The effects of closeness on the election of a pairwise majority rule winner

Mostapha Diss, Patrizia Pérez-Asurmendi, and Abdelmonaim Tlidi

Abstract Some studies have recently examined the effect of closeness on the probability of observing the monotonicity paradox in three-candidate elections under Scoring Elimination Rules. It has been shown that the frequency of such paradox significantly increases as elections become more closely contested. In this paper we consider the effect of closeness on one of the most studied notions in Social Choice Theory: The election of the Condorcet winner, i.e., the candidate who defeats any other opponent in pairwise majority comparisons, when she exists. To be more concrete, we use the well known concept of the Condorcet efficiency, that is, the conditional probability that a voting rule will elect the Condorcet winner, given that such a candidate exists. Our results, based on the Impartial Anonymous Culture (IAC) assumption, show that closeness has also a significant effect on the Condorcet efficiency of different voting rules in the class of Scoring and Scoring Elimination Rules.

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1 Introduction

In the literature of Social Choice Theory a wide variety of voting rules have been proposed to determine the winner of an election with more than two competing candidates. It is well known that the use of different voting rules may lead to different winners. Thus, a great deal of controversy surrounds the discussion of which voting rule should be implemented. To overcome such discussion many different criteria have been proposed in the literature. The most widely used one is the *Condorcet criterion* which states that the candidate who is able to defeat all others in pairwise majority comparisons, i.e., the *Condorcet winner*, should be the one elected in the voting process. Unfortunately it is well known that the existence of the Condorcet winner is not guaranteed. Therefore, it has become common to consider the *Condorcet efficiency* (CE) as a measure of partial fulfillment of the Condorcet criterion. Note that the CE of a voting rule is defined as the conditional probability that the given voting rule elects the Condorcet winner, given that such a candidate exists. As the CE involves the computation of probabilities of electoral events, it is usually required to make some assumptions on the likelihood of the different possible individual votes that could be observed. One of the most widespread used assumptions is the *Impartial Anonymous Culture* (IAC) introduced by Kuga and Nagatani (1974) and later developed by Gehrlein and Fishburn (1976).

The CE has been extensively studied under the IAC assumption. In particular, a whole body of literature can be found regarding the CE of the *Weighted Scoring Rules* (WSRs). Under those rules each candidate is awarded with a number of points according to her relative position within each individual voter's preference ranking and the winner is the candidate with the highest total score. The *Plurality Rule* (PR), the *Negative Plurality Rule* (NPR), and the *Borda Rule* (BR) are well known examples of WSRs. It is also usual to find the IAC assumption in many papers dealing with the CE of the *Weighted Scoring Elimination Rules* (WSERs) which also constitute an important class of voting rules. Those rules also give points to candidates according to their rank in voters' preference orders and eliminate the candidate(s) with the lowest number of points. The number of rounds is determined by the number of candidates and the implemented method. The elected candidate is the majority winner between the two remaining candidates in the last round. The scoring rules that follow this process are called *Weighted Scoring Elimination Rules* (WSERs). The *Plurality Elimination Rule* (PER), the *Negative Plurality Elimination Rule* (NPER) and the *Borda Elimination Rule* (BER) are widely known examples of WSERs.

Related literature

Many research papers have already analyzed the CE on various voting rules taking into account different assumptions on individuals' preferences. Taking into account the aim of this paper, we will only recall some relating results. The interested reader can find an exhaustive review of this topic in the recent books of Gehrlein and

Lepelley (2011, 2017). First of all, Gehrlein (1982, 1992) has calculated the CE values of BR, PR, PER, NPR and NPER in three-candidate elections under the IAC assumption for large electorates (Table 1).

Table 1 Condorcet efficiency values under the IAC assumption for large electorates

Voting rules	IAC Condorcet efficiency
PR	0.8815[a]
BR	0.9111[b]
NPR	0.6296[a]
PER	0.9685[a]
NPER	0.9704[a]

a. Gehrlein (1982); b. Gehrlein (1992).

Also in the context of three-candidate elections under the IAC hypothesis, Gehrlein and Lepelley (2001) have obtained a closed form representation of the CE of BR as a function of the number of voters, and Cervone et al. (2005) have developed a representation for the CE of every WSR. As a quite natural extension, many studies have been carried out to deal with the effect of some additional assumptions on the CE of several voting rules. For instance, Lepelley (1995) provided an exact representation for the CE of WSRs when voters are endowed with *single-peaked preferences*. Intuitively, voters are said to have single-peaked preferences if there is an ideal outcome that they prefer the most, and alternatives that are further away from this ideal outcome (according to some linear ordering) are less preferred (see for instance, Brown et al., 2014). Gehrlein et al. (2012) and Gehrlein and Lepelley (2015) dealt with the CE in the presence of degrees of *group mutual coherence* which measures a voting situation's propensity to specific underlying rational behavior models that may govern voter preferences. Finally, notice that many researchers have reconsidered the CE of voting rules by using a modified IAC model. For instance, we refer the reader to Diss and Gehrlein (2015), among others.

It is important to mention that many studies have focused on other assumptions on the individuals' preferences when calculating the CE of several voting rules. For instance, the *Impartial Culture* condition is used in Diss and Merlin (2010), Diss et al. (2010), Gehrlein and Fishburn (1978a,b), Gehrlein and Lepelley (2014), and Gehrlein and Valognes (2001), among others. The *Dual Culture* condition is assumed for instance in Gehrlein (1999) and Gehrlein and Roy (2014) while the *Maximal Culture* condition is considered in Gehrlein and Lepelley (1999), among others. We again refer the reader to the books of Gehrlein and Lepelley (2011, 2017) for more information on other research papers related to those assumptions and their refinements.

Our contribution

The central concern of this paper is to deal with the problem of the CE of many common voting rules when the results of an election are closely contested. Specifically, we focus on the following WSRs and WSERs voting rules: PR, PER, NPR, NPER, and BR. Up to our knowledge, too little attention has been paid to this issue in the literature. Recently, Miller (2017) has studied, in the context of three-candidate elections, the effect of closeness on the probability of occurrence of monotonicity paradox that is, getting more points from voters can make a candidate a loser and getting fewer points can make a candidate a winner. The obtained results have revealed that such probability can be very high under PER when the results of the elections become very close. Lepelley et al. (2018) have extended the previous study to other WSERs and showed that the probability of occurrence of monotonicity paradox remains also very high for BER and NPER. To measure the election closeness, Miller (2017) and Lepelley et al. (2018) have considered the ratio between the score of the last ranked candidate and the sum of the scores of all competing candidates. It seems from these research papers that closeness deserves more consideration in Social Choice Theory.

