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Moderately Discontinuos Algebraic Topology for Metric Subanalytic Germs

Topología Algebraica Moderadamente Discontinua para Gérmenes Métricos Subanalíticos

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"Time flies like an arrow;
fruit flies like a banana."

Attributed to Anthony Oettinger
and to Groucho Marx

"

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John Cage

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Contents

1 Introduction

We have developed both a homology theory and a homotopy theory in the context of metric subanalytic germs (see Definition 2.1). The former is called *MD homology*. It is covered in Chapter 2, which contains a paper that is joined work with my PhD advisors Javier Fernández de Bobadilla and María Pe Pereira and with Edson Sampaio. The latter is called *MD homotopy* and is covered in Chapter 3. Both theories are functors from a category of germs of metric subanalytic spaces (resp. germs of metric subanalytic spaces that are punctured in a way that will be defined) to a category of commutative diagrams of groups. For the concrete definition of the domain categories see Definition 2.10 and Definition 3.47 respectively; for the target categories see Definition 2.42 and Definition 3.52 respectively. Similarly to classical homology and homotopy theories, the groups appearing in the target category are abelian in the homology theory for any degree and in the homotopy theory for degree $n > 1$. Both theories serve as a bi-Lipschitz subanalytic invariant. Therefore, in the context of real or complex analytic germs equipped with the inner or the outer metric, they are analytic invariants.

The MD homology shares several properties with the singular homology: it is invariant by suitable metric homotopies (see Definition 2.75 and Theorem 2.76 as well as Definition 2.79 and Theorem 2.80); it allows a relative and absolute Mayer-Vietoris long exact sequences (see Theorem 2.91) for a suitable cover of the metric subanalytic germ (see Definition 2.88); and as a consequence we have a certain theorem of excision (see Corollary 2.92) and a Čech spectral sequence (see Theorem 2.93). The MD homotopy has several of the properties of the ordinary homotopy theory of punctured topological spaces: it admits a Hurewicz homomorphism from the MD homotopy to the MD homology (see Proposition 3.40); in degree one, the Hurewicz homomorphism is an isomorphism when abelianizing the domain (see Theorem 3.55); and when the metric subanalytic germ fulfils a certain condition that softens the one of path-connectedness (see Definifion 3.43), it is independent from the choice of base point (see Proposition 3.46).

In our theories the role of simplices (resp. loops and homotopies) is taken by 1-parameter families of simplices (resp. loops and homotopies) with the following property that we call *linearly vertex approaching* (*l.v.a.*, for short). Measuring in the outer distance, when the family parameter approaches zero, the simplex (resp. loop or homotopy) approaches the vertex of the germ at a rate of order one.

The abbreviation MD stands for *moderately discontinuous*. The motivation behind that name in the context of the MD homology is the following. In singular homology an n-chain is a cycle, if its boundary can be written as a sum $\sum_i a_i\sigma_i$, where a_i are elements of the ring of coefficients and σ_i are $(n-1)$-simplices, whose summands cancel in pairs. In the MD-homology an n-chain is a cycle, if its boundary can be written as a sum $\sum_i a_i\sigma_i$, where a_i are elements of the ring of coefficients and σ_i are l.v.a. families of $(n-1)$-simplices, whose summands can be ordered into pairs as follows. For a pair of summands $a_i\sigma_i$ and $a_{i'}\sigma_{i'}$, we have $a_i = a_{i'}$ and the two l.v.a. families of simplices σ_i and $\sigma_{i'}$ approach each other faster than at rate t^b, where t is the family parameter and b is a fixed parameter in $(0,\infty]$. A simple cycle in MD homology is illustrated in Example 2.34. That is how we gain a homology group for any parameter $b \in (0,\infty]$.

In the context of the MD homotopy, we integrate the concept of moderate discontinuities in l.v.a. families of loops from $[0,1]^n$ into the metric subanalytic germ as follows. Broadly speaking, we partition $[0,1]^n$ into a finite number of closed sets and define a l.v.a. family of continuous maps on one of those sets for each set. Then fixing two sets with non-empty intersection, the associated l.v.a. families of continuous maps fulfil the following: restricting to the boundary between the two sets yields two l.v.a. families of continuous functions that approach each other faster than at rate t^b, where t is the family parameter and $b \in (0,\infty]$. Again, we gain a homotopy group for every parameter $b \in (0,\infty]$.

Until here, for any $b \in (0,\infty]$ we have constructed functors that take values in the category of groups. We call them *b-MD homology* and *b-MD homotopy*, respectively. But both our invariants are further enriched by group homomorphisms from the b_1-MD homology/homotopy group to the b_2-MD homology/homotopy group for any $b_1 \geq b_2$. We call those homomorphisms *connecting homomorphisms*. That is why the target categories of the MD homology functor and the MD homotopy functor do not only consist of a family of groups, but also of homomorphisms between those groups; and that is how the morphisms in the target categories become commutative diagrams of groups. Furthermore, the functoriality of the b-MD homology and the b-MD homotopy for a fixed $b \in (0,\infty]$ can be improved by augmenting the domain category by allowing uncommon morphisms that are moderately discontinuous in a way similar to the one described above for l.v.a. families of loops. Those morphisms are called *b-maps* (see Definition 2.59).

There are already two different homology theories in the context of Lipschitz geometry (see [42] [2], and [40]). Those two homology theories are of different nature than the MD homology theory. In those two theories the groups in the underlying chain complex get restricted. That has as a consequence that for a chain it is harder to be a boundary, since the boundary is the image under the boundary operator of a smaller group. Furthermore there are chains that get discarded before taking the homology. In the MD homology the groups of the chain complex get quotient by a certain equivalence relation that we call b-equivalence relation for any parameter $b \in (0,\infty]$. That has as a consequence that for a chain it is easier to be a cycle, since the kernel of the boundary operator increases by quotienting the target. Furthermore there are chains that get identified already in the chain complex before taking the homology.

What we particularly like about the MD homology theory, is that it provides those computational tools mentioned above similarly to the tools in singular homology that make it relatively well computable. We have given examples of computations of both the MD homology and the MD homotopy. In particular, we have given a concrete formula for the MD homology of complex plane algebraic curve germs equipped with the outer metric (see Proposition 2.105). That formula reveals how the MD homology recovers both, all Puiseux pairs of the branches of the curve, and the set of all contact numbers between two branches (see Corollary 2.108). In [38] (see also [29] and [15]), it is shown that the geometric type of a complex plane algebraic curve germ equipped with the outer metric coincides with its embedded topological type. Therefore, the MD homology is a complete invariant of irreducible complex plane algebraic curve germs equipped with the outer metric.

Throughout the thesis, we have chosen to work in the setting of subanalytic germs, but oberve the following remark that is explained in Appendix A in more detail:

Remark A.8. *One could define MD homology and MD homotopy in the context of any O-minimal structure over the reals copying this thesis word by word and replacing subanalytic (or globally subanalytic) by the definable sets and definable maps of that O-minimal structure.*

Observe that this work has laid the ground for possible future work in different directions. For example, it is interesting to explore how strong our invariants are in obstructing Lipschitz equisingularity (see [26]). Its relation with Zariski equisingularity is also worth exploring because of the work done in [28] and [31]. In [22], [30] and [39] subanalytic spaces are decomposed into pieces which are simple from the outer Lipschitz viewpoint. It would be interesting to study the relation of such decompositions for subanalytic germs with our invariants.

1.1. Summary of Chapter 2

In Section 2.1, we define the domain category of the MD homology functor. The objects are pairs of subanalytic germs (X, Y, x_0) equipped with a metric d. We write (X, Y, x_0, d). The morphisms from (X, Y, x_0, d) to (X', Y', x_0', d') are subanalytic map germs of pairs from (X, Y) to (X', Y') that are Lipschitz with respect to d and d' and that are furthermore l.v.a. as mentioned above:

Definition 2.7. *A map germ $f : (X, x_0) \to (Y, y_0)$ is said to be* linearly vertex approaching *(l.v.a. for brevity) if there exists $K \geq 1$ such that*

$$\frac{1}{K}||x - x_0|| \leq ||f(x) - y_0|| \leq K||x - x_0||$$

for every x in some representative of (X, x_0). The constant K is called the l.v.a constant for f.

Our n-simplices are also defined as continuous subanalytic map germs that are l.v.a from the germ of

$$\hat{\Delta}_n := (\{(tx, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : x \in \Delta_n, t \in [0, 1)\},$$

where Δ_n denotes the standard n-simplex, into the metric subanalytic germ (X, x_0, d_X). But they are not defined to be Lipschitz. We call them *l.v.a. n-simplices.* Our n-chains are finite formal sums of n-simplices. We call the abelian groups of n-chains the *pre-chain complex* and denote it by $MDC_\bullet^{\text{pre},\infty}(X, x_0, d_X)$. It is associated to the pair $(X, \emptyset)$.

In the sequel, we develop two different equivalence relations on the pre-chain complex (see Subsection 2.2.2 to Subsection 2.2.4): one by homological subdivisions and one by b-equivalences. The one by homological subdivisions are simply meant to be a technical tool. They simplify constructions for which in the singular homology theory barycentric subdivisions would be used. The relevant identification of n-chains is the one by the b-equivalence relation:

Definition 2.25. *Let $b \in (0, \infty)$. Let σ_1, σ_2 be n-simplices in $MDC_\bullet^{\text{pre},\infty}(X, x_0, d_X)$. We say that σ_1 and σ_2 are* b-equivalent *(we write $\sigma_1 \sim_b \sigma_2$), if*

$$\lim_{t \to 0^+} \frac{\max\{d(\sigma_1(tx, t), \sigma_2(tx, t)); x \in \Delta_n\}}{t^b} = 0.$$

We extend the relation to $MDC_n^{\text{pre},\infty}(X; A)$ by linearity.

For $b = \infty$ we do not impose any b-equivalence relation. The quotient of the pre-chain complex by a combination of both equivalence relations leads to one chain complex for every $b \in (0, \infty]$ whose homology we call the b-MD homology.

In Subsection 2.2.5 we explain how for a pair (X, Y, x_0, d_X) of metric germs, the b-MD homology of Y can be considered a subcomplex of the b-MD homology of X and define the relative chain complex and its homology accordingly. That immediately leads to a long exact relative b-MD homology sequence.

Notice that until here everything has been done for a fixed $b \in (0, \infty]$ and that the b_1-MD chain complex is richer than the b_2-MD chain complex for $b_1 \geq b_2$ since the b_1-equivalence relation is more restrictive than the b_2-equivalence relation. That leads to a natural projection from the former to the latter, which induces a homomorphism in homology. Those homomorphisms are the connecting homomorphisms of the MD-homology that carry an important amount of information of the invariant. Accordingly, in Subsection 2.2.6, the MD homology is defined as a functor from the category of metric subanalytic germs into a category that can be described as follows (see Definition 2.42): the objects are families of abelian groups with family parameter $b \in (0, \infty]$ together with families of group homomorphisms with pairs $(b_1, b_2) \in (0, \infty]^2$, such that $b_1 \geq b_2$, as family parameters; the morphisms are families of commutative diagrams of abelian groups and group homomorphisms with the same pairs as parameters.

In Subsection 2.3.2 we introduce the notion of a point in our category of metric subanalytic germs (see Definition 2.50). Concretely, we mark an object in our category as the object that takes the role of the one-point space in the topological category. Later we introduce a concept that mimics the notion of points in a fixed topological space (see Definition 2.57). That notion is in line with our definition of the object that mimics the one-point set. But be aware: we get a different notion of point for any parameter $b \in (0, \infty]$. The b-MD homology of the point for any parameter $b \in (0, \infty]$ coincides with the singular homology of the one-point space (see Proposition 2.51).

In Subsection 2.3.4 it becomes clear why we have chosen to work with subanalytic simplices. We introduce the notion of *small chains* with respect to a finite closed subanalytic cover of the germ: they are chains for which the image of each simplex is contained in one of the elements of the cover. The two equivalence relations on the chain complex of small chains are defined in the same way as on the chain complex $MDC_\bullet^{\mathrm{pre},\infty}(X, \emptyset, x_0, d_X)$. Thanks to the subanalytic Hauptvermutung we get an isomorphism from the chain complex of small chains onto our usual chain complex for any $b \in (0, \infty]$. That isomorphism turns out to be an important computational tool. Observe also that if we did not include the homological subdivision equivalence relation in our definition of the chain complex, then we would not get that isomorphism. In that case, one could only hope for a quasi-isomorphism and would expect a far harder proof.

We show that the functoriality of the b-MD homology can be improved by augmenting the class of morphisms in the category of metric subanalytic germs by including b-maps as described above. By allowing moderate discontinuities, b-maps and sections for b-maps are easy to find and therefore provide an important computational tool.

In the sequel, we provide the main computational tools in singular homology theory adapted to the MD homology theory. The invariance by homotopies is given for two different notions of homotopy, one of them being the following:

Definition 2.75 (Metric homotopy). *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs. Let $f, g : (X, x_0, d_X) \to (Y, y_0, d_Y)$ be Lipschitz l.v.a. subanalytic maps. A continuous subanalytic map $H : X \times I \to Y$ is called a* metric homotopy between f and g, *if there is a uniform constant $K \geq 0$ such that for any s the mapping $H_s := H(-, s)$ is Lipschitz l.v.a. subanalytic with Lipschitz l.v.a. constant K and $H_0 = f$ and $H_1 = g$.*

In Theorem 2.91 we construct a Mayer-Vietoris long exact sequence for the b-MD homology groups with respect to a cover of the germ, if the cover fulfils a certain metric condition with respect to $b \in (0, \infty]$ (see Definition 2.88). To obtain a relative Mayer-Vietoris long exact sequence, we have to adapt the notion of the chain complex relative to a subgerm (see Definition 2.85). That is why we only get an adapted version of the excision theorem (see Corollary 2.92). The Čech spectral sequence comes along easily (see Theorem 2.93). Still, it provides a powerful tool as can be seen for example in the proof, in which we show that the ∞-MD homology coincides with the simplicial homology of the link (see Theorem 2.101).

Finally in Section 2.9, we compute the MD homology of complex plane algebraic curve germs. We fully describe it via the Eggers-Wall tree of the curve. Thereby we show that it fully recovers the set of all Puiseux pairs of all branches. The example of complex plane algebraic curve germs is a striking example of how important the connecting homomorphisms are since the MD homology groups do not give any information about the Puiseux pairs. In fact, for an irreducible complex plane algebraic curve germ, the MD homology group for any parameter b of degree zero or one coincides with the ring of coefficients and for greater degree is trivial. The MD homology also recovers the set of all contact numbers between two branches. But it does not tell us which branch a Puiseux pair belongs to and which pair of branches a contact number corresponds to. That becomes clear in Example 2.109.

1.2. Summary of Chapter 3

Analogously to the cone over the standard n-simplex $\hat{\Delta}_n$ as defined above, we define the cone over the n-cube I^n to be the germ of

$$C(I^n) := \{(yt, t) \in I^n \times \mathbb{R} : y \in I^n, t \in [0, 1)\}.$$

But the moderate discontinuities in MD homotopy are constructed in a different way than the ones in MD homology. Our loops and homotopies gain the possibility of being moderately discontinuous by defining them to be weak b-maps. Weak b-maps are very similar to b-maps dropping the Lipschitz condition. By not having the Lipschitz condition weak b-maps cannot be composed such as b-maps. But still, they can be concatenated and therefore serve their purpose of providing moderate discontinuous loops and homotopies. Both the definition of b-maps and weak b-maps rely on our notion of points, which we define to be subanalytic l.v.a. arcs:

Definition 3.4. *Let $q : [0, \epsilon) \to C(I^n)$ be a continuous path germ. We write $q(s) = (\alpha(s), t(s)) \in C(I^n)$. We call q a* point *in $(C(I^n), \underline{0})$, if there is a representative $[0, \epsilon')$ of the germ $[0, \epsilon)$ and a $K \geq 1$ such that*

$$\frac{1}{K} s \leq \tau(s) \leq Ks$$

for all $s < \epsilon'$.

Points in any metric subanalytic germ are defined analogously. Two points p_1 and p_2 are called b-equivalent in a space with distance d, if we have

$$\lim_{t \to 0} \frac{d(p_1(t), p_2(t))}{t^b} = 0.$$

The definition of weak b-maps is the following:

Definition 3.7. *Let (X, x_0, d_X) be a metric subanalytic germ and let $(Z, \underline{0})$ be a subanalytic subgerm of $C(I^n)$. Let $b \in (0, \infty)$. A* weak b-moderately discontinuous subanalytic map (weak b-map, for abbreviation) *from $(Z, 0)$ to (X, x_0, d_X) is a finite collection $\{(C_j, f_j)\}_{j \in J}$, where $\{C_j\}_{j \in J}$ is a finite closed subanalytic cover of $(Z, \underline{0})$ and $f_j : C_j \to X$ are continuous l.v.a. subanalytic maps for which for any $j_1, j_2 \in J$ and any point q in $C_{j_1} \cap C_{j_2}$, the points $f_{j_1} \circ q$ and $f_{j_2} \circ q$ are b-equivalent.*

Two weak b-maps $\{(C_j, f_j)\}_{j \in J}$ and $\{(C'_k, f'_k)\}_{k \in K}$ are called equivalent*, if for any $j \in J$ and $k \in K$ and any point q contained in the intersection $C_j \cap C'_k$, the points $f_j \circ q$ and $f'_k \circ q$ are b-equivalent in X.*

The b-MD homotopy groups are defined analogously to the ordinary homotopy groups of punctured topological spaces with the following difference: we use weak

b-maps with their b-equivalence relation instead of continuous maps; and we use our notion of points with their b-equivalence relation. Points are of relevance for example in the definition of homotopies relative to a subspace (see Definition 3.14).

Thanks to the definition we have found, the properties mentioned above, that the b-MD homotopy theory shares with the ordinary homotopy theory, come along easily: the existence of the Hurewicz homomorphism; the fact that the Hurewicz homomorphism is an isomorphism in degree one when abelianizing the domain; the independence of the base point, when the germ is b-path connected (see Definition 3.43); and the fact that the higher degree b-MD homotopy groups are abelian. Those properties are proven in Subsection 3.1.3 and Subsection 3.1.4 and in Section 3.2.

Until now we have focused on the b-MD homotopy groups for a fixed $b \in (0, \infty]$. But the MD homotopy also provides the connecting homomorphisms the MD homology provides. That is, we have a homomorphism from the n-th b_1-MD homotopy group to the n-th b_2-MD homotopy group for any $b_1 \geq b_2$. Therefore observe that a weak b_1-map is also a weak b_2-map for $b_1 \geq b_2$. Consequently the target category of the MD homotopy is defined analogously to the one of the MD homology. The difference is that for degree one any group is allowed as opposed to restricting to abelian groups. Functoriality is shown in Subsection 3.1.5. The domain category is the category of punctured metric subanalytic germs. A punctured metric subanalytic germ is a metric subanalytic germ together with a fixed point defined as above. In the same way as for the b-MD homology, we improve functoriality for a fixed $b \in (0, \infty]$ by allowing b-maps as morphisms.

Among others, in Section 3.3, we show that the ∞-homotopy coincides with the ordinary homotopy of the link, just as in the case of the MD homology. In Section 3.4 we conjecture that the MD homotopy detects the existence of fast loops as an obstruction to metrical conicalness.

Moderately Discontinuous Metric Homology 2

The homology theory we have developed in this chapter shares several properties with the singular homology: it is invariant by suitable metric homotopies (see Definition 2.75 and Theorem 2.76 as well as Definition 2.79 and Theorem 2.80); it allows a relative and absolute Mayer-Vietoris long exact sequences (see Theorem 2.91) for a suitable cover of the metric subanalytic germ (see Definition 2.88); and as a consequence we have a certain theorem of excision (see Corollary 2.92) and a Ĉech spectral sequence (see Theorem 2.93). In Subsection 2.3.2 we compute the homology we have developed for the object in its domain category that corresponds to the one-point space in the topological category.

2.1. Pairs of metric subanalytic germs

As usual in algebraic topology, our invariant will be a functor from a category of geometric nature to a category of an algebraic nature. We start defining precisely the geometric category.

Definition 2.1. *A subanalytic germ* (X, x_0) *is a germ* (X, x_0) *of a subanalytic set* $X \subset \mathbb{R}^m$ *such that* $x_0 \in \overline{X}$ *(where* $\overline{X}$ *denotes the closure of* X *in* $\mathbb{R}^m$*). We say that* x_0 *is the* vertex of (X, x_0).

A metric subanalytic set (X, d_X) *is a subanalytic set* X *in some* $\mathbb{R}^m$*, together with a subanalytic metric* d_X *that induces the same topology on* X *as the restriction of the standard topology on* $\mathbb{R}^m$*.*

A metric subanalytic germ (X, x_0, d_X) *is a subanalytic germ* (X, x_0) *where* (X, d_X) *is a metric subanalytic set. We omit* x_0 *and* d_X *in the notation when it is clear from the context.*

A metric subanalytic subgerm *of a metric subanalytic germ* (X, x_0, d_X) *is a metric subanalytic germ* (Y, x_0, d_Y) *with* $Y \subseteq X$ *and* d_Y *equal to the restriction* $d_X|_Y$ *of the metric* d_X *to* Y*, that is, the restriction to* $Y \times Y$ *of* $d_X : X \times X \to \mathbb{R}$*.*

A pair of metric subanalytic germs (X, Y, x_0, d_X) *is the metric subanalytic germ* (X, x_0, d_X) *together with the subgerm* $(Y, x_0, d_X|_Y)$*.*

Given two germs (X, x_0) *and* (Y, y_0)*, a* subanalytic map germ $f : (X, x_0) \to (Y, y_0)$ *is a subanalytic continuous map* $f : X \to Y$ *that admits a continuous and subanalytic extension to a map germ* $\overline{f} : (X \cup \{x_0\}, x_0) \to (Y \cup \{y_0\}, y_0)$*.*

Remark 2.2. *Notice that in our definition, for a subanalytic germ (X, x_0) it is possible that $x_0 \notin X$. These sets play an important role (see for example Definition 2.54 or 2.88).*

Example 2.3. *A subanalytic germ $(X, x_0) \subset (\mathbb{R}^m, x_0)$ with the outer metric (the metric induced by restriction of the euclidean metric in $\mathbb{R}^m$) is a metric subanalytic germ. We denote the associated metric subanalytic germ by (X, x_0, d_{out}).*

We denote by (X, x_0, d_{in}) the metric subanalytic germ with the inner metric (defined to be the infimum of the lengths of the rectifiable paths between two points). This distance is not known to be subanalytic. However, according to [23] there is a subanalytic distance d' on X such that the identity $Id : (X, x_0, d_{in}) \to (X, x_0, d')$ is bi-Lipschitz. This allows us to apply the theory to the germ (X, x_0, d_{inn}) in the following way: our homology can be calculated for (X, x_0, d'). Moreover if d'' is a different choice of subanalytic metric with the same property than d', then the identity map is a subanalytic bi-Lipschitz homeomorhism between (X, x_0, d') and (X, x_0, d''). Hence the invariant calculated to each of the two subanaytic metric germs is the same. See Remark 2.11) for an extension of this idea.

Some basic examples are the following:

Definition 2.4 (Standard b-cones and straight cones)**.** *Let $L \subset \mathbb{R}^k$ be a subanalytic set and $b \in \mathbb{Q} \cap (0, +\infty)$. Consider the subanalytic set*

$$C_L^b = \{(t^b x, t) \in \mathbb{R}^k \times \mathbb{R};\ x \in L \ \textit{and}\ t \in [0, +\infty)\}.$$

The outer (respectively inner) standard b-cone over L *is the triple $(C_L^b, (\underline{0}, 0), d_{out})$ (respectively $(C_L^b, (\underline{0}, 0), d_{in})$), where d_{out} denotes the outer metric and d_{in} denotes the inner metric.*

When $b = 1$, we say C_L^1 is a straight cone over L *and we denote it by $(C(L), d_{out}) := (C_L^1, (\underline{0}, 0), d_{out})$ and $(C(L), d_{in}) := (C_L^1, (\underline{0}, 0), d_{in})$.*

By C_L^b or $C(L)$ we always mean the germ $(C_L^b, (\underline{0}, 0))$ or $(C(L), (\underline{0}, 0))$.

Remark 2.5. *We can assume that we are always working with bounded representatives of germs, and in particular with globally subanalytic sets (see Remark A.6). Recall that the collection of all globally subanalytic sets forms an O-minimal structure (see Remark A.6). Therefore, references for O-minimal structures such as [9] can also be applied to our category.*

Remark 2.6. *We recall that the link of a subanalytic germ is well defined as a topological space as the intersection of X with a small enough sphere centered at x_0; we denote it by $Link(X, x_0)$ or simply L_X. Moreover, the conical structure theorem says, given a subanalytic germ (X, x_0) and a family of subanalytic subgerms $(Z_1, 0)$,...,$(Z_k, 0) \subseteq (X, 0)$, that there exists a subanalytic homeomorphism $h : C(L_X) \to (X, x_0)$ such that $||x_0 - h(tx, t)|| = t$ and such that $h(C(L_{Z_i})) = Z_i$ with L_{Z_i} in L_X (see Theorem 4.10, 5.22, 5.23 in [9]). We say that the conical structure h is compatible with the family $\{Z_i\}$. The conical structure is why we say that x_0 is the vertex of (X, x_0).*

Let us add that in Proposition 1 of [11], when X is semialgebraic it is proved that the link is well defined up to semialgebraic homeomorphisms. However, this fact is not used along this thesis.

Definition 2.7. *A map germ $f : (X, x_0) \to (Y, y_0)$ is said to be* linearly vertex approaching *(l.v.a. for brevity) if there exists $K \geq 1$ such that*

$$\frac{1}{K}||x - x_0|| \leq ||f(x) - y_0|| \leq K||x - x_0||$$

for every x in some representative of (X, x_0). The constant K is called the l.v.a constant for f.

Remark 2.8. *Let (X, x_0) be a subanalytic germ with compact link. Consider any subanalytic map germ $f : (X, x_0) \to (Y, y_0)$ that is a homeomorphism onto its image. Let $\{Z_j\}_{j\in J}$ be a finite collection of closed subanalytic subsets of X. There is a subanalytic homeomorphism germ $\phi : (X, x_0) \to (X, x_0)$ such that $\phi(Z_j) = Z_j$ for all $j \in J$ and such that $||f\circ\phi(x) - y_0|| = ||x - x_0||$, which is stronger than l.v.a.*

Proof. Let $h : C(L_X) \to (X, x_0)$ be a subanalytic homemomorphism defining the conical structure compatible with the Z_i (which means that $h(C(L_{Z_i})) = Z_i$) and such that $||h(tx, t) - x_0|| = t$ (see Remark 2.6).

Consider the mapping $g : C(L_X) \to C(L_X)$ that sends $(xt, t) \mapsto (x \cdot ||f \circ h(xt, t) - x_0||, ||f \circ h(xt, t) - x_0||)$. It is clearly subanalytic in the coordinates $y = xt$ and t for $t \neq 0$ and therefore it extends continuously and subanalytically to the closure $C(L_X)$. Note that g is a homeomorphism.

To finish, it is clear that $\phi := h \circ g^{-1} \circ h^{-1}$ satisfies the statement. □

Remark 2.8 can also be shown adapting the following result of Shiota's:

Corollary 2 of [36]. *Let f_1 and f_2 be subanalytic functions on X with*

$$f_1^{-1}(0) = f_2^{-1}(0), \quad \{f_1 < 0\} = \{f_2 < 0\}, \quad \{f_1 > 0\} = \{f_2 > 0\}.$$

Then there exists a subanalytic homeomorphism ϕ of X such that

$$f_1\circ\phi = f_2$$

on a neighborhood of $f_1^{-1}(0)$.

We assume $x_0 = 0$. If the family $\{Z_j\}_{j\in J}$ is empty, we simply apply Corollary 2 of [36] to the functions $||x||$ and $||f(x)||$. The proof of Corollary 2 [36] only uses the subanalytic triangulation Theorem (Theorem 1 of [36]) together with Lemmata 10 and 11 in the same paper, which are stated in the presence of the family $\{Z_j\}_{j\in J}$. Notice that the subanalytic triangulation Theorem (Theorem 1 of [36]) is valid when the family $\{Z_j\}_{j\in J}$ is non-empty (this is Theorem II of Chapter II of [37]). So Remark 2.8 is true without the assumption that f is a homeomorphism onto its image.

We have preferred to give a simple proof of the case that is used in this thesis for the sake of completeness, i.e. including the assumption that f is a homeomorphism onto its image.

Definition 2.9. *Let* (X, x_0, d_1) *and* (Y, y_0, d_2) *be two metric subanalytic germs. A* Lipschitz linearly vertex approaching subanalytic map germ *(*Lipschitz l.v.a. subanalytic map *for short)*

$$f : (X, x_0, d_X) \to (Y, y_0, d_Y)$$

is a l.v.a subanalytic map germ such that there exists $K \geq 1$ *and a representative* X *of the germ such that*

$$d_Y(f(x), f(\tilde{x})) \leq K d_X(x, \tilde{x}) \ \forall x, \tilde{x} \in X.$$

Any such K *which also serves as a l.v.a. constant for* f *will be called a* Lipschitz l.v.a. constant for f*.*

A Lipschitz l.v.a. subanalytic map of pairs *is a map germ of pairs*

$$f : (X, Y, x_0, d_X) \to (X', Y', x_0', d_{X'})$$

such that $f : (X, x_0, d_X) \to (X', x_0', d_{X'})$ *is a Lipschitz l.v.a. subanalytic map.*

Given a subanalytic subgerm $(Y, x_0, d_X|_Y) \subset (X, x_0, d_X)$, the inclusion is an example of a Lipschitz l.v.a. map.

Definition 2.10. *The* category of pairs of subanalytic metric subanalytic germs *has pairs of metric subanalytic germs as objects and Lipschitz l.v.a. subanalytic maps of pairs as morphisms.*

Remark 2.11. *Equivalently, we can work with the bigger category of subanalytic germs* (X, x_0, d_X) *which are endowed with a metric* d_X *that induce the same topology as the euclidean metric, and such that there exists a subanalytic metric* d' *that is bi-Lipschitz equivalent to* d_X*, which means that the identity* $(X, x_0, d_X) \to (X, x_0, d')$ *is bi-Lipschitz. Hence we are not asking* d_X *to be a subanalytic metric. Then, a subanalytic germ* (X, x_0) *with the inner metric belongs to this category, see Example 2.3.*

2.2. Definition of the Moderately Discontinuous Metric Homology

The Moderately Discontinuous Metric Homology (Moderately Discontinuous Homology, or MD-Homology, for short) is a functor from the category of pairs of metric subanalytic germs to an algebraic category whose objects are diagrams of groups. Its definition needs a series of steps.

2.2.1. The pre-chain group $MDC_\bullet^{\text{pre},\infty}((X, x_0); A)$.

Notation 2.12. *For any* $n \in \mathbb{N}_0$*, we denote by* $\Delta_n \subset \mathbb{R}^{n+1}$ *the standard* n*-simplex*

$$\Delta_n := \{(p_0, ..., p_n) \in (\mathbb{R}_{\geq 0})^{n+1} : \sum_{i=0}^{n} p_i = 1\}$$

oriented as follows: the standard orientation on $\mathbb{R}^{n+1}$ *orients the convex hull of* $\Delta_n \cup \underline{0}$*, where* $\underline{0}$ *denotes the origin, which in turn induces an orientation on* Δ_n*. We denote by*

$i^k_n : \Delta_{n-1} \to \Delta_n$ *the map sending* $p_0, ..., p_{n-1}$ *to* $p_0, ..., p_{k-1}, 0, p_k, ..., p_{n-1}$. *The image of* i^k_n *is the k-th facet of* Δ_n.

Consider the oriented germ of the cone over Δ_n *and denote it as*

$$\hat{\Delta}_n := (\{(tx, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : x \in \Delta_n, t \in [0, 1)\},$$

and let $j^k_n : \hat{\Delta}_{n-1} \to \hat{\Delta}_n$ *be the map sending* $(tx, t) \mapsto (ti^k_n(x), t)$. *The* k-th facet *of* $\hat{\Delta}_n$ *is the image of* j^k_n. *More generally, a face of* $\hat{\Delta}_n$ *is the cone over a face of* Δ_n.

We will usually use $\hat{\Delta}_n$ *to denote the germ* $(\hat{\Delta}_n, (\underline{0}, 0))$.

The following definition is coherent with Definition 2.7:

Definition 2.13. *A* linearly vertex approaching (subanalytic) n-simplex *is a continuous subanalytic map germ* $\sigma : \hat{\Delta}_n \to (X, x_0)$ *such that there is a* $K \geq 1$ *such that*

$$\frac{1}{K} t \leq ||\sigma(xt, t) - x_0|| \leq Kt$$

for any $x \in \Delta_n$ *and any small enough* t. *We will say simply a* l.v.a. simplex.

Similarly, a map $\nu : \hat{\Delta}_n \to \hat{\Delta}_n$, *expressed as* $\nu(xt, y) = (\nu_1(xt, t)\nu_2(xt, t), \nu_2(xt, t))$ *in the coordinates* (xt, t) *of* $\hat{\Delta}_n$, *is linearly vertex approaching, if there is a* $K \geq 1$ *such that* $\frac{1}{K} t \leq \nu_2(xt, t) \leq Kt$.

