Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 00 (1900), 000–000
DOI: 10.4171/ZAA/XXX

The equivalence theorem for logarithmic interpolation spaces in the quasi-Banach case

Blanca F. Besoy and Fernando Cobos

Abstract. We study the description by means of the J-functional of logarithmic interpolation spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ in the category of the p-normed quasi-Banach couples $(0 < p \le 1)$. When (A_0, A_1) is a Banach couple, it is known that the description changes depending on the relationship between q and $\mathbb{A}$. In our more general setting, the parameter p also has an important role as the results show.

Keywords. Logarithmic interpolation spaces; J-functional; K-functional; characteristic function

Mathematics Subject Classification (2010). 46M35,46B70

1. Introduction

The real interpolation method $(A_0, A_1)_{\theta,q}$ is an useful tool to work in PDEs, Harmonic Analysis, Approximation Theory, Function Spaces and Operator Theory. See the books by Butzer and Berens [9], Bergh and Löfström [4], Triebel [32], König [25], Bennett and Sharpley [3] and Brudnyĭ and Krugljak [8]. The most common definition of the real method is given in terms of Peetre's K-functional. But there is an equivalent description using the J-functional, which is also very useful. For instance, to establish the reiteration theorem, to study duality problems or interpolation of compactness by the real method, the J-representation of the K-spaces $(A_0, A_1)_{\theta,q}$ plays a central role (see [4] and the papers by Cwikel and Peetre [17] and Cobos, Kühn and Schonbek [15]).

Logarithmic perturbations $(A_0, A_1)_{\theta,q,\mathbb{A}}$ of the real method are also very useful in applications. Here $0 \le \theta \le 1$, $0 < q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and the

B.F. Besoy: Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain; blanca.f.besoy@ucm.es.
F. Cobos: Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain; cobos@mat.ucm.es (Corresponding author).

space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is defined similarly to $(A_0, A_1)_{\theta,q}$ but inserting the weight

$$\ell^{\mathbb{A}}(t) = \begin{cases} (1 + |\log t|)^{\alpha_0} & \text{if } 0 < t \leq 1, \\ (1 + |\log t|)^{\alpha_\infty} & \text{if } 1 < t < \infty, \end{cases}$$

with the K-functional. See the papers by Gustavsson [23], Doktorskii [18], Evans and Opic [21], Evans, Opic and Pick [22], Edmunds and Opic [20] and Cobos and Segurado [16]. For suitable choices of $\mathbb{A}$ we can allow that θ takes the extreme values 1 and 0. Spaces $(A_0, A_1)_{j,q,\mathbb{A}}$ are very close to A_j, $j = 0, 1$. They are also connected with the so-called limiting interpolation spaces for ordered couples which have been studied by Cobos, Fernández-Cabrera, Kühn and Ullrich [12], Cobos and Kühn [14], Cobos and Domínguez [10] and Cobos, Domínguez and Triebel [11], among other authors.

For couples of Banach spaces the description of $(A_0, A_1)_{1,q,\mathbb{A}}$ in terms of the J-functional has been studied by Cobos and Segurado [16] in the case $1 \leq q \leq \infty$ and by the present authors [5] when $0 < q \leq 1$. It turns out that if $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$, to go from the K-representation of $(A_0, A_1)_{1,q,\mathbb{A}}$ to the J-representation, one should correct the exponents of the logarithm by adding 1 when $1 \leq q \leq \infty$, but if $0 < q < 1$ then one should add $1/q$ and moreover the Gagliardo completions of A_0 and A_1 are involved. The extreme case $q = \infty$, $\alpha_0 = 0$ and $\alpha_\infty > 0$ has been studied by Besoy, Cobos and Fernández-Cabrera [7]. Then the J-representation is of another nature.

When $0 < q < 1$ the space $(A_0, A_1)_{1,q,\mathbb{A}}$ is not Banach but quasi-Banach. In fact, see for example the books by Bergh and Löfström [4], Triebel [32] and König [25] or the paper by Nilsson [27], because of applications it is suitable to work with the real method defined for p-normed quasi-Banach couples ($0 < p \leq 1$). Accordingly, we consider here the logarithmic spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ in that category of couples and investigate their description in terms of the J-functional.

Our results disclose that now the correction factor in the exponents of the logarithm depends on the relation among p, q and $\mathbb{A}$. Sometimes, although $(A_0, A_1)_{1,q,\mathbb{A}}$ admits a description as a J-space $(A_0, A_1)_{\Gamma;J}$, the sequence lattice Γ is not a logarithmic sequence space of the kind $\ell_q(2^{-m}\ell^{\mathbb{M}}(2^m))$ but a sequence of a different scale. In such cases we determine the best $\mathbb{M}$ and $\mathbb{B}$ such that

$$(A_0, A_1)_{\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m));J} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)_{\ell_q(2^{-m}\ell^{\mathbb{M}}(2^m));J} \, .$$

We start by recalling the basic properties of $(A_0, A_1)_{1,q,\mathbb{A}}$ in Section 2, where we also determine its characteristic function. Logarithmic J-spaces are introduced and studied in Section 3. Finally, in Section 4, we establish the equivalence theorems and the embedding theorems.

2. Preliminaries

Let $(A, \|\cdot\|_A)$ be a quasi-Banach space with constant $c_A \geq 1$ in the quasi-triangle inequality. Let $0 < p \leq 1$ such that $c_A = 2^{1/p-1}$. As it is shown in [26, §15.10] or [25, Proposition 1.c.5], there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality. We say that $(A, |||\cdot|||)$ is a p-normed quasi-Banach space. Note that if $0 < r < p$ then $(A, |||\cdot|||)$ is also an r-normed space. Conversely, if $(A, |||\cdot|||)$ is a p-normed space then it is quasi-normed with constant $2^{1/p-1}$.

Let $\mathcal{A}$ be a Hausdorff topological vector space and let A_j, $j = 0, 1$, be (p-normed) quasi-Banach spaces such that $A_j \hookrightarrow \mathcal{A}$, where $\hookrightarrow$ means continuous embedding. Then we say that $\overline{A} = (A_0, A_1)$ is a *(p-normed) quasi-Banach couple.*

For $t > 0$ and $a \in A_0 + A_1$, *Peetre's K-functional* is defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\}.$$

It is useful to consider also

$$K_p(t, a) = K_p(t, a; A_0, A_1) = \inf\{\left(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p\right)^{1/p} : a = a_0 + a_1, a_j \in A_j\}.$$

Peetre's J-functional is given by

$$J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in A_0 \cap A_1.$$

The functionals $K(t, \cdot)$ and $K_p(t, \cdot)$ are quasi-norms in $A_0 + A_1$ and $J(t, \cdot)$ is a quasi-norm in $A_0 \cap A_1$. We can take the same constant $c \geq 1$ in the quasi-triangle inequality for any $t > 0$. Moreover, $K(1, \cdot)$, $J(1, \cdot)$ coincide with the usual quasi-norms $\|\cdot\|_{A_0+A_1}$, $\|\cdot\|_{A_0\cap A_1}$ of $A_0 + A_1$ and $A_0 \cap A_1$, respectively. Note also that $K(t, \cdot)$ and $K_p(t, \cdot)$ are equivalent quasi-norms

$$K(t, a) \leq K_p(t, a) \leq 2^{1/p-1}K(t, a),\ a \in A_0 + A_1.$$

Another useful property is that if $\|\cdot\|_{A_j}$ is a p-norm for $j = 0, 1$, then $K_p(t, \cdot)$ and $J(t, \cdot)$ are p-norms.

Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. The *logarithmic interpolation space* $\overline{A}_{\theta,q,\mathbb{A}} = (A_0, A_1)_{\theta,q,\mathbb{A}}$ consists of all those $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{\overline{A}_{\theta,q,\mathbb{A}}} = \|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\sum_{m=-\infty}^{\infty} [2^{-\theta m}\ell^{\mathbb{A}}(2^m)K(2^m, a)]^q\right)^{1/q}$$

(the sum should be replaced by the supremum when $q = \infty$). Here $\ell(t) = 1 + |\log t|$ and

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 \leq t < \infty. \end{cases}$$

See [5, 16, 21, 22]. When $\mathbb{A} = (0,0)$ and $0 < \theta < 1$ then $(A_0, A_1)_{\theta,q,(0,0)} = (A_0, A_1)_{\theta,q}$ is the *real interpolation space* (see [3, 4, 8, 9, 32]). For $\mathbb{A} \neq (0,0)$ and $0 < \theta < 1$ the space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is a special case of the real method with a function parameter (see [23, 24, 31]). Note also that the equality

$$K(t, a; A_0, A_1) = tK(t^{-1}, a; A_1, A_0), \; a \in A_0 + A_1,$$

implies that $(A_0, A_1)_{0,q,(\alpha_0,\alpha_\infty)} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)}$.

Subsequently, we are interested in the spaces that arise when $\theta = 1$. On the contrary to the case of the real method where $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q}$, now it may happen that $(A_0, A_1)_{1,q,\mathbb{A}} = \{0\}$. To avoid it, we assume that

$$\begin{cases} \alpha_0 + 1/q < 0 & \text{if } 0 < q < \infty, \\ \alpha_0 \leq 0 & \text{if } q = \infty, \end{cases} \tag{2.1}$$

(see [22, Theorem 2.2]). Under this assumption, the space $(A_0, A_1)_{1,q,\mathbb{A}}$ is an *intermediate space* with respect to $\overline{A}$ in the sense that the following embeddings hold

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow A_0 + A_1.$$

Moreover, if $\overline{B} = (B_0, B_1)$ is another quasi-Banach couple and the operator $T \in \mathcal{L}(A_0 + A_1, B_0 + B_1)$ is a linear operator such that $T : A_j \to B_j$ is bounded for $j = 0, 1$, then the restriction $T : \overline{A}_{1,q,\mathbb{A}} \to \overline{B}_{1,q,\mathbb{A}}$ is also bounded. So, in the terminology of [4, 32], the logarithmic method with parameters $1, q, \mathbb{A}$ is an *interpolation functor* in the category of all quasi-Banach couples.

If A is a quasi-Banach space intermediate with respect to $\overline{A}$ then we write A° for the closure of $A_0 \cap A_1$ in A.

It follows from the fundamental lemma (see [4, Lemma 3.3.2]) that $a \in (A_0 + A_1)^\circ$ if and only if

$$\min(1, 1/t)K(t, a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty.$$

Assume that, in addition to (2.1), we have

$$\begin{cases} \alpha_\infty + 1/q \geq 0 & \text{if } 0 < q < \infty, \\ \alpha_\infty > 0 & \text{if } q = \infty. \end{cases} \tag{2.2}$$

Then, given any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, we have

$$\left(\sum_{m=-\infty}^{\infty} [2^{-m} \ell^{\mathbb{A}}(2^m) K(2^m, a)]^q \right)^{1/q} < \infty \qquad \text{but} \qquad \sum_{m=1}^{\infty} \ell^{\alpha_\infty q}(2^m) = \infty$$

and $\sum_{m=-\infty}^{0} [2^{-m} \ell^{\alpha_0}(2^m)]^q = \infty$. Therefore, $\min(1, 1/t)K(t, a) \to 0$ as $t \to 0$ and as $t \to \infty$. Whence

$$(A_0, A_1)_{1,q,\mathbb{A}} \subseteq (A_0 + A_1)^\circ. \tag{2.3}$$

Remark 2.1. Note that if (2.2) does not hold then (2.3) may fail. An example can be found in [5, Example 2.1].