In what follows, we derive an exact representation for the CE as a function of the same closeness index that has been used by Miller (2017) and Lepelley et al. (2018). Our results show that closeness also matters in our context since it affects negatively the CE of the five considered voting rules. However, the effect of the closeness varies through the different considered voting rules. Specifically, the CE of BR remains relatively stable with the range of closeness index when compared to the one of PR, PER, NPR, and NPER.

The paper is organized as follows. In Section 2, we introduce the basic notations and definitions. In Section 3, we derive the analytical representations for the CE of the five considered WSRs and WSERs and discuss our results. Section 4 is devoted to summarize our findings, and to end, the proofs are presented in the Appendix.

2 Preliminaries

Throughout this paper, we consider n voters in three-candidate elections and denote by $\mathbb{C} = \{a, b, c\}$ the set of candidates. Each voter is endowed with a linear preference ordering on the candidates, i.e., she is able to rank the set of candidates from the most desirable one to the least desirable one. We also assume that voters vote according to their true preferences, which means that the strategic behaviors are not allowed in this paper. In this setting, there are six possible linear preference rankings that voters might have on $\mathbb{C}$:

Ranking	**#**	**Ranking**	**#**	**Ranking**	**#**	
$a \succ b \succ c$	n_1	$a \succ c \succ b$	n_2	$b \succ a \succ c$	n_3	(1)
$b \succ c \succ a$	n_4	$c \succ a \succ b$	n_5	$c \succ b \succ a$	n_6	

The notation $a \succ b \succ c$ in (1) means that the most preferred candidate of a voter is a, the middle-ranked candidate is b, and the least preferred one is c. A voting situation can be defined by the 6-tuple $\tilde{n} = (n_1, n_2, \ldots, n_6)$, where n_i denotes the number of voters endowed with the associated i^{th} preference ranking, such that $\sum_{i=1}^{6} n_i = n$. Let us denote by $a\mathbf{M}b$ the event that candidate a defeats b in a pairwise majority comparison, i.e., when more voters are endowed with $a \succ b$ in their preference rankings than with $b \succ a$. A Condorcet winner exists in a voting situation if there is a candidate who would be able to defeat any other opponent in pairwise majority comparisons. For instance, candidate a is a Condorcet winner if both $a\mathbf{M}b$ and $a\mathbf{M}c$ hold, which is equivalent to respectively $n_1 + n_2 + n_5 > n_3 + n_4 + n_6$ and $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$ following our notation in (1). It is well known that such a candidate does not necessarily exist which means that cycles of types $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$ or $b\mathbf{M}a$, $a\mathbf{M}c$ and $c\mathbf{M}b$ can be observed in our framework. The Condorcet efficiency of any given voting rule is the conditional probability that such a voting rule selects the Condorcet winner, given that such a candidate exists.

In our setting of three-candidate elections, Weighted Scoring Rules (WSRs) can be represented by the vector of weights $(1, \lambda, 0)$ such that $0 \leq \lambda \leq 1$. In other words, each of the n voters assigns 1 point to her most preferred candidate, λ points to her middle-ranked candidate, and 0 points to her least preferred candidate. A candidate's score is the total number of points summed over all voters, and the winner is the candidate with the the highest total score from the voters. Let $S(a)$, $S(b)$ and $S(c)$ be the scores of candidates a, b, and c, respectively, under the WSR with weights $(1, \lambda, 0)$. Taking into account our notation in (1), the scores of the candidates a, b, and c are the following:

$$S(a) = n_1 + n_2 + \lambda\,(n_3 + n_5) \tag{2}$$

$$S(b) = n_3 + n_4 + \lambda\,(n_1 + n_6) \tag{3}$$

$$S(c) = n_5 + n_6 + \lambda\,(n_2 + n_4) \tag{4}$$

To illustrate how a WSR works, let us assume that candidate a is the winner under the WSR with weights $(1, \lambda, 0)$. In such a case, $S(a)$ has to be greater than both $S(b)$ and $S(c)$, i.e., $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_3 + n_4 + \lambda(n_1 + n_6)$ and $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_5 + n_6 + \lambda\,(n_2 + n_4)$, respectively. Moreover, Weighted Scoring Elimination Rules (WSERs) can also be represented by the vector of weights $(1, \lambda, 0)$ in three-candidate elections. The two-stage election process works as follows: at the first step, the lowest scored candidate under the corresponding WSR is eliminated; in the second step, the candidate, with the highest number of votes between the two remaining candidates, wins. For instance, assuming that candidate c is the last ranked one under the WSR with weights $(1, \lambda, 0)$, the candidate a will be the winner under the corresponding WSER if the three following inequalities hold: $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_5 + n_6 + \lambda\,(n_2 + n_4)$, $n_3 + n_4 + \lambda\,(n_1 + n_6) > n_5 + n_6 + \lambda\,(n_2 + n_4)$ and $n_1 + n_2 + n_5 > n_3 + n_4 + n_6$.

In this paper, we focus on the following well known WSRs and WSERs:

1. Plurality Rule (PR) which is the WSR with $\lambda = 0$. PR counts the number of times each candidate is first ranked, and the winner is the candidate that gets the highest number of first ranks.
2. Plurality Elimination Rule (PER) which is the WSER with $\lambda = 0$. In the first step, the candidate with the fewest number of votes under PR is eliminated; in the second step, the winner is the candidate with the highest number of votes between the two remaining candidates.
3. Negative Plurality Rule (NPR) which is the WSR with $\lambda = 1$. NPR counts the number of times each candidate is ranked last, and the winner is the candidate that gets the fewest number of last ranks.
4. Negative Plurality Elimination Rule (NPER) which is the WSER with $\lambda = 1$. This two stage voting rule operates in the same fashion as PER, with NPR being used in the first step to determine the candidate to be eliminated from further consideration.
5. Borda Rule (BR) which is the WSR with $\lambda = \frac{1}{2}$. Under this voting rule, voters assign one point to their most preferred candidate, one-half point to their middle-ranked candidate and zero points to their least preferred candidate. The winner is the candidate who receives the greatest total number of points form the voters.