Definition 2.14. *Given a subanalytic germ* (X, x_0) *and an abelian group* A, *a* linearly vertex approaching n-chain in (X, x_0) *(l.v.a. n-chain, for brevity) is a finite formal sum* $\sum_i a_i \sigma_i$, *where* $a_i \in A$ *and* σ_i *is a l.v.a subanalytic n-simplex in* (X, x_0). *We define* $MDC^{\text{pre},\infty}_n((X, x_0); A)$ *to be the abelian group of n-chains. Given a subanalytic germ* (X, x_0) *and an abelian group* A, *a* linearly vertex approaching n-chain in (X, x_0) *(l.v.a. n-chain, for brevity) is a finite formal sum* $\sum_i a_i \sigma_i$, *where* $a_i \in A$ *and* σ_i *is a l.v.a subanalytic n-simplex in* (X, x_0). *We define* $MDC^{\text{pre},\infty}_n((X, x_0); A)$ *to be the abelian group of n-chains.*

We define the boundary of σ *to be the formal sum*

$$\partial \sigma = \sum_{k=0}^{n} (-1)^k \sigma \circ j^k_n.$$

The boundary extends linearly to n-chains and defines a complex $MDC^{\text{pre},\infty}_\bullet((X, x_0); A)$ *whose components are the groups* $MDC^{\text{pre},\infty}_n((X, x_0); A)$ *for* $n \geq 0$.

Often, when it is clear from the context we will skip the coefficients group A and/or the vertex in the notation.

2.2.2. The homological subdivision equivalence relation in $MDC^{\text{pre},\infty}_\bullet((X, x_0); A)$

As in Singular Homology Theory, in order to prove Excision and Mayer-Vietoris we will need to subdivide simplices. In Singular Homology, the standard procedure is to devide a simplex into a chain of smaller simplices by taking barycentric subdivisions. The existence of a Lebesgue number in that context guarantees that iterating that procedure enough times yields a chain for which all of its simplices are contained in one of the open sets of the cover. In our theory, the role of open subgerms are taken by subgerms whose representatives are open and whose closure contains the vertex, but that do not contain the vertex themselves. Observe that those are the complements of subgerms whose representatives are closed and contain the vertex. Therefore an open cover of germs does not cover the image of a l.v.a. simplex. As a result we do not get a Lebesgue number. That is why we do not adapt the procedure used in Singular Homology, but build a chain complex that incorporates the subdivisions from the beginning.

Observe also that by incorporating subdivisions from the beginning and therefore not depending on the Lebesgue number we achieve that Mayer-Vietoris, and therefore also the Excision Theorem and the Čech Theorem, can be shown directly for closed covers. In fact, that is what we do in Section 2.6. In Subsection 2.6.5 we then show that the proofs for closed covers can be adapted to open covers.

Given a finite simplicial complex K we denote by $|K|$ the geometric realization of K. We call the subsets of $|K|$, that correspond to a simplex in K, the *faces of* $|K|$. Let Z be a subanalytic set. A *subanalytic triangulation* is a finite simplicial complex K of closed simplices and a subanalytic homeomorphism $\alpha : |K| \to Z$.

Remark 2.15. *Given a finite family $\mathcal{S}$ of closed subanalytic subsets of Z, there exists a subanalytic triangulation $\alpha : |K| \to Z$ compatible with $\mathcal{S}$, that is, such that every subset of $\mathcal{S}$ is a union of images of simplices of $|K|$. See for example Theorem 4.4. in [9] or Theorem II.2.1. in [37].*

By a subanalytic triangulation of a subanalytic germ (X, x_0) we mean a subanalytic triangulation of a representative of it, which is compatible with the vertex.

Given two subanalytic triangulations $\alpha : |K| \to Z$ and $\alpha' : |K'| \to Z$, we say that α' *refines* α if the image by α of any simplex of $|K|$ is the union of images by α' of simplices of $|K'|$.

Given a subanalytic triangulation $\alpha : |K| \to Z$, a simplex of $|K|$ is called *maximal* if it is not strictly contained in another simplex. We consider the collection $\mathcal{T} := \{T_i\}_{i \in I}$ of subsets of Z that are images of the maximal simplices of $|K|$. We call it the *collection of maximal triangles.*

Given two simplicial complexes K and K', a continuous mapping $f : |K| \to |K'|$ *preserves the simplicial structure* if it takes faces to faces.

In the next definition we will need a representative of the germ $\hat{\Delta}_n$. By abuse of notation we denote it also by $\hat{\Delta}_n$, and consider the representative

$$\{(tx, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : x \in \Delta_n, t \in [0, 1/2]\}.$$

Definition 2.16. *A* homological subdivision *of $\hat{\Delta}_n$ is a finite family $\{\rho_i\}_{i \in I}$ of injective l.v.a. subanalytic map germs $\rho_i : \hat{\Delta}_n \to \hat{\Delta}_n$ for which there is a subanalytic triangulation $\alpha : |K| \to \hat{\Delta}_n$ with the following properties:*

- *the triangulation α is compatible with the collection of all faces of $\hat{\Delta}_n$;*
- *all maximal triangles of α meet the vertex of $\hat{\Delta}_n$;*

- *the collection $\{T_i\}$ of maximal triangles of α is also indexed by I;*
- *for any $i \in I$, the image of ρ_i is T_i and $\alpha^{-1}|_{T_i} \circ \rho_i$ is a homeomorphism that takes faces of $\hat{\Delta}_n$ to faces of $|K|$.*

For a homological subdivision $\{\rho_i\}_{i\in I}$, the sign of ρ_i for any $i \in I$ is defined to be 1, if ρ_i is orientation preserving, and -1, if it is orientation reversing. We denote it by $sgn(\rho_i)$.

Note that this implies that, whenever the sum $\sum_{i\in I} sgn(\rho_i)\rho_i$ is a cycle in the singular homology $H_{n+1}(\hat{\Delta}_n, \partial\hat{\Delta}_n \cup \hat{\Delta}_n^{\geq\epsilon}; \mathbb{Z})$ for $\epsilon = min\{\epsilon_i\}$ where $\hat{\Delta}_n^{\geq\epsilon}$ denotes the set $\{(tx,t) \in \mathbb{R}^{n+1} \times \mathbb{R} : x \in, t \in [\epsilon, 1)\}$, then $\sum_{i\in I} sgn(\rho_i)\rho_i$ represents the fundamental class. However, in general $\sum_{i\in I} sgn(\rho_i)\rho_i$ does not have to represent a cycle in $H_{n+1}(\hat{\Delta}_n, \partial\hat{\Delta}_n \cup \hat{\Delta}_n^{\geq\epsilon}; \mathbb{Z})$.

Definition 2.17 (Immediate equivalences)**.** *Two chains*

$$\sum_{j\in J} a_j\sigma_j, \sum_{k\in K} b_k\tau_k \in MDC_n^{\mathrm{pre},\infty}(X; A)$$

are called immediately equivalent *(and we denote it by $\sum_{j\in J} a_j\sigma_j \to_\infty \sum_{k\in K} b_k\tau_k$), if for any $j \in J$ there are homological subdivisions $\{\rho_{ji}\}_{i\in I_j}$ such that we have the equality*

$$\sum_{j\in J}\sum_{i\in I_j} sgn(\rho_{ji}) a_j \sigma_j \circ \rho_{ji} = \sum_{k\in K} b_k\tau_k$$

in $MDC_n^{\mathrm{pre},\infty}(X; A)$.

Remark 2.18. *The immediate equivalences can be defined as well by imposing $\sigma \to_\infty \sum_{i\in I} sgn(\rho_i)\sigma\circ\rho_i$ for any l.v.a n-simplex σ and any subdivision $\{\rho_i\}_{i\in I}$, and extending the immediate equivalences by linearity.*

Remark 2.19. *Any l.v.a. subanalytic homeomorphism $\mu : (\hat{\Delta}_n, 0) \to (\hat{\Delta}_n, 0)$ which preserves the simplicial structure is a homological subdivision of $\hat{\Delta}_n$ for which the index set I has just one element. As a consequence, for any n-simplex σ, we have $\sigma \to_\infty \sigma\circ\mu$, if μ is orientation preserving, and $\sigma \to_\infty -\sigma\circ\mu$, if μ is orientation reversing.*

Definition 2.20 (The homological subdivision equivalence relation)**.** *The* subdivision equivalence relation *in $MDC_n^{\mathrm{pre},\infty}(X; A)$ (denoted by $\sim_{S,\infty}$) is the equivalence relation generated by immediate equivalences. That is $z \sim_{S,\infty} z'$ if there exists a sequence $w_1, ..., w_k$ such that $z = w_1$, $z' = w_k$ and for any $1 \leq i < k$ we have either the immediate equivalence $w_i \to_\infty w_{i+1}$ or $w_{i+1} \to_\infty w_i$.*

Lemma 2.21. *Given any three chains $w_1, w_2, w_3 \in MDC_n^{\mathrm{pre},\infty}(X; A)$, and immediate equivalences $w_3 \to_\infty w_1$ and $w_3 \to_\infty w_2$ there exists an element $w_4 \in MDC_n^{\mathrm{pre},\infty}(X; A)$ and two immediate equivalences $w_1 \to_\infty w_4$ and $w_2 \to_\infty w_4$.*

Proof. Since the immediate equivalences are compatible with linear combinations (see Remark 2.18) we may assume that w_3 is equal to an n-simplex σ. Then there exist two subdivisions $\{\rho_i\}_{i\in I}$ and $\{\rho'_{i'}\}_{i'\in I'}$ of $\hat{\Delta}_n$ such that we have the equalities

$$w_1 = \sum_{i\in I} sgn(\rho_i)\sigma\circ\rho_i, \qquad w_2 = \sum_{i'\in I'} sgn(\rho'_{i'})\sigma\circ\rho'_{i'}. \tag{2.1}$$

Let $\alpha : |K| \to \hat{\Delta}_n$ and $\alpha' : |K'| \to \hat{\Delta}_n$ be the subanalytic triangulations associated with the homological subdivisions $\{\rho_i\}_{i\in I}$ and $\{\rho'_{i'}\}_{i'\in I'}$. By the subanalytic Hauptvermutung (Chapter II, Theorem II in [37])) there is a subanalytic triangulation $\beta : |L| \to \hat{\Delta}_n$ refining α and α'. Let $\{T_j\}_{j\in J}$ be the collection of maximal triangles of the triangulation β. Let $\{\nu_j\}_{j\in J}$ be a collection of orientation preserving l.v.a. subanalytic homeomorphisms $\nu_j : \hat{\Delta}_n \to T_j$ preserving the simplical structure.

Consider the splitting $J = \coprod_{i\in I} J_i$, where $j \in J_i$ if and only if T_j is included in the image of ρ_i.

Then the collection $\{\rho_i^{-1}\circ\nu_j\}_{j\in J_i}$ is a homological subdivision of $\hat{\Delta}_n$ and we have the immediate equivalence

$$\sigma\circ\rho_i \to_\infty \sum_{j\in J_i} \sigma\circ\rho_i\circ\rho_i^{-1}\circ\nu_j = \sum_{j\in J_i} \sigma\circ\nu_j. \tag{2.2}$$

The splitting $J = \coprod_{i'\in I'} J'_{i'}$ is defined considering the analogous interaction between the triangulations α' and β. By the same kind of arguments we have the immediate equivalence

$$\sigma\circ\rho'_i \to_\infty \sum_{j\in J'_i} \sigma\circ\nu_j. \tag{2.3}$$

Defining $w_4 := \sum_{j\in J} \sigma\circ\nu_j$ and using Equations (2.1), (2.2) and (2.3) we complete the proof. □

Corollary 2.22. *We have the equivalence $w \sim_{S,\infty} z$ if there exist sequences of immediate equivalences $z = z_0 \to_\infty z_1 \to_\infty ... \to_\infty z_l$ and $w = w_0 \to_\infty w_1 \to_\infty ... \to_\infty w_m = z_l$.*

Proof. The sequence $z = x_1, ..., x_k = w$ predicted in Definition 2.20 is *monotonous* at the i-th position if we have either $x_{i-1} \to_\infty x_i \to_\infty x_{i+1}$ or $x_{i+1} \to_\infty x_i \to_\infty x_{i-1}$. The sequence $x_1, ..., x_k$ *has a roof* at the i-th position if we have $x_i \to_\infty x_{i-1}$ and $x_i \to_\infty x_{i+1}$. The sequence $x_1, ..., x_k$ *has a valley* at the i-th position if we have $x_{i-1} \to_\infty x_i$ and $x_{i+1} \to_\infty x_i$. Repeated applications of the previous lemma allow to replace every roof by a valley. □

2.2.3. ∞-Moderately discontinuous homology

Lemma 2.23. *The homological subdivision equivalence relation is compatible with the boundary operator ∂ in $MDC_n^{\mathrm{pre},\infty}(X;A)$ in the following sense: given a simplex $\sigma \in MDC_n^{\mathrm{pre},\infty}(X;A)$ and $\{\rho_i\}_{i\in I}$ a homological subdivision of $\hat{\Delta}_n$, then $\partial\sigma$ and $\sum_{i\in I} \partial(sgn(\rho_i)\sigma\circ\rho_i)$ are $\sim_{S,\infty}$-equivalent.*

Proof. A homological subdivision $\{\rho_i\}_{i\in I}$ of $\hat{\Delta}_n$ induces a homological subdivision $\{\rho_i^k\}_{i\in I_k}$ of the k-th facet of $\hat{\Delta}_n$. So $\partial(\sum_{i\in I} sgn(\rho_i)\sigma \circ \rho_i)$ splits as the sum, with appropriate signs, of the facets of σ (expressed after the corresponding homological subdivision) and the sum of the interior facets of all $\sigma \circ \rho_i$ which cancels in pairs.

□

Definition 2.24 (∞-MD Homology). *We define the* ∞-moderately discontinuous chain complex of (X, x_0, d_X) *with coefficients in A (∞-MD complex for short) to be the quotient of $MDC^{\text{pre},\infty}_\bullet((X, x_0, d_X); A)$ by the homological subdivision equivalence relation. We denote it by $MDC^{\infty}_\bullet((X, x_0, d_X); A)$. Its homology is called the* ∞-moderately discontinuous homology with coefficients in A *and is denoted by $MDH^{\infty}_\bullet((X, x_0, d_X); A)$.*

Note that this homology does not depend on a metric. As we will see in Section 2.8, the ∞-moderately discontinuous homology coincides with the homology of the link of the germ (X, x_0).

2.2.4. b-Moderately discontinuous homology

Given a metric subanalytic germ (X, x_0, d_X), for each $b \in (0, +\infty)$, we define the following equivalence relation in the set of l.v.a. subanalytic n-simplices:

Definition 2.25. *Let $b \in (0, \infty)$. Let σ_1, σ_2 be n-simplices in $MDC^{\text{pre},\infty}_\bullet(X, x_0, d_X)$. We say that σ_1 and σ_2 are* b-equivalent *(we write $\sigma_1 \sim_b \sigma_2$) if*

$$\lim_{t\to 0^+} \frac{\max\{d_X(\sigma_1(tx, t), \sigma_2(tx, t)); x \in \Delta_n\}}{t^b} = 0.$$

We extend the relation to $MDC^{\text{pre},\infty}_n(X; A)$ by linearity.

Remark 2.26. *The quotient of the free group $MDC^{\text{pre},\infty}_\bullet((X, x_0, d_X); A)$ by the $\sim_b$-equivalence relation is the free group generated by the $\sim_b$-equivalence classes of simplices with coefficients in A. As a consequence we have the following: let $w = \sum_{j\in J} b_j\tau_j$ and $w' = \sum_{j\in J'} b'_j\tau'_j$ be chains in $MDC^{\text{pre},\infty}_\bullet((X, x_0, d_X); A)$. Split the index sets $J = \coprod_{k\in K} J_k$ and $J' = \coprod_{k\in K} J'_k$ in the unique way that satisfies the following properties:*

- *any two $j_1, j_2 \in J$ belong to the same J_k if and only if we have $\tau_{j_1} \sim_b \tau_{j_2}$,*
- *any two $j'_1, j'_2 \in J$ belong to the same J'_k if and only if we have $\tau'_{j'_1} \sim_b \tau'_{j'_2}$,*
- *for any $k \in K$ and $j \in J_k$ and $j' \in J'_k$ we have $\tau_j \sim_b \tau'_{j'}$.*

Then $w \sim_b w'$ if and only if for any $k \in K$ we have the equality

$$\sum_{j\in J_k} b_j = \sum_{j'\in J'_k} b_{j'}. \tag{2.4}$$

The following arc interpretation of the b-equivalence relation will be useful later.

Lemma 2.27. *Let σ_1, σ_2 be n-simplices in $MDC^{\text{pre},\infty}_\bullet(X, x_0, d_X)$. Then we have that the following statements are equivalent:*

(i) $\sigma_1 \sim_b \sigma_2$;

(ii) *for any subanalytic continuous arc* $\gamma : [0, \epsilon) \to \hat{\Delta}_n$ *such that* $\gamma(0)$ *is equal to the vertex and* $\gamma(t)$ *is different to the vertex for* $t \neq 0$ *we have the equality*

$$\lim_{t \to 0^+} \frac{d(\sigma_1(\gamma(t)), \sigma_2(\gamma(t)))}{\gamma_2(t)^b} = 0, \tag{2.5}$$

where $\gamma(t) = (\gamma_2(t)\gamma_1(t), \gamma_2(t))$ *is the expression of the arc in the coordinates* (tx, t) *of* $\hat{\Delta}_n$*;*

(iii) *for any subanalytic l.v.a. continuous arc* $\gamma : [0, \epsilon) \to \hat{\Delta}_n$ *we have the equality*

$$\lim_{t \to 0^+} \frac{d(\sigma_1(\gamma(t)), \sigma_2(\gamma(t)))}{t^b} = 0. \tag{2.6}$$

Proof. If we have the equivalence $\sigma_1 \sim_b \sigma_2$ it is obvious that the limit vanishes for any arc as in the statement of (ii). Let $\gamma(t) = (\gamma_2(t)\gamma_1(t), \gamma_2(t))$ be any subanalytic l.v.a continuous arc. Then the limit $\lim_{t \to 0^+} \frac{\gamma_2(t)}{t}$ is finite, and therefore condition (ii) implies condition (iii).

So, to finish the proof, we only need to prove that (iii) $\Rightarrow$ (i). Assume that the condition on arcs in (iii) is satisfied. The function

$$t \mapsto \max\{d(\sigma_1(tx, t), \sigma_2(tx, t)); x \in \Delta_n\}$$

is subanalytic. Therefore it admits an expansion of the form

$$\max\{d(\sigma_1(tx, t), \sigma_2(tx, t)); x \in \Delta_n\} = Ct^{b'} + o(t^{b'})$$

for a certain $b' \in \mathbb{Q}$ and $C > 0$. Then the subset

$$Z := \{(tx, t) \in \hat{\Delta}_n : d(\sigma_1(tx, t), \sigma_2(tx, t)) \geq (C/2)t^{b'}\}$$

is subanalytic and contains sequences converging to the vertex of $\hat{\Delta}_n$. Therefore, by the subanalytic Curve Selection Lemma there exists a subanalytic continuous arc $\gamma : [0, \epsilon) \to Z$ such that $\gamma(0)$ is equal to x_0 and $\gamma(t)$ is different to the vertex for $t \neq 0$. By Remark 2.8 we can assume that $||\gamma(t) - x_0|| = t$. Thus, we have the following inequality

$$\lim_{t \to 0^+} \frac{d(\sigma_1(\gamma(t)), \sigma_2(\gamma(t)))}{t^{b'}} \geq C/2.$$

The equivalence $\sigma_1 \sim_b \sigma_2$ holds if and only if we have the strict inequality $b' > b$. The previous inequality implies that if $b' \leq b$ then the arc γ contradicts the arc condition in the statement of (iii). $\square$

Lemma 2.28. *Let* σ, σ' *be simplices in* $MDC_n^{\mathrm{pre},\infty}(X; A)$ *such that* $\sigma \sim_b \sigma'$*. If* $\{\rho_i\}_{i \in I}$ *is a homological subdivision of* $\hat{\Delta}_n$*, then we have*

$$\sum_{i \in I} sgn(\rho_i)\sigma \circ \rho_i \sim_b \sum_{i \in I} sgn(\rho_i)\sigma' \circ \rho_i.$$

Proof. Let γ be any subanalytic l.v.a continuous arc in $\hat{\Delta}_n$. Since ρ_i is l.v.a then $\rho_i \circ \gamma$ is also a subanalytic l.v.a continuous arc. Since we have the equivalence $\sigma \sim_b \sigma'$, Lemma 2.27 implies the vanishing of the limit

$$\lim_{t \to 0^+} \frac{d(\sigma(\rho_i(\gamma(t))), \sigma'(\rho_i(\gamma(t))))}{t^b} = 0.$$

Again by Lemma 2.27 this implies the equivalence $\sigma \circ \rho_i \sim_b \sigma' \circ \rho_i$. □

In order to define the complex of b-moderately discontinuous chains we introduce the b-subdivision equivalence relation.

Definition 2.29 (The b-subdivision equivalence relation)**.** *Two chains*

$$\sum_{j \in J} a_j \sigma_j, \sum_{k \in K} b_k \tau_k \in MDC_n^{\text{pre},\infty}(X;A)$$

are called b-immediately equivalent *(and we denote it by $\sum_{j \in J} a_j \sigma_j \to_b \sum_{k \in K} b_k \tau_k$), if for any $j \in J$ there is a homological subdivision $\{\rho_i\}_{i \in I_j}$ such that we have the b-equivalence*

$$\sum_{j \in J} \sum_{i \in I_j} sgn(\rho_i) a_j \sigma_j \circ \rho_i \sim_b \sum_{k \in K} b_k \tau_k$$

in $MDC_n^{\text{pre},\infty}(X;A)$.

The b-subdivision equivalence relation *in $MDC_n^{\text{pre},\infty}(X;A)$ is the equivalence relation generated by the b-immediate equivalences, and is denoted by $\sim_{S,b}$. The equivalence classes are called b-moderately discontinuous chains or b-chains.*

We denote by $MDC_n^b(X;A)$ the quotient group of $MDC_n^{\text{pre},\infty}(X;A)$ by the $\sim_{S,b}$-equivalence relation. It is the group of b-moderately discontinuous chains.

Proposition 2.30. *$w \sim_{S,b} z$ if and only if there exist sequences of b-immediate equivalences $w = w_0 \to_b w_1 \to_b \ldots \to_b w_l$ and $z = z_0 \to_b z_1 \to_b \ldots \to_b z_m = w_l$.*

Proof. The proof is an adaptation of the proofs of Lemma 2.21 and Corollary 2.22, taking into account Lemma 2.28. □

Remark 2.31. *Often we will need to define homomorphisms*

$$h : MDC_\bullet^b((X, x_0, d_X); A) \to G,$$

where G is an abelian group. The usual procedure is to define first a homomorphism $\bar{h} : MDC_\bullet^{\text{pre},\infty}((X, x_0, d_X); A) \to G$, and check that it descends to a well defined h. It is convenient to record that $\bar{h}$ descends if and only if the following two conditions hold:

- *For any $\sigma, z \in MDC_\bullet^{\text{pre},\infty}((X, x_0, d_X); A)$, where σ is a simplex and z is a chain such that we have the immediate equivalence $\sigma \to_\infty z$, we have the equality $h(\sigma) = h(z)$.*

- *For any two simplices* $\sigma, \sigma' \in MDC_\bullet^{\text{pre},\infty}((X, x_0, d_X); A)$ *such that we have the equivalence* $\sigma \sim_b \sigma'$, *we have the equality* $h(\sigma) = h(\sigma')$.

Lemma 2.32. *The boundary operator* ∂ *in* $MDC_\bullet^{\text{pre},\infty}(X; A)$ *descends to a well defined boundary operator in* $MDC_\bullet^b(X; A)$.

Proof. We have to check the conditions of Remark 2.31. The first condition is exactly Lemma 2.23. The second condition is similar to the proof of Lemma 2.28. □

Definition 2.33 (b-Moderately discontinuous homology)**.** *We define the* b-moderately discontinuous chain complex of (X, x_0, d_X) *with coefficients in* A *(the* b*-MD complex for short) to be the complex* $MDC_\bullet^b((X, x_0, d_X); A)$ *with the boundary operator defined in the previous lemma. Its homology is called the* b-moderately discontinuous homology with coefficients in A *(*b-MD homology*, for short) and is denoted by* $MDH_\bullet^b((X, x_0, d_X); A)$.

For $b_1 \geq b_2$ the chain complex $MDC_\bullet^{b_1}((X, x_0, d_X); A)$ is richer than the chain complex $MDC_\bullet^{b_2}((X, x_0, d_X); A)$. But for the b-MD homology the situation is more complex:

Example 2.34. *We take the straight cone over the circle and for every level* (xt_1, t_1), *where* $t_1 \in (0, \epsilon)$ *is fixed, we remove an open segment of length* t_1^b *from the circle as illustrated in Figure 2.1. We denote the result by* X *and we equip* X *with the outer metric* d_X. *We denote the boundary arcs of* X *by* $x_1(t)$ *and* $x_2(t)$ *respectively and parametrize both such that* $\|x_k(t)\| = t$. *For* $b' < b$, *the l.v.a. simplex* $\sigma : \hat{\Delta}_1 \to X$ *defined as follows is a cycle in the chain group* $MDC_1^{b'}((X, 0, d_X); A)$. *Let* $\mathbb{S}_t$ *denote the sphere in* $\mathbb{R}^3$ *of radius* t. *For a fixed* $t \in (0, \epsilon]$, *we define* $\sigma(t(-), t) : \Delta_1 \to X \cap \mathbb{S}_t$ *to be a continuous subanalytic function for which* $\sigma(t(0, 1), t) = x_1(t)$ *and* $\sigma(t(1, 0), t) = x_2(t)$ *and such that* σ *is continuous.*

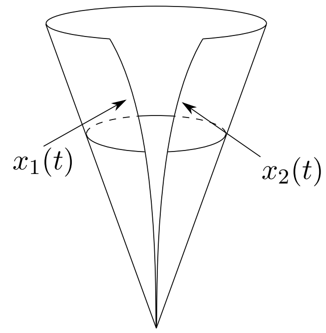


Figure 2.1.: The subanalytic metric germ X.

The following consequence of Proposition 2.30 will be used repeatedly:

Lemma 2.35. *Given an element* $z \in MDC_\bullet^{\text{pre},\infty}((X, x_0, d_X); A)$, *the class* $[z]$ *in* $MDC_\bullet^b((X, x_0, d_X); A)$ *vanishes if and only if there exists a sequence of immediate equivalences* $z = z_0 \to_\infty z_1 \to_\infty \ldots \to_\infty z_r$ *such that* $z_r \sim_b 0$. *Notice that, by Remark 2.26, the chain* $z_r = \sum_{i \in I} a_i \sigma_i$ *is as follows: consider the subdivision of the index* *set $I = \coprod_{j \in J} I_j$ so that i, i' belong to the same I_j if and only if $\sigma_i \sim_b \sigma_{i'}$. Then for any $j \in J$ we have the equality*

$$\sum_{i \in I_j} a_i = 0. \tag{2.7}$$

Proof. If there exists a sequence $z = z_0 \to_\infty z_1 \to_\infty ... \to_\infty z_r \sim_b 0$, then the class $[z] \in MDC^b_\bullet((X, x_0, d_X); A)$ vanishes obviously. Let us prove the converse. Suppose that $z \in MDC^{\mathrm{pre},\infty}_\bullet((X, x_0, d_X); A)$ is such that its class $[z]$ vanishes in $MDC^b_\bullet((X, x_0, d_X); A)$. By Proposition 2.30 there exists a sequence of b-immediate equivalences

$$z = z_0 \to_b z_1 \to_b ... \to_b z_r = 0, \tag{2.8}$$

We proceed by induction over r. If $r = 1$, there is nothing to show.

For the induction step we prove the following: if z_1', z_1 and z_2 are chains in the complex $MDC^{\mathrm{pre},\infty}_\bullet((X, x_0, d_X); A)$ such that $z_1' \sim_b z_1$ and $z_1 \to_\infty z_2$, then there is a $z_2' \in MDC^{\mathrm{pre},\infty}_\bullet((X, x_0, d_X); A)$ such that $z_1' \to_\infty z_2'$ and $z_2' \sim_b z_2$. To show that, write $z_1 = \sum_{i \in I} a_i \sigma_i$ and let $\{\rho_{i,l}\}_{l \in L_i}$ be homological subdivisions for which

$$z_2 = \sum_{i \in I} \sum_{l \in L_i} a_i sgn(\rho_{i,l}) \sigma_i \circ \rho_{i,l}.$$

Let $z_1' = \sum_{j \in J} a_j' \sigma_j'$. Let $I = \coprod_{k \in K} V_k$ and $J = \coprod_{k \in K} V_k'$ be the splitting in accordance with Remark 2.26 applied to z_1 and z_1'. For any $j \in J$, choose a fixed $k_j \in V_k'$. Then it is

$$z_1' = \sum_{k \in K} \sum_{j \in V_k'} a_j' \circ \sigma_{j_k}' + \tilde{z}_1 = \sum_{k \in K} \sum_{i \in V_k} a_i \circ \sigma_{j_k}' + \tilde{z}_1$$

where $\tilde{z}_1 \sim_b 0$. Set

$$z_2' := \sum_{k \in K} \sum_{i \in V_k} \sum_{l \in L_i} sgn(\rho_{i,l}) a_i \circ \sigma_{j_k}' \circ \rho_{i,l} + \tilde{z}_1.$$

By Lemma 2.28, it is $z_2' \sim_b z_2$.

Now suppose $r > 1$. By what we have just shown and the induction hypothesis, sequence (2.8) can be transformed into

$$z = z_0 \to_\infty z_0' \to_\infty z_1' \to_\infty ... \to_\infty z_r' \sim_b z_r \sim_b 0.$$

□

Remark 2.36. *For the outer metric, in a joint paper with Javer Fernández de Bobadilla, María Pe Pereira and Edson Sampaio (preprint), we proved that the MD Homology for $b = 1$ coincides with the singular homology of the tangent cone.*

Moreover, considering X equipped with the inner metric, it would be interesting to study the relation of our invariants for $b = 1$ with the topology of the Gromov tangent cone of [1]. The obstruction of the Gromov Tangent Cone analyzed in Section 3 of [1] resembles our equivalence relation for points at $b = 1$, since it involves identifying subanalytic arcs that approach each other at speed larger than 1. We thank A. Parusinski *for pointing out a possible relation with that article.*

2.2.5. Relative b-Moderately Discontinuous Homology

In our setting relative homology exists in two different levels of generality. Let us start with the less general one, which is analogue to the classical Singular Homology Theory (See Subsection 2.6.1 for the other one, which is essential for the formulation of the relative Mayer-Vietoris Theorem in our theory).

Consider $b \in (0, +\infty]$. Given a subanalytic subgerm $(Y, x_0, d_{X_{|Y}}) \hookrightarrow (X, x_0, d_X)$, we denote by $K_\bullet$ the minimal subcomplex of $MDC^b_\bullet(X, x_0, d_X)$ which contains the classes $[\sigma]$, where σ is a l.v.a. simplex in Y. An easy application of Lemma 2.35 shows that the obvious epimorphism of complexes $MDC^b_\bullet(Y, x_0, d_{X_{|Y}}) \to K_\bullet$ is an isomorphism. Therefore we have an inclusion of complexes

$$MDC^b_\bullet(Y, x_0, d_{X_{|Y}}) \hookrightarrow MDC^b_\bullet(X, x_0, d_X). \tag{2.9}$$

Definition 2.37. *Consider $b \in (0, +\infty]$. Given a subanalytic subgerm $(Y, x_0, d_{X_{|Y}}) \hookrightarrow (X, x_0, d_X)$, we define the* complex of relative b-moderately discontinuous chains with coefficients in A, *denoted by $MDC^b_\bullet((X, Y, x_0, d_X); A)$ as the following quotient:*

$$MDC^b_\bullet((X, x_0, d_X); A) \Big/ MDC^b_\bullet((Y, x_0, d_X|_Y); A),$$

which makes sense by inclusion (2.9).

The b-moderately discontinuous homology $MDH^b_*((X, Y, x_0, d_X); A)$ with coefficients in A *is the homology of the complex $MDC^b_\bullet((X, Y, x_0, d_X); A)$.*

We abbreviate calling these complexes and graded abelian groups the b-MD complex *and* b-MD homology *of the pair (X, Y, x_0, d_X). When it is clear from the context we will denote it simply by $MDC^\infty_\bullet(X, Y; A)$ and similarly for homology.*

Notation 2.38. *Denote by $Kom(Ab)^-$ the category of complexes of abelian groups bounded from the right. Denote by $\mathcal{D}(Ab)^-$ the bounded above derived category of abelian groups. It is the localization at quasi-isomorphisms of the category whose objects are complexes bounded from the right and whose morphisms are homotopy classes of morphisms of complexes. Since we will deal with homology we will index the complexes as $... \to C_k \to C_{k-1} \to ....$ There is a functor denoted by H_* from $Kom(Ab)^-$ to the category $GrAb$ of graded abelian groups, which consists in taking the homology of a complex.*

At this point we check functoriality for the first time:

Proposition 2.39. *For every $b \in (0, +\infty]$, the assignments*

$$(X, Y, x_0, d_X) \mapsto MDC^b_\bullet((X, Y, x_0, d_X); A) \quad \textit{and}$$

$$(X, Y, x_0, d_X) \mapsto MDH^b_\bullet((X, Y, x_0, d_X); A)$$

are functors from the category of pairs of metric subanalytic germs to $Kom(Ab)^-$ resp. $GrAb$.