Subsequently, if X and Y are quantities depending on certain parameters some of them being the significant parameters in our reasoning, we write $X \lesssim Y$ if $X \le cY$ with a constant $c > 0$ independent of the significant parameters. We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

In order to give an example let (Ω, μ) be a σ-finite measure space and let $0 < r < \infty$. According to [4, Theorem 5.2.1] we have that

$$K(t, f; L_r(\Omega), L_\infty(\Omega)) = \Big(\int_0^{t^r} \big(f^*(s)\big)^r ds \Big)^{1/r},$$

where f^* is the non-increasing rearrangement of f. Let $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (2.1). By a change of variable and using Hardy's inequality we obtain

$$\begin{aligned} \|f\|_{(L_r(\Omega), L_\infty(\Omega))_{1,q,\mathbb{A}}} &\sim \Big(\int_0^\infty \Big[t^{-1} \ell^{\mathbb{A}}(t) \Big(\int_0^{t^r} (f^*(s))^r ds \Big)^{1/r} \Big]^q \frac{dt}{t} \Big)^{1/q} \\ &\sim \Big(\int_0^\infty \Big[u^{-1/r} \ell^{\mathbb{A}}(u) \Big(\int_0^{u} (f^*(s))^r ds \Big)^{1/r} \Big]^q \frac{du}{u} \Big)^{1/q} \\ &\sim \Big(\int_0^\infty [\ell^{\mathbb{A}}(u) f^*(u)]^q \frac{du}{u} \Big)^{1/q}. \end{aligned}$$

This yields that $(L_r(\Omega), L_\infty(\Omega))_{1,q,\mathbb{A}}$ coincides, with equivalence of quasi-norms, with the *generalized Lorentz-Zygmund space* $L_{\infty,q;\mathbb{A}}(\Omega)$. We refer to [19, 28] for properties of these function spaces.

If $(A, \|\cdot\|_A)$ is a quasi-Banach space and $s > 0$, we write sA for the space A endowed with the quasi-norm $s\,\|\cdot\|_A$.

The characteristic function $\Phi_{\mathcal{F}}$ of an interpolation functor $\mathcal{F}$ is the function defined by

$$\mathcal{F}(\mathbb{R}, (1/t)\mathbb{R}) = (1/\Phi_{\mathcal{F}}(t))\mathbb{R},\ t > 0$$

(see [8, 24, 29]). Next we describe the characteristic function $\Phi_{q,\mathbb{A}}$ of $(\cdot,\cdot)_{1,q,\mathbb{A}}$.

In what follows, we put $\ell\ell(t) = 1 + \log(1 + |\log t|)$ and define $\ell\ell^{\mathbb{A}}(t)$ similarly to $\ell^{\mathbb{A}}(t)$.

If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we write $\mathbb{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$ and $\lambda\mathbb{A} = (\lambda\alpha_0, \lambda\alpha_\infty)$.

The following auxiliary result is useful.

Lemma 2.2. *Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (2.1). Put*

$$\mathrm{v}_{q,\mathbb{A}}(2^k) = \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{m-k}) 2^{-m} \ell^{\mathbb{A}}(2^m)]^q \right)^{1/q},\ k \in \mathbb{Z}.$$

Then we have for $0 < q < \infty$

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k}\ell^{\mathbb{A}+1/q}(2^k) & \text{if } \alpha_\infty + 1/q > 0, \\ 2^{-k}\ell^{\mathbb{A}+1/q}(2^k)\ell\ell^{(0,1/q)}(2^k) & \text{if } \alpha_\infty + 1/q = 0, \\ 2^{-k}\ell^{(\alpha_0+1/q,0)}(2^k) & \text{if } \alpha_\infty + 1/q < 0, \end{cases}$$

and for $q = \infty$

$$\mathrm{v}_{\infty,q}(2^k) \sim \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k) & \text{if } \alpha_\infty \geq 0, \\ 2^{-k}\ell^{(\alpha_0,0)}(2^k) & \text{if } \alpha_\infty < 0. \end{cases}$$

Proof. If $0 < q < \infty$ then the result follows proceeding as in [5, Lemma 3.1]. Suppose $q = \infty$. We have

$$\begin{aligned} \mathrm{v}_{\infty,q}(2^k) &= \sup_{m\in\mathbb{Z}}\{\min(1, 2^{m-k})2^{-m}\ell^{\mathbb{A}}(2^m)\} \\ &= \max\{\sup_{m\leq k}\{2^{-k}\ell^{\mathbb{A}}(2^m)\}, \sup_{m\geq k}\{2^{-m}\ell^{\mathbb{A}}(2^m)\}\} \\ &\sim 2^{-k}\sup_{m\leq k}\{\ell^{\mathbb{A}}(2^m)\}. \end{aligned}$$

If $k \leq 0$, since $\alpha_0 \leq 0$, we get

$$\mathrm{v}_{\infty,q}(2^k) \sim 2^{-k}\sup_{m\leq k}\{(1-\log 2^m)^{\alpha_0}\} = 2^{-k}\ell^{\alpha_0}(2^k).$$

If $k > 0$, we have

$$\begin{aligned} \mathrm{v}_{\infty,\mathbb{A}}(2^k) &\sim 2^{-k}\max\left(\sup_{m\leq 0}\{(1-\log t)^{\alpha_0}\}, \sup_{0\leq m\leq k}\{(1+\log t)^{\alpha_\infty}\}\right) \\ &= 2^{-k}\max(1, \sup_{0\leq m\leq k}\{(1+\log 2^m)^{\alpha_\infty}\}) \\ &= \begin{cases} 2^{-k}\ell^{\alpha_\infty}(2^k) & \text{if } \alpha_\infty \geq 0, \\ 2^{-k} & \text{if } \alpha_\infty < 0. \end{cases} \end{aligned}$$

This completes the proof. □

Proposition 2.3. *Let* $(A, \|\cdot\|_A)$ *be a* p*-normed quasi-Banach space and let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $0 < q \leq \infty$ *satisfying (2.1). Then, with equivalence of quasi-norms, we have*

$$(A, 2^{-k}A)_{1,q,\mathbb{A}} = \mathrm{v}_{q,\mathbb{A}}(2^k)A,$$

where the constants in the equivalence depend on p *but they are independent of* $k \in \mathbb{Z}$ *and of the concrete* p*-normed space* A*.*

Proof. Let $0 < p \le 1$ such that A is p-normed. Take any $a \in A$ and $m \in \mathbb{Z}$. It is not hard to check that $K_p(2^m, a) = \min(1, 2^{m-k}) \|a\|_A$. Therefore

$$\begin{aligned} \|a\|_{(A,2^{-k}A)_{1,q,\mathbb{A}}} &\sim \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{m-k}) 2^{-m} \ell^{\mathbb{A}}(2^m)]^q \right)^{1/q} \|a\|_A \\ &= \mathrm{v}_{q,\mathbb{A}}(2^k) \|a\|_A . \end{aligned}$$

□

Corollary 2.4. *Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $0 < q \le \infty$ *satisfying (2.1). Then* $\Phi_{q,\mathbb{A}}(2^k) \sim \mathrm{v}_{q,\mathbb{A}}(2^k)^{-1}$.

3. J-spaces

Next we recall the general real interpolation method realized by means of the J-functional. See the monographs by Peetre [30] and Brudnyĭ and Krugljak [8]. Since we are interested in quasi-Banach couples, following the paper by Nilsson [27], we work with the general J-method described in discrete way.

By a *quasi-Banach sequence lattice* $(\Gamma, \|\cdot\|_\Gamma)$ we mean a quasi-Banach space of real valued sequences with $\mathbb{Z}$ as index set and satisfying the following two properties:

i) Γ contains all sequences with only finitely many non-zero coordinates.
ii) Whenever $|\xi_m| \le |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \le \|(\eta_m)\|_\Gamma$.

For $0 < q < \infty$, the space ℓ_q of q-summable sequences and the space ℓ_∞ of bounded sequences are examples of quasi-Banach sequence lattices. Another example is the space $\ell_q(2^{-m})$, formed by all sequences (ξ_m) such that $(2^{-m}\xi_m) \in \ell_q$.

Let $0 < p \le 1$. The quasi-Banach sequence lattice $(\Gamma, \|\cdot\|_\Gamma)$ is said to be (p,J)-*non-trivial* if $\Gamma \hookrightarrow \ell_p + \ell_p(2^{-m})$, that is,

$$\sup \left\{ \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{-m}) |\xi_m|]^p \right)^{1/p} : \|(\xi_m)\|_\Gamma \le 1 \right\} < \infty$$

(see [27, p. 294]). Clearly, if Γ is (p,J)-non-trivial then Γ is (r,J)-non-trivial for any $p \le r \le 1$.

Let Γ be a (p,J)-non-trivial quasi-Banach sequence lattice and let $\overline{A} = (A_0, A_1)$ be a p-normed quasi-Banach couple. The *J-space* $\overline{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is formed by all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $(u_m) \subseteq A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. The quasi-norm on $\overline{A}_{\Gamma;J}$ is

$$\|a\|_{\overline{A}_{\Gamma;J}} = \|a\|_{(A_0,A_1)_{\Gamma;J}} = \inf\{\|(J(2^m, u_m))\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m\}$$

(see [27]). It holds

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\Gamma;J} \hookrightarrow A_0 + A_1. \tag{3.1}$$

Moreover, $(\cdot,\cdot)_{\Gamma;J}$ is an interpolation functor in the category of all p-normed quasi-Banach couples.

Remark 3.1. If Γ is not (p, J)-non-trivial then the second embedding in (3.1) may fail. Indeed, consider the p-normed quasi-Banach couple $(\ell_p, \ell_p(2^{-m}))$. If Γ is not (p, J)-non-trivial then given any $N \in \mathbb{N}$, there is $x = (x_m) \in \Gamma$ and $L_N \in \mathbb{N}$ such that

$$\|x\|_\Gamma \le 1 \qquad \text{and} \qquad \Big(\sum_{|m| \le L_N} \big[\min(1, 2^{-m})\, |x_m|\big]^p \Big)^{1/p} > N.$$

Put $y_m = x_m$ if $|m| \le L_N$ and $y_m = 0$ otherwise, and let $y = (y_m)$. Let $e_k = (\delta_m^k)$ where δ_m^k is the Kronecker's delta and write $u_k = y_k e_k$, $k \in \mathbb{Z}$. Then we have that $y = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $\ell_p + \ell_p(2^{-m})$) with $J(2^k, u_k; \ell_p, \ell_p(2^{-m})) = |y_k|$. Hence

$$\|y\|_{(\ell_p,\ell_p(2^{-m}))_{\Gamma;J}} \le \|y\|_\Gamma \le \|x\|_\Gamma \le 1$$

and

$$\|y\|_{\ell_p+\ell_p(2^{-m})} \sim \Big(\sum_{|m| \le L_N} \big[\min(1, 2^{-m})\, |x_m|\big]^p \Big)^{1/p} > N.$$

This yields that $(\ell_p, \ell_p(2^{-m}))_{\Gamma;J}$ is not continuously embedded in $\ell_p + \ell_p(2^{-m})$.