As mentioned before, BER is another well known WSER. More precisely, in a three-candidate election, BER is defined as a two-step voting rule with BR being used in the first stage following the same reasoning as PER and NPER. However, as it is shown in Gehrlein and Lepelley (2015), this rule is the only WSER that guarantees the selection of the Condorcet winner when such a candidate exists. Consequently, the study of this rule is out of the scope of this paper.

In order to measure the closeness of an election, recall that the closeness index that we consider is the ratio between the score of the last ranked candidate and the sum of the scores of all competing candidates. Notice that the considered index increases when elections become closer, reaching the value of $\frac{1}{3}$ when the three candidates obtain approximately the same score. Clearly, PR and PER share the same closeness index since the two voting rules use the same weight, $\lambda = 0$. This is also true when considering NPR and NPER where $\lambda = 1$.

Without loss of generality, assume that candidate c is last ranked under the considered voting rule and let α_1, α_2, and α_3 denote the closeness indices of PR/PER, NPR/NPER, and BR, respectively. Taking into account our notation in (1), the closeness indices α_1, α_2, and α_3 are computed as follows:

$$\alpha_1 = \frac{n_5 + n_6}{n} \tag{5}$$

$$\alpha_2 = \frac{n_2 + n_4 + n_5 + n_6}{2\,n} \tag{6}$$

$$\alpha_3 = \frac{2\,(n_5 + n_6) + n_2 + n_4}{3\,n} \tag{7}$$

Recall that the objective of this paper is to derive the effect of closeness on the CE of PR, PER, NPR, NPER, and BR. To find our probabilities, we need to assume a probability distribution that underlies how individual preferences are considered. The probabilities that we investigate are driven by the well known Impartial Anonymous Culture (IAC) condition (Gehrlein and Fishburn, 1976). In our three-candidate setting, it states that all voting situations $\tilde{n} = (n_1, n_2, \ldots, n_6)$, such that $\sum_{i=1}^{6} n_i = n$, for a specified number of voters n are equally likely to be observed. As long as we take into account only voting events where elections are supposed to be closely contested, we define the α_i-IAC assumption, where α is the closeness index and $i = 1, 2, 3$, based on the IAC condition as follows: all possible voting situations $\tilde{n} = (n_1, n_2, \ldots, n_6)$ having a concrete value of α_i are equally likely to be observed. To derive our probabilities, we use the *parameterized Barvinok's algorithm* (see for instance, Verdoolaege et al., 2004; Bruynooghe et al., 2005; Lepelley et al., 2008). This algorithm allows us to compute the number of integer solutions for systems of inequalities with parameters. The representation of this number is given by *quasi-polynomials* with periodic coefficients (see for instance, Ehrhart, 1962, 1967). Further results based on this algorithm has been provided by Bubboloni et al. (2019); Diss (2015); Diss et al. (2012); Diss and Pérez-Asurmendi (2016), among others. For our concern of large electorates, it is possible to obtain the representation of the CE of the considered WSRs and WSERs as a function of the corresponding closeness index α_i, with $i = 1, 2, 3$.

3 Results and discussions

Propositions 1 provides the CE of PR under the α_1-IAC assumption as a function of its closeness index α_1.

Proposition 1 Consider a three-candidate election with large electorates and α_1 the proportion of points obtained by the last ranked candidate over the total number of points under PR. Then, the CE of PR under the α_1-IAC assumption is given as follows:

$$CE_{PR}^{\infty}(\alpha_1) = \begin{cases} \dfrac{72\,\alpha_1{}^3 - 22\,\alpha_1{}^2 - 27\,\alpha_1 + 8}{8(11\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1)} & \text{for } 0 \leq \alpha_1 < \dfrac{1}{4} \\[2ex] \dfrac{(2\,\alpha_1 - 1)\left(12\,\alpha_1{}^2 - 15\,\alpha_1 + 5\right)}{4(18\,\alpha_1{}^3 - 18\,\alpha_1{}^2 + 6\,\alpha_1 - 1)} & \text{for } \dfrac{1}{4} \leq \alpha_1 \leq \dfrac{1}{3} \end{cases}$$

Propositions 2 provides the CE of PER under the α_1-IAC assumption as a function of its closeness index α_1.

Proposition 2 Consider a three-candidate election with large electorates and α_1 the proportion of points obtained by the last ranked candidate over the total number of points under PR. Then, the CE of PER under the α_1-IAC assumption is given as follows:

$$CE_{PER}^{\infty}(\alpha_1) = \begin{cases} \dfrac{10\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1}{11\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1} & \text{for } 0 \le \alpha_1 < \dfrac{1}{4} \\[2ex] \dfrac{(2\,\alpha_1 - 1)\left(6\,\alpha_1{}^2 - 3\,\alpha_1 + 1\right)}{18\,\alpha_1{}^3 - 18\,\alpha_1{}^2 + 6\,\alpha_1 - 1} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases}$$

Notice that $CE_{PR}^{n}(0) = 1$ for all n. To show this, let us assume, without loss of generality, that a is the Condorcet winner and c is the last ranked candidate under PR. If $\alpha_1 = 0$, then $n_5 = n_6 = 0$. Therefore, the scores of the candidates a and b under PR are $S(a) = n_1 + n_2$ and $S(b) = n_3 + n_4$, respectively. Since a beats b in pairwise majority comparisons ($n_1 + n_2 > n_3 + n_4$), we have $S(a) > S(b)$. In such a case, candidate a will be elected under PR with absolute certainty. Similarly, it is also possible to show that $CE_{PER}^{n}(0) = 1$ for any given number of voters n. In Figure 1, we represent graphically the results from Propositions 1 and 2. Specifically, we illustrate the CE of PR and PER according to their closeness index α_1. Clearly, closeness significantly affects the CE of both voting rules. The CEs of PR and PER tends to dramatically decline as the election becomes closely contested. Notice that, in both cases, the decrease is stronger when α_1 belongs to the interval $[\frac{1}{4}, \frac{1}{3}]$. Nevertheless, the decrease is larger for PR than for PER. In the case of PR, the CE tends to a value of $\frac{1}{3}$ whereas in the case of PER, the CE tends to a value of $\frac{2}{3}$. In other words, PER remains more Condorcet consistent than PR over all the range of the closeness index α_1.