Proof. A Lipschitz l.v.a. subanalytic map $f : (X, x_0, d_X) \to (X', x'_0, d_{X'})$ induces morphisms $MDC^{pre,\infty}_\bullet(X, x_0.d_x) \to MDC^b_\bullet(X', x'_0, d_{X'})$ for every $b \in (0, +\infty]$, by taking $\sigma \mapsto f \circ \sigma$ for every l.v.a. simplex σ and extending by linearity. One can check that it descends to a well defined morphism from $MDC^b_\bullet((X, x_0, d_X); A)$ to $MDC^b_\bullet((X', x'_0, d_{X'}); A)$ because it satisfies the two conditions of Remark 2.31, which are straightforward.

If f takes a subanalytic subgerm Y into a subanalytic subgerm Y', then the homomorphism defined above transforms the subcomplex $MDC^b_\bullet((Y, x_0, d_X|_Y); A)$ into $MDC^b_\bullet((Y', x'_0, d_{X'}|_{Y'}); A)$, and hence descends to the relative homology groups. □

Notation 2.40. *Given a Lipschitz l.v.a. subanalytic map*

$$f : (X, Y, x_0, d_X) \to (X', Y', x'_0, d_{X'})$$

we denote by f_* the induced map at the level of b-MD chains for every $b \in (0, +\infty]$.

2.2.6. The final definition of Moderately Discontinuous Homology

In this section we introduce the complete definition of Moderately Discontinuous Chain Complexes/Homology as a functor from the category of pairs of metric subanalytic germs, to a category of diagrams of complexes/groups.

The starting observation is the following: for $b_1 \geq b_2$ with $b_i \in (0, +\infty]$ there are natural epimorphisms (see Section 2.3.3 for the associated long exact sequence):

$$h^{b_1,b_2} : MDC^{b_1}_\bullet((X, Y, x_0, d_X); A) \to MDC^{b_2}_\bullet((X, Y, x_0, d_X); A) \tag{2.10}$$

which induces a map in homology:

$$h^{b_1,b_2}_* : MDH^{b_1}_\bullet((X, Y, x_0, d_X); A) \to MDH^{b_2}_\bullet((X, Y, x_0, d_X); A). \tag{2.11}$$

Notation 2.41. *We define the category $\mathbb{B}$, where the set of objects is $(0, \infty]$ and there is a unique morphism from b to b' if and only if $b \geq b'$.*

Definition 2.42 (Categories of $\mathbb{B}$-complexes and $\mathbb{B}$-graded abelian groups)**.** *The* category $\mathbb{B} - Kom(Ab)^-$ of $\mathbb{B}$-complexes is the category whose objects are functors from $\mathbb{B}$ to $Kom(Ab)^-$ and the morphisms are natural transformations of functors. The *category* $\mathbb{B} - \mathcal{D}(Ab)^-$ is the category whose objects are functors from $\mathbb{B}$ to $\mathcal{D}(Ab)^-$ and the morphisms are natural transformations of functors. The *category* $\mathbb{B} - GrAb$ *of* $\mathbb{B}$*-graded abelian groups* is the category whose objects are functors from $\mathbb{B}$ to the category $GrAb$ and the morphisms are natural transformations of functors. Concatenation of objects in $\mathbb{B} - Kom(Ab)^-$ with the homology functor H_* yields a functor $\mathbb{B} - H_* : \mathbb{B} - Kom(Ab)^- \to \mathbb{B} - GrAb$ which factorizes through $\mathbb{B} - \mathcal{D} : \mathbb{B} - \mathcal{D}(Ab)^- \to \mathbb{B} - GrAb$.

Proposition 2.43. *The assignments $(X, Y, x_0, d_X) \mapsto MDC^\star_\bullet((X, Y, x_0, d_X); A)$ and $(X, Y, x_0, d_X) \mapsto MDH^\star_*((X, Y, x_0, d_X); A)$ are functors from the category of pairs of metric subanalytic germs to $\mathbb{B} - Kom(Ab)^-$ and $\mathbb{B} - GrAb$ respectively.*

Proof. One only needs to check that the functoriality for Lipschitz l.v.a subanalytic maps for each $b \in \mathbb{B}$ commutes with the epimorphisms (2.10), (2.11) which is clear. □

It is interesting to record that in the case of complex (resp. real) analytic germs our homology theory gives complex (resp. real) analytic invariants.

Corollary 2.44. *Given a complex (resp. real) analytic germ (X, x_0), the $\mathbb{B}$-moderately discontinuous homology $MDH^b_\bullet(X, x_0, d_{out})$ and $MDH^b_\bullet(X, x_0, d_{inn})$ for the outer and inner metrics are complex (resp. real) analytic invariants.*

Proof. A real or complex analytic diffeomorphism is well known to be bi-Lipschitz both for the inner and outer metric and it is clearly l.v.a. □

2.2.7. Bi-Lipschitz invariance of b-MD homology with respect to the inner distance

We check that a subanalytic homeomorphism between two germs (X, x_0) and (Y, y_0) that is bi-Lipschitz for the inner metric is l.v.a.. Then we conclude that the MD Homology for d_{inn} is a bi-Lispchitz invariant.

Proposition 2.45. *Let (X, x_0) and (Y, y_0) be two germs of subanalytic sets. Let $d_{X,inn}$ (resp. $d_{Y,inn}$) be the inner distance of X (resp. Y). Then we have the following:*

(a) $d_{X,inn}$ (resp. $d_{Y,inn}$) induces the same topology on X (resp. Y) as the topology induced by the standard topology on $\mathbb{R}^m$;

(b) If there exists an inner bi-Lipschitz homeomorphism $h\colon (X, x_0) \to (Y, y_0)$ then there exists $K > 0$ satisfying the inequalities

$$\frac{1}{K}\|x - x_0\| \leq \|h(x) - y_0\| \leq K\|x - x_0\|.$$

In order to prove Proposition 2.45, we recall the following result.

Proposition 2.46 (Proposition 3 in [23]). *Let $X \subset \mathbb{R}^m$ be a subanalytic set and $\varepsilon > 0$. Then there exists a finite decomposition $X = \bigcup_{j=1}^k \Gamma_j$ such that:*

1. *each Γ_j is a subanalytic connected analytic submanifold of $\mathbb{R}^m$,*
2. *each $\overline{\Gamma}_j$ satisfies $d_{\overline{\Gamma}_\nu,inn}(p, q) \leq (1 + \varepsilon)\|p - q\|$ for any $p, q \in \overline{\Gamma}_j$.*

Proof of Proposition 2.45. Let us consider $X = \bigcup_{j=1}^k \Gamma_j$ as in Proposition 2.46 and $\varepsilon = 1$. Thus, if $x \in X$, there exists a j such that $x \in \Gamma_j$ and, moreover, we get

$$\frac{1}{2}\|x - x_0\| \leq d_{X,inn}(x, x_0) \leq d_{\Gamma_j,inn}(x, x_0) \leq 2\|x - x_0\|. \tag{2.12}$$

Since $\|x - y\| \leq d_{X,inn}(x, y)$ for any $x, y \in X$, to prove item (a) it is enough to prove that for any $x \in X$ and any ball $B_{inn,\eta}(x)$ with respect to the inner distance, we can find a ball $B_\delta(x)$ with respect to the outer distance such that $B_\delta(x) \subset B_{inn,\eta}(x)$.

But to do this, we just apply Proposition 2.46 to (X, x) and $\varepsilon = 1$, and we get that $B_{\eta/2}(x) \subset B_{inn,\eta}(x)$.

Obviously we have the same result for Y and, in particular, we have

$$\frac{1}{2}\|y - y_0\| \leq d_{Y,inn}(y, y_0) \leq 2\|y - y_0\|. \tag{2.13}$$

In order to get item (b), we just need to apply the Lipschitz properties of h and Eq. (2.12) in Eq. (2.13). □

Thus, by considering Remark 2.11, the following is immediate.

Corollary 2.47. *Let (X, x_0) and (Y, y_0) be two subanalytic germs. If there exists a subanalytic bi-Lipschitz homeomorphism $h\colon (X, x_0, d_{X,inn}) \to (Y, y_0, d_{Y,inn})$, then $(X, x_0, d_{X,inn})$ and $(Y, y_0, d_{Y,inn})$ have the same MD homology. In particular, h induces isomorphisms*

$$h_n\colon MDH^b_n(X, x_0, d_{X,inn}) \to MDH^b_n(Y, y_0, d_{Y,inn})$$

for all $b \in (0, +\infty]$ and $n \in \mathbb{N}$.

2.3. Basic properties of MD-Homology

In this section we prove properties of MD-Homology in analogy with usual homology theories (relative exact sequence, its value at a "point" and sufficiency of chains which are small with respect to a cover). The analogues of homotopy invariance, Mayer-Vietoris and Excision are more subtle and are treated later in the thesis. We introduce also a long exact sequence measuring the relation of the b-MD homologies for different b.

2.3.1. The relative MD-Homology sequence

The relative homology sequence comes quite easily from the definition.

Proposition 2.48. *Let (X, x_0, d_X) be a metric subanalytic germ. Let $Z \subset Y \subset X$ be subanalytic subgerms. For any $b \in \mathbb{B}$ there is a long exact sequence*

$$\begin{aligned} ... &\to MDH^b_n(Y, Z; A) \to MDH^b_n(X, Z; A) \to MDH^b_n(X, Y; A) \\ &\to MDH^b_{n-1}(Y, Z; A) \to MDH^b_{n-1}(X, Z; A) \to MDH^b_{n-1}(X, Y; A) \to ... \end{aligned} \tag{2.14}$$

This exact sequence is functorial in $Z \subset Y \subset X$ and in $b \in \mathbb{B}$.

Proof. The proof is obvious from the definitions. □

Similarly we obtain the spectral sequence of a filtration of pairs of metric subanalytic germs:

Proposition 2.49. *Let $Z_0 \subset Z_1 \subset ... \subset Z_r = X$ be a filtration by closed subanalytic subgerms of (X, x_0). Let Y be another closed subanalytic subgerm of (X, x_0, d_X). For each b, the induced filtration in $MDC^b_\bullet(X, Y; A)$ yields a spectral sequence abutting to $MDH^b_{p+q}(X, Y; A)$ with E^1 page equal to*

$$E[b]^1_{p,q} = MDH^b_{p+q}(Z_p \cup Y, Z_{p-1} \cup Y; A).$$

The spectral sequence is functorial in $b \in \mathbb{B}$.

2.3.2. The Moderately Discontinuous Homology of a "point"

Like in any homology theory the point plays a special role. In the next definition we clarify the notion of point in our category.

Definition 2.50. *A point in the category of metric subanalytic germs is a metric subanalytic germ isomorphic to $([0, \epsilon), 0, d)$, where d is the Euclidean metric.*

Proposition 2.51. *For any $b \in [1, \infty)$ the complex $MDC^b_\bullet((0, \epsilon); A)$ is quasi-isomorphic to the complex $A[0]$, i.e. $MDH^b_0((0, \epsilon); A) = A$ and $MDH^b_n((0, \epsilon); A) = 0$ for all $n > 0$.*

Proof. We show that the augmented chain complex of $C^b_\bullet([0, \epsilon); A)$ by A in degree -1 has trivial homology by constructing a chain homotopy H from the identity to the 0-map: denote by σ_0 the identity map on $[0, \epsilon)$. On degree -1, we define $H(a) = a\sigma_0$. For $n \in \mathbb{N}_0$, given $\sigma : \hat{\Delta}_n \to [0, \epsilon)$ in $MDC^{\text{pre},\infty}_n((0, \epsilon); A)$ define $H(\sigma) \in MDC^{\text{pre},\infty}_{n+1}((0, \epsilon); A)$ to be the suspension of σ by σ_0 given by the formula

$$H(\sigma)(ts_0, ..., ts_{n+1}, t) := (-1)^{n+1}(S\sigma(\frac{ts_0}{S}, ..., \frac{ts_n}{S}, t) + s_{n+1}(\sigma_0(t)))$$

where $(s_0, ..., s_{n+1})$ are barycentric coordinates in Δ_{n+1} and $S := s_0 + ... + s_n$. If $S = 0$, define $H(\sigma)(ts_0, ..., ts_{n+1}, t) := (-1)^{n+1}\sigma_0(t)$. Observe that for an n-simplex σ with $n \geq 1$ it is $H(\sigma \circ j^k_n) = -H(\sigma) \circ j^k_{n+1}$ for $k \leq n$ and $H(\sigma) \circ j^{n+1}_{n+1} = (-1)^{n+1}\sigma$. This defines the chain homotopy in the augmentation of $MDC^{\text{pre},\infty}_\bullet((0, \epsilon); A)$.

In order to finish the proof, we use Remark 2.31 in order to show that the chain homotopy descends to a chain homotopy defined in the augmentation of $MDC^b_\bullet((0, \epsilon); A)$.

Let $\{\rho_i\}_{i \in I}$ be a homological subdivision of $\hat{\Delta}_n$ associated with a triangulation $\alpha : |K| \to \hat{\Delta}_n$. Notice that Δ_{n+1} is the cone over Δ_n, with vertex $p = (0, ..., 0, 1)$; this allows us to see $\hat{\Delta}_{n+1}$ as the cone over $\hat{\Delta}_n$. Let $C(K)$ be the cone over the simplicial complex K and let $\beta : |C(K)| \to \hat{\Delta}_{n+1}$ be the triangulation obtained by taking the cone over the triangulation α. Define $\rho'_i : \hat{\Delta}_{n+1} \to \hat{\Delta}_{n+1}$ to be the cone over the mapping ρ_i. Then the collection $\{\rho'_i\}_{i \in I}$ is a homological subdivision of $\hat{\Delta}_{n+1}$ associated with the triangulation β, such that for any $i \in I$ we have the equality

$$H(\sigma \circ \rho_i) = H(\sigma) \circ \rho'_i.$$

This shows that the homotopy descends to $MDC^\infty_\bullet((0, \epsilon); A)$.

In order to prove that it descends to $MDC^b_\bullet((0, \epsilon); A)$ it only remains to show that it preserves the b-equivalence relation. Let σ_1 and σ_2 be b-equivalent l.v.a. n-simplices.

Then $H(\sigma_1)$ and $H(\sigma_2)$ are b-equivalent, since we have the inequality

$$
\begin{aligned}
&|(S\sigma_1(\frac{ts_0}{S},...,\frac{ts_n}{S},t)-(S\sigma_2(\frac{ts_0}{S},...,\frac{ts_n}{S},t)| \\
\leq & S\max\{|\sigma_1(u_0t,\dots u_nt,t)-\sigma_2(u_0t,\dots u_nt,t)| : (u_0,\dots,u_n)\in\Delta_n\}
\end{aligned}
$$

for every $(s_0,\dots,s_n)\in\Delta_n$ and $S\leq 1$. □

2.3.3. Relative homology with respect to $b\in[0,+\infty)$

In our theory we have also a notion of relative homology with respect to $b\in[0,+\infty)$.

Definition 2.52. *Let (X,Y) be a pair of metric subanalytic germs, A an abelian group and $b_1\geq b_2\in Obj(\mathbb{B})$. We define the chain complex $MDC_\bullet^{b_1,b_2}(X,Y;A)$ to be the kernel of the epimorphism*

$$
h^{b_1,b_2} : MDC_\bullet^{b_1}(X,Y;A)\to MDC_\bullet^{b_2}(X,Y;A)
$$

The n-th (b_1,b_2)-moderately discontinuous homology *is defined to be the homology of $MDC_\bullet^{b_1,b_2}(X,Y;A)$.*

Proposition 2.53. *The following long exact sequence is an immediate consequence of the last definition:*

$$
\begin{aligned}
... &\to MDH_n^{b_1,b_2}(X,Y;A)\to MDH_n^{b_1}(X,Y;A)\to MDH_n^{b_2}(X,Y;A)\to \\
&\to MDH_{n-1}^{b_1,b_2}(X,Y;A)\to MDH_{n-1}^{b_1}(X,Y;A)\to MDH_{n-1}^{b_2}(X,Y;A)\to ...
\end{aligned}
\tag{2.15}
$$

Its association to (X,Y) is functorial.

2.3.4. MD-chains which are small with respect to a subanalytic cover

We will need to use chains which are small with respect to covers as in the classical development of singular homology (see for example [17], Ch 15) as a technical tool.

Definition 2.54. *Let (X,x_0) be a subanalytic germ. A* finite closed subanalytic cover of X *is a finite collection of closed subanalytic subsets $\mathcal{C}:=\{C_i\}_{i\in I}$ of X such that $X=\bigcup_{i\in I}C_i$.*

Let (X,x_0,d_X) be a metric subanalytic germ. Given a finite closed subanalytic cover $\mathcal{C}$, a chain $\sum_{j\in J}a_j\sigma_j\in MDC_\bullet^{\mathrm{pre},\infty}(X,x_0;A)$ is called small with respect to $\mathcal{C}$, *if for any j the image of σ_j is contained in one of the subsets of the cover. We denote by $MDC_\bullet^{\mathrm{pre},\infty,\mathcal{C}}(X,x_0;A)$ the subcomplex of $MDC_\bullet^{\mathrm{pre},\infty}(X,x_0;A)$ formed by the chains which are small with respect to the cover.*

We define the complexes $MDC_\bullet^{\infty,\mathcal{C}}(X,x_0;A)$ and $MDC_\bullet^{b,\mathcal{C}}(X,x_0,d_X;A)$ by restricting the equivalence relations $\sim_{S,\infty}$ and $\sim_{S,b}$ to $MDC_\bullet^{\mathrm{pre},\infty,\mathcal{C}}(X,x_0;A)$.

Given a subanalytic subgerm $Y\subset X$ we define the complexes $MDC_\bullet^{\infty,\mathcal{C}}(X,Y;A)$ and $MDC_\bullet^{b,\mathcal{C}}(X,Y,d_X;A)$ as the quotients of $MDC_\bullet^{\infty,\mathcal{C}}(X,x_0;A)$ and $MDC_\bullet^{b,\mathcal{C}}(X,x_0,d_X;A)$ by $MDC_\bullet^{\infty,\mathcal{C}}(Y,x_0;A)$ and $MDC_\bullet^{b,\mathcal{C}}(Y,x_0,d_X|_Y;A)$ respectively (as in Definition 2.37, we may assume that the complexes we quotient by are subcomplexes).

Proposition 2.55. *Let $\mathcal{C}$ be a finite closed subanalytic cover of X. The natural morphism of complexes*

$$g : MDC_\bullet^{b,\mathcal{C}}(X,Y;A) \to MDC_\bullet^{b}(X,Y;A)$$

is an isomorphism.

Proof. By the 5-Lemma it is enough to prove the proposition for absolute homology, that is to prove the isomorphism $MDC_\bullet^{b,\mathcal{C}}(X;A) \to MDC_\bullet^{b}(X;A)$ for any metric subanalytic germ (X, x_0, d_X).

The surjectivity is proved as follows: let $\sigma : \hat{\Delta}_n \to (X, x_0)$ be an n-simplex. We consider the collection $\mathcal{D}$ of closed subanalytic subsets of $\hat{\Delta}_n$ given by the preimages by σ of the subsets of $\mathcal{C}$ together with the collection of all the faces of $\hat{\Delta}_n$. Let $\alpha : |K| \to \hat{\Delta}_n$ be a triangulation of a representative of $\hat{\Delta}_n$ compatible with $\mathcal{D}$ (see Remark 2.15). Let $\{T_i\}_{i\in I}$ be the collection of maximal triangles of α. By restricting the representative of $\hat{\Delta}_n$ we may assume that each maximal triangle T_i contains the vertex. For each $i \in I$ choose a subanalytic orientation preserving, homeomorphism $\rho_i : \hat{\Delta}_n \to T_i$ sending the vertex to the vertex, and which preserves the simplicial structure. By Remark 2.8 we may assume ρ_i to be l.v.a. Then the collection $\{\rho_i\}_{i\in I}$ is a homological subdivision of $\hat{\Delta}_n$, and we have the equivalence $\sigma \sim_{S,b} \sum_{i\in I} sgn(\rho_i)\sigma\circ\rho_i$. Since the chain on the right hand side is small with respect to $\mathcal{C}$ surjectivity is proven.

Injectivity is an immediate consequence of Lemma 2.35. □

Proposition 2.55 allows us to improve Remark 2.31 in the following manner:

Remark 2.56. *In order to define a homomorphism*

$$MDC_\bullet^{b}((X, x_0, d_X); A) \to G,$$

where G is an abelian group, we will often proceed as follows: We take a finite closed subanalytic cover $\mathcal{C}$ of X, define a homomorphism $\bar{h} : MDC_\bullet^{\mathrm{pre},\infty,\mathcal{C}}((C_i, x_0, d_X); A) \to G$, check that the two conditions of Remark 2.31 hold and compose with g^{-1} on the right, where g is the isomorphism of Proposition 2.55.

2.4. Moderately discontinuous functoriality

In this section we improve functoriality properties of the b-MD homology for a fixed b by allowing a certain class of non-continuous maps. This makes our theory quite flexible. The discontinuities that we allow are *moderated* in a Lipschitz sense. This may be seen as a motivation for the name of our homology.

2.4.1. Definition and functoriality of b-maps

Definition 2.57. *Let (X, x_0, d_X) be a metric subanalytic germ. In line with Definition 2.50, we define a* point in X *to be a continuous l.v.a. subanalytic map germ $p : [0, \epsilon) \to X$. For any subanalytic $Y \subseteq X$, we say that p* is contained in Y, *if $Im(p) \subseteq Y$. Observe that a point in X is the same as a l.v.a. 0-simplex of X.*

Two points p and q are called b-equivalent*, for $b \in (0, +\infty)$, and we write $p \sim_b q$, if*

$$\lim_{t \to 0} \frac{d_X(p(t), q(t))}{t^b} = 0.$$

We can restate the equivalence of (i) and (iii) of Lemma 2.27 as follows:.

Remark 2.58. *Let σ_1, σ_2 be n-simplices in $MDC_\bullet^{\mathrm{pre},\infty}(X, x_0, d_X)$. It follows from Lemma 2.27 that we have the equivalence $\sigma_1 \sim_b \sigma_2$ if and only if for any point p in $\hat{\Delta}_n$, $\sigma_1 \circ p$ and $\sigma_2 \circ p$ are b-equivalent.*

Definition 2.59. *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs, $b \in (0, \infty)$. A* b-moderately discontinuous subanalytic map (b-map, for abbreviation) *from (X, x_0, d_X) to (Y, y_0, d_Y) is a finite collection $\{(C_i, f_i)\}_{i \in I}$, where $\{C_i\}_{i \in I}$ is a finite closed subanalytic cover of X and $f_i : C_i \to Y$ is a Lipschitz l.v.a. subanalytic map satisfying the following: for any b-equivalent pair of points p and q contained in C_i and C_j respectively, the points $f_i \circ p$ and $f_j \circ q$ are b-equivalent in Y.*

Two b-maps $\{(C_i, f_i)\}_{i \in I}$ and $\{(C'_i, f'_i)\}_{i \in I'}$ are called b-equivalent *if for any b-equivalent pair of points p, q with $Im(p) \subseteq C_i$ and $Im(q) \subseteq C'_{i'}$, the points $f_i \circ p$ and $f'_{i'} \circ q$ are b-equivalent in Y.*

We make an abuse of language and we also say that a b-map from (X, x_0, d_X) to (Y, y_0, d_Y) is an equivalence class as above.

For $b = \infty$, a b-map from X to Y is a Lipschitz l.v.a. subanalytic map from X to Y.

Proposition 2.60 (Definition of composition of b-maps)**.** *Let $\{(C_i, f_i)\}_{i \in I}$ be a b-map from X to Y and let $\{(D_j, g_j)\}_{j \in J}$ be a b-map from Y to Z. Then the composition of the two b-maps is well defined by $\{(f_i^{-1}(D_j) \cap C_i, g_j \circ f_{i|f_i^{-1}(D_j) \cap C_i})\}_{(i,j) \in I \times J}$.*

Proof. Any pair of b-equivalent points p and q that are contained in $f_{i_1}^{-1}(D_{j_1}) \cap C_{i_1}$ resp. $f_{i_2}^{-1}(D_{j_2}) \cap C_{i_2}$ are sent by f_{i_1} resp. f_{i_2} to b-equivalent points in Y contained in D_{j_1} resp. D_{j_2}. Those are sent by g_{j_1} resp. g_{j_2} to b-equivalent points in Z.

Let $\{(\hat{C}_i, \hat{f}_i)\}_{i \in \hat{I}}$ and $\{(\hat{D}_j, \hat{g}_j)\}_{j \in \hat{J}}$ be b-equivalent to $\{(C_i, f_i)\}_{i \in I}$ and $\{(D_j, g_j)\}_{j \in J}$ respectively. Let p and q be b-equivalent points contained in $f_i^{-1}(D_j) \cap C_i$ and $\hat{f}_{\hat{i}}^{-1}(\hat{D}_{\hat{j}}) \cap \hat{C}_{\hat{i}}$ respectively. By the exact same reasoning p and q are sent to b-equivalent points in Z by $g_j \circ f_i$ and $\hat{g}_{\hat{j}} \circ \hat{f}_{\hat{i}}$ respectively. □

Corollary 2.61. *The category of metric subanalytic germs with b-maps is well defined.*

Definition 2.62. *A b-map between pairs of metric subanalytic germs (X, Y, x_0, d_X) and $(\tilde{X}, \tilde{Y}, \tilde{x}_0, d_{\tilde{X}})$ is a b-map from X to $\tilde{X}$ admitting a representative $\{(C_i, f_i)\}_{i \in I}$ for which the image of $C_i \cap Y$ under f_i is contained in $\tilde{Y}$ for any i.*

Let $\phi := \{(C_i, f_i)\}_{i \in I}$ be a b-map between two pairs (X, Y, x_0, d_X) and $(\tilde{X}, \tilde{Y}, \tilde{x}_0, d_{\tilde{X}})$. We are going to define a homomorphism

$$\phi_\bullet^b : MDC_\bullet^b((X, x_0, d_X); A) \to MDC_\bullet^b((\tilde{X}, \tilde{x}_0, d_{\tilde{X}}); A)$$

depending on $\{(C_i, f_i)\}_{i\in I}$ that clearly descends to a homomorphism on the relative chain complexes. Following Remark 2.56, to define $\phi^b_\bullet$, we define a homomorphism

$$\phi^{pre,b}_{\mathcal{C}} : MDC^{pre,\infty,\mathcal{C}}_\bullet((X, x_0, d_X); A) \to MDC^b_\bullet((\tilde{X}, \tilde{x}_0, d_{\tilde{X}}); A)$$

where $\mathcal{C}$ is a finite closed subanalytic refinement of $\{C_i\}_i$ as follows: the image of any $\sigma \in MDC^{\text{pre},\infty,\mathcal{C}}_\bullet((X, x_0, d_X); A)$ is contained in some C_i. We define the image of σ under $\phi^{pre,b}_{\mathcal{C}}$ to be $f_i \circ \sigma$ and extend this definition linearly.

There are five things to be checked to guarantee that $\phi^{pre,b}_{\mathcal{C}}$ and $\phi^b_\bullet$ are well-defined:

1. If the image of σ is also contained in a different C_j, $f_j \circ \sigma$ is b-subdivision equivalent to $f_i \circ \sigma$;
2. $\phi^{pre,b}_{\mathcal{C}}$ is compatible with the b-equivalence relation;
3. $\phi^{pre,b}_{\mathcal{C}}$ is compatible with ∞-immediately equivalences;
4. If $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are different refinements of $\{C_i\}_i$, $\phi^{pre,b}_{\mathcal{C}}$ and $\phi^{pre,b}_{\tilde{\mathcal{C}}}$ define the same ϕ;
5. If $\{(\tilde{C}_i, \tilde{f}_i)\}_{i\in\tilde{I}}$ is b-equivalent to $\{(C_i, f_i)\}_{i\in I}$, consider a refinement $\mathcal{C}$ that refines both $\{C_i\}_{i\in I}$ and $\{\tilde{C}_i\}_{i\in\tilde{I}}$. The image of any $\sigma \in MDC^{\text{pre},\infty,\mathcal{C}}_\bullet((X, Y, x_0, d_X); A)$ is contained both in some C_i and in some $\tilde{C}_j$. The two simplices $f_i \circ \sigma$ and $\tilde{f}_j \circ \sigma$ are b-subdivision equivalent.

For (1), we are going to show that $f_i \circ \sigma$ and $f_j \circ \sigma$ are b-equivalent, where σ is an n-simplex whose image is contained both in C_i and C_j. Let p be a point in $\hat{\Delta}_n$. By definition of b-map, $f_i \circ \sigma \circ p$ and $f_j \circ \sigma \circ p$ are b-equivalent. So the statement follows from Remark 2.58. For (5), we can use the exact same argument to show that $f_i \circ \sigma$ and $\tilde{f}_j \circ \sigma$ are b-equivalent. Statement (3) is obvious. For (2), let σ and σ' be l.v.a. simplices that are b-equivalent whose images are contained in C_{i_1} resp. C_{i_2}. We have to show that $f_{i_1} \circ \sigma$ and $f_{i_2} \circ \sigma'$ are b-equivalent. Suppose they were not. Then there would be a point p in $\hat{\Delta}_n$ for which $f_{i_1} \circ \sigma \circ p$ and $f_{i_2} \circ \sigma' \circ p$ are not b-equivalent. So $\sigma \circ p$ and $\sigma' \circ p$ would not be b-equivalent. To show (4), take a common refinement $\mathcal{D}$ of $\mathcal{C}$ and $\tilde{\mathcal{C}}$. Then, $\mathcal{D}$ defines the same $\phi^b_\bullet$ as $\mathcal{C}$ and the same as $\tilde{\mathcal{C}}$.

Then, we have proved the following:

Proposition 2.63 (Functoriality for b-maps)**.** *For a fixed $b \in (0, \infty]$, there are well defined functors*

$$(X, Y, x_0, d_X) \mapsto MDC^b_*((X, Y, x_0, d_X); A)$$
$$(X, Y, x_0, d_X) \mapsto MDH^b_*((X, Y, x_0, d_X); A)$$

from the category of pairs of metric subanalytic germs with b-maps to $Kom(Ab)^-$ and $GrAb$ respectively.

Corollary 2.64. *The b-moderately discontinuous homology is invariant by isomorphisms in the category of b-maps.*

2.4.2. A sufficient geometric condition

Notation 2.65. *Let (X, x_0, d_X) be a metric subanalytic germ. Denote by $L_{X,\epsilon}$ the link $\{x \in X : ||x - x_0|| = \epsilon\}$.*

Definition 2.66. *Let (X, x_0, d_X) be a metric subanalytic germ and $Y \subset X$ a subanalytic subgerm. Let $b \in (0, \infty)$. The b-*horn neighborhood of amplitude η of Y in X *is the subset*

$$\mathcal{H}_{b,\eta}(Y; X) := \bigcup_{y \in Y} B(y, \eta \|y - x_0\|^b),$$

where $B(y, \eta\|y - x_0\|^b) := \{x \in X : d_X(x, y) < \eta\|y - x_0\|^b\}$ denotes the ball in X centered in y of radius $\eta d_X(y, x_0)^b$. The ∞-horn neighborhood $\mathcal{H}_{b,\eta}(Y; X)$ is defined to be Y.

The importance of b-horn neighbourhoods in our theory is due to the following:

Remark 2.67. *Any l.v.a. simplex that is b-equivalent to a l.v.a. simplex whose image is contained in Y is contained in any b-horn neighborhood Y in X.*

We have the following geometric condition that is sufficient for a collection $\{(C_i, f_i)\}_{i \in I}$ to define a b-map:

Lemma 2.68. *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs, $b \in (0, \infty)$. Let $\{(C_i, f_i)\}_{i \in I}$ be a finite collection, where $\{C_i\}_{i \in I}$ is a finite closed subanalytic cover of X, the maps $f_i : C_i \to Y$ are Lipschitz l.v.a. subanalytic and admit an extension $\overline{f}_i : \mathcal{H}_{b,\eta}(C_i; X) \to Y$ that are Lipschitz (non-necessarily subanalytic) l.v.a. maps for some $\eta \in \mathbb{R}_{>0}$, and the following condition is satisfied for any pair of indices $i, j \in I$:*

$$\lim_{\epsilon \to 0} \frac{\sup\{d_Y(\overline{f}_i(x), \overline{f}_j(x)); x \in L_{X,\epsilon} \cap \mathcal{H}_{b,\eta}(C_i; X) \cap \mathcal{H}_{b,\eta}(C_j; X)\}}{\epsilon^b} = 0 \tag{2.16}$$

Then, $\{(C_i, f_i)\}_{i \in I}$ is a b-map.