Note that a direct consequence of the definition of $(A_0, A_1)_{\Gamma;J}$ is that

$$(A_0, A_1)_{\Gamma;J} \subseteq (A_0 + A_1)^\circ. \tag{3.2}$$

Next we take $\mathbb{A} \in \mathbb{R}^2$, $0 < q \le \infty$ satisfying (2.1) and consider the K-method $(\cdot,\cdot)_{1,q,\mathbb{A}}$. We want to describe $(\cdot,\cdot)_{1,q,\mathbb{A}}$ as a J-functor.

Remark 3.2. It follows from Remark 2.1 and (3.2) that if (2.2) fails then $(\cdot,\cdot)_{1,q,\mathbb{A}}$ does not admit a representation as a J-method.

If (2.1) and (2.2) holds, then for any p-normed quasi-Banach couple (A_0, A_1) we have with equivalent quasi-norms

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{\Lambda;J} \ \text{ where } \Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}} \tag{3.3}$$

(see [27, Theorem 3.19] and [6, Theorem 2.1]). Here $A_j^\sim$ stands for the *Gagliardo completion* of A_j, formed by all those $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{A_j^\sim} = \sup\{t^{-j} K(t, a) : t > 0\}, \ j = 0, 1$$

(see [3, 4]).

The following result gives a more precise description of the quasi-norm of the sequence lattice Λ of (3.3).

Lemma 3.3. *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (2.1) and (2.2). If $x = (x_m)_{m\in\mathbb{Z}}$ then*

$$\|x\|_{(\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}} \sim \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=m}^{0} 2^{-kp} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} + \Big(\sum_{m=0}^{\infty} \big[\ell^{\alpha_\infty}(2^m)(\sum_{k=m}^{\infty} 2^{-kp} |x_k|^p)^{1/p}\big]^q\Big)^{1/q}.$$

Proof. Since $K_p(2^k, x; \ell_p, \ell_p(2^{-m})) = (\sum_{m=-\infty}^{\infty}[\min(1, 2^{k-m}) |x_m|]^p)^{1/p}$ we have

$$\begin{aligned}
\|x\|_{(\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}} &\sim \Big(\sum_{m=-\infty}^{\infty} \big[2^{-m}\ell^{\mathbb{A}}(2^m)(\sum_{k=-\infty}^{\infty} \min(1, 2^{m-k})^p |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\sim \Big(\sum_{m=-\infty}^{0} \big[2^{-m}\ell^{\alpha_0}(2^m)(\sum_{k=-\infty}^{m} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=m}^{0} 2^{-kp} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=0}^{\infty} 2^{-kp} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{m=0}^{\infty} \big[2^{-m}\ell^{\alpha_\infty}(2^m)(\sum_{k=-\infty}^{0} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{m=0}^{\infty} \big[2^{-m}\ell^{\alpha_\infty}(2^m)(\sum_{k=0}^{m} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{m=0}^{\infty} \big[\ell^{\alpha_\infty}(2^m)(\sum_{k=m}^{\infty} 2^{-kp} |x_k|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\quad = S_1 + S_2 + S_3 + S_4 + S_5 + S_6.
\end{aligned}$$

Having in mind (2.1), we get $S_3 \sim (\sum_{k=0}^{\infty} 2^{-kp} |x_k|^p)^{1/p}$. Hence, the term in S_6 with $m = 0$ shows that $S_3 \lesssim S_6$. On the other hand,

$$S_4 \sim (\sum_{k=-\infty}^{0} |x_k|^p)^{1/p} \leq S_1.$$

Next we check that $S_1 \lesssim S_2$ and $S_5 \lesssim S_6$. Assume first that $0 < q \leq p$. Then

$$S_1 \leq \Big(\sum_{m=-\infty}^{0} 2^{-mq}\ell^{\alpha_0 q}(2^m) \sum_{k=-\infty}^{m} |x_k|^q \Big)^{1/q} = \Big(\sum_{k=-\infty}^{0} |x_k|^q \sum_{m=k}^{0} 2^{-mq}\ell^{\alpha_0 q}(2^m)\Big)^{1/q}$$

$$\lesssim \Big(\sum_{k=-\infty}^{0} |x_k|^q \, 2^{-kq} \ell^{\alpha_0 q}(2^k) \Big)^{1/q} \leq S_2.$$

Similarly,

$$S_5 \leq \Big(\sum_{m=0}^{\infty} 2^{-mq} \ell^{\alpha_\infty q}(2^m) \sum_{k=0}^{m} |x_k|^q \Big)^{1/q} = \Big(\sum_{k=0}^{\infty} |x_k|^q \sum_{m=k}^{\infty} 2^{-mq} \ell^{\alpha_\infty q}(2^m) \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=0}^{\infty} |x_k|^q \, 2^{-kq} \ell^{\alpha_\infty q}(2^k) \Big)^{1/q} \leq S_6.$$

Suppose now that $0 < p < q < \infty$. Let $0 < \varepsilon < 1$. Using Hölder's inequality we get

$$\Big(\sum_{k=-\infty}^{m} |x_k|^p \Big)^{q/p} \leq \Big(\sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q} \, |x_k|^q \Big) \Big(\sum_{k=-\infty}^{m} 2^{k(1-\varepsilon)pq/(q-p)} \Big)^{(q-p)/p}$$
$$\sim 2^{m(1-\varepsilon)q} \sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q} \, |x_k|^q \, .$$

Hence,

$$S_1 \lesssim \Big(\sum_{m=-\infty}^{0} 2^{-m\varepsilon q} \ell^{\alpha_0 q}(2^m) \sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q} \, |x_k|^q \Big)^{1/q}$$
$$= \Big(\sum_{k=-\infty}^{0} 2^{-k(1-\varepsilon)q} \, |x_k|^q \sum_{m=k}^{0} 2^{-m\varepsilon q} \ell^{\alpha_0 q}(2^m) \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=-\infty}^{0} 2^{-kq} \, |x_k|^q \, \ell^{\alpha_0 q}(2^k) \Big)^{1/q} \leq S_2.$$

As for S_5, for the interior sum we obtain

$$\Big(\sum_{k=0}^{m} |x_k|^p \Big)^{q/p} \leq \Big(\sum_{k=0}^{m} 2^{-kq\varepsilon} \, |x_k|^q \Big) \Big(\sum_{k=0}^{m} 2^{k\varepsilon pq/(q-p)} \Big)^{(q-p)/p}$$
$$\sim 2^{m\varepsilon q} \sum_{k=0}^{m} 2^{-kq\varepsilon} \, |x_k|^q \, .$$

Therefore,

$$S_5 \lesssim \Big(\sum_{m=0}^{\infty} 2^{-mq(1-\varepsilon)} \ell^{\alpha_\infty q}(2^m) \sum_{k=0}^{m} 2^{-kq\varepsilon} \, |x_k|^q \Big)^{1/q}$$

$$= \Big(\sum_{k=0}^{\infty} 2^{-kq\varepsilon} |x_k|^q \sum_{m=k}^{\infty} 2^{-mq(1-\varepsilon)} \ell^{\alpha_\infty q}(2^m) \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=0}^{\infty} 2^{-kq} |x_k|^q \, \ell^{\alpha_\infty q}(2^k) \Big)^{1/q} \le S_6.$$

The case $q = \infty$ can be treated analogously. This completes the proof. □

When $p = q$ the space Λ is a weighted ℓ_q-space as we show next.

Lemma 3.4. *Let* $0 < q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *satisfying* $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$. *Then we have, with equivalence of quasi-norms*

$$(\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}} = \begin{cases} \ell_q(2^{-m} \ell^{\mathbb{A}+1/q}(2^m)) & \text{if } 0 < \alpha_\infty + 1/q, \\ \ell_q(2^{-m} \ell^{\mathbb{A}+1/q}(2^m) \ell\ell^{(0,1/q)}(2^m)) & \text{if } 0 = \alpha_\infty + 1/q. \end{cases}$$

Proof. According to Lemma 3.3, we obtain

$$\|x\|_{(\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}}} \sim \left(\sum_{m=-\infty}^{0} \ell^{\alpha_0 q}(2^m) \sum_{k=m}^{0} 2^{-kq} |x_k|^q \right)^{1/q} + \left(\sum_{m=0}^{\infty} \ell^{\alpha_\infty q}(2^m) \sum_{k=m}^{\infty} 2^{-kq} |x_k|^q \right)^{1/q}.$$

Changing the order of summation, we derive

$$\|x\|_{(\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}}} \sim \left(\sum_{k=-\infty}^{0} 2^{-kq} |x_k|^q \sum_{m=-\infty}^{k} \ell^{\alpha_0 q}(2^m) \right)^{1/q} + \left(\sum_{k=0}^{\infty} 2^{-kq} |x_k|^q \sum_{m=0}^{k} \ell^{\alpha_\infty q}(2^m) \right)^{1/q}.$$

Since

$$\sum_{m=-\infty}^{k} \ell^{\alpha_0 q}(2^m) \sim \ell^{\alpha_0 q+1}(2^k) \text{ if } \alpha_0 + 1/q < 0$$

and

$$\sum_{m=0}^{k} \ell^{\alpha_\infty q}(2^m) \sim \begin{cases} \ell^{\alpha_\infty q+1}(2^k) & \text{if } 0 < \alpha_\infty + 1/q, \\ \ell\ell(2^k) & \text{if } 0 = \alpha_\infty + 1/q, \end{cases}$$

the result follows. □

A consequence of Lemma 3.4 is that the space $\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m))$ (respectively, $\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)\ell\ell^{(0,1/q)}(2^m)))$ is continuously embedded in $\ell_q+\ell_q(2^{-m})$, that is to say, it is a (q,J)-non-trivial lattice. Now we characterize this property for $p\neq q$ and $\mathbb{B}\neq(0,1/q)$.

In what follows, given $0<p\le 1$ and $0<q\le\infty$, we put

$$q^* = \begin{cases} \infty & \text{if } 0<q\le p,\\ \frac{pq}{q-p} & \text{if } 0<p<q<\infty,\\ p & \text{if } q=\infty. \end{cases}$$

Observe that if $p=1$ and $1\le q\le\infty$, then $q^*=q'$ where $1/q+1/q'=1$.