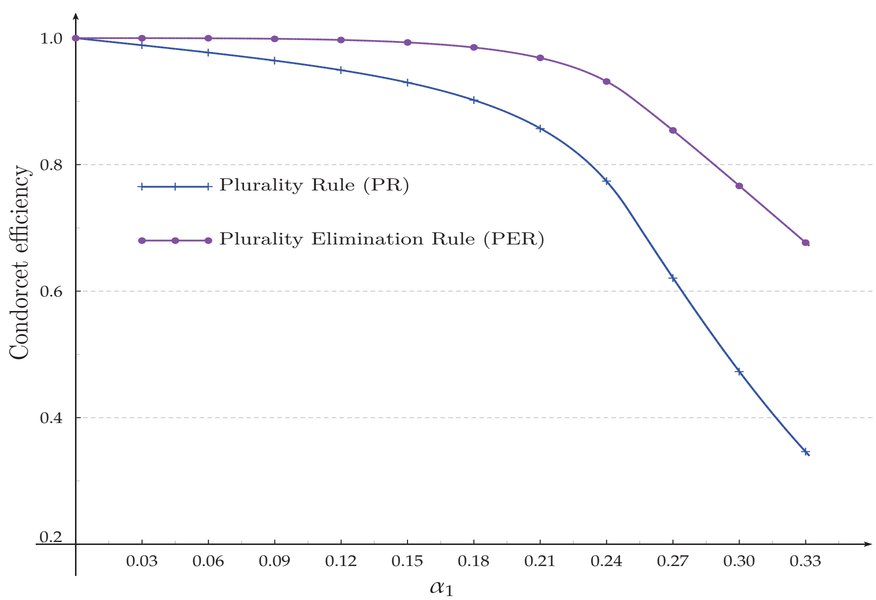

Fig. 1 Condorcet efficiency of PR and PER as a function of their closeness index for large electorates

Propositions 3 provides the CE of NPR under the α_2-IAC as a function of the closeness index α_2.

Proposition 3 Consider a three-candidate election with large electorates and α_2 the proportion of points obtained by the last ranked candidate over the total number of points under NPR. Then, the CE of NPR under the α_2-IAC assumption is given as follows:

$$CE^{\infty}_{NPR}(\alpha_2) = \begin{cases} \dfrac{5\,\alpha_2 - 4}{8\,(2\,\alpha_2 - 1)} & \text{for } 0 \le \alpha_2 < \dfrac{1}{4} \\[2ex] \dfrac{510\,\alpha_2{}^3 - 510\,\alpha_2{}^2 + 165\,\alpha_2 - 17}{2\,(144\,\alpha_2{}^3 - 144\,\alpha_2{}^2 + 48\,\alpha_2 - 5)} & \text{for } \dfrac{1}{4} \le \alpha_2 \le \dfrac{1}{3} \end{cases}$$

Propositions 4 gives the CE of NPER under the α_2-IAC as a function of the closeness index α_2.

Proposition 4 Consider a three-candidate election with large electorates and α_2 the proportion of points obtained by the last ranked candidate over the total number of points under NPR. Then, the CE of NPER under the α_2-IAC assumption is given as follows:

$$CE^{\infty}_{NPER}(\alpha_2) = \begin{cases} 1 & \text{for } 0 \le \alpha_2 < \dfrac{1}{4} \\[2ex] \dfrac{2\,(120\,\alpha_2{}^3 - 120\,\alpha_2{}^2 + 39\,\alpha_2 - 4)}{144\,\alpha_2{}^3 - 144\,\alpha_2{}^2 + 48\,\alpha_2 - 5} & \text{for } \dfrac{1}{4} \le \alpha_2 \le \dfrac{1}{3} \end{cases}$$

Notice that for any given number of voters n, $CE^{n}_{NPR}(0) = \frac{1}{2}$. Indeed, when $\alpha_2 = 0$ (i.e., $n_2 = n_4 = n_5 = n_6$) and $\lambda = 1$, the scores of candidates a, b, and c are given by $S(a) = S(b) = n_1 + n_3$ and $S(c) = 0$. Thus, NPR will elect a and b with the same probability due to the symmetry of IAC-like assumption with respect to candidates. This means that $CE^{n}_{NPR}(0) = \frac{1}{2}$. Notice also that for any given number of voters n, $CE^{n}_{NPER}(\alpha_2) = 1$ for $0 \le \alpha_2 < \frac{1}{4}$. To prove this statement, suppose that the Condorcet winner, say a, is not elected under NPER. This implies that a is eliminated in the first round under NPR; otherwise, she would win the election since she is supposed to be the Condorcet winner. In addition, α_2 is supposed to be less than $\frac{1}{4}$, which implies that $n_1 + n_2 + n_3 + n_5 < n_4 + n_6$ (i). Since $a\mathbf{M}b$, then we can show that $n_3 + n_4 + n_6 + n_3 < n_1 + n_2 + n_3 + n_5$ (ii). From (i) and (ii), we deduce that $n_3 + n_4 + n_6 + n_3 < n_4 + n_6$, which implies that $n_3 < 0$. Because of this contradiction, candidate a will be elected with absolute certainty under NPER. We plot the results from Propositions 3 and 4 in Figure 2 supplying a graphical representation of the CE of NPR and NPER as a function of the closeness index α_2. From Figure 2, it is clear that the performance of NPER in terms of the CE is significantly better than the one of NPR with independence of the value of α_2. More specifically, the CE of NPER takes values in the interval $]\frac{2}{3}, 1]$ whereas in the case of NPR the CE values are located in the range $]\frac{1}{3}, 0.6875]$. Recall that in the case of NPER the CE reaches the value of 1 over the range $0 \leq \alpha_2 \leq \frac{1}{4}$; for values of α_2 greater than $\frac{1}{4}$, that is, as elections are very close, we found that the CE decreases until the value of $\frac{2}{3}$. In the case of NPR, the behavior of the CE is slightly different for lower values of α_2. To be more concrete, it increases from 0 to 0.25 and decreases from 0.25 to $\frac{1}{3}$.