Proof. We suppose $x_0 = 0$. Let p and q be two b-equivalent points contained in C_i and C_j respectively. We have to show that $f_i \circ p$ and $f_j \circ q$ are b-equivalent.

Since p and q are b-equivalent, the image of q is contained in $\mathcal{H}_{b,\eta}(C_i; X)$. By the triangle inequality we have

$$\frac{d_X(f_i \circ p(t), f_j \circ q(t))}{t^b} \leq \frac{d_X(\overline{f}_i \circ p(t), \overline{f}_i \circ q(t))}{t^b} + \frac{d_X(\overline{f}_i \circ q(t), \overline{f}_j \circ q(t))}{t^b}.$$

Since $\overline{f}_i$ is Lipschitz and p and q are b-equivalent, the first summand of the right hand side converges to 0 as t approaches 0. The second summand converges to 0 by the equation (2.16).

□

Lemma 2.69. *Let $\{(C_i, f_i)\}_{i\in I}$ and $\{(C'_i, f'_i)\}_{i\in I'}$ be two collections fulfilling the conditions of Lemma 2.68. If for any $i \in I$ and $i' \in I'$, it is*

$$\lim_{\epsilon\to 0} \frac{\sup\{d_Y(\bar{f}_i(x), \bar{f}'_{i'}(x)); x \in L_{X,\epsilon} \cap \mathcal{H}_{b,\eta}(C_i; X) \cap \mathcal{H}_{b,\eta}(C'_{i'}; X)\}}{\epsilon^b} = 0, \tag{2.17}$$

the two b-maps defined by them are b-equivalent.

Proof. The proof is analogous to the one of Lemma 2.68. □

2.4.3. Applications using b-maps

Definition 2.70. *A* section of a b-map $\varphi : X \to Y$ (b-section for short) *is a b-map $\psi : Y \to X$ such that $\varphi\circ\psi = Id_Y$ in the category of b-maps.*

Remark 2.71. *Notice that admitting sections in the category of b-maps is much less restrictive than in the category of continuous subanalytic maps, since b-maps are only piecewise continuous, and piecewise univalued.*

Theorem 2.72. *Let $\varphi : X \to Y$ be a Lipschitz l.v.a. subanalytic map between two metric subanalytic germs so that there exists a finite closed subanalytic cover $\{Y_i\}_{i\in I}$ of Y so that*

$$\varphi|_{\varphi^{-1}(Y_i)} : \varphi^{-1}(Y_i) \to Y_i$$

admits a b-section $\{(Y_{i,j}, \psi_{i,j})\}_{j\in J_i}$ for any $i \in I$. Suppose that for any two points p and q in X for which $\varphi \circ p$ and $\varphi \circ q$ are b-equivalent in Y, p and q are b-equivalent in X. Then, φ induces an isomorphism

$$\varphi_* : MDC^b_\bullet(X; A) \to MDC^b_\bullet(Y; A).$$

Consequently φ_* induces an isomorphism in b-MD homology.

Proof. The b-sections glue to a global b-section $(Y_{i,j}, \psi_{i,j})_{i\in I,j\in J_i}$: let p_1 and p_2 be b-equivalent points in Y_{i_1,j_1} and Y_{i_2,j_2} respectively. Then, $\varphi \circ \psi_{i_l,j_l} \circ p_l$ is b-equivalent to p_l for $l = 1, 2$ and therefore $\varphi \circ \psi_{i_l,j_l} \circ p_1$ and $\varphi \circ \psi_{i_l,j_l} \circ p_2$ are b-equivalent. Therefore, by hypothesis so are $\psi_{i_l,j_l} \circ p_1$ and $\psi_{i_l,j_l} \circ p_1$.

To show that the global b-section is in fact the inverse of (X, φ), we have to show that $\{(\varphi^{-1}(Y_{i,j}), \psi_{i,j} \circ \varphi)\}_{i\in I,j\in J_i}$ is b-equivalent to (X, id_X). Let p and q be b-equivalent points in $\varphi^{-1}(Y_{i,j})$ and X respectively. Then $\varphi \circ \psi_{i,j} \circ \varphi \circ p$ is b-equivalent to $\varphi \circ p$, which is b-equivalent to $\varphi \circ q$ as φ is Lipschitz. Therefore, $\psi_{i,j} \circ \varphi \circ p$ is b-equivalent to q. □

Corollary 2.73. *Let $\varphi : (X, x_0, d_X) \to (Y, y_0, d_Y)$ be a Lipschitz l.v.a. subanalytic map between two metric subanalytic germs so that there exists a finite closed subanalytic cover $\{Y_i\}_{ß\in I}$ of Y and open sets U_i containing Y_i for every $i \in I$ such that there is a b-horn neighborhood $\mathcal{H}_{b,\eta}(Y_i; Y)$ contained in U_i, and*

$$\varphi|_{\varphi^{-1}(U_i)} : \varphi^{-1}(U_i) \to U_i$$

admits a section ψ_i in the category of Lipschitz l.v.a. subanalytic maps for any $i \in I$. Suppose that

$$\lim_{t\to 0^+} \frac{\sup\{diam(\varphi^{-1}(y)) : y \in L_{Y,t}\}}{t^b} = 0,$$

Then, φ fulfils the hypothesis of Theorem 2.72 and therefore induces an isomorphism in $Kom(Ab)^-$ and $GrAb$.

Proof. Let p_1 and p_2 be points in X for which $\varphi \circ p_1$ and $\varphi \circ p_2$ are b-equivalent and contained in Y_i and Y_j respectively. By Remark 2.67, $\varphi \circ p_2$ is contained in $\mathcal{H}_{b,\eta}(Y_i; Y)$. As ψ_i is Lipschitz, $\psi_i \circ \varphi \circ p_1$ and $\psi_i \circ \varphi \circ p_2$ are b-equivalent. Further, if K_l is a l.v.a. constant for $\varphi \circ p_l$, $l \in \{1, 2\}$, we have

$$\frac{d_X(p_l(t), \psi_i \circ \varphi \circ p_l(t))}{t^b} \leq K_l^b \frac{\sup\{diam(\varphi^{-1}(y)) : y \in L_{Y,\|\varphi\circ p_l(t)\|}\}}{\|\varphi \circ p_l(t)\|^b}$$

and therefore p_l and $\psi_i \circ \varphi \circ p_l$ are b-equivalent. Using the triangle inequality, we get that p_1 and p_2 are b-equivalent. □

The following corollary is an example of how b-maps and Theorem 2.72 can be used concretely.

Corollary 2.74. *Let X be a metric subanalytic germ such that*

$$\lim_{t\to 0^+} \frac{diam(L_{X,t})}{t^b} = 0$$

Then X has the b-MD homology of a point in the category of metric subanalytic germs (recall Definition 2.50).

Proof. Map X to $[0, 1)$ by outer distance to the vertex of X and use the previous corollary. Considering the trivial cover of X by the single open subset X, the required section is the parametrization of an arc in X by its distance to the origin. □

2.5. Metric homotopy and b-homotopy invariance

Now we prove the invariance of MD-Homology by different kinds of metric homotopies. Here the theory differs if we consider actual (Lipschitz l.v.a. subanalytic) maps or b-maps. For actual maps the notion of metric homotopy is simply a family of Lipschitz l.v.a subanalytic maps with uniform Lipschitz and l.v.a. constant. For b-maps the definition is slightly more elaborated.

In this section I denotes the unit interval $[0, 1]$.

2.5.1. Metric homotopy

Definition 2.75 (Metric homotopy)**.** *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs. Let $f, g : (X, x_0, d_X) \to (Y, y_0, d_Y)$ be Lipschitz l.v.a. subanalytic maps. A continuous subanalytic map $H : X \times I \to Y$ is called a* metric homotopy between f and

g, if there is a uniform constant $K \geq 0$ such that for any s the mapping $H_s := H(-, s)$ is Lipschitz l.v.a. subanalytic with Lipschitz l.v.a. constant K and $H_0 = f$ and $H_1 = g$.

Theorem 2.76. *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs. Let A be an abelian group.*

1. *Let $f, g : (X, x_0) \to (Y, y_0)$ be l.v.a. subanalytic maps such that there exists a continuous subanalytic mapping $H : X \times I \to Y$ with $H_0 = f$ and $H_1 = g$ such that there exists a uniform constant $K > 0$ such that for every s, the mapping H_s is l.v.a. for the constant K. Then we have that both $f^\infty, g^\infty : MDC^\infty_\bullet(X; A) \to MDC^\infty_\bullet(Y; A)$ are the same in $\mathcal{D}(Ab)^-$.*

2. *Let $f, g : (X, x_0, d_X) \to (Y, y_0, d_Y)$ be Lipschitz l.v.a. subanalytic maps that are metrically homotopic. Then*

$$f_\bullet, g_\bullet : MDC^\star_\bullet(X; A) \to MDC^\star_\bullet(Y; A)$$

   represent the same map in the category $\mathbb{B} - \mathcal{D}(Ab)^-$. As a consequence they induce the same homomorphism in MD homology.

Proof. Let us prove Assertion (2). The proof of Assertion (1) is completely similar, disregarding metric considerations.

A common proof for the analogue statement in singular homology uses the inclusions $x \to (x, 0)$ and $x \to (x, 1)$ from X to $X \times I$ and constructs a chain homotopy between the maps they induce on the singular chain complex; functoriality then yields the desired result. To prove Assertion (2), we imitate the idea behind that chain homotopy, but as $X \times I$ is not an object of the category of metric subanalytic germs, we directly construct a chain homotopy η^b from $f^b_\bullet$ to $g^b_\bullet$. Such a chain homotopy will be clearly compatible with the homomorphisms connecting the complexes for different b's.

Let H be a metric homotopy from f to g. In order to construct the chain homotopy $\eta^b_\bullet : MDC^b_\bullet(X; A) \to MDC^b_{\bullet+1}(Y; A)$, by Remark 2.31, it is enough to construct a homomorphism $h_\bullet : MDC^{pre,\infty}_\bullet(X; A) \to MDC^b_{\bullet+1}(Y; A)$ fulfilling the two conditions of the remark.

Define

$$\widehat{\Delta_n \times I} := \{(t(x, s), t) : (x, s) \in \Delta_n \times I, t \in [0, 1)\} \subset \mathbb{R}^{n+2}.$$

The parameter s is the "homotopy parameter", and the parameter t measures the proximity to the vertex, as usually along this thesis. We have the notion of l.v.a. maps from $\widehat{\Delta_n \times I}$ to a metric germ (X, x_0, d_X), in an analogous way with the case of maps from $\hat{\Delta}_n$. Moreover Definition 2.16 extends in an obvious way to a notion of homological subdivision of $\widehat{\Delta_n \times I}$.

Let $\sigma : \hat{\Delta}_n \to X$ be a l.v.a. simplex. Define $\hat{h}_n(\sigma) : \widehat{\Delta_n \times I} \to Y$ to be the continuous subanalytic extension of the map given by $(t(x, s), t) \mapsto H(\sigma(tx, t), s)$ for $t \neq 0$. The map $\hat{h}_n(\sigma)$ is subanalytic and l.v.a..

Let $\alpha_j : |K| \to \widehat{\Delta_n \times I}$ be triangulations of $\widehat{\Delta_n \times I}$ for $j = 1, 2$, and let $\{\rho_{j,i}\}_{i \in I_j}$ be an orientation preserving homological subdivisions of $\widehat{\Delta_n \times I}$ associated with each of the triangulations. For $j = 1, 2$ the sum $z_j := \sum_{i \in I_j} \hat{h}_n(\sigma) \circ \rho_{j,i}$ is an element of

$MDC_\bullet^{\text{pre},\infty}(Y, A)$. By choosing a common refinement of the subanalytic triangulations α_1 and α_2 and arguing like in the proof of Lemma 2.21, we show that there exists an element $z_3 \in MDC_\bullet^{\text{pre},\infty,}(Y, A)$ and immediate equivalences $z_1 \to_\infty z_3$ and $z_2 \to_\infty z_3$. This shows that the assignment $h_n(\sigma) := z_j$ in $MDC_\bullet^b(Y, A)$ gives, extending by linearity, a well defined homomorphism

$$h_n : MDC_n^{pre,\infty}(X; A) \to MDC_{n+1}^b(Y; A).$$

Now we check that the conditions of Remark 2.31 are satisfied.

If we have two b-equivalent simplices $\sigma \sim_b \sigma'$, in order to prove the equivalence $h_n(\sigma) \sim_b h_n(\sigma')$, using the arc characterization of Lemma 2.27, it is enough to prove that for any subanalytic l.v.a continuous arc $\gamma : [0, \epsilon) \to \widehat{\Delta}_n$, with coordinates $\gamma(t) = (\gamma_2(t)\gamma_1(t), \gamma_2(t))$, and for any subanalytic function $\rho : [0, \epsilon) \to I$, we have the vanishing of the limit

$$\lim_{t\to 0^+} \frac{d(H(\sigma(\gamma(t)), \rho(s)), \sigma'(\gamma(t)), \rho(s)))}{t^b} = 0.$$

Since the numerator is bounded by $Kd(\sigma(\gamma(t), \sigma'(\gamma(t))$, and we have $\sigma \sim_b \sigma'$ and γ is l.v.a. the limit vanishes as needed.

Let σ be a n-simplex, and $\{\rho_i\}_{i\in I}$ be a homological subdivision of $\hat{\Delta}_n$ associated with a subanalytic triangulation $\alpha : |K| \to \hat{\Delta}_n$. The triangulation α induces a decomposition of $\widehat{\Delta_n \times I}$ that can be refined to a subanalytic triangulation β of $\widehat{\Delta_n \times I}$. Let $\{\mu_k\}_{k\in K}$ be a homological subdivision associated to β. Then we have that $h_n(\sigma)$, previously defined, coincides with $\sum_k sgn(\mu_k)\hat{h}_n(\sigma)\circ\mu_k$.

Thus, we have constructed for every n a well defined map

$$\eta_n^b : MDC_n^b(X; A) \to MDC_{n+1}^b(Y; A).$$

In order to prove that it is a chain homotopy we have to check the equation $\partial\eta_n^b + \eta_{n-1}^b\partial = g_\bullet^b - f_\bullet^b$. For this we only need a cancelling of interior boundaries very similar to the proof of Lemma 2.23.

□

2.5.2. b-Homotopies.

For the definition of b-homotopies we need a notion of product of a metric subanalytic germ $(X, 0, d_X)$ with the interval I, which lives in the category of metric subanalytic germs. Moreover we need the hypothesis $d_{X,out} \leq d_X$ which in particular holds for the inner and the outer metrics.

Definition 2.77. *Let $(X, 0, d_X)$ be a metric subanalytic germ. For $x \in X$ we denote by $||x||$ the usual euclidean norm of x, which may differ from $d_X(x, 0)$. By $X \times_p I$, we denote the following metric subanalytic germ $(\tilde{X}, \tilde{v}, \tilde{d})$:*

$\tilde{X} := \{(x, \|x\|s) : x \in X, s \in I\} \subset X \times \mathbb{R}$

$\tilde{v} := (0, 0)$

$\tilde{d}((x_1, ||x_1||s_1), (x_2, ||x_2||s_2)) := \sup\{d_X(x_1, x_2),\ d_{X,\triangledown}((x_1, ||x_1||s_1), (x_2, ||x_2||s_2))\}$

where $d_{X,\triangledown}$ is defined as follows: let T denote the straight cone over the unit interval:

$$T := \{(d, ds) \in \mathbb{R}^2 : d \in [0,1], s \in [0,1]\}$$

Let $d_\triangledown$ denote the maximum metric on T. We define

$$d_{X,\triangledown}((x_1, \|x_1\|s_1), (x_2, \|x_2\|s_2)) := d_\triangledown((\|x_1\|, \|x_1\|s_1), (\|x_2\|, \|x_2\|s_2))$$

For a visualization of $d_{X,\triangledown}$, see Figure 2.2.

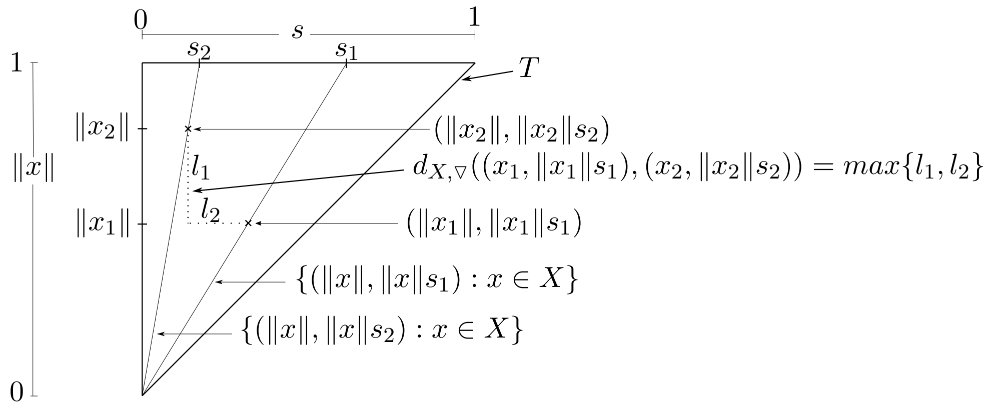


Figure 2.2.: The metric $d_{X,\triangledown}$.

Lemma 2.78. *Let d_X be a metric on a subanalytic germ (X, x_0) such that $d_{X,out} \leq d_X$. The following inequality holds*

$$d_{X,\triangledown}((x_1, \|x_1\|s), (x_2, \|x_2\|s)) \leq M\sqrt{2}d_X(x_1, x_2) \tag{2.18}$$

for any $x_1, x_2 \in X, s \in I$, where M is the bi-Lipschitz constant between the maximum and the Euclidean norm on T.

Moreover,

$$\tilde{d}((x_1, \|x_1\|s), (x_2, \|x_2\|s)) \leq M\sqrt{2}d_X(x_1, x_2). \tag{2.19}$$

Proof. We have the following easy chain of inequalities:

$$d_{X,\triangledown}((x_1, \|x_1\|s), (x_2, \|x_2\|s)) \leq M\sqrt{1+s^2}|\|x_1\| - \|x_2\|| \leq$$

$$\leq M\sqrt{1+s^2}\|x_1 - x_2\| \leq M\sqrt{1+s^2}d_X(x_1, x_2).$$

Notice that $s \leq 1$. To prove (2.19) we just use the previous inequality and the definition to get that $\tilde{d} \leq d_X \cdot max M\sqrt{1+s^2}, 1$. □

Definition 2.79 (b-homotopy)**.** *Let (X, x_0, d_X) and (Y, y_0, d_Y) be metric subanalytic germs. A* b-homotopy *is a b-map from $X \times_p I$ to Y.*

Theorem 2.80. *If there is a b-homotopy H with $H_0 = f$ and $H_1 = g$, then*

$$f_\bullet, g_\bullet : MDC^b_\bullet(X; A) \to MDC^b_\bullet(Y; A)$$

represent the same map in the category $\mathcal{D}(Ab)^-$. As a consequence they induce the same homomorphism in MD homology.

Proof. For this proof we can follow the classical proof for singular homotopy much more closely: denote by $i_s : X \to X \times_p I$ the inclusion given by $i_s(x) := (x, ||x||s)$ (which is a Lipschitz l.v.a. subanalytic map by (2.19)). It is enough to prove that i_0 and i_1 induce chain homotopic homomorphisms from $MDC^b_*(X)$ to $MDC^b_*(X \times_p I)$.

Given any l.v.a. simplex $\sigma : \hat{\Delta}_n \to X$ we define $\hat{\eta}_n(\sigma) : \widehat{\Delta_n \times I} \to X \times_p I$ to be the map $(t(x, s), t) \mapsto (\sigma(tx, t), ||\sigma(tx, t)||s)$.

In order to define the homomorphism $\eta_n : MDC^b_n(X) \to MDC^b_n(X \times_p I)$ we proceed as in the proof of Theorem 2.76: choose an orientation preserving homological subdivision $\{\rho_k\}_{k\in K}$ of $\widehat{\Delta_n \times I}$ associated with a triangulation and define $\eta_n(\sigma) := \sum_{k\in K} \hat{\eta}_n(\sigma)\circ\rho_i$. Independence of the subdivision and compatibility with immediate equivalences is checked in the same way. Compatibility with b-equivalences follows by the inequality (2.19).

Checking that the collection of maps η_n for n varying is a chain homotopy between the homomorphisms induced by i_0 and i_1 is like in Theorem 2.76.

□

Definition 2.81. *Let $\iota : X \hookrightarrow Y$ be a Lipschitz l.v.a. map of metric subanalytic germs which on the level of sets is an injection. A* b-retraction *is a b-map $r : Y \to X$ such that $r\circ\iota$ is the identity as a b-map. A* b-deformation retraction *is a b-retraction such that $\iota\circ r$ is b-homotopic to the identity. In those cases X is called a* b-retract *or* b-deformation retract of Y*, respectively. A metric subanalytic germ is called* b-contractible *if it admits $[0, \epsilon)$ as a b-deformation retract.*

The usual consequences of the existence of retracts and deformation retracts in topology hold trivially in our theory

Corollary 2.82. *If $\iota : X \hookrightarrow Y$ admits a b-retraction the connecting homomorphisms in the long exact sequence of relative b-MD homology vanishes. If $\iota : X \hookrightarrow Y$ admits a b-deformation retraction, ι induces a quasi-isomorphism of b-MD chain complexes. If X is b-contractible then it has the b-MD homology of the metric subanalytic germ $[0, \epsilon)$.*

Example 2.83. *Let $b \in (1, \infty)$. Let $(X, 0) = ([0, \epsilon), 0)$ be the point of our category as defined in Definition 2.50. Let $(Y, 0) = (C^b_{\mathbb{S}_1}, 0)$ be the b-cone over the unit cycle as defined in Definition 2.4. Fix $x_1 \in \mathbb{S}_1$ and define $\iota : (X, 0) \to (Y, 0)$ by $\iota(t) := (t^b x_1, t)$. Then the projection $r : (Y, 0) \to (X, 0)$ defined by $r(t^b x, t) := t$ is a b'-deformation* *retraction for any $b' < b$. Indeed the b-map $\{(C_i, f_i)\}_{i\in\{1,2\}}$ from $Y \times_p I$ to Y given by*

$$C_1 := \{((t^b x, t), s) \in Y \times_p I : s \in [0, \frac{1}{2}\}, \qquad C_1 := \{((t^b x, t), s) \in Y \times_p I : s \in [0, \frac{1}{2}\},$$
$$f_1((t^b x, t), s) := ((t^b x_1, t), s), \qquad f_2(t^b x, t), s) := (t^b x, t)$$

is a b'-homotopy from $\iota \circ r$ to the identity on Y.

2.6. Mayer-Vietoris and Excision

2.6.1. An extension of relative homology

For the proof and statement of the relative Mayer-Vietoris exact sequence we need to generalize the concept of relative homology.

Definition 2.84 (Category of pairs of metric subanalytic subgerms)**.** *A* pair of metric subanalytic subgerms $(Y_1, Y_2, x_0, d_X)_{rel\ X}$ *is given by two metric subanalytic subgerms (Y_i, x_0) of a certain metric subanalytic germ (X, x_0, d_X). Recall that on each Y_i we consider the restriction metric $d_X|_{Y_i}$.*

A Lipschitz l.v.a. subanalytic map between the pairs of subgerms $(Y_1, Y_2, x_0, d_X)_{rel\ X}$ *and $(Y_1', Y_2', x_0, d_{X'})_{rel\ X'}$ is a Lipschitz l.v.a. subanalytic map*

$$(Y_1 \cup Y_2, x_0, d_{X|_{Y_1\cup Y_2}}) \to (Y_1' \cup Y_2', x_0, d_{X'|_{Y_1'\cup Y_2'}})$$

that carries Y_i into Y_i'.

The category of pairs of metric subanalytic subgerms *has, as objects, pairs of metric subanalytic subgerms, and as morphisms, Lipschitz subanalytic l.v.a. maps between them, as defined above.*

Definition 2.85. *Consider $b \in (0, +\infty]$. Given a pair of subanalytic subgerms $(Y_1, Y_2, x_0, d_X)_{rel\ X}$, we identify $MDC^b_\bullet(Y_i, x_0, d_{X|_{Y_i}})$ with the subgroup of $MDC^b_\bullet(X, x_0, d_X)$ generated by all l.v.a. simplices in X that are b-equivalent to a representative fully contained in Y_i. We define the* complex of relative b-moderately discontinuous chains of the pair $(Y_1, Y_2, x_0, d_X)_{rel\ X}$ with coefficients in A, *denoting it by $MDC^b_\bullet((Y_1, Y_2, x_0, d_X); A)_{relX}$, as the quotient*

$$MDC^b_\bullet((Y_1, x_0, d_{X|_{Y_1}}); A) + MDC^b_\bullet((Y_2, x_0, d_{X|_{Y_2}}); A) \Big/ MDC^b_\bullet((Y_2, x_0, d_{X|_{Y_2}}); A).$$

The b-moderately discontinuous homology of the pair $(Y_1, Y_2, x_0, d_X)_{rel\ X}$ is denoted by $MDH^b_*((Y_1, Y_2, x_0, d_X); A)_{rel\ X}$ *and it is the homology of the complex defined above.*

We abbreviate calling these complexes and graded abelian groups the b-MD complex *and* b-MD homology *of the pair $(Y_1, Y_2, x_0, d_X)_{rel\ X}$.*

It is straightforward that a Lipschitz subanalytic l.v.a. map f between pairs of subanalytic subgerms of some (X, x_0, d_X), $(X', x_0', d_{X'})$ induces morphisms at the level of b-MD chains for every $b \in (0, +\infty]$ (we denote by f_* the morphism at the level of

b-MD chains similarly to Notation 2.40). Moreover, morphisms (2.10) and (2.11) also hold. So, the following proposition is obvious from the definitions:

Proposition 2.86. *The assignments*

$$(Y_1, Y_2, x_0, d_X)_{rel\ X} \mapsto MDC^{\star}_{\bullet}((Y_1, Y_2, x_0, d_X); A)_{rel\ X}$$

and

$$(Y_1, Y_2, x_0, d_X)_{rel\ X} \mapsto MDH^{\star}_{*}((Y_1, Y_2, x_0, d_X); A)_{rel\ X}$$

are functors from the category of pairs of metric subanalytic subgerms to $\mathbb{B}-Kom(Ab)^-$ *and* $\mathbb{B}-GrAb$ *respectively.*

We have also the obvious generalizations of the definitions of small chain complexes with respect to a finite closed subanalytic covering $\mathcal{C}$. We denote them by $MDC^{pre,+\infty,\mathcal{C}}_{\bullet}(Y_1, Y_2; A)_{relX}$, $MDC^{b,\mathcal{C}}_{\bullet}(Y_1, Y_2; A)_{relX}$. We also have the analogue to Proposition 2.55:

$$g : MDC^{b,\mathcal{C}}_{\bullet}(Y_1, Y_2, d_X; A)_{relX} \to MDC^{b}_{\bullet}(Y_1, Y_2, d_X, A)_{relX} \tag{2.20}$$

is an isomorphism for every $b \in (0, \infty]$.

Remark 2.87. *Note that when* $Y_2 \subset Y_1$ *then* $MDC^{b,\mathcal{C}}_{\bullet}(Y_1, Y_2, d_X; A)_{relX}$ *coincides with* $MDC^{b,\mathcal{C}}_{\bullet}(Y_1, Y_2, d_X; A)$.

2.6.2. b-covers

Definition 2.88. *Let* (X, x_0, d_X) *be a metric germ and* Y_1, Y_2 *subanalytic subgerms, consider* $b \in (0, \infty]$. *A collection* $\{U_i\}_{i\in I}$ *of subanalytic subgerms is called* a closed b-cover of (Y_1, Y_2), *if it is a finite closed cover of* Y_1 *and for any* i *there is a subanalytic subset* $\hat{U}_i \subseteq Y_1$ *such that*

- *for any two* b*-equivalent points* $p, q : [0, \epsilon) \to (Y_1, x_0)$, *if* p *has image in* U_i *then* q *has image in* $\hat{U}_i$.
- *For any finite* $J \subseteq I$ *there is a subanalytic retraction* $r_J : \cap_{i\in J}\hat{U}_i \to \cap_{i\in J}U_i$ *which induces an inverse in homology of the associated morphism of complexes:*

$$MDC^{b}_{\bullet}((\cap_{i\in J}U_i, Y_2, x_0, d_X); A)_{relX} \to MDC^{b}_{\bullet}((\cap_{i\in J}\hat{U}_i, Y_2, x_0, d_X); A)_{relX}.$$

We call the collection $\{\hat{U}_i\}_{i\in I}$ *a* b-extension of $\{U_i\}_{i\in I}$.

Observe that for $b = \infty$ any finite closed subanalytic cover of X is a closed b-cover.

The following remark is a consequence of the definition of b-horn neighborhood and of Theorem 2.76.

Remark 2.89. *In the terminology of the previous definition, when $Y_1 = X$ and $Y_2 = \emptyset$, the following two conditions imply the two conditions of the previous definition respectively:*

- *there is a b-horn neighborhood $\mathcal{H}_{b,\eta}(U_i; X)$ contained in $\hat{U}_i$ for any $i \in I$ (see Definition 2.66).*
- *For any finite $J \subseteq I$, the intersection $\cap_{i \in J} U_i$ is a b-deformation retract of $\cap_{i \in J} \hat{U}_i$ (see 2.81).*

Lemma 2.90. *Let (X, x_0, d_X) be a metric germ, $b \in (0, \infty]$ and $U \subset \hat{U} \subset X$ be subanalytic subsets such that for any two b-equivalent points $p, q : [0, \epsilon) \to (X, x_0)$, if p has image in U_i then q has image in $\hat{U}_i$. If $\sigma_i : \hat{\Delta}_n \to X$ are b-equivalent l.v.a simplices for $i = 1, 2$ and σ_1 is a simplex in U then σ_2 is a simplex in $\hat{U}$.*

Proof. Assume the contrary. Then $\sigma_2^{-1}(X \setminus \hat{U})$ is a subanalytic subset of $\hat{\Delta}_n$ having the vertex at its closure. By the subanalytic Curve Selection Lemma and Remark 2.8 there exists a l.v.a subanalytic map $\gamma : [0, \epsilon) \to \hat{\Delta}_n$ such that $\gamma(t)$ is in $\sigma_2^{-1}(X \setminus \hat{U})$ for $t > 0$. The arcs $p_i := \sigma_i \circ \gamma$ give a contradiction. □

2.6.3. The Mayer-Vietoris Exact Sequence

Theorem 2.91. *Let (X, x_0, d_X) be a metric germ, Y_1, Y_2 subanalytic subgerms and $\{U, V\}$ a closed b-cover of (Y_1, Y_2). The single complex associated with the Mayer-Vietoris double complex*

$$MDC^b_\bullet(U \cap V, Y_2)_{relX} \to MDC^b_\bullet(U, Y_2)_{relX} \oplus MDC^b_\bullet(V, Y_2)_{relX}$$

is quasi-isomorphic to $MDC^b_\bullet(Y_1, Y_2)_{relX}$. As a consequence there is a Mayer-Vietoris long exact sequence as follows:

$$\begin{aligned} ... \to MDH^b_n(U \cap V, Y_2)_{relX} \to MDH^b_n(U, Y_2)_{relX} \oplus MDH^b_n(V, Y_2)_{relX} \to \\ \to MDH^b_n(Y_1, Y_2)_{relX} \to MDH^b_{n-1}(U \cap V, Y_2)_{relX} \to ... \end{aligned} \tag{2.21}$$

Note that we have omitted the coefficient group A in the notation for brevity.

Proof. We omit the coefficient group A in the notation for brevity.

We have the following short exact sequence, where $\alpha(\sigma, \tau) := \sigma - \tau$ is extended linearly:

$$0 \to \mathrm{Ker}(\alpha) \to MDC^b_\bullet(U, Y_2)_{relX} \oplus MDC^b_\bullet(V, Y_2)_{relX} \xrightarrow{\alpha} MDC^b_\bullet(Y_1, Y_2)_{relX} \to 0 \tag{2.22}$$

Surjectivity follows from the fact that (2.20) is an isomorphism. As a consequence, the single complex associated with the double complex

$$d : \mathrm{Ker}(\alpha) \to MDC^b_\bullet(U, Y_2)_{relX} \oplus MDC^b_\bullet(V, Y_2)_{relX}$$

is quasi-isomorphic to $MDC^b_\bullet(Y_1, Y_2)_{relX}$.