Lemma 3.5. *Let $0<p\le 1$, $0<q\le\infty$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)$, $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$. The necessary and sufficient condition for the continuous embedding*

$$\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\hookrightarrow \ell_p+\ell_p(2^{-m})$$

is that

$$\begin{cases} \alpha_\infty>0, \text{ or } \alpha_\infty=0 \text{ and } \beta_\infty\ge 0 & \text{if } 0<q\le p,\\ \alpha_\infty+\frac{1}{q}-\frac{1}{p}>0, \text{ or } \alpha_\infty+\frac{1}{q}-\frac{1}{p}=0 \text{ and } \beta_\infty+\frac{1}{q}-\frac{1}{p}>0 & \text{if } p<q\le\infty. \end{cases} \tag{3.4}$$

Proof. Take any $x=(x_m)\in\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$. Using Hölder's inequality we get

$$\begin{aligned} \|x\|_{\ell_p+\ell_p(2^{-m})} &\sim \Big(\sum_{m=-\infty}^{\infty}\min(1,2^{-m})^p\,|x_m|^p\Big)^{1/p}\\ &\le \big\|(\min(1,2^{-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m))}\,\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))}\\ &= \Big(\sum_{m=-\infty}^{0}[2^m\ell^{-\alpha_0}(2^m)\ell\ell^{-\beta_0}(2^m)]^{q^*}+\sum_{m=0}^{\infty}[\ell^{-\alpha_\infty}(2^m)\ell\ell^{-\beta_\infty}(2^m)]^{q*}\Big)^{1/q^*}\\ &\qquad\times\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))}. \end{aligned}$$

The last sums are finite by (3.4). Therefore we obtain that

$$\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\hookrightarrow \ell_p+\ell_p(2^{-m}).$$

Conversely, if (3.4) does not hold then

$$\begin{cases} \alpha_\infty<0, \text{ or } \alpha_\infty=0 \text{ and } \beta_\infty<0 & \text{if } 0<q\le p,\\ \alpha_\infty+\frac{1}{q}-\frac{1}{p}<0, \text{ or } \alpha_\infty+\frac{1}{q}-\frac{1}{p}=0 \text{ and } \beta_\infty+\frac{1}{q}-\frac{1}{p}\le 0 & \text{if } p<q\le\infty, \end{cases}$$

and so $\|(\min(1,2^{-m}))\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m))} = \infty$. If follows that for any $N \in \mathbb{N}$, there is $L_N \in \mathbb{N}$ such that

$$N < \Big(\sum_{|m|\le L_N} [\min(1,2^{-m})2^m \ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q^*}\Big)^{1/q^*}.$$

Assume now that $0 < p < q < \infty$. Let

$$y_m = \begin{cases} \min(1,2^{-m})^{\frac{p}{q-p}} 2^{m\frac{q}{q-p}} \ell^{-\frac{q}{q-p}\mathbb{A}}(2^m)\ell\ell^{-\frac{q}{q-p}\mathbb{B}}(2^m) & \text{if } |m| \le L_N, \\ 0 & \text{otherwise,} \end{cases}$$

put $y = (y_m)$ and $x = \big(y_m/\|y\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))}\big)$. Then x belongs to the unit ball of $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ but

$$\begin{aligned} \|x\|_{\ell_p+\ell_p(2^{-m})} &\sim \Big(\sum_{m=-\infty}^{\infty} [\min(1,2^{-m})\,|x_m|]^p\Big)^{1/p} \\ &= \frac{\Big(\sum_{|m|\le L_N}[\min(1,2^{-m})2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q^*}\Big)^{1/p}}{\Big(\sum_{|m|\le L_N}[\min(1,2^{-m})2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q^*}\Big)^{1/q}} \\ &= \Big(\sum_{|m|\le L_N} [\min(1,2^{-m})2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q^*}\Big)^{1/q^*} > N. \end{aligned}$$

Hence, $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ is not continuously embedded in $\ell_p + \ell_p(2^{-m})$.

In the case $q = \infty$, where $q^* = p$, we put

$$y_m = \begin{cases} 2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m) & \text{if } |m| \le L_N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|(y_m)\|_{\ell_\infty(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))} = 1$ but

$$\|(y_m)\|_{\ell_p+\ell_p(2^{-m})} \sim \Big(\sum_{|m|\le L_N} [\min(1,2^{-m})2^m\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^p\Big)^{1/p} > N.$$

Finally, if $0 < q < p$ we have $q^* = \infty$. We may assume that L_N satisfies that

$$\min(1,2^{-L_N})2^{L_N}\ell^{-\mathbb{A}}(2^{L_N})\ell\ell^{-\mathbb{B}}(2^{L_N}) > N.$$

Let $e_{L_N} = (\delta_m^{L_N})$ and $x = 2^{L_N}\ell^{-\mathbb{A}}(2^{L_N})\ell\ell^{-\mathbb{B}}(2^{L_N})e_{L_N}$. Then we have

$$\begin{aligned} &\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))} = 1 \text{ and} \\ &\|x\|_{\ell_p+\ell_p(2^{-m})} = \min(1,2^{-L_N})2^{L_N}\ell^{-\mathbb{A}}(2^{L_N})\ell\ell^{-\mathbb{B}}(2^{L_N}) > N. \end{aligned}$$

This completes the proof. □

Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$ satisfying (3.4). Given any p-normed quasi-Banach couple (A_0, A_1), the *logarithmic J-spaces* are defined by

$$\overline{A}^J_{1,q,\mathbb{A}} = (A_0, A_1)^J_{1,q,\mathbb{A}} = (A_0, A_1)_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m));J}$$

and

$$\overline{A}^J_{1,q,\mathbb{A},\mathbb{B}} = (A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}} = (A_0, A_1)_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m));J} .$$

We close this section by computing the characteristic function of the interpolation functor $(\cdot,\cdot)^J_{1,q,\mathbb{A}}$.

Lemma 3.6. *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that*

$$\begin{cases} \alpha_\infty \geq 0 & \text{if } 0 < q \leq p, \\ \alpha_\infty + \frac{1}{q} - \frac{1}{p} > 0 & \text{if } 0 < p < q \leq \infty. \end{cases} \tag{3.5}$$

For $k \in \mathbb{Z}$, put

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \sup\left\{ \Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})\, |x_m|]^p \Big)^{1/p} : \|(x_m)\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} \leq 1 \right\}.$$

Then we have

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \big\|(\min(1, 2^{k-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))} .$$

Proof. Let $x = (x_m) \in \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))$ with $\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} \leq 1$. Applying Hölder's inequality we get

$$\Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})\, |x_m|]^p \Big)^{1/p} \leq \big\|(\min(1, 2^{k-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))} \|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))}$$

Therefore, $\mathrm{u}_{q,\mathbb{A},p}(2^k) \leq \big\|(\min(1, 2^{k-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))}$.

Conversely, if $0 < q \leq p$, so $q^* = \infty$, given any $\varepsilon > 0$ we can find $n \in \mathbb{Z}$ such that

$$\min(1, 2^{k-n})2^n\ell^{-\mathbb{A}}(2^n) \geq \big\|(\min(1, 2^{k-m}))\big\|_{\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))} - \varepsilon.$$

Take $x = 2^n\ell^{-\mathbb{A}}(2^n)e_n$. Clearly $\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} = 1$. Moreover

$$\begin{aligned} \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})\, |x_m|]^p \right)^{1/p} &= \min(1, 2^{k-n})2^n\ell^{-\mathbb{A}}(2^n) \\ &> \big\|(\min(1, 2^{k-m}))\big\|_{\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))} - \varepsilon. \end{aligned}$$

Whence $\mathrm{u}_{q,\mathbb{A},p}(2^k) = \big\|(\min(1, 2^{k-m}))\big\|_{\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))}$.

Suppose now $0 < p < q < \infty$. Let

$$y_m = \min(1, 2^{k-m})^{\frac{p}{q-p}} 2^{m\frac{q}{q-p}} \ell^{-\frac{q}{q-p}\mathbb{A}}(2^m), \ m \in \mathbb{Z},$$

and $y = (y_m)$. Then

$$\|y\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} = \big\|(\min(1, 2^{k-m}))\big\|^{p/(q-p)}_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))}.$$

Put

$$x_m = \frac{y_m}{\|(\min(1, 2^{k-m}))\|^{p/(q-p)}_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))}} \text{ and } x = (x_m).$$

Then, $\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} = 1$ and

$$\begin{aligned}\Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})\, |x_m|]^p\Big)^{1/p} &= \frac{\Big(\sum_{m=-\infty}^{\infty}[\min(1, 2^{k-m})2^m\ell^{-\mathbb{A}}(2^m)]^{q^*}\Big)^{1/p}}{\Big(\sum_{m=-\infty}^{\infty}[\min(1, 2^{k-m})2^m\ell^{-\mathbb{A}}(2^m)]^{q^*}\Big)^{1/q}} \\ &= \big\|(\min(1, 2^{k-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))}.\end{aligned}$$

Consequently,

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \big\|(\min(1, 2^{k-m}))\big\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))}.$$

Finally, assume that $q = \infty$ so $q^* = p$. Let $x_m = 2^m\ell^{-\mathbb{A}}(2^m)$, $m \in \mathbb{Z}$. We have $\|(x_m)\|_{\ell_\infty(2^{-m}\ell^{\mathbb{A}}(2^m))} = 1$ and

$$\Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})\, |x_m|]^p\Big)^{1/p} = \Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m})2^m\ell^{-\mathbb{A}}(2^m)]^p\Big)^{1/p}.$$

This yields that

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \big\|(\min(1, 2^{k-m}))\big\|_{\ell_p(2^m\ell^{-\mathbb{A}}(2^m))}.$$

□

Proposition 3.7. *Let A be a p-normed quasi-Banach space, let $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (3.5). For any $k \in \mathbb{Z}$, we have*

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} \hookrightarrow \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1}A,$$

being the norm of the embedding less than or equal to 1. Moreover, if in addition we suppose that $p = 1$ or $0 < q \le p$, then we have with equal quasi-norms

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1}A.$$

Proof. Observe that $J(2^m, u; A, 2^{-k}A) = \max(1, 2^{m-k}) \|u\|_A$. If $(u_m) \subseteq A$ and $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in A), we obtain

$$\begin{aligned}
\|a\|_A &\leq \Big(\sum_{m=-\infty}^{\infty} \|u_m\|_A^p \Big)^{1/p} = \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m}) J(2^m, u_m)]^p \right)^{1/p} \\
&\leq \left(\sum_{m=-\infty}^{\infty} \Big[\min(1, 2^{k-m}) \frac{J(2^m, u_m)}{\|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))}} \Big]^p \right)^{1/p} \\
&\qquad \times \|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} \\
&\leq \mathrm{u}_{q,\mathbb{A},p}(2^k) \, \|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} \, .
\end{aligned}$$

This yields that the embedding $(A, 2^{-k}A)^J_{1,q,\mathbb{A}} \hookrightarrow \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1} A$ has a norm less than or equal to 1.

Conversely, if $p = 1$, given any $\varepsilon > 0$ we can find $x = (x_m)$ such that $\|x\|_{\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))} \leq 1$ and

$$\mathrm{u}_{q,\mathbb{A},1}(2^k) - \varepsilon < \sum_{m=-\infty}^{\infty} \min(1, 2^{k-m}) \, |x_m| \, .$$

We can represent any $a \in A$ as $a = \sum_{m=-\infty}^{\infty} u_m$ with

$$u_m = \frac{\min(1, 2^{k-m}) \, |x_m|}{\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m}) \, |x_m|} a, \; m \in \mathbb{Z}.$$

Then

$$J(2^m, u_m) = \frac{|x_m|}{\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m}) \, |x_m|} \, \|a\|_A \leq \frac{|x_m|}{\mathrm{u}_{q,\mathbb{A},1}(2^k) - \varepsilon} \, \|a\|_A \, .$$

Whence

$$\|a\|_{(A, 2^{-k}A)^J_{1,q,\mathbb{A}}} \leq (\mathrm{u}_{q,\mathbb{A},1}(2^k) - \varepsilon)^{-1} \, \|a\|_A \, .$$

This yields that the embedding $\mathrm{u}_{q,\mathbb{A},1}(2^k)^{-1} A \hookrightarrow (A, 2^{-k}A)^J_{1,q,\mathbb{A}}$ has norm less than or equal to 1.

