *be the sequences of matrix orthogonal Laurent polynomials associated with* $L(z)\dfrac{d\theta}{|h(e^{i\theta})|^2}$. *Since* $L(z)$ *is a Hermitian Laurent polynomial matrix on* $\mathbb{T}$, *then* $\hat{\mathcal{X}}_n^{R[1]}(z) = \hat{\mathcal{X}}_n^{R[2]}(z) := \hat{\mathcal{X}}_n^{R}(z)$. *Using Theorem* 1, *we get*

$$L(z)\hat{\mathcal{X}}_n^{R}(z) =$$

$$\Theta_*\left|\begin{array}{cccc|c}
i^{1/2}l_{1,0}\mathcal{X}_n(i^{1/4}) & i^{1/2}l_{1,0}\mathcal{X}_{n+1}(i^{1/4}) & i^{1/2}l_{1,0}\mathcal{X}_{n+2}(i^{1/4}) & i^{1/2}l_{1,0}\mathcal{X}_{n+3}(i^{1/4}) & i^{1/2}l_{1,0}\mathcal{X}_{n+4}(i^{1/2}) \\
i^{1/2}l_{2,0}\mathcal{X}_n(-i^{1/4}) & i^{1/2}l_{2,0}\mathcal{X}_{n+1}(-i^{1/4}) & i^{1/2}l_{2,0}\mathcal{X}_{n+2}(-i^{1/4}) & i^{1/2}l_{2,0}\mathcal{X}_{n+3}(-i^{1/4}) & i^{1/2}l_{2,0}\mathcal{X}_{n+4}(-i^{1/4}) \\
i^{3/2}l_{3,0}\mathcal{X}_n(i^{3/4}) & i^{3/2}l_{3,0}\mathcal{X}_{n+1}(i^{3/4}) & i^{3/2}l_{3,0}\mathcal{X}_{n+2}(i^{3/4}) & i^{3/2}l_{3,0}\mathcal{X}_{n+3}(i^{3/4}) & i^{3/2}l_{3,0}\mathcal{X}_{n+4}(i^{3/4}) \\
i^{3/2}l_{4,0}\mathcal{X}_n(-i^{3/4}) & i^{3/2}l_{4,0}\mathcal{X}_{n+1}(-i^{3/4}) & i^{3/2}l_{4,0}\mathcal{X}_{n+2}(-i^{3/4}) & i^{3/2}l_{4,0}\mathcal{X}_{n+3}(-i^{3/4}) & i^{3/2}l_{4,0}\mathcal{X}_{n+4}(-i^{3/4}) \\
i^{5/2}l_{5,0}\mathcal{X}_n(i^{5/4}) & i^{5/2}l_{5,0}\mathcal{X}_{n+1}(i^{5/4}) & i^{5/2}l_{5,0}\mathcal{X}_{n+2}(i^{5/4}) & i^{5/2}l_{5,0}\mathcal{X}_{n+3}(i^{5/4}) & i^{5/2}l_{5,0}\mathcal{X}_{n+4}(i^{5/4}) \\
i^{5/2}l_{6,0}\mathcal{X}_n(-i^{5/4}) & i^{5/2}l_{6,0}\mathcal{X}_{n+1}(-i^{5/4}) & i^{5/2}l_{6,0}\mathcal{X}_{n+2}(-i^{5/4}) & i^{5/2}l_{6,0}\mathcal{X}_{n+3}(-i^{5/4}) & i^{5/2}l_{6,0}\mathcal{X}_{n+4}(-i^{5/4}) \\
i^{7/2}l_{7,0}\mathcal{X}_n(i^{7/4}) & i^{7/2}l_{7,0}\mathcal{X}_{n+1}(i^{7/4}) & i^{7/2}l_{7,0}\mathcal{X}_{n+2}(i^{7/4}) & i^{7/2}l_{7,0}\mathcal{X}_{n+3}(i^{7/4}) & i^{7/2}l_{7,0}\mathcal{X}_{n+4}(i^{7/4}) \\
i^{7/2}l_{8,0}\mathcal{X}_n(-i^{7/4}) & i^{7/2}l_{8,0}\mathcal{X}_{n+1}(-i^{7/4}) & i^{7/2}l_{8,0}\mathcal{X}_{n+2}(-i^{7/4}) & i^{7/2}l_{8,0}\mathcal{X}_{n+3}(-i^{7/4}) & i^{7/2}l_{8,0}\mathcal{X}_{n+4}(-i^{7/4}) \\
\hline
z^2x_n(z)I_2 & z^2x_{n+1}(z)I_2 & z^2x_{n+2}(z)I_2 & z^2x_{n+3}(z)I_2 & z^2x_{n+4}(z)I_2
\end{array}\right|.$$

Using (26), *each element of the above last quasi-determinant can be easily computed.*

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