Let $\{\hat{U},\hat{V}\}$ be a b-extension of $\{U,V\}$. In the analogue short exact sequence for $\{\hat{U},\hat{V}\}$, we denote the analogue of α by $\hat{\alpha}$. The inclusions $U \hookrightarrow \hat{U}$ and $V \hookrightarrow \hat{V}$ together induce a morphism

$$\iota_*^{U,V} MDC^b_\bullet(U,Y_2)_{relX} \oplus MDC^b_\bullet(V,Y_2)_{relX} \to MDC^b_\bullet(\hat{U},Y_2)_{relX} \oplus MDC^b_\bullet(\hat{V},Y_2)_{relX}$$

that restricts to a morphism $\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\hat{\alpha})$. This restriction admits the following factorization:

$$\mathrm{Ker}(\alpha) \xrightarrow{f} MDC^b(\hat{U}\cap\hat{V},Y_2)_{relX} \xrightarrow{g} \mathrm{Ker}(\hat{\alpha})$$

where $g(\hat{\sigma}) := (\hat{\sigma},\hat{\sigma})$ is extended linearly and f is defined as follows:

Let $([\sum_{i\in I} a_i\sigma_i],[\sum_{j\in J} b_i\psi_j])$ be an element of $MDC^b_\bullet(U,Y_2)_{relX} \oplus MDC^b_\bullet(V,Y_2)_{relX}$ such that $[\sum_{i\in I} a_i\sigma_i] + [\sum_{j\in J} b_i\psi_j] = 0$ in $MDC^b_\bullet(Y_1,Y_2)_{relX}$. After replacing the representatives by the ones obtained by sequences of $\to_\infty$-equivalences as in Lemma 2.35, consider splittings $I = I_0 \cup I_1 \cup ..., I_r$, $J = J_0 \cup J_1 \cup ..., J_r$ as above, which satisfy that

1. $[\sigma_i] \in \mathrm{Ker}(MDC^b_\bullet(U)_{relX} \to MDC^b_\bullet(U,Y_2)_{relX})$ for any $i \in I_0$,
2. $[\psi_j] \in \mathrm{Ker}(MDC^b_\bullet(V)_{relX} \to MDC^b_\bullet(V,Y_2)_{relX})$ for any $j \in J_0$,
3. $\sigma_i \sim_b \psi_j$ if $i \in I_k$ and $j \in J_k$ for a given $k \geq 1$,

and that for any $k \geq 1$ we have

$$\sum_{i\in I_k} a_i + \sum_{j\in J_k} b_k = 0.$$

If I_k and J_k are non-empty, there is a $\tau \in MDC^b_\bullet(V,Y_2)_{relX}$ in the same b-equivalence class as σ_i for any $i \in I_k$. Observe that any l.v.a. simplex b-equivalent to a l.v.a. simplex in V is contained in $\hat{V}$ by Lemma 2.90, so $\sigma_i \in MDC^b_\bullet(\hat{U}\cap\hat{V},Y_2)_{relX}$. We define

$$f(\sum_{i\in I} a_i[\sigma_i], \sum_{j\in J} b_i[\psi_j]) := \sum_{i\in I\setminus I_0} a_i[\sigma_i] = \sum_{j\in J\setminus J_0} b_j[\psi_j].$$

By hypothesis, there are retractions $r_U : \hat{U} \to U$ and $r_V : \hat{V} \to V$, whose induced maps provide an inverse to $\iota_*^{U,V}$ in the derived category. We denote the inverse by $r_{U,V}$. Then, in the derived category $r_{U,V} \circ g$ is a left-inverse for f. In the derived category, f also has a right-inverse: let ι denote the inclusion $U \cap V \hookrightarrow \hat{U}\cap\hat{V}$. Then in the derived category the isomorphism $\iota_* : MDC^b(U\cap V,Y_2)_{relX} \to MDC^b(\hat{U}\cap\hat{V},Y_2)_{relX}$ is the composition of the inclusion

$$h : MDC^b_\bullet(U\cap V,Y_2)_{relX} \hookrightarrow \mathrm{Ker}(\alpha)$$

and f.

We conclude that f is an isomorphism in the derived category, and, using it and the isomorphism ι_* we conclude that h is an isomorphism in the derived category. So in the derived category, the single complex associated with the double complex

$$MDC^b_\bullet(U\cap V,Y_2)_{relX} \to MDC^b_\bullet(U,Y_2)_{relX} \oplus MDC^b_\bullet(V,Y_2)_{relX}$$

is isomorphic to the double complex associated with d which is isomorphic to the complex $MDC^b_\bullet(Y_1, Y_2)_{relX}$.

□

As a consequence we obtain the Excision Theorem.

Corollary 2.92. *Let (X, x_0, d_X) be a metric germ. Let $U \subset X \backslash \{x_0\}$ and $K \backslash \{x_0\} \subset U$ such that $\{U, X \setminus K\}$ is a closed b-cover of (X, U). Then the inclusion induces a quasi-isomorphism $MDC^b_\bullet(X \setminus K, U; A)_{relX} \to MDC^b_\bullet(X, U; A)$. As a consequence for each n we have an isomorphism*

$$MDH^b_n(X \setminus K, U; A)_{relX} \xrightarrow{\simeq} MDH^b_n(X, U; A)$$

Proof. Apply Theorem 2.91 to the b-cover $\{U, X \setminus K\}$. □

2.6.4. The Čech homology complexes

Let $\mathcal{U} = \{U_i\}_{i \in \{1,\dots,r\}}$ be a finite closed subanalytic cover of X. Denote by $U_{i_1,\dots,i_r}$ the intersection $U_{i_1} \cap \dots \cap U_{i_r}$. The Čech double complex of b-MD homology of a pair (X, Y) associated with $\mathcal{U}$ with coefficients in A is defined by

$$MDC^b(\mathcal{U}, X, Y; A)_{p,q} := \bigoplus_{1 \leq i_0 < \dots < i_p \leq r} MDC^b_q(U_{i_0,\dots,i_p}, Y; A)_{relX},$$

with vertical differential equal to the b-MD differential and horizontal differential the usual Čech homology differential:

$$MDC^b_q(U_{i_0,\dots,i_p}, Y; A)_{relX} \to \oplus_{k=0}^p MDC^b_q(U_{i_0,\dots,\hat{i}_k,\dots,i_p}, Y; A)_{relX}$$

$$[\sigma] \mapsto \sum_{k=0}^p (-1)^k j^b_{i_k}([\sigma]),$$

where $j^b_{i_k}$ is the b-MD chain map associated to the inclusion $U_{i_0,\dots,i_p} \subset U_{i_0,\dots,\hat{i}_k,\dots,i_p}$.

Theorem 2.93. *Let Y_1, Y_2 be subanalytic subgerms of a metric germ (X, x_0, d_X). If for any two disjoint finite subsets $I, J \subset \{1, \dots, r\}$ we have that $\{(\cap_{æ \in J} U_j) \cap U_i\}_{i \in I}$ is a b-cover of $(\cup_{i \in I} U_i \cap (\cap_{æ \in J} U_j), Y_2)$ and $\cup_{i=1}^r U_i = Y_1$, then the single complex associated with the Čech complex $MDC^b_{\bullet,\bullet}(\mathcal{U}, Y_1, Y_2; A)$ is quasi-isomorphic to $MDC^b_\bullet(Y_1, Y_2; A)_{relX}$. Consequently there is a Čech spectral sequence abutting to $MDH^b_*(Y_1, Y_2; A)_{relX}$ with E^1 page*

$$E[b]^1_{p,q} := \bigoplus_{1 \leq i_0 < \dots < i_p \leq r} MDH^b_q(U_{i_0,\dots,i_p}, Y_2; A)_{relX}.$$

Proof. The case of a cover of 2 closed subsets is exactly Theorem 2.91. The general case runs by induction on the number of open subsets, applying Mayer-Vietoris for the decomposition $V \cup U_r$ with $V := U_1 \cup \dots \cup U_{r-1}$ and the induction step for the decompositions $U_1 \cup \dots \cup U_{r-1}$ and $(U_1 \cap U_r) \cup \dots \cup (U_{r-1} \cap U_r)$:

Let $\tilde{A}_\bullet$ and $\tilde{B}_\bullet$ denote the single complexes associated with the Čech complexes

$MDC^b_{\bullet,\bullet}(\{U_1 \cap U_r, \ldots, U_{r-1} \cap U_r\}, V \cap U_r, Y_2; A)$ and $MDC^b_{\bullet,\bullet}(\{U_1, \ldots, U_{r-1}\}, V, Y_2; A)$,

respectively. By induction hypothesis, we get that $\tilde{A}_\bullet$ and $\tilde{B}_\bullet$ are quasi-isomorphic to

$$A_\bullet := MDC^b_\bullet(V \cap U_r, Y_2; A)_{relX} \text{ and } B_\bullet := MDC^b_\bullet(V, Y_2; A)_{relX},$$

respectively.

We get the following diagram:

$$\begin{array}{ccc} S_\bullet(A_\bullet \to MDC_\bullet(U_r, Y_2; A)_{relX} \oplus B_\bullet) & \longrightarrow & MDC_\bullet(Y_1, Y_2; A)_{relX} \\ \uparrow & & \\ S_\bullet(\tilde{A}_\bullet \to MDC_\bullet(U_r, Y_2; A)_{relX} \oplus \tilde{B}_\bullet) & & \end{array}$$

where $S\bullet(D_{\bullet,\bullet})$ for a double complex $D_{\bullet,\bullet}$ denotes the single complex assiciated with that double complex. The vertical arrow represents the chain map resulting from the quasi-isomorphisms $\tilde{A}_\bullet \to A_\bullet$ and $\tilde{B}_\bullet \to B_\bullet$ and is therefore a quasi-isomorphism. The horizontal arrow is the result of applying the Mayer Vietoris Theorem (Theorem 2.91) to the cover $\{U_r, V\}$ and is therefore also a quasi-isomorphism.

Now the statement follows from the fact that the complex at the bottom of that commutative diagram by definition is isomorphic to the single complex associated with the Čech complex $MDC^b_{\bullet,\bullet}(\mathcal{U}, Y_1, Y_2; A)$. □

Definition 2.94. *The nerve of the cover $\mathcal{U} = \{U_i\}_{i \in \{1,\ldots,r\}}$ is the simplicial complex which assigns a p-simplex to each non-empty intersection $U_{i_0,\ldots,i_p}$, and identifies faces according to the inclusions $U_{i_0,\ldots,i_p} \subset U_{i_0,\ldots,\hat{i}_k,\ldots,i_p}$.*

Corollary 2.95. *In the setting of the last theorem, if $Y_1 = X$ and $Y_2 = \emptyset$ and for any finite set of indexes $U_{i_0,\ldots,i_p}$ is either empty or has the b-MD homology of a point, the b-MD homology of X coincides with the ordinary homology of the nerve of the cover with coefficients in A.*

Proof. In the spectral sequence of Theorem 2.93 we have $E[b]^1_{p,q} = 0$ if $q > 0$ and $E[b]^1_{p,0} = \bigoplus_{1 \leq i_0 < \ldots < i_p \leq r} MDH^b_0(U_{i_0,\ldots,i_p}, A)$, where $MDH^b_0(U_{i_0,\ldots,i_p}; A) \cong A$ if and only if $U_{i_0,\ldots,i_p}$ is not empty. □

2.6.5. Mayer-Vietoris and Čech spectral sequence for open coverings

The purpose of this section is to prove the validity of the Mayer-Vietoris sequence and the Čech spectral sequence for finite open subanalytic coverings.

Let (Y_1, Y_2) be subanalytic subgerms of a metric subanalytic germ (X, x_0, d_X). A collection $\{U_i\}_{i=1}^r$ of subanalytic subgerms is an open b-cover of (Y_1, Y_2) if it is a finite open subanalytic cover, that is the U_i are open subanalytic sets and $\cup_i U_i = Y_1 \setminus \{x_0\}$ and the conditions of Definition 2.88 are satisfied.

Theorem 2.96. *All the results in Sections 2.6.3 and 2.6.4 remain true replacing closed b-covers by open b-covers.*

Proof. Everything boils down to proving Theorem 2.91. In fact, Corollary 2.92 is a direct consequence of Theorem 2.91 and the proof of Theorem 2.93 consists of a repeated application of Theorem 2.91. In the proof of Theorem 2.91, the only place in which the hypothesis that the subsets of the b-cover are closed is used, is in showing the surjectivity of the last mapping of Sequence (2.22).

In the case that the cover $\{U, V\}$ is formed by closed subsets, surjectivity is direct from the fact that (2.20) is an isomorphism if the cover $\mathcal{C}$ is closed. In the case that the cover $\{U, V\}$ is formed by open subsets, surjectivity follows from the fact that (2.20) is an isomorphism for a closed cover together with Proposition 2.97. Indeed, in the case that the cover is formed by open subsets, we have that

$$MDC^b_\bullet(Y_1, Y_2)_{relX} \cong MDC^{b,\mathcal{C}}_\bullet(Y_1, Y_2)_{relX},$$

where $\mathcal{C}$ is a finite closed subanalytic cover refining the open cover. The existence of such a refinement is shown in the following proposition. □

Proposition 2.97. *Let $\mathcal{U} = \{U_1, ..., U_k\}$ a finite open subanalytic covering of a subanalytic germ $(X, 0)$. Then there exists a subanalytic closed set C_i contained in U_i for every i such that $\{C_1, .., C_k\}$ is a closed covering.*

The proof is obtained by repeatedly applying the following lemma:

Lemma 2.98. *Let $\mathcal{U} = \{U_1, ..., U_k\}$ be a finite open subanalytic covering of a subanalytic germ $(X, 0)$. There exists a closed set C_1 contained in U_1 such that $\{U_2, .., U_k, \mathring{C}_1\}$ is also an open covering of $(X, 0)$ where $\mathring{C}_1$ is the interior of C_1.*

Proof. Let L_X be the link of X. By the conical Structure Theorem (see Remark 2.6) we can take a subanalytic homeomorphism $h : C(L_X) \to X$ for a small enough representative for $(X, 0)$ compatible with the covering $\mathcal{U}$. That is, any U_i coincides with $h(L_i)$ for a certain subanalytic subset L_i of L_X.

We prove that given a finite open subanalytic covering $\mathcal{U} = \{U_1, ..., U_k\}$ of L_X, there exists a closed set D_1 contained in U_1 such that $\{U_2, .., U_k, \mathring{D}_1\}$ is also an open covering of L_X where $\mathring{D}_1$ is the interior of D_1.

To finish the proof we will consider the covering given by $C_i := h(C(D_i))$.

Let us prove the statement for a covering of L_X. We denote by $\partial_X Y$ the boundary set $\bar{Y} \setminus \mathring{Y}$ of Y in X.

Let K be $\partial_X U_1 \cap (U_2 \cup ... \cup U_k)$. Note that in fact $\partial_X U_1$ equals K.

Let $\theta : K \to \mathbb{R}$ be the function $\theta(x) := d_{out}(x, \partial_X(U_2 \cup ... \cup U_k)$. Choose another subanalytic function $\eta : K \to \mathbb{R}$ such that $\eta(x) < \theta(x)$ for every $x \in K$.

Let $\{K_i\}_{i \in I}$ be a stratification of K by C^r subanalytic submanifolds.

For every $i \in I$, consider the following subset in the normal bundle of K_i

$$W_i := \{(x, v) \in NK_i : ||v|| < \eta(x)\}.$$

Let V_i be a neighbourhood of K_i inside NK_i, whose existence follows the Definable Tubular Neighborhood Theorem (see Theorem 6.11 in [9]), such that $\pi|_{V_i}$ is a diffeomorphism and such that $\pi(V_i)$ is a subanalytic neighbourhood of K_i where $\pi : NK_i \to X$ is defined by $\pi(x, v) = x + v$.

Define $\mathcal{U}(K, \eta) := \cup_{i \in I} \pi(V_i \cap W_i)$. This is a globally subanalytic neighbourhood of K. By the definition of η, we have that the closure of $\mathcal{U}(K, \eta) \cap X$ is contained in $U_2 \cup ... \cup U_k$.

We define C_1 as $U_1 \setminus \mathcal{U}(K, \eta)$. This is a closed set since it coincides with $\overline{U}_1 \setminus \mathcal{U}(K, \eta)$. Moreover $\{\mathring{C}_1, U_2, ..., U_k\}$ covers X. □

2.7. Moderately Discontinuous Homology in degree 0

Definition 2.99. *Let (X, x_0) be a metric subanalytic germ. Two connected components X^1 and X^2 of $X \setminus \{x_0\}$ are b-equivalent if there exist two l.v.a. 0-simplices $\sigma_i : \hat{\Delta}_0 \to (X_i, x_0)$ which are b-equivalent. The equivalence classes are called b-connected components of X. The ∞-connected components are the usual connected components of $X \setminus \{x_0\}$.*

Proposition 2.100. *The b-moderately discontinuous homology $MDH_0^b(X; A)$ at degree 0 is isomorphic to $A^{r(b,X)}$, where $r(b, X)$ is the number of b-connected components of X. A basis is given by the choice of a 0-simplex in each b-connected component. For $b_1, b_2 \in (0, \infty]$, $b_1 \geq b_2$, the homomorphism $h_0^{b_1,b_2}$ is the projection that sends a base element α of $A^{r(b_1,X)}$ onto the base element of $A^{r(b_2,X)}$ that represents the b_2-connected component α lies in.*

Proof. Let $L := X \cap \mathbb{S}_\epsilon$ be the link of X (where $\epsilon > 0$ is small enough). Let $\theta : C_L^1 \to X$ be a subanalytic homeomorphism preserving the distance to the origin (this exists by Remark 2.6). Let $\tau : \Delta_n \to L$ be a subanalytic map. The *straight* n-simplex with respect to θ associated with τ is defined to be the map germ $\sigma : \hat{\Delta}_n \to X$ given by $\sigma(tz, t) := \theta(\tau(z), t)$.

Let x_1, x_2 be two points in the same connected component of L. Then there exists a subanalytic path $\gamma : [0, 1] \to L$ joining x_1 and x_2. The boundary operator "∂" applied to the straight simplex associated with γ is the difference of the straight simplices associated with x_i. So, we conclude that two straight 0-simplices in the same connected component of $X \setminus \{x_0\}$ are b-homologous for any b.

Let $\sigma : \hat{\Delta}_0 = [0, 1) \to (X, x_0)$ be any 0-simplex. Up to reparametrization (see Remark 2.8) we may assume that $||\sigma(t)|| = t$. We can express the restriction $\theta^{-1} \circ \sigma|_{\Delta_0 \times (0,1)}$ as a pair $\theta^{-1} \circ \sigma|_{\Delta_0 \times (0,1)}(t) = (\gamma(t), t)$, where $\gamma : (0, 1) \to L$ is the germ at 0 of a subanalytic path. We may choose the radius ϵ defining the link L small enough so that ϵ is in the domain of definition of the germ σ, and hence of γ. The map $\tau : \hat{\Delta}_1 \to (X, x_0)$, where $\tau(ts, t) := \theta(\gamma(\epsilon + s(t - \epsilon)), t)$, defines a 1-simplex whose boundary shows that σ is b-homologous to a straight simplex.

We have proven that all 0-simplices lying in the same connected component of $X \setminus \{x_0\}$ are b-homologous for any b.

After this the proof is obvious. □

2.8. The ∞-MD Homology: comparison with the homology of the link

Let $(X, Y, \{x_0\}, d_X)$ be a pair of closed metric subanalytic germs in $\mathbb{R}^n$. By Remark 2.15 there is a finite subanalytic triangulation $\alpha : |K| \to X \cap B_\epsilon$ of a representative $X \cap B_\epsilon$, which is compatible with Y and x_0. By choosing ϵ sufficiently small and intersecting with $\mathbb{S}_\epsilon$ we obtain a subanalytic triangulation $\beta : |L| \to X \cap \mathbb{S}_\epsilon$ compatible with $Y \cap \mathbb{S}_\epsilon$ such that (K, α) is the cone over (K, β). In other words: there exists a pair of simplicial complexes (L_1, L_2) and a subanalytic homeomorphism

$$h : (C(|L_1|), C(|L_2|)) \to (X, Y, \{x_0\}) \cap B_\epsilon$$

from the cones of the geometric realizations to the representative $(X, Y, \{x_0\}) \cap B_\epsilon$. By the reparametrization trick of Remark 2.8 we may assume that $||h(tx, t)|| = t$.

Denote by $C^{\mathrm{Simp}}_\bullet(L_1, L_2; A)$ the simplicial homology complex for the pair (L_1, L_2) with coefficients in A. The homeomorphism h induces a morphism of complexes

$$c : C^{\mathrm{Simp}}_\bullet(L_1, L_2; A) \to MDC^\infty_\bullet(X, Y; A). \tag{2.23}$$

Theorem 2.101. *The morphism (2.23) is a quasi-isomorphism. As a consequence we have an isomorphism between the singular homology $H_*(X \setminus \{x_0\}, Y \setminus \{x_0\}; A)$ and $MDH^\infty_*(X, Y, x_0; A)$.*

Proof. By using the relative homology sequence and the 5-lemma we reduce to the absolute case $Y = \emptyset$. The singular homology $H_*(X \setminus \{x_0\}; A)$ is isomorphic to the singular homology of the link, by homotopy invariance, and the later is isomorphic with the simplicial homology of L_1.

A simplex of L_1 is called *maximal* if it is not strictly contained in another simplex. The collection $\{Z_i\}_{i \in I}$ of maximal simplices forms a closed cover of $|L_1|$ such that any finite intersection is a simplex, and hence, contractible. Then the simplicial homology of L_1 coincides with the homology of the nerve of the cover.

The collection $\{h(C(Z_i))\}_{i \in I}$ is a closed subanalytic cover. Any finite intersection $\cap_{i \in J} h(C(Z_i))$ is of the form $h(C(T))$ where T is a simplex in L_1. An immediate application of Assertion (1) of Theorem 2.76 shows that $h(C(T))$ has the ∞-MD homology of a point. Since any closed subanalytic cover is an ∞-cover, by Corollary 2.95 the homology $MDH^\infty_*(X \setminus \{x_0\}; A)$ coincides with the homology with coefficients in A of the nerve of the cover. This concludes the proof. □

2.9. MD Homology of plane curves with the outer metric.

Throughout this subsection, whenever we say *curve germ*, we refer to a complex algebraic plane curve germ in the origin equipped with the outer geometry. We are going to recall the definition of the Eggers-Wall tree of a curve germ. It uses the following correspondence between Puiseux pairs and Puiseux exponents: let $(m_1, k_1) \dots (m_l, k_l)$ denote all Puiseux pairs of a curve germ in order. Then the Puiseux exponents of that curve germ are given by $\frac{m_i}{\prod_{j=1}^{i} k_j}$ for $i = 1, \ldots l$. We call (m_i, k_i) the *Puiseux pair corresponding to* $\frac{m_i}{\prod_{j=1}^{i} k_j}$. Recall the following definition of the contact number between two branches, which can be found for example on p. 68 of [41].

Definition 2.102. *Let C be a curve germ. Let $f_i = \sum_{j=1}^{\infty} \alpha_{i,j} x^{\frac{j}{\kappa_i}}$ be parametrizations of the branches C_i of C, where $i \in \{1, \ldots, n\}$. Let $i \neq k \in I$. The* contact number *$c(C_i, C_k)$ between C_i and C_k is defined as*

$$c(C_i, C_k) := \min\{j : \alpha_{i,j} \neq \alpha_{k,j}\}$$

Definition 2.103. *Let C be a curve germ. In this definition, we are defining the* Eggers-Wall tree $\mathcal{G}_C$ of C. *Depending on the context, $\mathcal{G}_C$ can be interpreted either as a graph or as a topological space with a finite number of special points; we call these special points in the topological space* vertices *as they correspond to the vertices in the graph.*

If C is irreducible, we define the Eggers-Wall tree $\mathcal{G}_C$ of C *to be the segment $[0, \infty]$ with a vertex at both ends and one vertex at each rational number in that segment that is a Puiseux exponent of C. Every vertex is decorated by the corresponding value in $\mathbb{Q} \cup \{\infty\}$. For two adjacent vertices at q_1 and q_2 respectively, with $q_1 < q_2$, the edge between them is weighted by the product $\prod_{i=0}^{l} k_i$, where $k_0 = 1$ and $(m_1, k_1), \ldots, (m_l, k_l)$ are all Puiseux pairs corresponding to Puiseux exponents less than or equal to q_1.*

If C is reducible, the Eggers-Wall tree $\mathcal{G}_C$ *is defined as follows. Let C_n denote one of its branches and let $\hat{C}_n$ denote the union of all the other branches. Let c be the greatest contact number that C_n has with any of the other branches and let C_k be one of the branches that C_n has that contact number with. If C has only two branches, we have $C_k = \hat{C}_n$. If $\mathcal{G}_{C_n}$ does not have a vertex at c, add it in the following manner: let q_1 be the greatest vertex in $\mathcal{G}_{C_n}$ smaller than c and q_2 the smallest one greater than c. We add c as a vertex in $\mathcal{G}_{C_n}$ and give both edges $\{q_1, c\}$ and $\{c, q_2\}$ the weight the edge $\{q_1, q_2\}$ had before. Then, we do the same for the segment in $\mathcal{G}_{\hat{C}_n}$ corresponding to $\mathcal{G}_{C_k}$, if it does not contain c as a vertex already. Now, glue the segment from 0 to c in $\mathcal{G}_{C_n}$ to the segment from 0 to c in $\mathcal{G}_{C_k}$ by the identity on $[0, c]$. As $\mathcal{G}_{C_k}$ is naturally embedded in $\mathcal{G}_{\hat{C}_n}$, we have glued $\mathcal{G}_{\hat{C}_n}$ and $\mathcal{G}_{C_n}$ to one graph $\mathcal{G}_C$.*

There is a natural map $r : \mathcal{G}_C \to [0, \infty]$ defined as follows: For a point $g \in \mathcal{G}_C$, let C_g be one of the branches of C for which g is in the image of the natural inclusion $\mathcal{G}_{C_g} \hookrightarrow \mathcal{G}_C$. We assign to g the point in the segment $[0, \infty]$ that is sent to g by that inclusion.

Example 2.104. *Let C be the curve with the following four branches:*

$$C_1 = \{(x, y) \in \mathbb{C}^2 : y = x^{\frac{3}{2}} + x^{\frac{5}{2}}\},$$
$$C_2 = \{(x, y) \in \mathbb{C}^2 : y = x^{\frac{3}{2}} + x^{\frac{11}{4}}\},$$
$$C_3 = \{(x, y) \in \mathbb{C}^2 : y = x^{\frac{3}{2}} + x^{\frac{11}{4}} + x^{\frac{37}{12}}\},$$
$$C_4 = \{(x, y) \in \mathbb{C}^2 : y = x^{\frac{5}{2}} + x^{\frac{11}{4}}\}.$$

We have visualized the Eggers-Wall tree $\mathcal{G}_C$ together with the function $r : \mathcal{G}_C \to [0, \infty]$ in Figure 2.3.

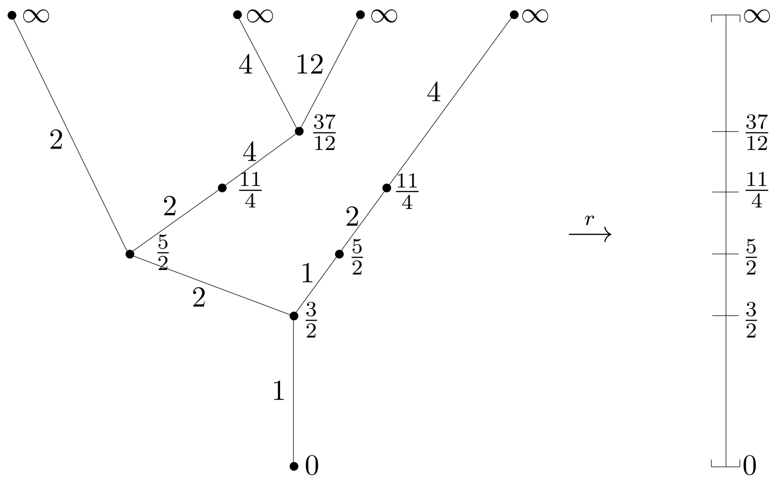


Figure 2.3.: The Eggers-Wall tree $\mathcal{G}_C$ and $r : \mathcal{G}_C \to [0, \infty]$.

Proposition 2.105. *Let C be a curve germ. Let $\mathcal{G}_C$ be the Eggers-Wall tree of C with $r : \mathcal{G}_C \to [0, \infty]$ as defined above. Let A be an abelian group. The MD homology of C with respect to A can be described as follows:*

1. *For any $b \in [1, \infty]$, it is $MDH_0^b(C; A) \cong MDH_1^b(C; A) \cong A^{l_b}$, where l_b is the number of points in $r^{-1}(b + \epsilon)$, where ϵ is so small that $r^{-1}((b, b + \epsilon])$ does not contain a vertex. For the case $b = \infty$, we consider $\infty + \epsilon = \infty$.*

2. *For any $b \in [1, \infty]$ and $n > 1$, it is $MDH_n^b(C; A) \cong \{0\}$.*

3. *For $b_1, b_2 \in [1, \infty]$ with $b_1 \geq b_2$, $h_0^{b_1,b_2}(C)$ and $h_1^{b_1,b_2}(C)$ are the homomorphisms given by multiplication with the following matrices M_0 and M_1, respectively: let ϵ be so small that $r^{-1}((b_1, b_1 + \epsilon])$ and $r^{-1}((b_2, b_2 + \epsilon])$ do not contain any vertices. Let $\mathcal{G}_1$,..., $\mathcal{G}_l$ be the connected components of $r^{-1}([b_2 + \epsilon, b_1 + \epsilon])$, where $l = l_{b_2}$. For $i \in \{1, \dots, l\}$, let p_i be the unique point in $r^{-1}(b_2 + \epsilon) \cap \mathcal{G}_i$. Further, let $p_{1,i}, \dots, p_{m_i,i}$ be the points in $r^{-1}(b_1 + \epsilon) \cap \mathcal{G}_i$. Notice that $\sum_{i=1}^{l} m_i = l_{b_1}$. We define*

$$M_0 := \begin{pmatrix} \overbrace{\begin{matrix} 1 & \dots & 1 \end{matrix}}^{m_1 \text{ times}} & \overbrace{\begin{matrix} 0 & \dots & 0 \end{matrix}}^{m_2 \text{ times}} & & \overbrace{\begin{matrix} 0 & \dots & 0 \end{matrix}}^{m_l \text{ times}} \\ \begin{matrix} 0 & \dots & 0 \end{matrix} & \begin{matrix} 1 & \dots & 1 \end{matrix} & & \begin{matrix} 0 & \dots & 0 \end{matrix} \\ \begin{matrix} 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \end{matrix} & \dots & \vdots \\ \vdots & \vdots & & \begin{matrix} 0 & \dots & 0 \end{matrix} \\ \begin{matrix} 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \end{matrix} & & \begin{matrix} 1 & \dots & 1 \end{matrix} \end{pmatrix}$$

Now, for $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, m_i\}$, let $k_{j,i} := \frac{w_{j,i}}{w_i}$, where $w_{j,i}$ and w_i are *the weights assigned to the edges on which $p_{j,i}$ respectively p_i lie. We define*

$$M_1 := \begin{pmatrix} k_{1,1} & \dots & k_{m_1,1} & 0 & \dots & 0 & & 0 & \dots & 0 \\ 0 & \dots & 0 & k_{1,2} & \dots & k_{m_2,2} & & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & & \vdots & \\ & \vdots & & & \vdots & & & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & & k_{1,l} & \dots & k_{m_l,l} \end{pmatrix}$$

The data used in the statement of this proposition is visualized in Example 2.106.

Proof. Let $f_i \in \mathbb{C}[[x^{\frac{1}{\kappa_i}}]]$, $i \in \{1, \dots, n\}$, be parametrizations of the branches of C. For $f_i = \sum_{j=1}^{\infty} \alpha_{i,j} x^{\frac{j}{\kappa_i}}$, $b \in [1, \infty)$, let $f_{i,b}$ be the truncation $\sum_{j=1}^{\lfloor b \rfloor} \alpha_{i,j} x^{\frac{j}{\kappa_i}}$, where $\lfloor b \rfloor$ denotes the greatest integer smaller than or equal to b. In the case of $b = \infty$, we set $f_{i,b} := f_i$.

The proof consists in an application of the Mayer-Vietoris Theorem, which resembles the computation of the singular homology of a circle in a certain way. The subsets involved in the Mayer-Vietoris decomposition are of the following form: we write $x \in \mathbb{C} \setminus \{0\}$ as $x = r_x e^{2\pi\varphi_x}$. Let $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}$ with $\phi_1 < \phi_2$ and $\phi_3 < \phi_4$ be fixed. For $b \in [1, \infty]$, we define the subgerm $(V_b, \underline{0})$ of $(\mathbb{C}^2, \underline{0})$ by

$$\begin{aligned} V_b := (\{(x,y) \in \mathbb{C}^2 : y = f_{i,b}(x), (x = 0 \text{ or } \exists n, m \in \mathbb{Z} : &\phi_1 < \varphi_x + 2\pi n < \phi_2 \\ &\text{or } \phi_3 < \varphi_x + 2\pi m < \phi_4)\}. \end{aligned}$$

Recall Definition 2.99. Because of Proposition 2.100, for $b_1 \geq b_2 \geq b_3$ we have the following:

- The map $H_0^{b_1}(V_{b_3}, \mathbb{Z}) \to H_0^{b_2}(V_{b_3}, \mathbb{Z})$ is an isomorphism, as for any $b \geq b_3$ the b-connected components of V_{b_3} are just its connected components.
- The map $H_0^{b_3}(V_{b_1}; \mathbb{Z}) \to H_0^{b_3}(V_{b_2}; \mathbb{Z})$ induced by the natural projection $V_{b_1} \to V_{b_2}$ is an isomorphism, as there is a $1:1$ correspondence between the b_3-connected components of V_b and the connected components of V_{b_3} for any $b \geq b_3$.