Suppose now that $0 < q \leq p$. Then $q^* = \infty$. Given any $\varepsilon > 0$ we can find $n \in \mathbb{Z}$ such that

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon < 2^n \ell^{-\mathbb{A}}(2^n) \min(1, 2^{k-n}).$$

For any $a \in A$ we obtain

$$J(2^n, a) = \frac{1}{\min(1, 2^{k-n})} \, \|a\|_A < \big(\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon\big)^{-1} 2^n \ell^{-\mathbb{A}}(2^n) \, \|a\|_A \, .$$

Consider the representation $a = \sum_{m=-\infty}^{\infty} \delta_m^n a$. We have

$$\|a\|_{(A,2^{-k}A)^J_{1,q,\mathbb{A}}} \le 2^{-n}\ell^{\mathbb{A}}(2^n) J(2^n, a) < \left(\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon\right)^{-1} \|a\|_A .$$

This completes the proof. □

Corollary 3.8. *Let A be a p-normed quasi-Banach space, let $k \in \mathbb{Z}, 0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (3.5). The following equalities hold with equivalence of quasi-norms:*

For $p = 1$,

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k)A & \text{if } 0 < q \le 1 \text{ and } \alpha_0 \le 0, \\ 2^{-k}\ell^{(0,\alpha_\infty)}(2^k)A & \text{if } 0 < q \le 1 \text{ and } \alpha_0 > 0, \\ 2^{-k}\ell^{\mathbb{A}-1/q^*}(2^k)A & \text{if } 1 < q \le \infty \text{ and } \alpha_0 < \frac{1}{q^*}, \\ 2^{-k}\ell^{\mathbb{A}-1/q^*}(2^k)\ell\ell^{(-1/q^*,0)}(2^k)A & \text{if } 1 < q \le \infty \text{ and } \alpha_0 = \frac{1}{q^*}, \\ 2^{-k}\ell^{(0,\alpha_\infty-1/q^*)}(2^k)A & \text{if } 1 < q \le \infty \text{ and } \alpha_0 > \frac{1}{q^*}. \end{cases}$$

For $0 < q \le p$,

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k)A & \text{if } \alpha_0 \le 0, \\ 2^{-k}\ell^{(0,\alpha_\infty)}(2^k)A & \text{if } \alpha_0 > 0. \end{cases}$$

The constants in the equivalence depend on p but they are independent of $k \in \mathbb{Z}$ and of the concrete p-normed space A.

Proof. Let $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$. Using Lemma 3.6 and a change of variables we can relate $\mathrm{u}_{q,\mathbb{A},p}$ with the function $\mathrm{v}_{q^*,-\tilde{\mathbb{A}}}$ defined in Lemma 2.2. We have

$$\begin{aligned} \mathrm{u}_{q,\mathbb{A},p}(2^k) &= \left\|(\min(1, 2^{k-m}))\right\|_{\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))} \\ &= \left\|(\min(1, 2^{k+m}))\right\|_{\ell_{q^*}(2^{-m}\ell^{-\tilde{\mathbb{A}}}(2^m))} = \mathrm{v}_{q^*,-\tilde{\mathbb{A}}}(2^{-k}). \end{aligned}$$

Now the result follows by applying Proposition 3.7 and Lemma 2.2. □

4. The equivalence theorems

In this section we investigate the description of $(A_0, A_1)_{1,q,\mathbb{A}}$ in terms of the J-functional and the logarithmic sequence spaces $\ell_q(2^{-m}\ell^{\mathbb{M}}(2^m))$. Fix $0 < p \le 1$. We assume that the functors $(\cdot,\cdot)_{1,q,\mathbb{A}}$ and $(\cdot,\cdot)^J_{1,q,\mathbb{M}}$ are well-defined in the category of p-normed quasi-Banach couples. This means that $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and q should satisfy (2.1), and $\mathbb{M} = (\mu_0, \mu_\infty)$, q and p should satisfy (3.4). In addition, according to (3.2) and Remark 2.1, $\mathbb{A}$ and q should also satisfy (2.2). So, the problem to study reads as follows:

Given $0 < p \leq 1$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ such that

$$\begin{cases} \alpha_0 + 1/q < 0 \leq \alpha_\infty + 1/q & \text{if } 0 < q < \infty, \\ \alpha_0 \leq 0 < \alpha_\infty & \text{if } q = \infty, \end{cases} \tag{4.1}$$

find $\mathbb{M} = (\mu_0, \mu_\infty) \in \mathbb{R}^2$ with

$$\begin{cases} \mu_\infty \geq 0 & \text{if } 0 < q \leq p, \\ \mu_\infty > \frac{1}{p} - \frac{1}{q} & \text{if } 0 < p < q \leq \infty, \end{cases} \tag{4.2}$$

and such that for any p-normed quasi-Banach couple $\overline{A} = (A_0, A_1)$ we have

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)^J_{1,q,\mathbb{M}} \tag{4.3}$$

with equivalence of quasi-norms where the constants are independent of $\overline{A}$.

Unfortunately, for some values of parameters p, q and $\mathbb{A}$ there is no $\mathbb{M}$ satisfying (4.3). Take, for example, $0 < p < q \leq 1$ and $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q \leq 1/p - 1/q$. Let $\mathbb{M} = (\mu_0, \mu_\infty)$ with $\mu_\infty > 1/p - 1/q$. If (4.3) holds then Proposition 2.3 and Corollary 3.8 with $A = \ell_1$ would yield

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \mathrm{v}_{q,\mathbb{A}}(2^k)$$

with constant in the equivalence independent of k. By Lemma 2.2 and Corollary 3.8 we get for $k \in \mathbb{Z}$, k positive, that $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim 2^{-k}\ell^{\alpha_\infty+1/q}(2^k)$ and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim 2^{-k}\ell^{\mu_\infty}(2^k)$. It follows that $\alpha_\infty + 1/q = \mu_\infty$. Hence $\alpha_\infty + 1/q > 1/p - 1/q$ which contradicts that $\alpha_\infty + 1/q \leq 1/p - 1/q$.

This example leads us to investigate the weaker questions of finding the best $\mathbb{M}$ and $\mathbb{B}$ such that

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{M}} \tag{4.4}$$

or

$$(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \tag{4.5}$$

for any p-normed quasi-Banach couple (A_0, A_1) and with constants in the embeddings independent of (A_0, A_1).

In the case $0 < p < q \leq 1$, $\alpha_0 + 1/q < 0$ and $1/p - 1/q < \alpha_\infty + 1/q$, if we assume that (4.4) holds for some $\mathbb{M} = (\mu_0, \mu_\infty)$ with $\mu_\infty > 1/p - 1/q$, then proceeding as above we obtain that $\mathrm{v}_{q,\mathbb{A}}(2^k)\ell_1 \hookrightarrow \mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1}\ell_1$. The values of $\mathrm{v}_{q,\mathbb{A}}(2^k)$ and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1}$ have been pointed out above. Since the embedding is valid for any positive k, we get $\mu_\infty \leq \alpha_\infty + 1/q$. Let now $k \in \mathbb{Z}$, k negative, then $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim 2^{-k}\ell^{\alpha_0+1/q}(2^k)$ while

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \begin{cases} 2^{-k}\ell^{\mu_0}(2^k) & \text{if } \mu_0 \leq 0, \\ 2^{-k} & \text{if } \mu_0 > 0. \end{cases}$$

The option 2^{-k} is not possible since $\alpha_0 + 1/q < 0$. Hence, we should have $\mu_0 \le \alpha_0 + 1/q$. In other words, for these p, q and $\mathbb{A}$, the best possible $\mathbb{M}$ would be $(\alpha_0 + 1/q, \alpha_\infty + 1/q)$. Next we show that (4.4) holds for this choice of $\mathbb{M}$. We shall use that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. This equality follows from the equivalence

$$K(t, a; A_0^\sim, A_1^\sim) \le K(t, a; A_0, A_1) \le \max\{c_{A_0}, c_{A_1}\} K(t, a; A_0^\sim, A_1^\sim)$$

which can be established by doing minor modifications in the arguments of [3, Theorem V.I.5].

Theorem 4.1. *Let $0 < p < q \le 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying*

$$\alpha_0 + 1/q < 0, \ \ \alpha_\infty + 1/q > 1/p - 1/q. \tag{4.6}$$

Then, for any p-normed quasi-Banach couple $\overline{A} = (A_0, A_1)$, we have

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}\,.$$

Proof. According to (3.3), we get

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}};J}\,.$$

Besides, since $p < q$, we have $\ell_p \hookrightarrow \ell_q$ and $\ell_p(2^{-m}) \hookrightarrow \ell_q(2^{-m})$. Therefore,

$$(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}} \hookrightarrow (\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}} = \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m))$$

where the last equality follows from Lemma 3.4. Consequently,

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}\,.$$

□

Next we consider the case $1 \le q \le \infty$. This time our arguments are based on decompositions of the type considered in [12] and [16].

Theorem 4.2. *Let $0 < p \le 1 \le q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying*

$$\alpha_0 + 1/q < 0 \text{ and } \alpha_\infty + 1/q > 1/p - 1\,. \tag{4.7}$$

Then, for any p-Banach couple $\overline{A} = (A_0, A_1)$, we have

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{A}+1}\,.$$

Proof. Take any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$. It follows from (4.7) (see (2.2)) that $a \in (A_0 + A_1)^\circ$. Hence,

$$\min(1, t^{-1})K(t, a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty.$$

For $\nu \in \mathbb{Z}$, let

$$\gamma_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

Choose $a_{0,\nu} \in A_0$, $a_{1,\nu} \in A_1$ such that $a = a_{0,\nu} + a_{1,\nu}$ and

$$\|a_{0,\nu}\|_{A_0}^p + \gamma_{\nu-1}^p \|a_{1,\nu}\|_{A_1}^p \le 2^p K_p(\gamma_{\nu-1}, a)^p.$$

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$. We have

$$\begin{aligned} \left\| a - \sum_{\nu=N}^{M} u_\nu \right\|_{A_0+A_1}^p &= \|a + a_{0,N-1} - a_{0,M}\|_{A_0+A_1}^p \\ &= \|a_{1,M} + a_{0,N-1}\|_{A_0+A_1}^p \\ &\le 2^p \left(\gamma_{M-1}^{-p} K_p(\gamma_{M-1}, a)^p + K_p(\gamma_{N-2}, a)^p\right) \to 0 \end{aligned}$$

as $M \to \infty$ and $N \to -\infty$. So $a = \sum_{\nu=-\infty}^{\infty} u_\nu$. Moreover, $\|u_\nu\|_{A_0+A_1} \to 0$ as $\nu \to -\infty$ and as $\nu \to \infty$. Besides,

$$\frac{J(\gamma_{\nu-1}, u_\nu)}{\gamma_{\nu-1}} \lesssim \frac{K_p(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}, \ \nu \in \mathbb{Z}. \tag{4.8}$$

Indeed,

$$\begin{aligned} \frac{J(\gamma_{\nu-1}, u_\nu)^p}{\gamma_{\nu-1}^p} &\le \gamma_{\nu-1}^{-p}\left(\|u_\nu\|_{A_0}^p + \gamma_{\nu-1}^p \|u_\nu\|_{A_1}^p \right) \\ &\le \gamma_{\nu-1}^{-p}\left(\|a_{0,\nu}\|_{A_0}^p + \|a_{0,\nu-1}\|_{A_0}^p + \gamma_{\nu-1}^p \|a_{1,\nu}\|_{A_1}^p + \gamma_{\nu-1}^p \|a_{1,\nu-1}\|_{A_1}^p \right) \\ &\le 2^p \gamma_{\nu-1}^{-p}\left(K_p(\gamma_{\nu-1}, a)^p + \frac{\gamma_{\nu-1}^p}{\gamma_{\nu-2}^p} K_p(\gamma_{\nu-2}, a)^p\right) \lesssim \frac{K_p(\gamma_{\nu-2}, a)^p}{\gamma_{\nu-2}^p} \end{aligned}$$

where we have used in the last inequality that the function $t^{-1}K(t, a)$ is non-increasing.