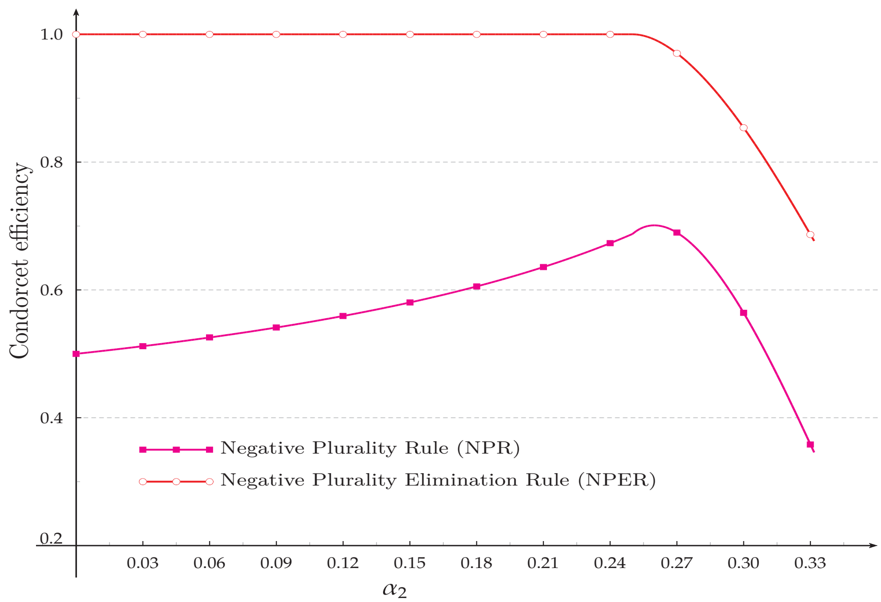

Fig. 2 Condorcet efficiency of NPR and NPER as a function of their closeness index for large electorates

Finally, Proposition 5 provides the CE of BR under the α_3-IAC assumption as a function of the closeness index α_3.

Proposition 5 Consider a three-candidate election with large electorates and α_3 the proportion of points obtained by the last ranked candidate over the total number of points under BR. Then, the CE of BR under the α_3-IAC assumption is given as follows:

$$CE^{\infty}_{BR}(\alpha_3) = \begin{cases} \dfrac{21\,\alpha_3 - 8}{2\,(9\,\alpha_3 - 4)} & \text{for } 0 \leq \alpha_3 < \dfrac{1}{9} \\[2ex] \dfrac{243\,\alpha_3{}^4 + 324\,\alpha_3{}^3 - 486\,\alpha_3{}^2 + 36\,\alpha_3 - 1}{648\,\alpha_3{}^3\,(9\,\alpha_3 - 4)} & \text{for } \dfrac{1}{9} \leq \alpha_3 < \dfrac{1}{6} \\[2ex] \dfrac{70227\,\alpha_3{}^4 - 46332\,\alpha_3{}^3 + 11178\,\alpha_3{}^2 - 1260\,\alpha_3 + 53}{75816\,\alpha_3{}^4 - 49248\,\alpha_3{}^3 + 11664\,\alpha_3{}^2 - 1296\,\alpha_3 + 54} & \text{for } \dfrac{2}{9} \leq \alpha_3 < \dfrac{5}{18} \\[2ex] \dfrac{20331\,\alpha_3{}^4 - 23436\,\alpha_3{}^3 + 9234\,\alpha_3{}^2 - 1500\,\alpha_3 + 89}{2\,(13608\,\alpha_3{}^4 - 15120\,\alpha_3{}^3 + 5832\,\alpha_3{}^2 - 936\,\alpha_3 + 55)} & \text{for } \dfrac{5}{18} \leq \alpha_3 < \dfrac{2}{9} \\[2ex] \dfrac{25587\,\alpha_3{}^2 - 11466\,\alpha_3 + 1339}{16\,(1647\,\alpha_3{}^2 - 732\,\alpha_3 + 85)} & \text{for } \dfrac{2}{9} \leq \alpha_3 \leq \dfrac{1}{3} \end{cases}$$

It can be noticed that $CE^{n}_{BR}(0) = 1$ for any given number of voters n. To show this statement, let us assume, without loss of generality, that candidate a is the Condorcet winner while c is the last ranked candidate under BR. If $\alpha_3 = 0$, it follows that $n_2 = n_4 = n_5 = n_6$. In such a case, $S(a) - S(b) = \frac{n_1 - n_3}{3} > 0$ because a beats c in pairwise majority comparisons ($n_1 > n_3$). Since candidate c is supposed to receive zero points, candidate a is elected under BR with absolute certainty. We represent graphically the results from Proposition 5 in Figure 3. As it can be seen, the CE of BR ranges within the interval $[0.8983, 1]$. The CE decreases when the closeness index takes values from 0 to 0.22 whereas it increases when the closeness index ranges from 0.22 to $\frac{1}{3}$.

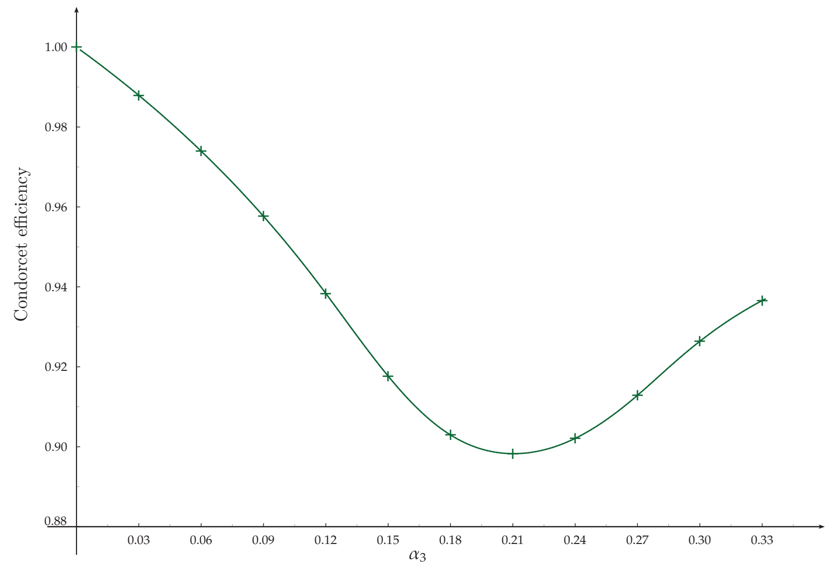

Fig. 3 Condorcet efficiency of BR as a function of its closeness index for large electorates

Finally, Table 2 summarizes some numerical values of the CE of the five considered WSRs and WSERs for different values of the closeness index α_i, with $i = 1, 2, 3$.