As a consequence, in the following commutative diagram we get the indicated isomorphisms:

$$\begin{array}{ccccc} MDH_0^{b_2}(V_\infty;\mathbb{Z}) & \xleftarrow{\varphi_V} & MDH_0^{b_1}(V_\infty;\mathbb{Z}) & \longleftarrow & MDH_0^{\infty}(V_\infty;\mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ MDH_0^{b_2}(V_{b_1};\mathbb{Z}) & \longleftarrow & MDH_0^{b_1}(V_{b_1};\mathbb{Z}) & \xleftarrow{\cong} & MDH_0^{\infty}(V_{b_1};\mathbb{Z}) \\ \downarrow \cong & & \downarrow & & \downarrow \psi_V \\ MDH_0^{b_2}(V_{b_2};\mathbb{Z}) & \xleftarrow{\cong} & MDH_0^{b_1}(V_{b_2};\mathbb{Z}) & \xleftarrow{\cong} & MDH_0^{\infty}(V_{b_2};\mathbb{Z}) \end{array}$$

So φ_V is the same as ψ_V up to concatenation with isomorphisms. By Theorem 2.101, up to isomorphisms the latter is the same as

$$\hat{\psi}_V : H_0(V_{b_1} \setminus \{\underline{0}\}, \mathbb{Z}) \to H_0(V_{b_2} \setminus \{\underline{0}\}, \mathbb{Z}),$$

where H_0 denotes the singular homology.

Now we introduce the specific b-cover that we use to apply the Mayer-Vietoris Theorem. Let $U_1 = V_\infty$ with $\phi_1 = \frac{1}{4}\pi$ and $\phi_2 = \frac{7}{4}\pi$ and $\phi_3 = \phi_4$; and let $U_2 = V_\infty$ with $\phi_1 = \frac{3}{2}\pi$ and $\phi_2 = \frac{5}{2}\pi$ and $\phi_3 = \phi_4$. We have that $U_1 \cap U_2 = V_\infty$ with $\phi_1 = \frac{1}{4}\pi$ and $\phi_2 = \frac{1}{2}\pi$ and $\phi_3 = \frac{3}{2}\pi$ and $\phi_4 = \frac{7}{4}\pi$. Note that $\{U_1, U_2\}$ is a b-cover of C for any $b \geq 1$. The n-th b-moderately discontinuous homology groups of U_1 and U_2 and $U_1 \cap U_2$ are trivial for any $b \geq 1$, if $n \geq 1$. So, by the Mayer-Vietoris Theorem (Theorem 2.91), for any $n > 1$, the n-th b-moderately discontinuous homology of C is trivial. This completes the proof of statement (2).

Furthermore, by the Mayer-Vietoris Theorem for $b_1 \geq b_2$ this gives us the following diagram with exact rows, in which we have omitted $\mathbb{Z}$:

$$\begin{array}{ccccccccc}
0 \to & MDH_1^{b_1}(C) & \to & MDH_0^{b_1}(U_1 \cap U_2) & \to & MDH_0^{b_1}(U_1) \oplus MDH_0^{b_1}(U_2) & \to & MDH_0^{b_1}(C) & \to 0 \\
& \downarrow h_1^{b_1,b_2} & & \downarrow \varphi_{U_1 \cap U_2} & & \downarrow \varphi_{U_1} \oplus \varphi_{U_2} & & \downarrow h_0^{b_1,b_2} & \\
0 \to & MDH_1^{b_2}(C) & \to & MDH_0^{b_2}(U_1 \cap U_2) & \to & MDH_0^{b_2}(U_1) \oplus MDH_0^{b_2}(U_2) & \to & MDH_0^{b_2}(C) & \to 0
\end{array}$$

We have shown above that we can replace $MDH_0^{b_i}(V;\mathbb{Z})$ by $H_0(V_{b_i};\mathbb{Z})$ for $i \in \{1,2\}$ and $V \in \{U_1 \cap U_2, U_1, U_2, C\}$, and that we can replace φ_V by $\hat{\psi}_V$ for $V \in \{U_1 \cap U_2, U_1, U_2\}$. Comparing the result with the analogous Mayer-Vietoris sequence in singular homology, we get that $MDH_1^{b_i}(C;\mathbb{Z}) \cong H_1(C_{b_i};\mathbb{Z})$ for $i \in \{1,2\}$ and that for $j \in \{0,1\}$ the homomorphism $h_j^{b_1,b_2}$ is the morphism induced on the j-th singular homology by the projection $\rho : C_{b_1} \to C_{b_2}$ which is the following covering map: the base space C_{b_2} is the disjoint union of l_{b_2} circles. The covering space C_{b_1} is the disjoint union of l_{b_1} circles. Let $l := l_{b_2}$. For $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m_i\}$), let $\rho_{i,j}$ be the $k_{i,j} : 1$ covering map from the circle to itself. For $i \in \{1, \ldots, l\}$, let $\rho_i : \coprod_{j=1}^{m_i} \mathbb{S}^1 \to \mathbb{S}^1$ be the morphism that all $\rho_{i,j}$ together induce on the coproduct $\amalg_{j=1}^{m_i} \mathbb{S}^1$. Concretely, ρ_i sends an element x in the j-th copy of $\mathbb{S}^1$ to $\rho_{i,j}(x)$. Then,

$$\rho : \coprod_{i=1}^{l} \coprod_{j=1}^{m_i} \mathbb{S}^1 \to \coprod_{i=1}^{l} \mathbb{S}^1$$

is is given by $\coprod_{i=1}^{l} \rho_i$. Concretely, ρ sends an element x in $\coprod_{j=1}^{m_i} \mathbb{S}^1$ by ρ_i into the i-th copy of $\mathbb{S}^1$ in $\coprod_{i=1}^{l} \mathbb{S}^1$. The proof of statement (1) is completed by the well-known fact of how the 0-th and first singular homology groups of C_b look like. The proof of statement (3) is completed by the well-known fact of how the morphism on the 0-th and first singular homology groups induced by ρ looks like. □

Example 2.106. *We continue Example 2.104 to visualize the data of the statement of Proposition 2.105. Let* $b_1 \in [\frac{11}{4}, \frac{37}{12})$ *and* $b_2 \in [\frac{3}{2}, \frac{5}{2})$*. In Figure 2.4, we have pictured* $\mathcal{G}_i$ *and* p_i *and* $p_{j,i}$ *for that choice of* b_1 *and* b_2*. There,* $\mathcal{G}_1$ *is the graph on the left hand side and* $\mathcal{G}_2$ *is the graph on the right hand side.*

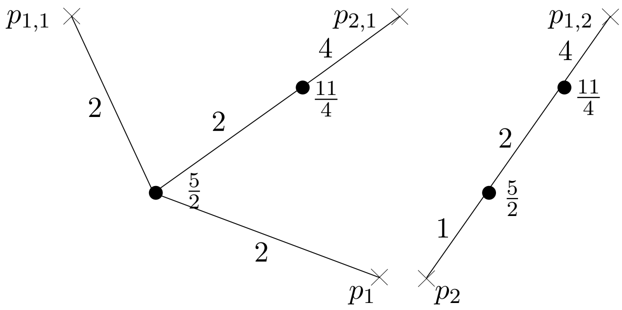


Figure 2.4.: The data of Propoistion 2.105.

It is

- $l_{b_1} = 3$ *and* $l := l_{b_2} = 2$,
- $m_1 = 2$ *and* $m_2 = 1$,
- $k_{1,1} = 1$ *and* $k_{2,1} = 2$ *and* $k_{1,2} = 4$.

Corollary 2.107. *Let* C *be an irreducible curve germ. We use the same notation as in Proposition 2.105. The MD homology of* C *with respect to* A *is as follows:*

1. *For any* $b \in [1, \infty]$*, it is* $MDH_0^b(C; A) \cong MDH_1^b(C; A) \cong A$ *and* $MDH_n^b(C; A) \cong \{0\}$*, if* $n > 1$*.*
2. *For any* $b_1, b_2 \in [1, \infty]$*,* $b_1 \geq b_2$*, it is* $h_0^{b_1,b_2} = id_A$*.*
3. *For* $b_1, b_2 \in [1, \infty]$*,* $b_1 \geq b_2$*, it is*
   - $h_1^{b_1,b_2} = id_A$*, if* $(b_2, b_1]$ *does not contain any Puiseux exponent,*
   - $h_1^{b_1,b_2}(x) = kx$*, if* $(b_2, b_1]$ *contains one Puiseux exponent with corresponding Puiseux pair* (m, k) *for some* $m \in \mathbb{N}$*.*

   If $(b_2, b_1]$ *contains more than one Puiseux exponent,* $h_1^{b_1,b_2}$ *can be determined by concatenation.*

Proof. This corollary follows directly from Proposition 2.105. □

By [38] (see also [29] and [15]), the classification of curve germs by its outer bi-Lipschitz geometry coincides with the classification of curve germs by its embedded topology. Therefore, we get the following corollary:

Corollary 2.108. *Let* C *be an irreducible curve germ. The MD homology of* C *with respect to* $\mathbb{Z}$ *detects all Puiseux pairs of* C *and therefore determines its outer geometry.*

Proof. We use the same notation as in Proposition 2.105. By Corollary 2.107, the set P of all Puiseux exponents of C can be described as follows:

$$P = \{b \in (1, \infty) : \text{there is no } \delta > 0 \text{ such that } h_1^{b,b-\delta} \text{ is an isomorphism}\}. \quad (2.24)$$

□

If C is reducible, equation (2.24) yields the set of all Puiseux exponents of all branches of C. Furthermore, b is a contact number between two branches of C if and only if $MDH_1^b(C;\mathbb{Z})$ is not isomorphic to $MDH_1^{b-\delta}(C;\mathbb{Z})$ for any δ. But this method of local analysis of the MD homology of the curve does not tell us which branch/branches those Puiseux exponents and contact numbers correspond to. For a small number of branches, such as two, this question can be answered by simple arithmetics, analysing the morphisms $h_1^{b_1,b_2}$ more globally. But in general, the MD homology might not be able to answer this question:

Example 2.109. *We use the same notation as in Proposition 2.105. Let C and D be the curves with the following five components respectively:*

$$\begin{aligned}
C_1 &= \{(x,y) \in \mathbb{C}^2 : y = x + x^2 + x^{\frac{5}{2}}\}, & D_1 &= \{(x,y) \in \mathbb{C}^2 : y = x + x^2\},\\
C_2 &= \{(x,y) \in \mathbb{C}^2 : y = x + 2x^2\}, & D_2 &= \{(x,y) \in \mathbb{C}^2 : y = x + 2x^2\},\\
C_3 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + x^2\}, & D_3 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + x^2\},\\
C_4 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + 2x^2\}, & D_4 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + 2x^2\},\\
C_5 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + 3x^2\}, & D_5 &= \{(x,y) \in \mathbb{C}^2 : y = 2x + 3x^2 + x^{\frac{5}{2}}\}
\end{aligned}$$

The embedded topological types of the two curves do not coincide since their Eggers-Wall trees are not isomorphic as trees. But their MD homology with respect to $\mathbb{Z}$ are isomorphic: we denote the morphisms $h_*^{b_1,b_2}$ of the MD homology of C and D by $h_*^{b_1,b_2}(C)$ and $h_*^{b_1,b_2}(D)$ respectively. Having a look at their Eggers-Wall trees, it becomes clear that the 0-th and first b-moderately discontinuous homology groups coincide for any b and so do the morphisms $h_0^{b_1,b_2}(D)$ and $h_0^{b_1,b_2}(D)$ for any $b_1 \geq b_2$. As the Eggers-Wall trees of C and D coincide on $r^{-1}([0,\frac{5}{2}))$ and $r^{-1}((\frac{5}{2},\infty])$, $h_1^{b_1,b_2}(C)$ and $h_1^{b_1,b_2}(D)$ also coincide, if $b_1, b_2 < \frac{5}{2}$ or $b_1, b_2 > \frac{5}{2}$. If $b_1 \geq \frac{5}{2}$ and $b_2 < \frac{5}{2}$, $h_1^{b_1,b_2}(C)$ and $h_1^{b_1,b_2}(D)$ are the same up to concatenation with isomorphisms on the right and on the left. For example, if $b_1 \geq \frac{5}{2}$ and $b_2 \in [1,2)$, $h_1^{b_1,b_2}(C)$ and $h_1^{b_1,b_2}(D)$ are given by matrix multiplication with $M_1(C)$ and $M_1(D)$, respectively, where

$$M_1(C) := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, M_1(D) := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

2.10. Finite generation, Bibrair's conjecture and rationality of jumps

The following conjecture was stated orally by Lev Birbrair in Oaxaca in fall 2018.

Conjecture 2.110 (Birbrair). *Let* $(X, x_0, d_{out}) \subset (\mathbb{R}^m, x_0)$ *be a subanalytic germ with the outer metric. Then for any* b *and* $\eta > 0$ *sufficiently small, we have that* $MDH_*^b(X, x_0, d_{out})$ *is isomorphic to the ordinary homology of the punctured cone* $\mathcal{H}_{b,\eta}(X, \mathbb{R}^m) \setminus \{x_0\}$.

Since $\mathcal{H}_{b,\eta}(X, \mathbb{R}^m)$ is subanalytic for $b \in \mathbb{Q}$, we know that its singular homology groups are finitely generated. Therefore, a positive answer to Conjecture 2.110, together with the fact (communicated to us by A. Parusinski) that any subanalytic subset with the inner metric admits a bi-Lipschitz subanalytic LNE re-embedding (see [8]) would imply immediately the following conjecture for $b \in \mathbb{Q}$:

Conjecture 2.111. *Let* $(X, x_0) \subset (\mathbb{R}^m, x_0)$ *be a subanalytic germ. Then for any* $b \in (0, \infty]$ *we have that* $MDH_*^b(X, x_0, d_{out})$ *and* $MDH_*^b(X, x_0, d_{in})$ *are finitely generated.*

Conjecture 2.111 can be shown for any $b \in (0, +\infty)$ by proving the following two conjectures besides Birbrair's conjecture:

Definition 2.112. *Let* (X, x_0, d_X) *be a subanalytic germ. An exponent* $b \in \mathbb{B}$ *is a* jumping exponent for X, *if for any* $\epsilon > 0$ *the homomorphism* $MDH_*^{b+\epsilon}(X, x_0, d_X) \to MDH_*^{b-\epsilon}(X, x_0, d_X)$ *is not an isomorphism.*

Conjecture 2.113 (Rationality). *Let* $(X, x_0) \subset (\mathbb{R}^m, x_0)$ *be a subanalytic germ and* d_X *be either the inner or the outer metric. Then the jumping exponents are rational numbers.*

Conjecture 2.114 (Finiteness). *Let* $(X, x_0) \subset (\mathbb{R}^m, x_0)$ *be a subanalytic germ and let* d_X *be either the inner or the outer metric. Then* X *has only a finite number of jumping exponents.*

Moderately Discontinuous Metric Homotopy 3

The homotopy theory we develop in this chapter is in concordance with the homology theory we developed in the previous chapter: it admits a Hurewicz morphism from the homotopy group of degree n to the homology group of degree n (see Proposition 3.40) that fulfils the Hurewicz Theorem in degree $n = 1$ (see Theorem 3.55). Similarly to the MD homology, it is a functor from a category of geometric nature (see Definition 3.47) to a category of an algebraic nature (see Definition 3.52).

Notation 3.1. *In this chapter $I := [0, 1]$ denotes the unit interval.*

Notation 3.2. *For any $n \in \mathbb{N}$, we denote by $\underline{0}$ the origin of $\mathbb{R}^n$.*

Notation 3.3. *For readability, in this chapter we denote (yt, t) in $C(I^n)$ by (y, t). But be aware that this does not provide a system of coordinates of $C(I^n)$. We also denote $(y_1, ..., y_n) \in I^n$ by $y_{1..n}$ or sometimes by $(y_{1..n-1}, y_n)$ and similarly. To recall the definition of $C(I^n)$, see Definition 2.4.*

We consider $C(I^n)$ to be equipped with the norm induced by the norm on $\mathbb{R}^{n+1}$. Therefore it makes sense to talk about l.v.a. maps from or into $C(I^n)$ (recall Definition 2.7). Recall the definition (Definition 2.50) of a point in the category of metric subanalytic germs.

Definition 3.4. *Let $q : [0, \epsilon) \to C(I^n)$ be a continuous path germ. We write $q(s) = (\alpha(s), t(s)) \in C(I^n)$. We call q a* point *in $(C(I^n), \underline{0})$, if there is a representative $[0, \epsilon')$ of the germ $[0, \epsilon)$ and a $K \geq 1$ such that*

$$\frac{1}{K} s \leq t(s) \leq Ks$$

for all $s < \epsilon'$.

If we even have the equality $t(s) = s$ for all $s < \epsilon'$, then we call q a normal point *in $(C(I^n), \underline{0})$,*

3.1. Definition of the moderately discontinuous metric homotopy

3.1.1. Weak b-maps

We are going to weaken the concept of b-maps (recall Definition 2.59). Recall that b-maps were introduced to augment the class of morphisms in the category of metric subanalytic germs. Weak b-maps are going to serve a different purpose. In fact, two weak b-maps cannot be composed. Weak b-maps are a means of weakening the notion of continuity in the concepts that provide the basis of classical homotopy theory: loops and homotopies. Indeed both, the role of loops and the one of homotopies in the classical homotopy theory will be taken by weak b-maps in our theory.

Before defining weak b-maps, we are going to prove a characterization theorem for b-maps on a convex metric subanalytic germ, namely Proposition 3.6. This will reveal the analogy between b-maps and weak b-maps.

Remark 3.5. *Let φ be a b-map (later we are going to refer to this remark also for weak b-maps) and let q be a point in its domain. When we write $\varphi \circ q$, we refer to the following: let $\{(C_j, f_j)\}_{j \in J}$ be a representative of φ. The collection $\{q^{-1}(C_j) : j \in J\}$ is a closed subanalytic cover of q's domain. We can deduce that there is a $j \in J$ for which q as a germ is contained in C_j. Then, $\varphi \circ q$ refers to $f_j \circ q$. Notice that this is well-defined up to b-equivalence of points (recall Definition 2.50).*

We state the following proposition for any convex metric subanalytic germ (Z, z_0) for the sake of generality. What we have in mind is $Z = C(I^n)$ for some $n \in \mathbb{N}$. Observe that, if a germ (Z, z_0) has a convex representative, then there is an $\epsilon' > 0$ such that the intersection of Z with any ball of radius smaller than ϵ' is convex.

Proposition 3.6. *Let (Z, z_0) and (X, x_0) be metric subanalytic germs. Let (Z, z_0) have a convex representative. Let Z be equipped with the outer metric (that coincides with the inner one).*

a) Let (Z_1, Z_2) be a finite subanalytic closed cover of Z. Let φ_k be a b-map from Z_k to X for $k = 1, 2$. For example, φ_k can be a subanalytic Lipschitz l.v.a. map from Z_k to Z. Then φ_1 and φ_2 glue to a global b-map if and only if for any point q in $Z_1 \cap Z_2$, the points $\varphi_1 \circ q$ and $\varphi_2 \circ q$ are b-equivalent.

b) Let $\varphi_1 = \{(C_j, f_j)\}_{j \in J}$ and $\varphi_2 = \{(D_k, g_k)\}_{k \in K}$ be two b-maps from Z to X. Then, φ_1 and φ_2 are equivalent if and only if for any $j \in J$ and $k \in K$ and any point q in the intersection $C_j \cap D_k$, the points $f_j \circ q$ and $g_k \circ q$ are b-equivalent.

Proof. First we show statement a). By definition, if φ_1 and φ_2 together form a b-map on Z, $\varphi_1 \circ q$ and $\varphi_2 \circ q$ have to be b-equivalent for any q in the intersection. To show the reverse implication, let q_1 and q_2 be two different points contained in Z_1 and Z_2 respectively that are b-equivalent. We define $l_t : [0, 1] \to Z$ by the formula $l_t(s) := q_2(t)s + (1 - s)q_1(t)$. Then, $l(s, t) := l_t(s)$ is subanalytic. As $Z_1 \cap \mathrm{Im}(l_t)$ and $Z_2 \cap \mathrm{Im}(l_t)$ are compact, the intersection $Z_1 \cap Z_2 \cap \mathrm{Im}(l_t)$ is non-empty for any

t. So $\mathrm{Im}(l) \cap Z_1 \cap Z_2$ is a non-empty subanalytic set with z_0 in its closure. So by the subanalytic Curve Selection Lemma, there is a continuous subanalytic germ

$$r : [0,1] \to \overline{\mathrm{Im}(l) \cap Z_1 \cap Z_2}$$

with $r(0) = z_0$ and $r(t) \neq z_0$ for any $t > 0$. By reparametrizing r, we can achieve that $r(t) \in \mathrm{Im}(l_t)$, so in particular that r is l.v.a. Then, r is b-equivalent to q_k for $k = 1, 2$. Using the triangle inequality, the fact that φ_k are Lipschitz on Z_k and the fact that $\varphi_1 \circ r$ and $\varphi_2 \circ r$ are b-equivalent, we get that $\varphi_1 \circ q_1$ and $\varphi_2 \circ q_2$ are b-equivalent.

For statement b), we can use the exact same line of argumentation. □

Proposition 3.6 motivates the following definition:

Definition 3.7 (Weak b-map). *Let (X, x_0, d_X) be a metric subanalytic germ and let $(Z, \underline{0})$ be a subanalytic subgerm of $C(I^n)$. Let $b \in (0, \infty)$. A* weak b-moderately discontinuous subanalytic map (weak b-map, for abbreviation) from $(Z, 0)$ to (X, x_0, d_X) *is a finite collection $\{(C_j, f_j)\}_{j \in J}$, where $\{C_j\}_{j \in J}$ is a finite closed subanalytic cover of $(Z, \underline{0})$ and $f_j : C_j \to X$ are continuous l.v.a. subanalytic maps for which for any $j_1, j_2 \in J$ and any point q in $C_{j_1} \cap C_{j_2}$, the points $f_{j_1} \circ q$ and $f_{j_2} \circ q$ are b-equivalent. We call $\{C_j\}_{j \in J}$ the* cover of the weak b-map $\{(C_j, f_j)\}_{j \in J}$.

Two weak b-maps $\{(C_j, f_j)\}_{j \in J}$ and $\{(C'_k, f'_k)\}_{k \in K}$ are called equivalent, *if for any $j \in J$ and $k \in K$ and any point q contained in the intersection $C_j \cap C'_k$, the points $f_j \circ q$ and $f'_k \circ q$ are b-equivalent in X.*

We make an abuse of language and we also say that a weak b-map from (Z, z_0) to (X, x_0, d_X) is an equivalence class as above.

For $b = \infty$, a weak b-map from Z to X is a continuous l.v.a. subanalytic map germ from (Z, z_0) to (X, x_0, d_X).

Notice that the f_i are not necessarily Lipschitz, unlike the case of b-maps.

Remark 3.8. *Definition 3.7 implies that statement a) of Proposition 3.6 also holds for weak b-maps: two weak b-maps φ_1 and φ_2 defined on Z_1 and Z_2 respectively glue to a global weak b-map if and only if for any point q in $Z_1 \cap Z_2$, the points $\varphi_1 \circ q$ and $\varphi_2 \circ q$ are b-equivalent.*

Remark 3.9. *Let $\varphi = \{(C_j, f_j)\}_{j \in J}$ be a weak b-map (or b-map) and $\{D_k\}_{k \in K}$ a refinement of $\{C_j\}_{j \in J}$. For $k \in K$, let $r(k) \in J$ be such that $D_k \subseteq C_{r(k)}$. Then $\{(D_k, f_{r(k)}|_{D_k})\}_{k \in K}$ is equivalent to φ.*

Remark 3.10. *Any weak b-map from Z to X has a representative $\{(C_j, f_j)\}_{j \in J}$, for which the interior of $C_{j_1} \cap C_{j_2}$ is empty for any $j_1, j_2 \in J$. This follows from Remark 3.9.*

Remark 3.11. *Let $b \geq b'$. Then, any weak b-map from $C(I^n)$ to (X, x_0, d_X) is also a weak b'-map. Since $C(I^n)$ is convex, by Theorem 3.6 this statement is also true for b-maps.*

Remark 3.12. *Let $\varphi = \{(C_j, f_j)\}_{j \in J}$ be a weak b-map (or b-map). Suppose there are $j_1, j_2 \in J$ with $C_{j_1} \subseteq C_{j_2}$. Then, $\{(C_j, f_j)\}_{j \in J \setminus \{j_1\}}$ is equivalent to φ.*

The definition of b-map and weak b-map is similar. Indeed, by Theorem 3.6, any weak b-map $\{(C_j, f_j)\}_{j\in J}$ defined on a convex subgerm of $C(I)$, for which all f_j are Lipschitz, is a b-map. But, as opposed to b-maps, two weak b-maps cannot be composed with each other. Nevertheless, a weak b-map can be composed with continuous l.v.a. subanalytic maps on its right and with b-maps on its left.as opposed to b-maps, two weak b-maps cannot be composed with each other. Nevertheless, a weak b-map can be composed with continuous l.v.a. subanalytic maps on its right and with b-maps on its left. This definition is analogous to the composition of two b-maps:

Definition 3.13. *Let Z and Z' be subanalytic convex subgerms of $C(I^n)$ and $C(I^m)$, respectively. Let $\varphi = \{(C_j, f_j)\}_{j\in J}$ be a weak b-map from Z to X. For a continuous l.v.a. subanalytic map ϕ from Z' to Z, we define $\varphi\circ\phi$ to be the weak b-map $\{(\phi^{-1}(C_j), \varphi_j\circ\phi)\}_{j\in J}$ from Z' to X. For a b-map $\psi = (D_k, g_k)_{k\in K}$ from X to X', we define $\psi\circ\varphi$ to be the weak b-map $\{(f_j^{-1}(D_k) \cap C_j, g_k \circ f_{j|f_j^{-1}(D_k)\cap C_j})\}_{(j,k)\in J\times K}$ from $C(I^n)$ to X'.*

3.1.2. Definition of the b-moderately discontinuous metric homotopy groups

We are going to define the b-moderately discontinuous metric homotopy groups for fixed $b \in (0, \infty]$. For that we need to weaken the concept of b-homotopies (recall Definition 2.79) from $C(I^n)$ to X.

Definition 3.14 (Weak b-homotopy (relative to W))**.** *Let (X, x_0, d_X) be a metric subanalytic germ and let φ_0 and φ_1 be weak b-maps from $C(I^n)$ to X. A* weak b-homotopy from φ_0 to φ_1 *is a weak b-map H from $C(I^{n+1})$ to X for which $H\circ\iota_k = \varphi_k$ for $k \in 0, 1$, where ι_k denotes the inclusion of $C(I^n)$ into $C(I^{n+1})$ given by $(y, t) \to ((y, k), t)$. We say that φ_0 and φ_1 are* weakly b-homotopically equivalent*.*

Let $(W, \underline{0}) \subseteq (C(I^n), \underline{0})$ be a subgerm. A weak b-homotopy relative to W from φ_1 to φ_2 *is a weak b-homotopy from φ_1 to φ_2 for which for any point q in $\rho^{-1}(W)$, the points $H\circ q$ and $\varphi_1\circ\rho\circ q$, where $\rho : C(I^{n+1}) \to C(I^n)$ denotes the projection $(y_{1..n+1}, t) \mapsto (y_{1..n}, t)$, are b-equivalent.*

Remark 3.15. *Let φ_0 and φ_1 be weak b-maps from $C(I^n)$ to X. Let $(W, 0) \subseteq (C(I^n), 0)$ be a subgerm. There is a necessary condition for φ_1 and φ_2 to admit a weak b-homotopy relative to W between them: for any point q in W, the points $\varphi_0\circ q$ and $\varphi_1\circ q$ are b-equivalent.*

The following definition describes the objects in the domain of the moderately discontinuous metric homotopy functor.

Definition 3.16. *A* pointed metric subanalytic germ *$((X, x_0, d_X), p(t))$ is a metric subanalytic germ (X, x_0, d_X) together with a point $p : [0, \epsilon) \to X$. We often suppress x_0 and d_X in the notation and simply write $(X, p(t))$ or (X, p).*

Definition 3.17 (b-MD n-loop)**.** *Let $((X, x_0, d_X), p(t))$ be a pointed metric subanalytic germ and $b \in (0, \infty]$ and $n \in \mathbb{N}$. A* b-moderately discontinuous n-loop (b-MD n-loop, for short) *is a weak b-map* φ *from* $C(I^n)$ *to* X *for which the following boundary condition holds: for any normal point* q *in* $C(\partial I^n)$, *the point* $\varphi\circ q$ *is b-equivalent to* p.

We denote the set of all b-MD n-loops in (X,p) *by* $MD\Gamma_n^b(X,p)$. *Observe that we suppress* x_0 *and* d_X *in the notation* $MD\pi_n^b(X,p)$, *even though they influence the set of b-MD n-loops in* (X,p).

With respect to the boundary condition for b-MD n-loops, observe that for a normal point q in $C(I^n)$, the point $f_j\circ q$ does not need to be a normal point in X. That boundary condition has a simple sufficient condition:

Example 3.18. *Let* φ *be a weak b-map from* $C(I^n)$ *to* X *that has a representative* $\{(C_j,f_j)\}_{j\in J}$ *as follows: for any* $j\in J$, *we have* $f_j(y,t)=p(t)$ *for any* $(y,t)\in C(\partial I^n)\cap C_j$. *Then* φ *is a b-MD n-loop.*

Remark 3.19. *Any two b-MD n-loops* φ_1 *and* φ_2 *in* $(X,p(t))$ *fulfil the necessary condition of Remark 3.15 to admit a weak b-homotopy relative to* $W=C(\partial I^n)$ *between them: for any point* q *in* $C(\partial I^n)$, *the points* $\varphi_0\circ q$ *and* $\varphi_1\circ q$ *are b-equivalent.*

Proof. Let q be a point in $C(\partial I^n)$. By Remark 2.8, there is a subanalytic homeomorphism $h:[0,\epsilon)\to[0,\epsilon)$, for which $\tilde q := q\circ h$ is a normal point. Since both q and $\tilde q$ are l.v.a., the homeomorphism h is also l.v.a.. Therefore, the b-equivalence between $\varphi_1\circ\tilde q$ and $\varphi_2\circ\tilde q$ implies the b-equivalence between $\varphi_1\circ q$ and $\varphi_2\circ q$.

□

Therefore the following definition is well-defined.

Definition 3.20. *The* n-th b-moderately discontinuous homotopy set (n-th b-MD homotopy set, for short), *denoted by* $MD\pi_n^b(X,p)$ *is the quotient of* $MD\Gamma_n^b(X,p)$ *by weak b-homotopies relative to* $C(\partial I^n)$.

We call the equivalence class in $MD\pi_n^b(X,p)$ *of an element* $\varphi\in MD\Gamma_n^b(X,p)$ *the* b-homotopy class of φ *and denote it by* $[\varphi]$.

In Definition 3.20 we have used that the relation by weak b-homotopies relative to ∂I is transitive by Remark 3.8 and clearly reflexive and symmetric. In order to give two simple but important examples of weak b-homotopies, we adapt the notion of l.v.a. to homotopies in the topological sense:

Definition 3.21. *Let* (Z,z_0) *and* (X,x_0) *be subanalytic germs and let* $\eta: Z\times I\to X$ *be a subanalytic continuous homotopy. We call* η l.v.a., *if there is a* $K\geq 1$ *such that*

$$\frac{1}{K}\|z-z_0\|\leq\|\eta(z,s)-x_0\|\leq K\|z-z_0\|$$

for any $(z,s)\in Z\times I$.

Example 3.22. *Let* $\varphi=\{(C_j,f_j)\}_{j\in J}$ *be a weak b-map from* $C(I^n)$ *to* (X,x_0). *Suppose there is a subanalytic l.v.a. homotopy* $\eta: C_{j_0}\times I\to X$ *relative to*

$$B_{j_0} := \cup_{j\in J\setminus\{j_0\}}C_j\cap C_{j_0}$$

from f_{j_0} *to some* $\hat{f}_{j_0} : C_{j_0} \to X$*. Relative to* B_{j_0} *means that* $H(x,s) = f_{j_0}(x)$ *for all* $x \in B_{j_0}$*. Then,* φ *is weakly b-homotopically equivalent to* $\{(C_j, \hat{f}_j)\}_{j\in J}$*, where* $\hat{f}_j = f_j$ *for* $j \neq j_0$*. Observe that implicitly we have used that for any* $s \in I$ *we have* $\eta((*,0),s) = x_0$*, where* $(*,0)$ *is the vertex of* $C(I^n)$*. That is a consequence of* η *being l.v.a..*

Example 3.23. *Let* $\eta : C(I^n) \times I \to C(I^n)$ *be a subanalytic l.v.a. homotopy. We write* η_s *for the map* $C(I^n) \to C(I^n)$ *sending* $(y_{1..n}, t)$ *to* $\eta((y_{1..n},t),s)$*. Assume that* $\eta_0 = id_{C(I^n)}$*. Since* η *is l.v.a., we can define* $\hat{\eta} : C(I^{n+1}) \to C(I^n)$ *by the formula* $\hat{\eta}(y_{1..n+1}, t) := \eta((y_{1..n},t), y_{n+1})$*. Then* $\varphi\circ\hat{\eta}$ *defines a weak b-homotopy from* φ *to* $\varphi\circ\eta_1$*.*

We are going to equip $MD\pi_n^b(X,p)$ with a group structure.