For $\nu \in \mathbb{Z}$, put $I_\nu = [\gamma_{\nu-1}, \gamma_\nu)$. If $2^m \in I_\nu$, let w_m be u_ν divided by the number of m such that $2^m \in I_\nu$. That is

$$w_m = \begin{cases} \frac{u_\nu}{2^{-\nu-1}} & \text{if } 2^m \in I_\nu \text{ and } \nu < 0, \\ u_\nu & \text{if } 2^m \in I_\nu \text{ and } \nu = 0, 1, \\ \frac{u_\nu}{2^{\nu-2}} & \text{if } 2^m \in I_\nu \text{ and } \nu > 1. \end{cases}$$

This sequence also satisfies that $(w_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m=-\infty}^{\infty} w_m$ because for some $0 \leq d, f < 1$ we have

$$\begin{aligned}\Big\| a - \sum_{m=N}^{M} w_m \Big\|_{A_0+A_1}^p &= \Big\| a - \sum_{\nu=P}^{Q} u_\nu - d u_{P-1} - f u_{Q+1} \Big\|_{A_0+A_1}^p \\ &\leq \Big\| a - \sum_{\nu=P}^{Q} u_\nu \Big\|_{A_0+A_1}^p + \|u_{P-1}\|_{A_0+A_1}^p + \|u_{Q+1}\|_{A_0+A_1}^p \to 0\end{aligned}$$

as $N \to -\infty$ and $M \to \infty$. Consequently, using (4.8), we derive if $1 \leq q < \infty$

$$\begin{aligned}\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}} &\leq \Big(\sum_{m=-\infty}^{\infty} 2^{-mq} \ell^{\mathbb{A}q+q}(2^m) J(2^m, w_m)^q \Big)^{1/q} \\ &\sim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} 2^{-|\nu| q} \frac{J(2^m, u_\nu)^q}{2^{mq}} \ell^{\mathbb{A}q+q}(2^m) \Big)^{1/q} \\ &\lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} 2^{-|\nu| q} \frac{J(\gamma_{\nu-1}, u_\nu)^q}{\gamma_{\nu-1}^q} 2^{|\nu|(\mathbb{A}q+q)} \Big)^{1/q} \\ &\lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} \frac{K_p(\gamma_{\nu-2}, a)^q}{\gamma_{\nu-2}^q} 2^{|\nu| \mathbb{A}q} \Big)^{1/q} \\ &\sim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_{\nu-2}} \frac{K_p(\gamma_{\nu-2}, a)^q}{\gamma_{\nu-2}^q} 2^{|\nu| \mathbb{A}q} \Big)^{1/q} \\ &\lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_{\nu-2}} \frac{K_p(2^m, a)^q}{2^{mq}} \ell^{\mathbb{A}q}(2^m) \Big)^{1/q} \\ &= \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}}.\end{aligned}$$

The case $q = \infty$ can be treated similarly. □

Next we show that the embedding in Theorem 4.2 is the best possible. We rely on the characteristic functions of the functors.

Let $0 < p < 1 < q < \infty$ and $\mathbb{A}, \mathbb{M} \in \mathbb{R}^2$ satisfying (4.1) and (4.2). If (4.4) holds then Proposition 2.3 and Corollary 3.8 with $A = \ell_1$ yield that

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \lesssim \mathrm{v}_{q,\mathbb{A}}(2^k). \tag{4.9}$$

Take any $k \in \mathbb{Z}$, k negative. Using Lemma 2.2 we have that $\mathrm{v}_{q,\mathbb{A}}(2^k) = 2^{-k} \ell^{\alpha_0+1/q}(2^k)$ and by Corollary 3.8 we get

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \begin{cases} 2^{-k} \ell^{\mu_0-1+1/q}(2^k) & \text{if } \mu_0 < 1 - 1/q, \\ 2^{-k} \ell\ell^{-1+1/q}(2^k) & \text{if } \mu_0 = 1 - 1/q, \\ 2^{-k} & \text{if } \mu_0 > 1 - 1/q. \end{cases}$$

Hence, (4.9) is only possible if $\mu_0 \le \alpha_0 + 1$. Consider now k positive. Then

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k}\ell^{\alpha_\infty+1/q}(2^k) & \text{if } \alpha_\infty + 1/q > 0, \\ 2^{-k}\ell\ell^{1/q}(2^k) & \text{if } \alpha_\infty + 1/q = 0 \end{cases}$$

and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim 2^{-k}\ell^{\mu_\infty-1+1/q}(2^k)$. Since $\mu_\infty > 1/p - 1/q > 1 - 1/q$, we derive that if (4.9) holds, then $\mu_\infty \le \alpha_\infty + 1$ and that there is no $\mathbb{M}$ if $0 \le \alpha_\infty + 1/q \le 1/p - 1$. Consequently, the embedding in Theorem 4.2 is optima. Note that these arguments also explain the assumption on α_∞ in the statement of the theorem. The cases $p = 1 < q \le \infty$, $0 < p < 1$ with $q = \infty$ and $0 < p < 1 = q$ can be treated analogously.

Now we turn our attention to embeddings of the type (4.5).

Theorem 4.3. *Let $0 < p \le 1$, $0 < p < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying*

$$\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q\,. \tag{4.10}$$

Then, for any p-Banach couple $\overline{A} = (A_0, A_1)$, we have

$$(A_0, A_1)^J_{1,q,\mathbb{A}+1/p} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}\,.$$

Proof. Take any $a \in (A_0, A_1)^J_{1,q,\mathbb{A}+1/p}$ and let $(u_m) \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ and $\|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{A}+1/p}(2^m))} \le 2\,\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1/p}}$. Then

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} &\le \Big(\sum_{k=-\infty}^{\infty} \big[2^{-k}\ell^{\mathbb{A}}(2^k)\big(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p\big)^{1/p}\big]^q\Big)^{1/q} \\
&\sim \|(J(2^m, u_m))\|_{(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}} \\
&\sim \Big(\sum_{k=-\infty}^{0} \big[\ell^{\alpha_0}(2^k)\big(\sum_{m=k}^{0} 2^{-mp} J(2^m, u_m)^p\big)^{1/p}\big]^q\Big)^{1/q} \\
&\quad + \Big(\sum_{k=0}^{\infty} \big[\ell^{\alpha_\infty}(2^k)\big(\sum_{m=k}^{\infty} 2^{-mp} J(2^m, u_m)^p\big)^{1/p}\big]^q\Big)^{1/q}
\end{aligned}$$

where we have used Lemma 3.3 in the last equivalence. Suppose now that $q < \infty$. To estimate the first term take any $0 < \varepsilon < -(\alpha_0 + 1/q)$. Using Hölder's inequality, for the interior sum of the first term, we get

$$\begin{aligned}
\Big(\sum_{m=k}^{0} 2^{-mp} J(2^m, u_m)^p\Big)^{q/p} &\le \Big(\sum_{m=k}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p + \varepsilon q}(2^m)\Big) \\
&\quad \times \Big(\sum_{m=k}^{0} \ell^{-qp(\alpha_0+1/p+\varepsilon)/(q-p)}(2^m)\Big)^{(q-p)/p}
\end{aligned}$$

$$\lesssim \ell^{-\alpha_0 q-\varepsilon q-1}(2^k)\Big(\sum_{m=k}^{0} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_0 q+q/p+\varepsilon q}(2^m)\Big)$$

where we have used the condition on ε in the last estimate. Whence, changing the order of summation, we derive

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{0}\big[\ell^{\alpha_0}(2^k)(\sum_{m=k}^{0} 2^{-mp} J(2^m,u_m)^p)^{1/p}\big]^q\Big)^{1/q}\\
&\quad\le \Big(\sum_{k=-\infty}^{0} \ell^{-\varepsilon q-1}(2^k)(\sum_{m=k}^{0} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_0 q+q/p+\varepsilon q}(2^m))\Big)^{1/q}\\
&\quad= \Big(\sum_{m=-\infty}^{0} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_0 q+q/p+\varepsilon q}(2^m)\sum_{k=-\infty}^{m} \ell^{-\varepsilon q-1}(2^k)\Big)^{1/q}\\
&\quad\sim \Big(\sum_{m=-\infty}^{0} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_0 q+q/p}(2^m)\Big)^{1/q}\\
&\quad\lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1/p}}\,.
\end{aligned}$$

For the second term, we choose now any $0<\varepsilon<\alpha_\infty+1/q$ and we proceed similarly. We get

$$\begin{aligned}
\Big(\sum_{m=k}^{\infty} 2^{-mp} J(2^m,u_m)^p\Big)^{q/p} &\le \Big(\sum_{m=k}^{\infty} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_\infty q+q/p-\varepsilon q}(2^m)\Big)\\
&\quad\times \Big(\sum_{m=k}^{\infty} \ell^{-pq(\alpha_\infty+1/p-\varepsilon)/(q-p)}(2^m)\Big)^{(q-p)/p}\\
&\sim \ell^{-\alpha_\infty q-1+\varepsilon q}(2^k)\Big(\sum_{m=k}^{\infty} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_\infty q+q/p-\varepsilon q}(2^m)\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty}\big[\ell^{\alpha_\infty}(2^k)(\sum_{m=k}^{\infty} 2^{-mp} J(2^m,u_m)^p)^{1/p}\big]^q\Big)^{1/q}\\
&\quad\lesssim \Big(\sum_{k=0}^{\infty} \ell^{-1+\varepsilon q}(2^k)(\sum_{m=k}^{\infty} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_\infty q+q/p-\varepsilon q}(2^m))\Big)^{1/q}\\
&\quad= \Big(\sum_{m=0}^{\infty} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_\infty q+q/p-\varepsilon q}(2^m)\Big(\sum_{k=0}^{m} \ell^{-1+\varepsilon q}(2^k)\Big)\Big)^{1/q}\\
&\quad\lesssim \Big(\sum_{m=0}^{\infty} 2^{-mq} J(2^m,u_m)^q \ell^{\alpha_\infty q+q/p}(2^m)\Big)^{1/q}
\end{aligned}$$

$$\lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1/p}}\,.$$

Consequently,

$$(A_0,A_1)^J_{1,q,\mathbb{A}+1/p} \hookrightarrow (A_0,A_1)_{1,q,\mathbb{A}}\,.$$

The case $q=\infty$ can be treated analogously. □

Next we show that the embedding of Theorem 4.3 is the best possible. For this aim we need some preparation.