Table 2 Computed values of the CE of PR, BR, NPR, PER, and NPER for large electorates

α_i	$CE^{\infty}_{PR}(\alpha_1)$	$CE^{\infty}_{PER}(\alpha_1)$	$CE^{\infty}_{NPR}(\alpha_2)$	$CE^{\infty}_{NPER}(\alpha_2)$	$CE^{\infty}_{BR}(\alpha_3)$
0	1	1	0.5	1	1
0.02	0.9925	1.0000	0.5078	1	0.9921
0.04	0.9850	0.9999	0.5163	1	0.9835
0.06	0.9772	0.9997	0.5256	1	0.9740
0.08	0.9689	0.9993	0.5357	1	0.9634
0.1	0.9598	0.9985	0.5469	1	0.9516
0.12	0.9494	0.9971	0.5592	1	0.9384
0.14	0.9370	0.9948	0.5729	1	0.9243
0.16	0.9218	0.9911	0.5882	1	0.9118
0.18	0.9020	0.9852	0.6055	1	0.9030
0.20	0.8750	0.9756	0.6250	1	0.8988
0.22	0.8357	0.9596	0.6473	1	0.8987
0.24	0.7736	0.9315	0.6731	1	0.9021
0.25	0.7273	0.9091	0.6875	1	0.9050
0.26	0.6737	0.8821	0.7011	0.9917	0.9087
0.28	0.5698	0.8254	0.6606	0.9390	0.9175
0.3	0.4731	0.7665	0.5640	0.8537	0.9264
0.32	0.3857	0.7067	0.4320	0.7457	0.9336
$(\frac{1}{3})^-$	0.3333	0.6667	0.3333	0.6667	0.9375

4 Conclusion

The main purpose of this study has been to provide new evidence of the effect of election closeness on the theoretical probability of electoral events. We have considered the impact of election closeness on the probability of selecting the Condorcet winner, when such a candidate exists, under several well known voting rules. In other words, the main aim of our study is to measure at which extend the CE of a given voting rule changes when the elections become more closely contested. For this purpose, we have focussed on five popular WSRs and WSERs in the context of three-candidate elections. The first three rules, PR, NPR, and BR, choose the winner in one step while the last two rules, PER and NPER, do in a two-step iterative process. Election closeness has been measured in our paper by an index calculated as a proportion of points obtained by the last ranked candidate divided by the aggregated scores of all competing candidates under the given WSR/WSER. We followed an IAC-like assumption, by considering that every voting situation, with a given value of election closeness index, is equally likely to occur. As a result, we calculate the CE of the considered WSRs and WSERs for large electorates as a function of the corresponding closeness index. We show that the CE of some WSRs and WSERs may significantly decrease as the results of elections become very close. However, such reduction varies depending on the considered voting rule; the CE does not substantially decrease under BR as it does in the case of the other analyzed WSRs and WSERs.

Finally, we believe that many extensions of our paper can be considered since many open questions still remain unanswered. given that we have only studied the CE of some common WSRs and WSERs, the extension of our results to other voting rules remains open. In addition, we have analyzed in this paper the performance of several voting rules according to their CE but it is worthy to analyze the impact of election closeness on other interesting voting paradoxes. The reader can find an overview about different voting paradoxes that can be considered for this exciting topic in Gehrlein and Lepelley (2011, 2017) and Nurmi (1999).

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5 Appendix

We only provide the proof of Proposition 1 which will allow the reader to understand the steps taken into account in order to derive the analytical representations for the CE under the α_i-IAC assumption. Complete proofs for the other voting rules are available upon request.

Let us assume, without loss of generality, that candidate c is the last ranked one under PR. In such a case, the closeness index is given by $\alpha_1 = \frac{n_5+n_6}{n} = \frac{k}{n}$ where $k = n_5 + n_6$ is the score of candidate c under PR. Recall that the CE of a giving voting rule is a conditional probability. In our setting, in order to compute the CE of PR under the α_1-IAC assumption, we first need to count the number of voting situations for which the Condorcet winner exists under the α_1-IAC assumption when candidate c is the last ranked one under PR. In order to accomplish this goal, we need to consider the three following independent events:

X_1 = "a is the Condorcet winner and c is last ranked under PR".

X_2 = "b is the Condorcet winner and c is last ranked under PR".

X_3 = "c is the Condorcet winner and c is last ranked under PR".

Let us denote by $|D_{X_j}(k,n)|$ the number of voting situations for which event X_j is observed under the α_1-IAC assumption, i.e., when $\alpha_1 = \frac{k}{n}$ takes a given value. The number $|D_{X_j}(k,n)|$ depends on the number of voters n and the score k of the candidate c under PR. Using those notations, the number of voting situations for which the Condorcet winner exists under the α_1-IAC assumption when candidate c is the last ranked one under PR can be written as follows:

$$|D_{X_1}(k,n)| + |D_{X_2}(k,n)| + |D_{X_3}(k,n)| \tag{8}$$

Due to the symmetry of IAC-like assumptions with respect to candidates, we can easily show that $|D_{X_1}(k,n)| = |D_{X_2}(k,n)|$. This means that the number of voting situations in (8) can also be written as follows:

$$2\,|D_{X_1}(k,n)| + |D_{X_3}(k,n)| \tag{9}$$

Thus, all that we have to do is to calculate $|D_{X_1}(k,n)|$ and $|D_{X_3}(k,n)|$. Notice first that $|D_{X_1}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_5 + n_6 = k \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1, \ldots, 6\} \\ k \geq 0 \end{cases} \tag{10}$$