Notation 3.24. *Let* $n \in \mathbb{N}$ *and let* $0 \le \alpha_1 \le \alpha_2 \le 1$ *and* $0 \le \alpha_1' < \alpha_2' \le 1$*. Then,* $\phi_{\alpha_1,\alpha_2}^{\alpha_1',\alpha_2'}$ *denotes the continuous subanalytic l.v.a. homeomorphism from* $C([\alpha_1',\alpha_2'] \times I^{n-1})$ *to* $C([\alpha_1,\alpha_2] \times I^{n-1})$ *that linearly transforms the former into the latter. This is defined by the formula*

$$\phi_{\alpha_1,\alpha_2}^{\alpha_1',\alpha_2'}(y_{1..n},t) := ((\alpha_2 - \frac{\alpha_2-\alpha_1}{\alpha_2'-\alpha_1'}(\alpha_2' - y_1), y_{2..n}), t)$$

We suppress n *in the notation. When* $\alpha_1' = 0$ *and* $\alpha_2' = 1$*, we simply write* ϕ_{α_1,α_2}*.*

Remark 3.25. *Let* $n \in \mathbb{N}$*,* $0 \le \alpha_1 < \alpha_2 \le 1$ *and* $0 \le \beta_1 < \beta_2 \le 1$*. Then we have* $\phi_{\beta_1,\beta_2}\circ\phi_{\alpha_1,\alpha_2} = \phi_{\gamma_1,\gamma_2}$*, where* $\gamma_1 = \alpha_1(\beta_2-\beta_1)+\beta_1$ *and* $\gamma_2 = \alpha_2(\beta_2-\beta_1)+\beta_1$*.*

Definition 3.26 (Concatenation of weak b-maps $(\cdot)$)**.** *Let* φ_1 *and* φ_2 *be b-MD n-loops. By the same argument as the one in Remark 3.19 and by Remark 3.8,* $\varphi_1\circ\phi_{0,\frac{1}{2}}^{-1}$ *and* $\varphi_2\circ\phi_{\frac{1}{2},1}^{-1}$ *glue to a weak b-map on* $C(I^n)$*, which we call the* concatenation of φ_1 and φ_2*. We denote it by* $\varphi_1 \cdot \varphi_2$*.*

If H_k *are weak b-homotopies from* φ_k *to* $\hat{\varphi}_k$ *for* $k = 1,2$*, then by Remark 3.8* $H_1\circ\phi_{0,\frac{1}{2}}^{-1}$ *and* $H_2\circ\phi_{\frac{1}{2},1}^{-1}$ *glue to a weak b-homotopy. So we can define the* concatenation of $[\varphi_1]$ and $[\varphi_2]$ *to be* $[\varphi_1\cdot\varphi_2]$*.*

Notation 3.27. *Let* φ *be a b-MD n-loop and* $a \in \mathbb{N}$*. The notation* φ^a *stands for the result of concatenating* φ *with itself* a *times.*

Remark 3.28. *In order to concatenate any two weak b-maps* φ_1 *and* φ_2 *(in that order) from* $C(I^n)$ *to* X*, it is enough to ask that for any subanalytic continuous path* $\mu : [0,\epsilon) \to I^{n-1}$ *that the two points* $\varphi_1\circ q_1$ *and* $\varphi_2\circ q_0$ *are b-equivalent, where* $q_k(t) := (k, \mu(t), t) \in C(I^n)$*. If* $n = 1$*, we define* $q_k(t) := (k,t)$*.*

The following remark gives an intuition of how to decompose b-maps defined on $C(I)$.

Remark 3.29. *a) Let* φ *be a weak b-map from* $C(I^n)$ *to* X*. Let* $y_0 \in (0,1)$*. By Example 3.23,* $(\varphi\circ\phi_{0,y_0})\cdot(\varphi\circ\phi_{y_0,1})$ *is weakly b-homotopically equivalent to* φ*. If* $y_0 = \frac{1}{2}$*, then we have* $(\varphi\circ\phi_{0,y_0})\cdot(\varphi\circ\phi_{y_0,1}) = \varphi$*.*

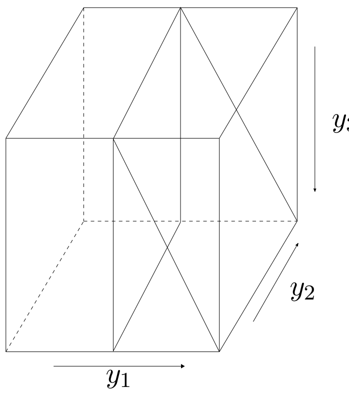
b) Let φ be a weak b-map from $C(I)$ to X. Let q be a point in $C(I)$ different from $q_k(t) := (k,t) \in C(I)$ for $k = 0,1$. Let A_1 and A_2 be the regions of $C(I)$ enclosed by q_0 and q and by q and q_1, respectively. Then there are l.v.a. homeomorphisms ψ_1 and ψ_2 from $C(I)$ to A_k, such that $(\varphi \circ \psi_1) \cdot (\varphi \circ \psi_2)$ is weakly b-homotopically equivalent to φ.

We are going to show that concatenation equips $MD\pi_n^b(X,p)$ with a well-defined group structure. That can be done in the same way as for the ordinary homotopy groups of a punctured topological space.

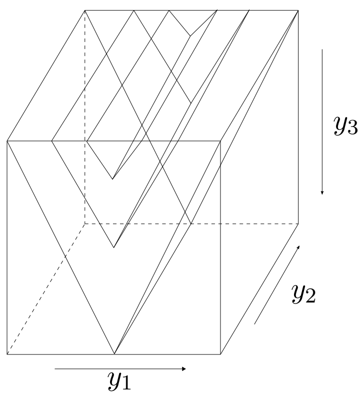
Notation 3.30. *Let $((X,x_0,d_X),p(t))$ be a pointed metric subanalytic germ and $n \in \mathbb{N}$. We denote by $c_{p,n}$ the weak b-map from $C(I^n)$ to X defined by $c_{p,n}(y,t) = p(t)$.*

Lemma 3.31 (Existence of unit element)**.** *Let φ be a b-MD n-loop in (X,p). Then, $[\varphi] \cdot [c_{p,n}] = [c_{p,n}] \cdot [\varphi] = [\varphi]$.*

Proof. We have illustrated the homotopy used in the analogous proof for the ordinary homotopy theory in Figure 3.1a. There, φ lies on the top of I^2 and $c_{p,n}$ on the bottom.



(a) Unit element.



(b) Inverse element.

Figure 3.1.: Group axioms.

The same idea can be adapted to the MD homotopy. That can be formalized as follows. Write $\varphi = \{(C_j, f_j)\}_{j \in J}$. Let $r : I \to I$ be defined by $r(y_{n+1}) := \frac{1}{2} + \frac{1}{2} y_{n+1}$. We define

$$\begin{aligned}
&\hat{C}_j := \{(y_{1..n+1}, t) \in C(I^{n+1}) : (y_{1..n}, t) \in \phi_{0,r(y_{n+1})}(C_j)\},\\
&\hat{f}_j : \hat{C}_j \to X, (y_{1..n+1}, t) \mapsto f_j \circ \phi^{-1}_{0,r(y_{n+1})}(y_{1..n}, t),\\
&T := \{(y_{1..n+1}, t) \in C(I^{n+1}) : (y_{1..n}, t) \in \operatorname{Im}(\phi_{r(y_{n+1}),1})\},\\
&g : T \to X, (x,t) \mapsto p_1(t).
\end{aligned}$$

By Remark 3.8 and the same argument as the one of Remark 3.19, the weak b-maps $\{(\hat{C}_j, \hat{f}_j)\}_{j\in J}$ and (T, g) glue to a weak b-homotopy from $\varphi \cdot c_{p_1,n}$ to φ. The weak b-homotopy from $c_{p_1,n} \cdot \varphi$ to φ is constructed analogously. $\square$

Lemma 3.32 (Associativity)**.** *Let φ_1, φ_2 and φ_3 be b-MD n-loops. It is $([\varphi_1] \cdot [\varphi_2]) \cdot [\varphi_3] = [\varphi_1] \cdot ([\varphi_2] \cdot [\varphi_3])$.*

Proof. By Remark 3.25, $(\varphi_1 \cdot \varphi_2) \cdot \varphi_3$ is given by glueing together $\varphi_1 \circ \phi^{-1}_{0,\frac{1}{4}}$, $\varphi_2 \circ \phi^{-1}_{\frac{1}{4},\frac{1}{2}}$ and $\varphi_3 \circ \phi^{-1}_{\frac{1}{2},1}$. In the same way $\varphi_1 \cdot (\varphi_2 \cdot \varphi_3)$ is given by gluing together $\varphi_1 \circ \phi^{-1}_{0,\frac{1}{2}}$, $\varphi_2 \circ \phi^{-1}_{\frac{1}{2},\frac{3}{4}}$ and $\varphi_3 \circ \phi^{-1}_{\frac{3}{4},1}$. There is a weak b-homotopy between them as a result of Example 3.23 applied to the continuous homotopy illustrated in Figure 3.2.

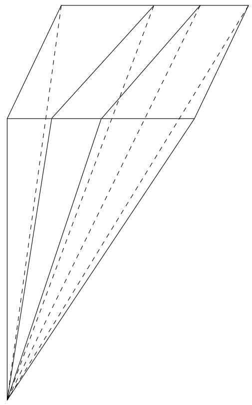


Figure 3.2.: Associativity.

$\square$

Notation 3.33. *Let φ be a weak b-map from $C(I^n)$ to X. We denote the weak b-map $\{(\overleftarrow{C_j}, \overleftarrow{f_j})\}$ by $\overleftarrow{\varphi}$, where $\overleftarrow{C_j}$ and $\overleftarrow{f_j}$ are the result of mirroring C_j and f_j respectively at the y_1-axis:*

$$\overleftarrow{C_j} := \{(y_{1..n}, t) \in I(C^n) : (1 - y_1, y_{2..n}, t) \in C_j\},$$
$$\overleftarrow{f_j}(y_{1..n}, t) := f_j(1 - y_1, y_{2..n}, t)$$

Lemma 3.34 (Existence of inverse elements)**.** *Let φ be a b-MD n-loop in (X, p).*
Then we have $[\varphi] \cdot [\overleftarrow{\varphi}] = [\overleftarrow{\varphi}] \cdot [\varphi] = [c_{p,n}]$.

Proof. We have illustrated the homotopy used in the analogous proof for the ordinary homotopy theory in Figure 3.1b. Again, φ lies on the top of I^2 and $c_{p,n}$ on the bottom. The intersection of any prism similar to the ones drawn in that figure with an affine plane spanned by y_1 and y_3 gives a triangle. The homotopy is constant on the two sides of any such triangle that lie in the interior of I^2. The same idea can be adapted to the MD homotopy. That can be formalized in a similar way as we formalized the proof of Lemma 3.31. □

Notation 3.35. *Let φ be a b-MD n-loop and $a \in \mathbb{N}$. The notation φ^{-a} stands for the result of concatenating $\overleftarrow{\varphi}$ with itself a times.*

The following small adaptation of the proof of Lemma 3.34 will be used repeatedly:

Lemma 3.36. *Let $\varphi = \{(C_j, f_j)\}_{j \in J}$ be a weak b-map from $C(I)$ to X. Let $q_0(t) := (0,t) \in C(I)$ be the left lateral point of $C(I)$ and $q_1(t) := (1,t) \in C(I)$ the right lateral point. Let $p_k := \varphi \circ q_k$. We define the constant weak b-maps $l = (C(I), g_0)$ and $r = (C(I), g_1)$ by $g_k(y,t) := p_k(t)$. Then there is a weak b-homotopy relative to $C(\partial I)$ from $\varphi \cdot \overleftarrow{\varphi}$ to l and from $\overleftarrow{\varphi} \cdot \varphi$ to r.*

Proof. The proof is analogous to the one of Lemma 3.34. □

Proposition 3.37. *The concatenation defined in Definition 3.26 defines a group structure on $MD\pi_n^b(X,p)$.*

Proof. The statement was shown in Lemma 3.31, Lemma 3.32 and Lemma 3.34. □

Definition 3.38. *From now on we call $MD\pi_n^b(X,p)$ the n-th* b-MD homotopy group *of (X,p). We also call $MD\pi_1^b(X,p)$ the* b-MD fundamental group *of (X,p).*

3.1.3. The Hurewicz homomorphism

In the same way as in the topological homotopy and homology theories, for the b-MD homology and homotopy theories there is a Hurewicz homomorphism relating those theories .

Let $((X, x_0, d), p)$ be a pointed metric subanalytic b-connected germ. We define an auxiliary map

$$\zeta_{n,b} : \{\varphi : \varphi \text{ is a weak } b\text{-map from } C(I^n) \text{ to } X\} \to MDC_n^b(X;\mathbb{Z}) \tag{3.1}$$

as follows: let $\varphi = \{(C_j, f_j)\}_{j \in J}$ be a weak b-map from $C(I^n)$ to X. Let $\{\rho_k\}_{k \in K}$ be an orientation preserving homological subdivision (recall Definition 2.16) of $C(I^n)$ whose associated triangulation is compatible with $\{C_j\}_{j \in J}$. For every $k \in K$, let $r(k) \in J$ such that the image of ρ_k is contained in $C_{r(k)}$. We define the map ζ by the formula

$$\zeta_{n,b}(\varphi) := \sum_{k \in K} f_{r(k)} \circ \rho_k. \tag{3.2}$$

We have defined $\zeta_{n,b}$ on its general domain instead of defining it on the domain of b-MD n-loops and make the following lemma for that general domain. The reason for that is that we need that generality in the proof of the Hurewicz Theorem of degree one (Theorem 3.55) further ahead.

Lemma 3.39. *Let $((X, x_0, d), p)$ be a pointed metric subanalytic b-connected germ. Let $\zeta_{n,b}$ be as defined above. Then $\zeta_{n,b}$ has the following properties:*

1. *The map $\zeta_{n,b}$ is well-defined independent from the choice of the homological subdivision.*

2. *It is $\zeta_{n,b}(\varphi_1) + \zeta_{n,b}(\varphi_2) = \zeta_{n,b}(\varphi_1 \cdot \varphi_2)$.*

3. *It is $\zeta_{n,b}(\overleftarrow{\varphi}) = -\zeta_{n,b}(\varphi)$.*

Proof. All the stated properties follow from the homological subdivision equivalence in $MDC_n^b((X, \mathrm{Im}(p)); \mathbb{Z})$. In particular, for property (3) recall Remark 2.19. □

Proposition 3.40 (Hurewicz homomorphism). *Let $((X, x_0), p)$ be a pointed metric subanalytic b-connected germ. Let $b \in (0, \infty]$ and $n \in \mathbb{N}$. Then the restriction of $\zeta_{n,b}$ to the space of b-MD n-loops induces a homomorphism*

$$\overline{\zeta_{n,b}} : MD\pi_n^b(X, p) \to MDH_n^b(X; \mathbb{Z}),$$

which we call the Hurewicz morphism.

Proof. By Lemma 3.39, if $\overline{\zeta_{n,b}}$ is well-defined, then it is a homomorphism. We consider $MDC_n^b(X, \mathrm{Im}(p); \mathbb{Z})$ to be a subcomplex of $MDC_n^b(X; \mathbb{Z})$ in the way we explained at the beginning of Subsection 2.2.5.

We show that $\overline{\zeta_{n,b}}$ is well-defined. For that, first we show that the image of a b-MD n-loop φ under $\overline{\zeta_{n,b}}$ is a cycle. Observe that the long exact relative b-MD homology sequence (recall Proposition 2.48) gives us an isomorphism $\psi_p : MDH_n^b(X; \mathbb{Z}) \xrightarrow{\sim} MDH_n^b(X, \mathrm{Im}(p); \mathbb{Z})$ as follows: we apply the long exact relative b-MD homology sequence to the subanalytic subgerms $\emptyset \subset \mathrm{Im}(p) \subset X$ and use Proposition 2.51 to fill in the b-MD homology of the point. It is clear that $\psi_p \circ \overline{\zeta_{n,b}}(\varphi)$ is trivial. Therefore $\overline{\zeta_{n,b}}(\varphi)$ is trivial.

Now we show that $\overline{\zeta_{n,b}}$ respects the equivalence by weak b-homotopies. Let φ_1 and φ_2 be b-MD n-loops and let η be a weak b-homotopy between them. Choose an orientation preserving homological subdivision of $C(I^{n+1})$ compatible with η's cover. Via that subdivision we can construct an element $z \in MDC_{n+1}^b(X, \mathbb{Z})$ for which $\delta(z) = \zeta_{n,b}(\varphi_1) - \zeta_{n,b}(\varphi_1) + r$, where r is an element of $MDC_n^b(\mathrm{Im}(p), \mathbb{Z})$. Then $\psi_p \circ \overline{\zeta_{n,b}}(r)$ is trivial and therefore $\overline{\zeta_{n,b}}(r)$ is trivial.

□

In the Hurewicz Theorem for degree one (see Theorem 3.55 in Section 3.2), we show that the Hurewicz morphism defines an isomorphism from the abelianization of $MD\pi_1^b(X, p)$ to $MDH_1^b((X, \mathrm{Im}(p)); \mathbb{Z})$.

3.1.4. Other properties

Apart from the Hurewicz morphism whose existence we have shown above, the b-MD homotopy groups share more of the classical properties of the ordinary homotopy groups of a punctured topological space. We are going to show that the b-MD homotopy groups of degree $n > 1$ are abelian for any $b \in (0, \infty]$; and that, if the metric subanalytic germ is b-path connected (see Definition 3.43), the b-MD homotopy groups are independent from the choice of base point. We have not written down the definition of b-MD homotopy groups relative to a metric subanalytic subgerm yet. That definition can be done analogously to the absolute one. We think that it will be straightforward to prove the existence of the long exact sequence of relative b-MD homotopy groups.

Proposition 3.41. *For $n > 1$, $MD\pi_n^b(X, p)$ is abelian for any $b \in (0, \infty]$.*

To prove the proposition, we can proceed exactly as in a proof for the ordinary higher homotopy groups, in which the following argument from [14] is used:

Lemma 3.42 (Eckmann-Hilton argument)**.** *Let X be a set and let $\circ$ and $\otimes$ be two binary operations on X that are both unital. Further suppose that*

$$(x_1 \circ x_2) \otimes (x_3 \circ x_4) = (x_1 \otimes x_3) \circ (x_2 \otimes x_4)$$

Then, $\circ$ and $\otimes$ coincide and are commutative.

Proof of Proposition 3.41. We have defined $\cdot$ by glueing two b-MD n-loops along the y_1-axis. One can easily check that the binary operation defined by glueing two b-MD n-loops the same way along the y_2-axis fulfils the hypothesis of Lemma 3.42 together with $\cdot$. □

Just as the ordinary homotopy group is independent from the choice of base point, if the topological space is path-connected, the b-MD homotopy group is independent from the choice of base point, if the metric subanalytic germ is b-path connected (see Definition 3.43). Notice that the first time the word *point* was used in the last sentence it referred to the set-theoretical notion of point.

Definition 3.43. *Let (X, x_0, d) be a metric subanalytic germ. It is called* b-path connected*, if for any two points $p_1(t)$ and $p_2(t)$ in X, there is a weak b-map $\eta : C(I) \to X$ for which $\eta(0, t)$ and $\eta(1, t)$ are b-equivalent to $p_1(t)$ and $p_2(t)$, respectively. We say that η* connects *p_1* and *p_2.*

The concept of b-path connectedness is related to the concept of b-connected components (recall Definition 2.99) as follows.

Definition 3.44. *Let (X, x_0, d) be a metric subanalytic germ. It is called* b-connected*, if it only has one b-connected component.*

Lemma 3.45. *Let (X, x_0, d_X) be a metric b-connected subanalytic germ. Then (X, x_0, d_X) is b-path connected.*

Proof. We recall the notion of straight points. Let $h : (X, x_0) \xrightarrow{\sim} C(L_X)$ be a subanalytic isomorphism for which it is $\|h(x) - \underline{0}\| = \|x - x_0\|$. A point $\tilde{p}$ is called a straight point with respect to h, if $p(t) = h^{-1}(x, t)$ for some x in the link L_X.

Let $p_1(t)$ and $p_2(t)$ be two points in X. Since X is b-connected, we can assume that $p_1((0, \epsilon))$ and $p_2((0, \epsilon))$ lie in the same connected component of $X \setminus \{x_0\}$. Then, showing the statement is an easy adaptation of the proof of Proposition 2.100. First we can show that there is a continuous subanalytic l.v.a. map connecting p_k with a straight point. Then we can show that for any two straight points there is a weak b-map from $C(I)$ to X connecting those two straight points. Concretely:

We assume that $p_1((0,\epsilon))$ and $p_2((0,\epsilon))$ lie in the same connected component of $X \setminus \{x_0\}$. Let $x_k : [0,\epsilon] \to L$ and $\tau_k : [0,\epsilon] \to [0,\epsilon]$ be such that $h \circ p_k(t) = (x_k(t), \tau_k(t))$ for $k = 1, 2$. Observe that $t = 0$ implies $\tau_k(t) = 0$. We define continuous subanalytic l.v.a. maps $\eta_k : C(I) \to X$ for $k = 1, 2$ by the formula

$$\eta(y,t) := h^{-1}(x_k(t + (\epsilon - t)y), \tau_k(t)).$$

Then η_k connects the point p_k with the straight point $\tilde{p}_k : t \mapsto h^{-1}(x_k(\epsilon), \tau_k(t))$ for $k = 1, 2$. Since $x_1(\epsilon)$ and $x_1(\epsilon)$ are in the same conneted component of $L_{X,\epsilon}$, there is a subanalytic path γ from one to the other. So we can define η_3 connecting $\tilde{p}_1$ and $\tilde{p}_2$ by the formula

$$\eta_3(y,t) := h^{-1}(\gamma(y), \tau_1(t) + (\tau_2(t) - \tau_1(t))y).$$

The concatenation $\eta_1 \cdot \eta_3 \cdot \overleftarrow{\eta_2}$ yields η as stated.

If $p_1((0,\epsilon))$ and $p_2((0,\epsilon))$ lie in different connected components of $X \setminus \{x_0\}$, there are $\tilde{p}_k$ in the connected component of p_k such that $\tilde{p}_1$ is b-equivalent to $\tilde{p}_2$. Then the b-map from p_1 to $\tilde{p}_1$ as obtained above and the one from $\tilde{p}_2$ to p_2 can be concatenated. □

Proposition 3.46 (Independence of base point)**.** *Let (X, x_0, d_X) be a metric subanalytic germ. Let $p_1(t)$ and $p_2(t)$ be points in X. Let η be any weak b-map from $C(I)$ to X connecting p_1 and p_2 as in Lemma 3.45. Let $\hat{\eta}$ be the weak b-map from $C(I)$ to X defined by the formula $\hat{\eta} := \eta \circ \rho$, where $\rho : C(I^n) \to C(I)$ is the projection $\rho(y_{1..n}, t) := (y_1, t)$. Then the homomorphism*

$$\zeta : MD\pi_n^b(X, p_1) \to MD\pi_n^b(X, p_2)$$

defined by $\zeta(\varphi) := \overleftarrow{\hat{\eta}} \cdot \varphi \cdot \hat{\eta}$ is an isomorphism. Moreover, its inverse is $\zeta^{-1}(\varphi) := \hat{\eta} \cdot \varphi \cdot \overleftarrow{\hat{\eta}}$.

Proof. The statement is obvious. □

3.1.5. Functoriality

The MD homotopy functor is defined in the following domain:

Definition 3.47. *The* category of pointed metric subanalytic germs *has pointed metric subanalytic germs (recall Definition 3.16) as objects and subanalytic Lipschitz l.v.a. maps $f : ((X, x_0, d_X), p) \to ((X', x'_0, d_{X'}), p')$ for which $f \circ p = p'$ as morphisms.*

Similarly to the case of the MD homology, the domain of the b-MD homotopy can also be augmented:

Definition 3.48. *The* category of pointed metric subanalytic germs *with b-maps has pointed metric subanalytic germs (recall Definition 3.16) as objects and b-maps ψ : $((X, x_0, d_X), p) \to ((X', x'_0, d_{X'}), p')$ for which $\psi \circ p$ is b-equivalent to p' as morphisms.*

The target category is defined analogously to the target category of the MD-homology functor (recall Definition 2.42)).

Notation 3.49. *We denote the category of groups by $\mathcal{G}$ and the category of abelian groups by $\mathcal{AG}$.*

Proposition 3.50. *Let $n \in \mathbb{N}$ and $b \in (0, \infty]$. There are functorial assignments $((X, x_0, d_X), p) \mapsto MD\pi_1^b(X, p)$ (resp. $((X, x_0, d_X), p) \mapsto MD\pi_n^b(X, p)$ for $n > 1$) from the category of pointed metric subanalytic germs with b-maps to $\mathcal{G}$ (resp. $\mathcal{AG}$).*

Proof. We assign to a b-map $\psi : ((X, , x_0, d_X), p_X) \to ((X', x'_0, d_{X'}), p_{X'})$ the group homomorphisms $\psi_* : MD\pi_n^b(X, q_X) \to MD\pi_n^b(X', q_{X'})$ that sends φ to $\psi \circ \varphi$. It is easy to see that ψ_* is well-defined □

Proposition 3.51. *Let $n \in \mathbb{N}$ and $b \in (0, \infty]$. There are functorial assignments $((X, x_0, d_X), p) \mapsto MD\pi_1^b(X, p)$ (resp. $((X, x_0, d_X), p) \mapsto MD\pi_n^b(X, p)$ for $n > 1$) from the category of pointed metric subanalytic germs with b-maps to $\mathcal{G}$ (resp. $\mathcal{AG}$).*

Proof. We assign to a b-map $\psi : ((X, , x_0, d_X), p_X) \to ((X', x'_0, d_{X'}), p_{X'})$ the group homomorphisms $\psi_* : MD\pi_n^b(X, q_X) \to MD\pi_n^b(X', q_{X'})$ that sends φ to $\psi \circ \varphi$. It is easy to see that ψ_* is well-defined. □

Definition 3.52. *The* category $\mathbb{B} - \mathcal{G}$ (resp. $\mathbb{B} - \mathcal{AG}$) of $\mathbb{B}$-groups (resp. $\mathbb{B}$-abelian groups) *is the category whose objects are functors from $\mathbb{B}$ to $\mathcal{G}$ (resp. $\mathcal{AG}$) and the morphisms are natural transformations of functors.*

Proposition 3.53. *Let $n \in \mathbb{N}$. There are functorial assignments $((X, x_0, d_X), p) \mapsto MD\pi_1^\star(X, p)$ (resp. $((X, x_0, d_X), p) \mapsto MD\pi_n^\star(X, p)$ for $n > 1$) from the category of pointed metric subanalytic germs to $\mathbb{B} - \mathcal{G}$ (resp. $\mathbb{B} - \mathcal{AG}$).*

Proof. We assign to an object (X, p) the groups $MD\pi_n^b(X, p)$ for any $b \in (0, \infty]$ together with the morphisms $\eta_{b,b'} : MD\pi_n^b(X, p) \to MD\pi_n^{b'}(X, p)$ for any $b \geq b'$ that we get from Remark 3.11. To a morphism $g : ((X, , x_0, d_X), p_X) \to ((X', x'_0, d_{X'}), p_{X'})$ between pointed metric subanalytic germs we assign the group homomorphisms $g_*^b : MD\pi_n^b(X, q_X) \to MD\pi_n^b(X', q_{X'})$ for any $b \in (0, \infty]$ that sends φ to $g \circ \varphi$. It is easy to see that g_*^b is well-defined and that it commutes with the $\eta_{b,b'}$. □

Notation 3.54. *We denote the group homomorphisms $MD\pi_*^b(X, q_X) \to MD\pi_*^b(X', q_{X'})$ induced by $g : (X, p_X) \to (X', p_{X'})$ by g_*^b.*

3.2. The Hurewicz Theorem in degree 1

In this chapter we adapt an important computational tool in the context of the ordinary homotopy and homology theory of punctured topological spaces to our theory: the Hurewicz Theorem in degree 1. For the statement in topology see for example Theorem 4.29 of [33]. We follow the line of proof given in [32]. Recall the definition of ζ in equation (3.1) and (3.2).

Theorem 3.55 (Hurewicz Theorem)**.** *Let (X, x_0, d) be a b-path connected metric subanalytic germ and let p be any point in X. Let $b \in (0, \infty]$ and $n \in \mathbb{N}$ and let $\Psi_{n,b}$ denote the Hurewicz morphism (recall Proposition 3.40). Let $C := [MD\pi_1^b(X, p), MD\pi_1^b(X, p)]$ denote the commutator of $MD\pi_1^b(X, p)$. Then,*

$$\overline{\Psi_{n,b}} : MD\pi_1^b(X, p)/C \to MDH_1^b(X, \mathbb{Z})$$

defined by the formula

$$\overline{\Psi_{n,b}}([\varphi]C) := \overline{\zeta_{n,b}}([\varphi])$$

is a group isomorphism.

Proof. We fix a l.v.a. homeomorphism $\mu : \hat{\Delta}_1 \to C(I)$. Via μ we identify $\hat{\Delta}_1$ with $C(I)$ and subanalytic l.v.a. maps on the domain $\hat{\Delta}_1$ with the corresponding maps on $C(I)$ and vice versa. Recall from Lemma 3.45 that we say that *a b-map φ from $C(I)$ to X connects two points p_0 and p_1 in X*, if the two lateral points $\varphi(0,t)$ and $\varphi(1,t)$ of φ are b-equivalent to p_0 and p_1, respectively.

We are going to construct the inverse of Ψ and denote it by Γ. For any b-equivalence class $[r(t)]$ of a point $r(t)$ in X, we fix a weak b-map $\tau_{[r]}$ connecting r and p. This exists due to Lemma 3.45. For the equivalence class [p], we choose $\tau_{[p]}$ to be the constant map $c_{p,q} : (y,t) \mapsto p(t)$. For a b-MD 1-simplex σ and $k \in \{0,1\}$, we denote by p_k the the two lateral points $t \mapsto \sigma(kt,t)$ of σ. We define

$$\Gamma([\sigma]) := [\overleftarrow{\tau}_{[p_0]} \cdot \sigma \cdot \tau_{[p_1]}]C.$$

Since the target of Γ is the quotient of $MD\pi_1^b(X,p)$ by its commutator, we can extend this definition linearly by $\Gamma(a_1\sigma_1 + a_2\sigma_2) := [\Gamma(\sigma_1)^{a_1} \cdot \Gamma(\sigma_2)^{a_2}]C$; in particular it is $\Gamma(-\sigma) := [\overleftarrow{\Gamma(\sigma)}]C$.

We have defined a group homomorphism from $MDC_1^{pre,\infty}(X,\mathbb{Z})$ to $MD\pi_1^b(X,p)$. To show that it descends to a morphism on $MDC_1^b(X,\mathbb{Z})$, we have to check the two conditions of Remark 2.31. Clearly, if σ_1 and σ_2 are b-equivalent l.v.a. 1-simplices, they are sent to the same weak b-map. If σ is a l.v.a. 1-simplex and z a l.v.a. 1-chain with $\sigma_1 \to_\infty z$, then σ and z are sent to weakly b-homotopically equivalent b-MD 1-loops.

To show that Γ descends to a morphism on $MDH_1^b(X,\mathbb{Z})$, let υ be a b-MD 2-simplex. Let p_1 be the point $t \mapsto \upsilon \circ j_2^0((0,1),t)$ (j_2^0 is defined in Notation 2.12). By Lemma 3.36, we have

$$\Gamma(\partial\upsilon) = [\overleftarrow{\tau}_{p_1} \cdot \upsilon\circ j_2^0 \cdot \overleftarrow{\upsilon\circ j_2^1} \cdot \upsilon\circ j_2^2 \cdot \tau_{p_1}]C.$$

So there is a continuous weak b-homotopy from $\Gamma(\partial\upsilon)$ to the constant weak b-map $c_{p,1}$.

We show that Γ is the inverse of $\overline{\Psi_{n,b}}$. Since we have chosen $\tau_{[p]}$ to be $(y,t) \mapsto p(t)$, we get $\Gamma\circ\overline{\Psi_{n,b}} = id_{MD\pi_1^b(X,p)/C}$. To show that $\overline{\Psi_{n,b}}\circ\Gamma = id_{MDH_1^b(X,\mathbb{Z})}$, let $z = \sum_{l=1}^n a_l\sigma_l \in MDH_n^b(X,\mathbb{Z})$. For every $l \in L$, let $p_{l,0}$ and $p_{l,1}$ denote the two lateral points of σ_l. We have

$$\overline{\Psi_{n,b}}(\Gamma(z)) = \overline{\Psi_{n,b}}((\overleftarrow{\tau_{[p_{1,0}]}} \cdot \sigma_1 \cdot \tau_{[p_{1,1}]})^{a_1} \cdot \ldots \cdot (\overleftarrow{\tau_{[p_{n,0}]}} \cdot \sigma_n \cdot \tau_{[p_{n,1}]})^{a_n}).$$

By Lemma 3.39, the right side is equal to

$$\sum_{l=1}^n a_l(-\tau_{[p_{l,0}]} + \sigma_l + \tau_{[p_{l,1}]}) = \sum_{l=1}^n a_l\sigma_l + \Sigma,$$

where Σ is a sum that cancels in pairs as z is a cycle. □

3.3. Basic computations

This section is divided into the computation of the ∞-MD homotopy group for any degree n and some basic computations for the b-MD fundamental group

3.3.1. The ∞-MD homotopy groups

In this subsection we use the existence of subanalytic representatives in any topological homotopy class.