Lemma 4.4. *Let $0<p\leq 1$, $0<q\leq\infty$ and take any $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$ with*

$$\begin{cases} \beta_\infty \geq 0 & \text{if } 0<q\leq p,\\ \beta_\infty > \frac{1}{p}-\frac{1}{q} & \text{if } 0<p<q\leq\infty. \end{cases} \tag{4.11}$$

Then

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p,\ell_p(2^{-m}))^J_{1,q,\mathbb{B}}.$$

Proof. Take any $x=(x_m)\in\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))$ and $e_k=(\delta^k_m)$, $k\in\mathbb{Z}$. Then

$$J(2^k,x_ke_k;\ell_p,\ell_p(2^{-m})) = |x_k| \text{ and } x=\sum_{m=-\infty}^{\infty} x_me_m.$$

Whence

$$\begin{aligned}\|x\|_{(\ell_p,\ell_p(2^{-m}))^J_{1,q,\mathbb{B}}} &\leq \big\|(J(2^k,x_ke_k;\ell_p,\ell_p(2^{-m})))\big\|_{\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))}\\ &= \|x\|_{\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))}\,.\end{aligned}$$

□

Proposition 4.5. *Let $0<p\leq 1$, $0<q\leq\infty$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)$, $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$ satisfying (2.1) and (4.11). Then the embedding*

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}$$

is a necessary and sufficient condition for $(A_0,A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0,A_1)_{1,q,\mathbb{A}}$ for any p-Banach couple (A_0,A_1).

Proof. The condition is necessary by Lemma 4.4. Let us check that the condition is sufficient. Take any p-Banach couple (A_0,A_1). For any $a\in(A_0,A_1)^J_{1,q,\mathbb{B}}$ we can find a representation $a=\sum_{m=-\infty}^{\infty}u_m$ with $(u_m)\subseteq A_0\cap A_1$ satisfying that $\|(J(2^m,u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))}\leq 2\,\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{B}}}$. Since

$$K(2^k,a)\leq K_p(2^k,a)\leq \Big(\sum_{m=-\infty}^{\infty} K_p(2^k,u_m)^p\Big)^{1/p}$$

$$\le \Big(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p \Big)^{1/p},$$

we obtain

$$\begin{aligned}\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} &\le \big\|(K(2^k, a))\big\|_{\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k))} \\ &\le \left\| \Big(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p \Big)^{1/p} \right\|_{\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k))} \\ &\sim \big\|(K(2^k, (J(2^m, u_m)); \ell_p, \ell_p(2^{-m}))\big\|_{\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k))} \\ &= \|(J(2^m, u_m))\|_{(\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}} .\end{aligned}$$

Now using the condition we derive that

$$\begin{aligned}\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} &\lesssim \|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))} \\ &\le 2\, \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{B}}} .\end{aligned}$$

□

The next result refers to the operators

$$\begin{aligned} H_1 x &= \Big(\sum_{n=1}^{k} (1 + \log 2^k)^{\alpha_0 p} (1 + \log 2^n)^{-\beta_0 p} x_n \Big)_{k\in\mathbb{N}}, \\ H_2 x &= \Big(\sum_{n=k}^{\infty} (1 + \log 2^k)^{\alpha_\infty p} (1 + \log 2^n)^{-\beta_\infty p} x_n \Big)_{k\in\mathbb{N}}. \end{aligned}$$

Here $x = (x_n)_{n\in\mathbb{N}}$. We consider H_1 and H_2 acting on the space $\boldsymbol{\ell}_{\mathbf{r}}$ of r-summable sequences with $\mathbb{N}$ as index set.

Lemma 4.6. *Let $0 < p \le 1$, $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (2.1) and (4.11). Assume also that*

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}.$$

Then H_1 and H_2 are bounded on $\boldsymbol{\ell}_{\mathbf{q/p}}$.

Proof. Take any $x = (x_n) \in \boldsymbol{\ell}_{\mathbf{q/p}}$. For $k \in \mathbb{Z}$, let

$$y_k = \begin{cases} |x_{-k}|^{1/p}\, 2^k (1 - \log 2^k)^{-\beta_0} & \text{if } k \le -1, \\ 0 & \text{if } k \ge 0, \end{cases}$$

and put $y = (y_k)_{k\in\mathbb{Z}}$. Then $\|y\|_{\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))} = \|x\|^{1/p}_{\boldsymbol{\ell}_{\mathbf{q/p}}} < \infty$. By the assumption and applying Lemma 3.3, we obtain

$$\|x\|^{1/p}_{\boldsymbol{\ell}_{\mathbf{q/p}}} = \|y\|_{\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))} \gtrsim \|y\|_{(\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}}$$

$$
\begin{aligned}
&\sim \Big(\sum_{k=-\infty}^{0} \big[\ell^{\alpha_0}(2^k)(\sum_{m=k}^{0} 2^{-mp}\,|y_m|^p)^{1/p}\big]^q\Big)^{1/q} \\
&\qquad + \Big(\sum_{k=0}^{\infty} \big[\ell^{\alpha_\infty}(2^k)(\sum_{m=k}^{\infty} 2^{-mp}\,|y_m|^p)^{1/p}\big]^q\Big)^{1/q} \\
&= \Big(\sum_{k=-\infty}^{-1} \big[(1-\log 2^k)^{\alpha_0}(\sum_{m=k}^{-1}(1-\log 2^m)^{-\beta_0 p}\,|x_{-m}|)^{1/p}\big]^q\Big)^{1/q} \\
&= \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_0}(\sum_{m=1}^{k}(1+\log 2^m)^{-\beta_0 p}\,|x_{m}|)^{1/p}\big]^q\Big)^{1/q} \\
&\geq \|H_1 x\|^{1/p}_{\boldsymbol{\ell}_{\mathbf{q/p}}}.
\end{aligned}
$$

Now we return our attention to H_2. Let $z = (z_k)_{k\in\mathbb{Z}}$ where

$$
z_k = \begin{cases} 0 & \text{if } k \leq 0, \\ |x_k|^{1/p}\, 2^k (1+\log 2^k)^{-\beta_\infty} & \text{if } k \geq 1. \end{cases}
$$

Using again the assumption and Lemma 3.3, we derive

$$
\begin{aligned}
\|x\|^{1/p}_{\boldsymbol{\ell}_{\mathbf{q/p}}} &= \|z\|_{\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))} \gtrsim \|z\|_{(\ell_p,\ell_p(2^{-m}))_{1,q,\mathbb{A}}} \\
&\gtrsim \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_\infty}(\sum_{m=k}^{\infty} 2^{-mp}\,|z_m|^p)^{1/p}\big]^q\Big)^{1/q} \\
&= \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_\infty}(\sum_{m=k}^{\infty}(1+\log 2^m)^{-\beta_\infty p}\,|x_m|)^{1/p}\big]^q\Big)^{1/q} \\
&\geq \|H_2 x\|^{1/p}_{\boldsymbol{\ell}_{\mathbf{q/p}}}.
\end{aligned}
$$

□

We shall also need some results on matrix transformations of $\boldsymbol{\ell}_{\mathbf{r}}$-spaces established by G. Bennett [1, 2]. Let (a_n), (b_n) be sequences of non-negative numbers and let $\mathbf{M} = (a_{nk})$ with

$$
a_{nk} = \begin{cases} a_n b_k & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}
$$

Put $Tx = \Big(\sum_{k=1}^{\infty} a_{nk}x_k\Big)_{n\in\mathbb{N}} = \Big(a_n \sum_{k=1}^{\infty} b_k x_k\Big)_{n\in\mathbb{N}}$ for the operator defined by $\mathbf{M}$. Let $1 \leq r \leq \infty$ and $1/r + 1/r' = 1$. According to [1, Theorem 2] and

[2, Theorem 1], the operator $T : \boldsymbol{\ell}_{\mathbf{r}} \to \boldsymbol{\ell}_{\mathbf{r}}$ is bounded if and only if

$$\begin{cases} \sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^r\right)^{1/r} \left(\sum_{k=1}^{N} b_k^{r'}\right)^{1/r'} < \infty & \text{for } 1 < r < \infty, \\ \sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^r\right)^{1/r} b_N < \infty & \text{for } r = 1, \\ \sup_{N\in\mathbb{N}} a_N \sum_{k=1}^{N} b_k < \infty & \text{for } r = \infty. \end{cases} \tag{4.12}$$

Put $\widehat{\mathbf{M}} = (\widehat{a}_{nk})$ with

$$\widehat{a}_{nk} = \begin{cases} 0 & \text{if } k < n, \\ a_k b_n & \text{if } k \geq n, \end{cases}$$

and let $\widehat{T}x = \left(\sum_{k=1}^{\infty} \widehat{a}_{nk} x_k\right)_{n\in\mathbb{N}} = \left(b_n \sum_{k=n}^{\infty} a_k x_k\right)_{n\in\mathbb{N}}$. Since $\widehat{T}$ is the adjoint of the operator T, it follows from (4.12) that $\widehat{T} : \boldsymbol{\ell}_{\mathbf{r}} \to \boldsymbol{\ell}_{\mathbf{r}}$ is bounded if and only if

$$\sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^{r'}\right)^{1/r'} \left(\sum_{k=1}^{N} b_k^{r}\right)^{1/r} < \infty \text{ for } 1 < r < \infty. \tag{4.13}$$

Furthermore, a direct computation shows that $\widehat{T} : \boldsymbol{\ell}_{\infty} \to \boldsymbol{\ell}_{\infty}$ is bounded if and only if

$$\sup_{N\in\mathbb{N}} b_N \sum_{k=N}^{\infty} a_k < \infty. \tag{4.14}$$

Observe that H_1 is the transformation defined by the matrix $\mathbf{M} = (a_{nk})$ given by the sequences

$$a_n = (1 + \log 2^n)^{\alpha_0 p} \text{ and } b_k = (1 + \log 2^k)^{-\beta_0 p}$$

while H_2 is the transformation defined by $\widehat{\mathbf{M}}$ with

$$a_k = (1 + \log 2^k)^{-\beta_\infty p} \text{ and } b_n = (1 + \log 2^n)^{\alpha_\infty p}.$$

Now we are ready to show that the embedding of Theorem 4.3 is the best possible.