As noticed before, we compute the number of voting situations that fulfill these conditions using the Parametrized Barvinok's algorithm. Such algorithm allows to quantify the number of integer solutions for systems of (in)equalities with parameters. In our study, given the two parameters n and k, the number of voting situations for the system (10) is provided by bivariate quasi-polynomials in n and k with 2-periodic coefficients meaning that such coefficients depend on the parity of the parameters n and k. We represent these coefficients by a list of 2-rational numbers enclosed in square brackets. To illustrate, the coefficient $[a, b]_n$ will be either a when n is even or b when n is odd. The program indicates that the corresponding quasi-polynomial for the system (10) is given as follows:

1. If $\frac{n}{4} \leq k \leq \frac{n-2}{3}$:

$$|D_{X_1}(k,n)| = \frac{3}{2}\,k^4 + f_1\,k^3 + f_2\,k^2 + f_3\,k + f_4$$

where,

$$f_1 = -2\,n + \left[\frac{7}{2}, 2\right]_n$$
$$f_2 = \frac{9}{8}\,n^2 + \left[-\frac{7}{2}, -\frac{3}{2}\right]_n + \left[1, \frac{7}{8}\right]_n$$
$$f_3 = -\frac{1}{3}\,n^3 + \left[\frac{9}{8}, \frac{1}{8}\right]_n n^2 + \left[-\frac{2}{3}, -\frac{7}{6}\right]_n n + \left[0, -\frac{5}{8}\right]_n$$
$$f_4 = \frac{1}{24}\,n^4 + \left[-\frac{1}{8}, \frac{1}{24}\right]_n n^3 + \left[\frac{1}{12}, \frac{5}{24}\right]_n n^2 + \left[0, -\frac{1}{24}\right]_n n + \left[0, -\frac{1}{4}\right]_n$$

2. If $0 \leq k \leq \frac{n-4}{4}$:

$$|D_{X_1}(k,n)| = \frac{5}{6}\,k^4 + g_1\,k^3 + g_2\,k^2 + g_3\,k + g_4$$

where,

$$g_1 = -\frac{1}{3}\,n + \left[\frac{19}{6}, \frac{10}{3}\right]_n$$
$$g_2 = -\frac{1}{4}\,n^2 + \left[-\frac{11}{4}, -3\right]_n\, n + \left[\frac{5}{3}, \frac{17}{12}\right]_n$$
$$g_3 = \frac{1}{12}\,n^3 + \left[\frac{1}{8}, \frac{1}{4}\right]_n\, n^2 + \left[-3, -\frac{11}{4}\right]_n\, n + \left[-\frac{5}{3}, -\frac{19}{12}\right]_n$$
$$g_4 = \frac{1}{12}\,n^3 + \left[\frac{3}{8}, \frac{1}{2}\right]_n\, n^2 + \left[-\frac{7}{12}, -\frac{1}{12}\right]_n\, n + \left[-1, -\frac{1}{2}\right]_n$$

$|D_{X_3}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_5 + n_6 = k \\ -n_1 + n_2 - n_3 - n_4 + n_5 + n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1,\ldots,6\} \\ k \geq 0 \end{cases} \tag{11}$$

Using again the Parametrized Barvinok's algorithm, the program indicates that the corresponding quasi-polynomial for the system (11) is given as follows:

1. If $\frac{n-1}{4} \leq k \leq \frac{n-2}{3}$:

$$|D_{X_3}(k,n)| = \frac{3}{2}k^4 + h_1\,k^3 + h_2\,k^2 + h_3\,k + h_4$$

   where,

$$h_1 = -2\,n + \left[\frac{7}{2}, 2\right]_n$$
$$h_2 = \frac{3}{4}\,n^2 + \left[-\frac{7}{2}, -3\right]_n\, n + \left[\frac{5}{2}, -\frac{1}{4}\right]_n$$
$$h_3 = -\frac{1}{12}\,n^3 + n^2 + \left[-\frac{5}{3}, -\frac{11}{12}\right]_n\, n + \left[\frac{1}{2}, -1\right]_n$$
$$h_4 = -\frac{1}{12}\,n^3 + \frac{1}{4}\,n^2 + \left[-\frac{1}{6}, \frac{1}{12}\right]_n\, n + \left[0, -\frac{1}{4}\right]_n$$

2. If $0 \leq k \leq \frac{n-2}{4}$:

$$|D_{X_3}(k,n)| = \frac{1}{6}\,k^4 + \left[\frac{1}{6}, \frac{2}{3}\right]_n\, k^3 + \left[-\frac{1}{6}, \frac{5}{6}\right]_n\, k^2 + \left[-\frac{1}{6}, \frac{1}{3}\right]_n\, k$$

In order to calculate the CE of PR under the α_1-IAC assumption, the three following independent events have to be taken into consideration:

Y_1 = "a is the Condorcet winner, a is chosen under PR, and c is last ranked under PR".
Y_2 = "b is the Condorcet winner, b is chosen under PR, and c is last ranked under PR".
Y_3 = "c is the Condorcet winner, c is chosen under PR, and c is last ranked under PR".

It follows that the CE of PR under the α_1-IAC assumption is given in general by the following function in n and k:

$$\frac{|D_{Y_1}(k,n)| + |D_{Y_2}(k,n)| + |D_{Y_3}(k,n)|}{2\,|D_{X_1}(k,n)| + |D_{X_3}(k,n)|} \tag{12}$$

We can show that $|D_{Y_3}(k,n)| = 0$ because when candidate c is chosen under PR it cannot be last ranked by this voting rule. Again, due to the symmetry of IAC-like assumptions with respect to candidates, we can also show that $|D_{Y_1}(k,n)| = |D_{Y_2}(k,n)|$. It follows that the CE of PR under the α_1-IAC assumption in (12) can also be calculated as follows:

$$\frac{2\,|D_{Y_1}(k,n)|}{2\,|D_{X_1}(k,n)| + |D_{X_3}(k,n)|} \tag{13}$$

$|D_{Y_1}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 > 0 \\ n_5 + n_6 = k \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1,\ldots,6\} \\ k \geq 0 \end{cases} \tag{14}$$

The program indicates that the corresponding quasi-polynomial of the system (14) is given as follows:

1. If $k \leq (n-4)/4$:

$$|D_{Y_1}(k,n)| = \frac{3}{4}k^4 + F_1(k,n)k^3 + F_2(k,n)k^2 + F_3(k,n)k + F_4(k,n)$$

where,

$$F_1(k,n) = -\frac{11}{48}n + \left[\left[\frac{19}{6}, \frac{53}{16}\right]_n, \left[\frac{79}{24}, \frac{51}{16}\right]_n\right]_k$$

$$F_2(k,n) = -\frac{9}{32}n^2 + \left[\left[-\frac{21}{8}, -\frac{23}{8}\right]_n, \left[-\frac{45}{16}, -\frac{43}{16}_n\right]_k\right]n + \left[\left[2, \frac{53}{32}\right]_n, \left[\frac{7}{4}, \frac{59}{32}\right]_n\right]_k$$

$$F_3(k,n) = \frac{1}{12}n^3 + \left[\left[\frac{1}{16}, \frac{3}{16}\right]_n, \left[\frac{1}{8}, \frac{1}{8}\right]_n\right]_k n^2 + \left[\left[-\frac{19}{6}, -\frac{35}{12}\right]_n, \left[-\frac{149}{48}, -\frac{137}{48}\right]_n\right]_k n + \left[\left[-5/3, -\frac{27}{16}\right]_n, \left[-\frac{43}{24}, -\frac{21}{16}\right]_n\right]_k$$

$$F_4(k,n) = \frac{1}{12}n^3 + \left[\left[\frac{3}{8}, \frac{1}{2}\right]_n, \left[\frac{13}{32}, \frac{13}{32}\right]_n\right]_k n^2 + \left[\left[-\frac{7}{12}, -\frac{1}{12}\right]_n, \left[-\frac{25}{48}, -\frac{19}{48}\right]_n\right]_k n + \left[\left[-1, -\frac{1}{2}\right]_n, \left[-1, -\frac{23}{32}\right]_n\right]_k$$

2. If $\frac{n-3}{4} \leq k \leq \frac{n-3}{3}$:

$$|D_{Y_1}(k,n)| = \frac{3}{4}k^4 + G_1(k,n)k^3 + G_2(k,n)k^2 + G_3(k,n)k + G_4(k,n)$$

where,

$$G_1(k,n) = -\frac{25}{16}n + \left[\left[\frac{1}{2}, -\frac{11}{16}\right]_n, \left[\frac{5}{8}, -\frac{13}{16}\right]_n\right]_k$$

$$G_2(k,n) = \frac{39}{32}n^2 + \left[\left[-\frac{5}{8}, \frac{9}{8}\right]_n, \left[-\frac{13}{16}, \frac{21}{16}\right]_n\right]_{kn} + \left[\left[0, \frac{5}{32}\right]_n, \left[-\frac{1}{4}, \frac{11}{32}\right]_n\right]_k$$

$$G_3(k,n) = -\frac{5}{12}n^3 + \left[\left[\frac{5}{16}, -\frac{9}{16}\right]_n, \left[\frac{3}{8}, -\frac{5}{8}\right]_n\right]_k n^2 + \left[\left[\frac{1}{6}, -\frac{1}{12}\right]_n, \left[\frac{11}{48}, -\frac{1}{48}\right]_n\right]_k n + \left[\left[0, \frac{1}{16}\right]_n, \left[-\frac{1}{8}, \frac{7}{16}\right]_n\right]_k$$

$$G_4(k,n) = \frac{5}{96}n^4 + \left[-\frac{1}{16}, \frac{1}{12}\right]_n n^3 + \left[\left[-\frac{1}{12}, -\frac{1}{48}\right]_n, \left[-\frac{5}{96}, -\frac{11}{96}\right]_n\right]_k n^2 + \left[\left[0, -\frac{1}{12}\right]_n, \left[\frac{1}{16}, -\frac{19}{48}\right]_n\right]_k n + \left[\left[0, -\frac{1}{32}\right]_n, \left[0, -\frac{1}{4}\right]_n\right]_k$$

3. Otherwise, $|D_{Y_1}(k,n)| = 0$.

Notice that it is possible to represent the above results as functions of the closeness index α_1. If we assume large electorates and replace k by $\alpha_1\, n$ in the above results, we obtain functions in α_1 by only considering the terms of higher degree in each function. Let us then denote by $|D^{\infty}_{X_j}(\alpha_1)|$ the number of voting situations for which event X_j is observed under the α_1-IAC assumption with large electorates. It follows that:

$$|D_{X_1}^{\infty}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1\left(10\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1\right)\,n^4}{12} & \text{for } 0 \le \alpha_1 \le \dfrac{1}{4} \\ \dfrac{(3\,\alpha_1 - 1)\,(2\,\alpha_1 - 1)\left(6\,\alpha_1{}^2 - 3\,\alpha_1 + 1\right)\,n^4}{24} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases} \tag{15}$$

$$|D_{X_3}^{\infty}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1^4\,n^4}{6} & \text{for } 0 \le \alpha_1 \le \dfrac{1}{4} \\ \dfrac{\alpha_1\,(3\,\alpha_1 - 1)\left(6\,\alpha_1{}^2 - 6\,\alpha_1 + 1\right)\,n^4}{12} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases} \tag{16}$$

$$|D_{Y_1}^{\infty}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1\left(72\,\alpha_1{}^3 - 22\,\alpha_1{}^2 - 27\,\alpha_1 + 8\right)\,n^4}{96} & \text{for } 0 \le \alpha_1 \le \dfrac{1}{4} \\ \dfrac{(3\,\alpha_1 - 1)\,(2\,\alpha_1 - 1)\,n^4}{96} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases} \tag{17}$$

By replacing (15), (16), and (17) in (13) we derive the CE of PR as a function of the closeness index α_1 for large electorates as follows:

$$CE_{PR}^{\infty}(\alpha_1) = \begin{cases} \dfrac{72\,\alpha_1{}^3 - 22\,\alpha_1{}^2 - 27\,\alpha_1 + 8}{8(11\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1)} & \text{for } 0 \le \alpha_1 < \dfrac{1}{4} \\ \dfrac{(2\,\alpha_1 - 1)\left(12\,\alpha_1{}^2 - 15\,\alpha_1 + 5\right)}{4(18\,\alpha_1{}^3 - 18\,\alpha_1{}^2 + 6\,\alpha_1 - 1)} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases}$$

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