Proposition 3.56. *Let $((X, x_0, d_X), p)$ be a pointed metric subanalytic germ. Fix $\epsilon > 0$ small enough. There is a group isomorphism*

$$\zeta : MD\pi_n^\infty(X, p) \xrightarrow{\sim} \pi_n(L_{X,\epsilon}, p(\epsilon)),$$

where $\pi_n(L_{X,\epsilon}, p(\epsilon))$ denotes the standard n-th homotopy group of the link $L_{X,\epsilon}$ of X.

Proof. Let $h : (X, \mathrm{Im}(p), x_0) \xrightarrow{\sim} (C(L_{X,\epsilon}), p(\epsilon))$ be a subanalytic homeomorphism. It exists since the subanalytic conical structure theorem is compatible with subgerms. We can assume that $h(x) = (\alpha(x), \|x - x_0\|)$ for some $\alpha(x) \in L$. Let $\alpha : L_{X,\epsilon} \to X$ and $\tau : (0, \epsilon] \to X$ such that $h(x) = (\alpha(x), \tau(x))$ for $x \neq x_0$. We fix $t_0 \in (0, \epsilon]$ and define the image of a ∞-MD n-loop $\varphi : C(I^n) \to X$ under ζ to be φ_{t_0}, defined by the formula $\varphi_{t_0}(y) := \alpha(\varphi(y, t_0))$. It is clear that φ_{t_0} and $\varphi_{\tilde{t}_0}$ are homotopic in the topological sense for any $t_0, \tilde{t}_0 \in (0, \epsilon)$. Furthermore, if $H : C(I^{n+1}) \to X$ is a weak ∞-homotopy between φ and $\tilde{\varphi}$, then H_{t_0} defined in the same way as φ_{t_0} is a homotopy from φ_{t_0} to $\tilde{\varphi}_{t_0}$.

Now we define the inverse υ of ζ. Let $[\gamma] \in \pi_n(L_{X,\epsilon})$ and let $\tilde{\gamma} \in [\gamma]$ be a subanalytic representative. We define the image of $[\gamma]$ under υ to be the mapping $(y, t) \mapsto h^{-1}(\tilde{\gamma}(y), t)$.

We call a b-MD n-loop $\{(C_j, f_j)\}_{j \in J}$ straight, if for any $j \in J$ we have $h \circ f_j(y, t) = (\alpha(y), t)$ for some subanalytic $\alpha : I^n \to L$. Then υ is the inverse of ζ due to the existence of straight representatives in the weak ∞-homotopy class of any ∞-MD n-loop. □

Corollary 3.57. *Let $((X, x_0, d_X), p(t))$ be a pointed metric subanalytic germ whose link $(L_{X,\epsilon})$ is path-connected. Let $b \in (0, \infty]$. If the ordinary fundamental group $\pi_1(L_{X,\epsilon}, p(\epsilon))$ is abelian and the group homomorphism $MD\pi_1^\infty(X, p) \to MD\pi_1^b(X, p)$ is surjective, then $MD\pi_1^b(X, p) = MDH_1^b(X, \mathbb{Z})$.*

Proof. Since $MD\pi_1^\infty(X, p) \to MD\pi_1^b(X, p)$ is surjective and $MD\pi_1^\infty(X, p)$ is abelian, so is $MD\pi_1^b(X, p)$. Now the statement follows from the Hurewicz Theorem (Theorem 3.55). □

3.3.2. Computations for the MD fundamental group

As expected, the MD fundamental group of the point is trivial.

Proposition 3.58. *Let $([0, \epsilon), d, 0))$ be the point of our category (recall Definition 2.50). Let $p(t)$ be a l.v.a. subanalytic continuous path in $[0, \epsilon)$. The MD-fundamental group of $([0, \epsilon), p)$ is trivial.*

Proof. We are going to show that $MD\pi_1^\infty([0,\epsilon),p) \to MD\pi_1^b([0,\epsilon),p)$ is surjective and apply Corollary 3.57. Let $\varphi = \{(C_j, f_j)\}_{j\in\{1,\ldots,r\}}$ be a b-MD 1-loop. We can assume that $J = \{1,\ldots,r\}$ and that every C_j is the straight cone over a closed interval $[a_{j-1}, a_j]$ and that $a_0 < a_1 < \ldots < a_r$. In particular we have that φ is weakly b-homotopically equivalent to $f_1 \cdot \ldots \cdot f_r$. For $j \in \{1,\ldots,r-1\}$, we define the constant weak b-map g_j on $C(I)$ by $g_j(y,t) := f_j(a_j,t)$. Then φ is weakly b-homotopically equivalent to the result of alternatingly concatenating f_j and g_j as follows:

$$f_1 \cdot g_1 \cdot f_2 \cdot \ldots \cdot g_{r-1} \cdot f_r \tag{3.3}$$

We replace every g_j in (3.3) by

$$\tilde{g}_j : (y,t) \mapsto f_j(a_j,t) + y(f_{j+1}(a_j,t) - f_j(a_j,t)).$$

We can do that, because they define the same b-map since the points $p_1(t) := f_j(a_j,t)$ and $p_2(t) := f_{j+1}(a_j,t)$ are b-equivalent. As a result we get a continuous b-MD 1-loop that is weakly b-homotopically equivalent to φ. □

Proposition 3.59. *Let $L \subset \mathbb{R}^m$ be a subanalytically path-connected subanalytic set. Let $b \in \mathbb{Q} \cap [1,\infty)$ and let $(C_L^b, \underline{0}, d)$ be the b-cone over L as in Definition 2.4, where d is the outer metric. Let p be a point in C_L^b and let $t_0 > 0$ be small.*

a) If $b' < b$, then $MD\pi_1^{b'}(C_L^b, p)$ is trivial.

b) Suppose that L is compact. If $b' \geq b$, then the morphism

$$\eta_{\infty,b'} : MD\pi_1^\infty(C_L^b, p) \to MD\pi_1^{b'}(C_L^b, p)$$

is surjective.

Proof. Statement a) follows from the functoriality of the b'-MD homotopy with respect to b'-maps as follows. We define $\psi : L_X^b \to [0,\epsilon)$ to be the b'-map $\psi = (C_L^b, f)$, where $f : C_L^b \to [0,\epsilon)$ is defined by $f(xt^b, t) = t$. We show that ψ has an inverse as a b'-map. For that, we choose a point $\tilde{p}(t)$ in C_L^b that is a normal point in the following sense: if we write $\tilde{p}(t) = (\alpha_{\tilde{q}}(t)\tau_{\tilde{q}}(t)^b, \tau_{\tilde{q}}(t))$, then it is $\tau_{\tilde{q}}(t) = t$. The inverse of ψ is the b'-map $([0,\epsilon), g)$, where $g : [0,\epsilon) \to C_L^b$ is defined by $g(t) = \tilde{p}(t)$.

For Statement b), let $\varphi = \{(C_j, f_j)\}_{j\in J}$ be a b'-MD 1-loop. We can assume that $J = \{1,\ldots,r\}$ and that every C_j is the straight cone over a closed interval $[a_{j-1}, a_j]$ and that $a_0 < a_1 < \ldots < a_r$. In particular we have that φ is weakly b-homotopically equivalent to $f_1 \cdot \ldots \cdot f_r$. We write $f_m(y,t) = (\alpha_m(y,t)\tau_m(y,t)^b, \tau_m(y,t))$ for $m = j, j+1$. We write $f_m(y,t) = (\alpha_m(y,t)\tau_m(y,t)^b, \tau_m(y,t))$ for $m = j, j+1$. Since f_m is l.v.a., the development of $\tau_m(y,t)$ as a fractional power series around 0 has to have a term of degree one for $m = j, j+1$. Let $o_{>b'}(\tau_m(a_j,t))$ be the sum of all terms in $\tau_m(a_j,t)$ of degree greater than b'. We can assume that $\tau_m(a_j,t)$ does not have terms of degree greater than b'. From a computation using that $\{\|x\| : x \in L\}$ is bounded and that for $\tau_1, \tau_2 > 0$ small enough it is $|\tau_1^b - \tau_2^b| \leq |\tau_1 - \tau_2|$, we get that the function

$$(y,t) \mapsto (\alpha_m(y,t)(\tau_m(y,t) - o_{>b'}(\tau_m(a_j,t)))^b, \tau_m(y,t) - o_{>b'}(\tau_m(a_j,t)))$$

is b'-equivalent to f_m for $m = j, j+1$. Therefore f_m can be replaced by that function.

We know that the points $p_j(t) := f_j(a_j, t)$ and $p_{j+1}(t) := f_{j+1}(a_j, t)$ are b'-equivalent. So both,

$$\lim_{t\to 0} \frac{|\tau_j(a_j,t) - \tau_{j+1}(a_j,t))|}{t^{b'}} = 0 \tag{3.4}$$

and

$$\lim_{t\to 0} \frac{d_{\text{out}}(\alpha_j(a_j,t)\tau_j(a_j,t)^b, \alpha_{j+1}(a_j,t)\tau_{j+1}(a_j,t)^b)}{t^{b'}} = 0 \tag{3.5}$$

and tend to zero. From equation (3.4) we can deduce that

$$\tau_j(a_j,t) = \tau_{j+1}(a_j,t), \tag{3.6}$$

since $\tau_j(a_j,t)$ and $\tau_{j+1}(a_j,t)$ do not have terms of degree greater than b'. Therefore from equation (3.5) we can deduce that $\alpha_j(a_j,t)$ and $\alpha_{j+1}(a_j,t)$ tend to the same point in L, using that L is compact. Therefore, by the Monotonicity Theorem, there is a $t_0 > 0$ such that

$$d(\alpha_j(a_j,t_1), \alpha_{j+1}(a_j,t_1)) \leq d(\alpha_j(a_j,t_2), \alpha_{j+1}(a_j,t_2)) \tag{3.7}$$

for all $t_1, t_2 \in [0, t_0]$ with $t_1 \leq t_2$. For $m = j, j+1$, we define the subanalytic l.v.a. maps $g_m : C(I) \to C_L^b$ by the formula

$$g_m(y,t) := (\alpha_m(a_j, t - yt)\tau_m(a_j,t)^b, \tau_m(a_j,t)).$$

We have that g_j and g_{j+1} coincide as weak b'-maps. That follows from equation (3.6) and inequality (3.7) applied to $t_1 = t - yt$ and $t_2 = t$.

By Lemma 3.36, the concatenation $f_1 \cdot ... \cdot f_r$ is weakly b-homotopically equivalent to

$$f_1 \cdot ... \cdot f_j \cdot g_j \cdot \overleftarrow{g_{j+1}} \cdot f_{j+1} \cdot ... \cdot f_r$$

We have connected the moderate discontinuity between f_j and f_{j+1} in a continuous way. Repeating this procedure for all $j \in J \setminus \{r\}$ yields a weak b-map $(C(I), \hat{f})$ without moderate discontinuities. It is in the image of $\eta_{\infty,b}$.

□

Corollary 3.60. *Let L be a compact path-connected subanalytic space with abelian ordinary fundamental group. Let C_L^b denote the b-cone over L equipped with the outer distance. Let p be a point in C_L^b. Then we have*

$$MD\pi_1(C_L^b, p) \cong MDH_1(C_L^b; \mathbb{Z})$$

for every $b \in (0, \infty]$.

Proof. The statement follows from Corollary 3.57 and Proposition 3.59. □

Example 3.61. *Let $\mathbb{S}_1$ be the circle. We are going to show that the b'-MD fundamental group of $C_{\mathbb{S}_1}^b$ is*

1. $\mathbb{Z}$, *if* $b' \in [b, \infty]$,

2. *trivial, if* $b' \in [1, b)$.

Proof. Statement (2) follows from part a) of Proposition 3.59. To show statement (1), we use Corollary 3.60. To compute the first b'-MD homology group of $C^b_{\mathbb{S}_1}$, we use Corollary 2.95 that relates the MD homology with the nerve of a cover. To construct the cover we choose three open segments S_j of $\mathbb{S}_1$, where $j \in \{1, 2, 3\}$, such that each segment overlaps with one of the other two segments at one of its ends and with the third segment at the other of its ends. We ask the overlap of the segments to be small enough so that the intersection of the three segments is empty. Now we define the cover $\{U_j\}_{j \in \{1,2,3\}}$ of $C^b_{\mathbb{S}_1}$ by $U_j := \{(xt^b, t) \in C^b_{\mathbb{S}_1} : x \in S_j\}$. That cover fulfils the hypothesis of Corollary 2.95: each U_j and any intersection of two of the U_j's is b'-contractible and the intersection of the three of them is empty; furthermore for any subset $J \subseteq \{1, 2, 3\}$, the collection $\{U_j\}_{j \in J}$ is a b'-cover of $\cup_{j \in J} U_j$. Therefore the first b'-MD homology of $C^b_{\mathbb{S}_1}$ coincides with the homology of the nerve of the cover $\{U_j\}_{j \in \{1,2,3\}}$ which is $\mathbb{Z}$. □

3.4. Detection of fast loops

In [4] the concept of fast loops plays an important role in the classification of normal complex algebraic surface germs. There a fast loop is defined as follows:

Definition 3.62. *Let* (X, x_0, d) *be a metric germ. For* $\epsilon > 0$*, let* $L_{X,\epsilon}$ *denote the intersection of* X *with the sphere in* $\mathbb{R}^n$ *of radius* ϵ*. Let* $\beta > 1$*. A smooth family of closed curves* $\gamma : [0, 1] \times [0, \epsilon_0) \to X$ *is called* β-fast loop in X*, if*

- $\gamma_\epsilon(t) := \gamma(t, \epsilon)$ *is a loop contained in* $L_{X,\epsilon}$ *for any* ϵ*,*
- γ_ϵ *is not contractible in* $L_{X,\epsilon}$*,*
- $\lim_{\epsilon \to 0} \frac{length(\gamma_\epsilon)}{\epsilon^\beta} = 0$*, where* $length(\gamma_\epsilon)$ *is defined as follows:*

$$length(\gamma_\epsilon) := \sup\{\sum_{j=1}^{n} d(\gamma_\epsilon(t_{j-1}), \gamma_\epsilon(t_j)) : n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1\}$$

When it is not important what the specific paremeter β *looks like, we simply call* γ *a* fast loop in X*.*

The concept was first introduced in [7], where it was shown, that at least under certain conditions, the existence of a fast loop in a metric germ X is an obstruction for X to be metrically conical. The existence of fast loops has the following impact on the MD fundamental group:

Proposition 3.63. *Let* (X, x_0, d) *be a metric subanalytic germ with path-connected link. Let* p *be a point in* X*. Let there be a subanalytic fast loop* $\gamma_\epsilon(s)$ *in* X*. Then the morphism*

$$\eta_{\infty,b'} : MD\pi_1^\infty(X, p) \to MD\pi_1^b(X, p)$$

is not injective for any $b \in (0, \beta)$.

Proof. By Proposition 3.46 we can assume that $p(t) := \gamma_t(0)$. We define the ∞-MD 1-loop $\varphi = (C(I), f)$ by the formula $f(y,t) := \gamma_t(y)$. By definition of fast loop and Proposition 3.56, we know that φ is a non-trivial element of $MD\pi_1^\infty(X,p)$. Let $b \in [1, \beta)$. Then φ is trivial as an element of $MD\pi_1^b(X,p)$. □

We conjecture the following extension of Proposition 3.59:

Conjecture 3.64. *Let* $L \subset \mathbb{R}^m$ *be a compact path-connected subanalytic set. Let* $b \in \mathbb{Q} \cap [1, \infty)$ *and let* $(C_L^b, \underline{0}, d)$ *be the b-cone over L as in Definition 2.4, where d is the inner or outer metric. Let p be a point in* C_L^b *and let* $t_0 > 0$ *be small. If* $b' \geq b$, *then the morphism*

$$\eta_{\infty,b'} : MD\pi_1^\infty(C_L^b, p) \to MD\pi_1^{b'}(C_L^b, p)$$

is an isomorphism. In particular $MD\pi_1^{b'}(C_L^b, p)$ *is isomorphic to* $\pi_1(L, p(t_0))$, *where* π_1 *denotes the ordinary fundamental group.*

If Conjecture 3.64 is true, then it implies that the MD fundamental group detects the existence of subanalytic fast loops as an obstruction to metrical conicalness:

Conjecture 3.65. *Let* (X, x_0, d) *be a metric subanalytic germ with path-connected link* L_X. *The MD fundamental group of* (X,p), *where p is any point in X, captures the existence of subanalytic fast loops in the following sense. Let* $C_{L_X}^1$ *denote the straight cone over* L_X. *Let* p_L *be any point in* $C_{L_X}^1$. *If there is a subanalytic fast loop in X, then the MD fundamental group of* (X,p) *is different to the one of* $(C_{L_X}^1, p_L)$.

Proof. The statement follows immediately from Proposition 3.63 and Conjecture 3.64. □

3.5. Alternative setting

We could have defined the MD homotopy theory without requiring the f_j's in a weak b-map $\{(C_j, f_j)\}_{j \in J}$ to be subanalytic. That would define a bi-Lipschitz invariant as opposed to the subanalytic bi-Lipschitz invariant we have defined. The main reasons why we have defined the MD homotopy theory in the subanalytic setting is to guarantee that we have a Hurewicz morphism as in topology.

Appendices

Subanalytic geometry and O-minimal structures A

This appendix collects a few facts on subanalytic geometry that I have learned from Edson Sampaio.

Definition A.1. *A subset $X \in \mathbb{R}^m$ is called* semianalytic *at $x \in \mathbb{R}^m$ if there exists an open neighborhood U of x in $\mathbb{R}^m$ such that $U \cap X$ can be written as a finite union of sets of the form $\{x \in \mathbb{R}^m \mid p(x) = 0, q_1(x) > 0, \ldots, q_k(x) > 0\}$, where $p, q_1, \ldots, q_k$ are analytic functions on U. A subset $X \subset \mathbb{R}^m$ is called* semianalytic *if X is semianalytic at each point $x \in \mathbb{R}^m$.*

Definition A.2. *A subset $X \subset \mathbb{R}^m$ is called* subanalytic at *$x \in \mathbb{R}^m$ if there exists an open neighborhood U of x in $\mathbb{R}^m$ and a relatively compact semianalytic subset $S \subset \mathbb{R}^m \times \mathbb{R}^k$, for some m, such that $U \cap X = \pi(S)$ where $\pi : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ is the orthogonal projection map. A subset $X \subset \mathbb{R}^m$ is called* subanalytic *if X is subanalytic at each point of $\mathbb{R}^m$.*

Definition A.3. *Let $X \subset \mathbb{R}^m$ be a subanalytic set. A map $f : X \to \mathbb{R}^k$ is called a* subanalytic map *if its graph is subanalytic.*

Definition A.4. *A subset $X \subset \mathbb{R}^m$ is called* globally subanalytic *if its image under the map*

$$(x_1, \ldots, x_m) \mapsto \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \ldots, \frac{x_m}{\sqrt{1 + x_m^2}}\right)$$

from $\mathbb{R}^m$ to $\mathbb{R}^m$ is subanalytic.

Remark A.5. *In [12], a globally subanalytic set is called a finitely subanalytic set.*

Remark A.6.

1. *Any globally subanalytic set is a subanalytic set;*
2. *Any bounded subanalytic set is a globally subanalytic set;*
3. *The collection of all globally subanalytic sets forms an O-minimal structure (see Theorem in [12]).*

For completeness of statement 3 of the previous remark, we remind the definition of O-minimal structures:

Definition A.7 (O-minimal structure)**.** *An* O-minimal structure over the reals *is a collection $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$, that satisfies the following axioms:*

1. *For any $n \in \mathbb{N}$, the collection $\mathcal{S}_n$ contains all algebraic subsets of $\mathbb{R}^n$.*

2. *For any $n \in \mathbb{N}$, the collection $\mathcal{S}_n$ is a Boolean subalgebra of the powerset of $\mathbb{R}^n$, i.e.*

   - $\mathbb{R}^n \in \mathcal{S}_n$;
   - $\emptyset \in \mathcal{S}_n$;
   - *for any $A, B \in \mathcal{S}_n$ we have $A \cap B \in \mathcal{S}_n$;*
   - *for any $A, B \in \mathcal{S}_n$ we have $A \cup B \in \mathcal{S}_n$;*
   - *for any $A \in \mathcal{S}$, its complement A^C is in $\mathcal{S}_n$.*

3. *For any $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, we have $A \times B \in \mathcal{S}_{m+n}$.*

4. *If $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection forgetting the last coordainate and $A \in \mathcal{S}_{n+1}$, then we have $p(A) \in \mathcal{S}_n$.*

5. *The elements of $\mathcal{S}_1$ are precisely all finite unions of points and intervals.*

Any set $A \in \mathcal{S}_n$ for $n \in \mathbb{N}$ is called definable. *For any two definable sets A and B of a given O-minimal structure, a map $f : A \to B$ is called* definable*, if its graph is definable.*

When working with subanalytic germs, one can take into account only bounded representatives. By statement 2 of Remark A.6, those representatives are globally subanalytic. That is the setting we have chosen to work in in this thesis. The theorems and facts we use in the thesis in this context are the following:

- Let A and B be bounded subanalytic sets. Let $f : A \to B$ be a subanalytic map. Then it follows directoy from the axioms of O-minimal structures and Definition A.3 that the images and preimages of globally subanalytic sets under f are globally subanalytic (see Remark 2 of [25]). We use the statement about preimages for example in Proposition 2.55.

- The Monotonicity Theorem (see Theorem 2.1 in [9]), which states the following: if $f : (a, b) \to C$ is a subanalytic function with $C \subset \mathbb{R}$ bounded, then there is a finite subdivision $a = a_0 < a_1, \cdots < a_k = b$ such that, on each interval (a_{a_i}, a_{i+1}), the function f is continuous and either constant or strictly monotone. We use it for example in Proposition 3.59.

- The Conical Structure Theorem for sets of an O-minimal structure. We use it often, for example in Remark 2.8. We have stated it for subanalytic germs in Remark 2.6:

  Remark 2.6. *We recall that the link of a subanalytic germ is well defined as a topological space as the intersection of X with a small enough sphere centered at x_0; we denote it by $Link(X, x_0)$ or simply L_X. Moreover, the conical structure theorem says, given a subanalytic germ (X, x_0) and a family of subanalytic subgerms $(Z_1, 0)$,...,$(Z_k, 0) \subseteq (X, 0)$, that there exists a subanalytic homeomorphism*

$h : C(L_X) \to (X, x_0)$ such that $||x_0 - h(tx, t)|| = t$ and such that $h(C(L_{Z_i})) = Z_i$ with L_{Z_i} in L_X (see Theorem 4.10, 5.22, 5.23 in [9]). We say that the conical structure h is compatible with the family $\{Z_i\}$. The conical structure is why we say that x_0 is the vertex of (X, x_0).

- The Curve Selection Lemma (see Theorem 3.2 of [9]) for sets of an O-minimal structure which states the following. Let $A \subseteq \mathbb{R}^n$ be a defineble set and $b \in \bar{A}$. Then there is a continuous definable map $\gamma : [0, 1) \to \mathbb{R}^n$ such that $\gamma(0) = b$ and $\gamma((0, 1)) \subseteq A$. We use it for example in Lemma 2.27.

- Triangulability as explained in Remark 2.15:

  Remark 2.15. *Given a finite family $\mathcal{S}$ of closed subanalytic subsets of Z, there exists a subanalytic triangulation $\alpha : |K| \to Z$ compatible with $\mathcal{S}$, that is, such that every subset of $\mathcal{S}$ is a union of images of simplices of $|K|$. See for example Theorem 4.4. in [9] or Theorem II.2.1. in [37].*

  We use it for example in Proposition 2.55.

- The subanalytic Hauptvermutung (see Chapter II, Theorem II in [37])), which states that for any two subanalytic triangulations of a subanalytic space there is a subanalytic triangulation refining them. It is used for example in Lemma 2.21.

Since all the mentioned statemenets are true for O-minimal structures over the reals in general, we observe the following:

Remark A.8. *One could define MD homology and MD homotopy in the context of any O-minimal structure over the reals copying this thesis word by word and replacing subanalytic (or globally subanalytic) by the definable sets and definable maps of that O-minimal structure.*

Remark A.9. *To give an example of Remark A.8: one could take $\mathbb{R}_{exp}$, which is the smallest O-minimal structure over the reals generated by the exponential function.*

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Summary in English (formality)

This thesis is called *Moderately Discontinuous Algebraic Topology for Metric Subanalytic Germs.*

Introduction

We have developed both a homology theory and a homotopy theory in the context of metric subanalytic germs (see Definition 2.1). The former is called *MD homology* and is covered in Chapter 2, which contains a paper that is joined work with my PhD advisors Javier Fernández de Bobadilla and María Pe Pereira and with Edson Sampaio. The latter is called *MD homotopy* and is covered in Chapter 3. Both theories are functors from a category of germs of metric subanalytic spaces (resp. germs of metric subanalytic spaces that are punctured in a way that will be defined) to a category of commutative diagrams of groups. For the concrete definition of the domain categories see Definition 2.10 and Definition 3.47 respectively; for the target categories see Definition 2.42 and Definition 3.52 respectively. Similarly to classical homology and homotopy theories, the groups appearing in the target category are abelian in the homology theory for any degree and in the homotopy theory for degree $n > 1$.

Objectives and results

The main objective was to construct an analytic invariant of real or complex analytic germs that would also contain information about the bi-Lipschitz geometry of the germ. We also had the objective to provide computational tools for that invariant. An optional objective was to concretely compute the invariant for some real or complex analytic germs.

The realization of those objectives is given by the MD homology and the MD homotopy as described above. The MD homology shares several of the properties with the singular homology: it is invariant by suitable metric homotopies (see Definition 2.75 and Theorem 2.76 as well as Definition 2.79 and Theorem 2.80); it allows a relative and absolute Mayer-Vietoris long exact sequences (see Theorem 2.91) for a suitable cover of the metric subanalytic germ (see Definition 2.88); and as a consequence we have a certain theorem of excision (see Corollary 2.92) and a Čech spectral sequence (see Theorem 2.93).

The MD homotopy shares several of the properties of the ordinary homotopy theory of punctured topological spaces: it admits a Hurewicz homomorphism from the MD homotopy to the MD homology (see Proposition 3.40); in degree one, the Hurewicz homomorphism is an isomorphism when abelianizing the domain (see Theorem 3.55); and when the metric subanalytic germ fulfils a certain condition that softens the one of path-connectedness (see Definifion 3.43), it is independent from the choice of base point (see Proposition 3.46).

The fact that the MD homology provides those computational tools mentioned above similarly to the tools in singular homology make it relatively well computable. We have given examples of computations of both the MD homology and the MD homotopy. In particular, we have given a concrete formula for the MD homology of complex plane algebraic curve germs equipped with the outer metric (see Proposition 2.105). That formula reveals how the MD homology recovers both, all Puiseux pairs of the branches of the curve, and the set of all contact numbers between two branches (see Corollary 2.108). In [38] (see also [29] and [15]), it is shown that the geometric type of a complex plane algebraic curve germ equipped with the outer metric coincides with its embedded topological type. Therefore, the MD homology is a complete invariant of irreducible complex plane algebraic curve germs equipped with the outer metric.

Conclusions

Both the MD homology and the MD homotopy fulfil the main objective of constructing an analytic invariant of real or complex analytic germs. Indeed, both theories serve as a bi-Lipschitz subanalytic invariant. Therefore, in the context of real or complex analytic germs equipped with the inner or the outer metric, they are analytic invariants. Both theories also provide several powerful computational tools as mentioned above and therefore meet the second objective. In the context of the MD homology we have also attained the optional objective of computing the invariant for an important group of complex analytic germs: for all complex plane algebraic curve germs.

Both theories seem very promising since they are rich invariants, as can be seen in the context of complex plane algebraic curve germs, and also well computable thanks to their various computational tools. Furthermore they provide a new and innovative approach to studying algebraic germs. Therefore we have the hope that the work done in this thesis might lay the ground for a new series of research in that direction.

Resumen en español (formalidad)

Esta tesis se llama *Topología Algebraica Moderadamente Discontinua para Gérmenes Métricos Subanalíticos.*

Introducción

Hemos desarrollado tanto una teoría de homología como una teoría de homotopía en el contexto de gérmenes subanalíticos métricos (véase Definition 2.1). La teoría de homología se llama *MD homología.* La desarrollamos en el capítulo 2, que contiene un artículo que es trabajo conjunto con mis directores de tesis Javier Fernández de Bobadilla y María Pe Pereira y con Edson Sampaio. La teoría de homotopía se llama *MD homotopía* y la desarrollamos en el capítulo 3. Ambas teorías son funtores de una categoría de gérmenes de espacios métricos subanalíticos (resp. gérmenes de espacios métricos subanalíticos puntuados de una manera que definamos) a una categoría de diagramas comutativos de grupos. Para la definición concreta de la categoría del dominio véase Definition 2.10 y Definition 3.47 respectivamente; para la definición de la categoría de llegada véase Definition 2.42 y Definition 3.52 respectivamente. Como pasa también en el contexto de las teorías de homología y homotopía clásicas, los grupos que aparecen en la categoría de llegada son abelianos en la teoría de homología de cualquier grado y en la teoría de homotopía para grado $n > 1$.

Objetivos y resultados

El objetivo principal era construir un invariante analítico de gérmenes reales o complejos que también contuviera información sobre la geometría bilipschitz del gérmen. Otro objetivo era dar herramientas computacionales para este invariante. Como objetivo opcional teníamos el cálculo concreto del invariante para algunos gérmenes analíticos reales o complejos.

La realización de estos objetivos viene dada por el desarrollo de la MD homología y la MD homotopía que hemos descrito arriba. La MD homología comparte varias propiedades con la homología singular: es invariante bajo homotopías métricas adecuadas (véase Definition 2.75 y Theorem 2.76 y también Definition 2.79 y Theorem 2.80); permite una sucesión exacta larga de Mayer-Vietoris tanto absoluta como relativa; y como consecuencia tenemos cierto teorema de excisión (véase Corollary 2.92) y una sucesión espectral de Čech (véase Theorem 2.93).

La MD homotopía comparte varias propiedades con la homotopía habitual de espacios topológicos puntuados: admite un homomorfismo de Hurewicz de la MD homotopía a la MD homología (véase Proposition 3.40); en grado uno, el homomorfismo de Hurewicz es un isomorfismo después de abelianizar el dominio (véase Theorem 3.55); y cuando el gérmen métrico subanalítico cumple cierta condición que suaviza el concepto de conexo por caminos, es independiente del punto base (véase Proposition 3.46).

Las herramientas de la MD homología descritas arriba parecidas a las herramientas en homología singular facilitan mucho el cálculo concreto de la MD homología. Hemos dado ejemplos tanto de cálculos de la MD homología como de cálculos de la MD homotopía. En particular hemos dado una formula concreta de la MD homología de gérmenes de curvas complejas algebraicas planas equipadas con la métrica externa (véase Proposition 2.105). Esta fórmula demuestra que la MD homología recupera tanto todos los pares de Puiseux de todas las ramas de la curva como el conjunto de todos los números de contacto entre dos ramas (véase Corollary 2.108). En [38] (véase también [29] and [15]), está demostrado que el tipo geométrico de gérmenes de curvas complejas algebraicas planas equipadas con la métrica externa coincide con su tipo topológico. Por lo tanto la MD homología es un invariante completo de gérmenes de curvas complejas algebraicas planas irreducibles equipadas con la métrica externa.

Conclusiones

Tanto la MD homología como la MD homotopía cumplen el objetivo principal de construir un invariante analítico de gérmenes analíticos reales o complejos. De hecho ambas teorías son invariantes bilipschitz subanalíticos. Como consecuencia son invariantes analíticos en el contexto de gérmenes reales o complejos analíticos equipados con la métrica interna o externa. Ambas teorías además ofrecen varias herramientas computacionales potentes y por lo tanto cumplen el segundo objetivo. En el contexto de la MD homotopía también logramos el objetivo opcional de calcular el invariante para un grupo importante de gérmenes analíticos complejos: para todos los gérmenes de curvas complejas algebraicas planas irreducibles equipadas con la métrica externa.

Ambas teorías parecen muy prometedoras ya que son invariantes bastante ricos como se puede ver en el contexto de gérmenes de curvas complejas algebraicas planas irreducibles equipadas con la métrica externa y porque además se dejan calcular relativamente bien gracias a sus herramientas computacionales. Además esta aproximación al estudio de gérmenes analíticos es totalmente nueva. Por lo tanto tenemos la esperanza de que el trabajo hecho en esta tesis haya sido el inicio de una seria de investigación en esta dirección.

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