Proposition 4.7. *Let $0 < p \leq 1$, $0 < p < q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (4.10) and (4.11) and consider the p-Banach couple $(\ell_p, \ell_p(2^{-m}))$. If $(\ell_p, \ell_p(2^{-m}))^J_{1,q,\mathbb{B}} \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$ then $\beta_0 \geq \alpha_0 + 1/p$ and $\beta_\infty \geq \alpha_\infty + 1/p$.*

Proof. By the assumptions on parameters and Lemma 4.4 we have that

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}.$$

Hence, according to Lemma 4.6, operators H_1 and H_2 are bounded on $\boldsymbol{\ell}_{\mathbf{q/p}}$. If $q<\infty$, applying (4.12) to H_1 we obtain that

$$\sup_{N\in\mathbb{N}}\Big(\sum_{n=N}^{\infty}(1+\log 2^n)^{\alpha_0 q}\Big)^{p/q}\Big(\sum_{k=1}^{N}(1+\log 2^k)^{-\beta_0\frac{pq}{q-p}}\Big)^{\frac{q-p}{p}}<\infty.$$

Using that $\alpha_0+1/q<0$, it follows that

$$\Big(\sum_{k=1}^{N}(1+\log 2^k)^{-\beta_0\frac{pq}{q-p}}\Big)^{(q-p)/p}\lesssim(1+\log 2^n)^{-\alpha_0-1/q}.$$

Whence

$$\left.\begin{cases}(1+\log 2^N)^{-\beta_0+1/p-1/q} & \text{if } \beta_0<1/p-1/q\\ (1+\log(1+\log 2^N)) & \text{if } \beta_0=1/p-1/q\\ 1 & \text{if } \beta_0>1/p-1/q.\end{cases}\right\}\lesssim(1+\log 2^n)^{-\alpha_0-1/q}.$$

Since $-\alpha_0-1/q>0$, it follows that

$$\beta_0\geq 1/p-1/q,\ \text{ or } \beta_0<1/p-1/q \text{ and} -\beta_0+1/p-1/q\leq-\alpha_0-1/q.$$

Consequently, in both cases $\beta_0\geq\alpha_0+1/p$.

On the other hand, since H_2 is also bounded on $\boldsymbol{\ell}_{\mathbf{q/p}}$, according to (4.13), we obtain

$$\sup_{N\in\mathbb{N}}\Big(\sum_{n=N}^{\infty}(1+\log 2^n)^{-\beta_\infty\frac{pq}{q-p}}\Big)^{(q-p)/q}\Big(\sum_{k=1}^{N}(1+\log 2^k)^{\alpha_\infty q}\Big)^{p/q}<\infty.$$

Having in mind that $\beta_\infty>1/p-1/q$ and $\alpha_\infty+1/q>0$, we get

$$(1+\log 2^N)^{\alpha_\infty+1/q}\lesssim(1+\log 2^N)^{\beta_\infty-1/p+1/q}.$$

This yields that $\beta_\infty\geq\alpha_\infty+1/p$.

The proof for $q=\infty$ is similar but using now (4.14). □

The next result complements Theorem 4.3. It corresponds to the limit case $\alpha_\infty+1/q=0$. One can establish it by using similar arguments to those of Theorem 4.3.

Theorem 4.8. *Let $0<p\leq 1$, $0<p<q<\infty$ and $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ satisfying*

$$\alpha_0+1/q<0=\alpha_\infty+1/q.$$

Then, for any p-Banach couple $\overline{A}=(A_0,A_1)$, we have

$$(A_0,A_1)^J_{1,q,\mathbb{A}+1/p,(0,1/p)}\hookrightarrow(A_0,A_1)_{1,q,\mathbb{A}}\,.$$

Using the previous ideas based on matrix transformations of $\boldsymbol{\ell}_{\mathbf{r}}$-spaces, one can also show that the embedding in Theorem 4.8 is the best possible.

Writing down Theorems 4.2 and 4.3 for $1 \le q \le \infty$ and $\overline{A}$ a Banach couple, so $p = 1$, we obtain the following result of Cobos and Segurado [16, Theorem 3.5].

Corollary 4.9. *Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$ such that*

$$\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q\,.$$

Then, for any Banach couple (A_0, A_1), we have with equivalence of norms $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)^J_{1,q,\mathbb{A}+1}$.

We finish the paper with the case $0 < q \le p \le 1$ where we have also equality.

Theorem 4.10. *Let $0 < q \le p \le 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$. Then, for any p-Banach couple $\overline{A} = (A_0, A_1)$, we have with equivalence of quasi-norms*

i) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}$ if $\alpha_\infty + 1/q > 0$,

ii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)}$ if $\alpha_\infty + 1/q = 0$.

Proof. Recall that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. Applying (3.3), using that $q \le p$ and Lemma 3.4 we get that

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow \begin{cases} (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q} & \text{if } \alpha_\infty + 1/q > 0, \\ (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)} & \text{if } \alpha_\infty + 1/q = 0. \end{cases}$$

In order to check the converse embedding, consider the function $\mathrm{v}_{q,\mathbb{A}}(\cdot)$. By Lemma 2.2, we have

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k} \ell^{\mathbb{A}+1/q}(2^k) & \text{if } \alpha_\infty + 1/q > 0, \\ 2^{-k} \ell^{\mathbb{A}+1/q}(2^k) \ell\ell^{(0,1/q)}(2^k) & \text{if } \alpha_\infty + 1/q = 0. \end{cases} \tag{4.15}$$

Take any $a \in (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. Since (2.2) holds, we know that $(A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}} \subseteq (A_0^\sim + A_1^\sim)^\circ$. Using [27, Theorem 3.2], we can find $(u_m) \subseteq A_0^\sim \cap A_1^\sim$ with $a = \sum_{m=-\infty}^{\infty} u_m$ in $A_0^\sim + A_1^\sim$ and

$$\Big(\sum_{m=-\infty}^{\infty} [\min(1, 2^{k-m}) J(2^m, u_m)]^q \Big)^{1/q} \le c K(2^k, a), \;\; k \in \mathbb{Z}. \tag{4.16}$$

Therefore, if $\alpha_\infty + 1/q > 0$, according to (4.15) and (4.16), we have

$$\|a\|_{(A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}} \le \|(J(2^m, u_m))\|_{\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m))}$$

$$\sim \Big(\sum_{m=-\infty}^{\infty} J(2^m, u_m)^q \sum_{k=-\infty}^{\infty} \big[\min(1, 2^{k-m}) 2^{-k} \ell^{\mathbb{A}}(2^k) \big]^q \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=-\infty}^{\infty} 2^{-kq} \ell^{\mathbb{A}q}(2^k) K(2^k, a)^q \Big)^{1/q}$$
$$= \|a\|_{\left(A_0^{\sim}, A_1^{\sim}\right)_{1,q,\mathbb{A}}}.$$

The case $\alpha_\infty + 1/q = 0$ can be treated analogously. □

Writing down Theorem 4.10 for (A_0, A_1) a Banach couple, we recover a previous result of the authors [5, Theorem 3.2].

Acknowledgement. The authors have been supported in part by MTM2017-84508-P (AEI/ FEDER, UE). B. F. Besoy has also been supported by FPU grant FPU16/02420 of the Spanish Ministerio de Educación, Cultura y Deporte.

The authors would like to thank the referee for his/her useful comments which have led to improve the paper.

References

[1] Bennett, G., Some elementary inequalities. *Q. J. Math.* 38 (1987), 401 – 425.

[2] Bennett, G., Some elementary inequalities, III. *Q. J. Math.* 42 (1991), 149 – 174.

[3] Bennett, C. and Sharpley, R., *Interpolation of Operators.* Boston: Academic Press 1988.

[4] Bergh, J. and Löfström, J., *Interpolation Spaces. An Introduction.* Berlin: Springer 1976.

[5] Besoy, B.F. and Cobos, F., Duality for logarithmic interpolation spaces when $0 < q < 1$ and applications. *J. Math. Anal. Appl.* 466 (2018), 373 – 399.

[6] Besoy, B.F. and Cobos, F., Logarithmic interpolation methods and measure of non-compactness. *Quart. J. Math.* 00 (2019) 1 – 23; doi:10.1093/qmathj/haz041.

[7] Besoy, B.F., Cobos, F. and Fernández-Cabrera, L.M., On the structure of a limit class of logarithmic interpolation spaces. Preprint, Madrid (2019).

[8] Brudnyĭ, Y. and Krugljak, N., *Interpolation Functors and Interpolation Spaces Vol. 1.* Amsterdam: North-Holland 1991.

[9] Butzer, P.L. and Berens, H., *Semi-Groups of Operators and Approximation.* Berlin: Springer 1967.

[10] Cobos, F. and Domínguez, O., On Besov spaces of logarithmic smoothness and Lipschitz spaces. *J. Math. Anal. Appl.* 425 (2015), 71 – 84.

[11] Cobos, F., Domínguez, O. and Triebel, H., Characterizations of logarithmic Besov spaces in terms of differences, Fourier-analytical decompositions, wavelets and semi-groups. *J. Funct. Anal.* 270 (2016), 4386 – 4425.

[12] Cobos, F., Fernández-Cabrera, L.M., Kühn, T. and Ullrich, T., On an extreme class of real interpolation spaces. *J. Funct. Anal.* 256 (2009), 2321 – 2366.

[13] Cobos, F., Fernández-Cabrera, L.M. and Mastyło, M., Abstract limit J-spaces. *J. Lond. Math. Soc.* 82 (2010), 501 – 525.

[14] Cobos, F. and Kühn, T., Equivalence of K- and J-methods for limiting real interpolation spaces. *J. Funct. Anal.* 261 (2011), 3696 – 3722.

[15] Cobos, F., Kühn, T. and Schonbek, T., One–sided compactness results for Aronszajn-Gagliardo functors. *J. Funct. Anal.* 106 (1992), 274 – 313.

[16] Cobos, F. and Segurado, A., Description of logarithmic interpolation spaces by means of the J–functional and applications. *J. Funct. Anal.* 268 (2015), 2906 – 2945.

[17] Cwikel, M. and Peetre, J., Abstract K– and J–spaces. *J. Math. Pures Appl.* 60 (1981), 1 – 50.

[18] Doktorskii, R.Y., Reiteration relations of the real interpolation method. *Soviet Math. Dokl.* 44 (1992), 665 – 669.

[19] Edmunds, D.E. and Evans W.D., *Hardy Operators, Function Spaces and Embeddings*. Berlin: Springer 2004.

[20] Edmunds D.E. and Opic B., Limiting variants of Krasnosel'skiĭ's compact interpolation theorem. *J. Funct. Anal.* 266 (2014), 3265 – 3285.

[21] Evans, W.D. and Opic, B., Real interpolation with logarithmic functors and reiteration. *Canad. J. Math.* 52 (2000), 920 – 960.

[22] Evans, W.D., Opic, B. and Pick, L., Real interpolation with logarithmic functors. *J. Inequal. Appl.* 7 (2002), 187 – 269.

[23] Gustavsson, J., A function parameter in connection with interpolation of Banach spaces. *Math. Scand.* 42 (1978), 289 – 305.

[24] Janson, S., Minimal and maximal methods of interpolation. *J. Funct. Anal.* 44 (1981), 50 – 73.

[25] König, H., *Eigenvalue distributions of compact operators*. Basel: Birkhäuser 1986.

[26] Köthe, G., *Topological Vector Spaces I*. Berlin: Springer 1969.

[27] Nilsson, P., Reiteration theorems for real interpolation and approximation spaces. Ann. Mat. Pura Appl. 132 (1982), 291 – 330.

[28] Opic, B. and Pick, L., On generalized Lorentz-Zygmund spaces. *Math. Inequal. Appl.* 2 (1999) 391 – 467.

[29] Ovchinnikov, V.I., The Method of Orbits in Interpolation Theory. *Math. Rep.* 1 (1984), 349 – 515.

[30] Peetre, J., *A Theory of Interpolation of Normed Spaces*. Notes Mat. 39. Brasilia: Lecture Notes 1963.

[31] Persson, L-E., Interpolation with a parameter function. *Math. Scand.* 59 (1986), 199 – 222.

[32] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators.* Amsterdam: North-Holland 1978.

Received