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10 March 2017

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Abstract

Aparicio et al. (2017) recently extended the Camanho and Dyson (2006) Malmquist-type index (CDMI) for determining group performance in cross-sectional studies to panel or pseudo-panel databases. In that paper, it was shown that the pseudo-panel Malmquist index (PPMI) can be easily interpreted as the ratio of aggregated productivity changes in two groups of decision making units over time if and only if a new -difficult to interpret- term, the so called 'divergence component' (DC), is equal to one. The aim of this paper is twofold. First, based upon considering a baseline group technology we define a new base-group base-period PPMI where the DC always vanished. Second, when more than two groups are analyzed we show that under this framework the new base-group base-period PPMI, the new base-group CDMI and the components of both indexes satisfy the circular relation. As a consequence, the complicated 'adjusted index' defined in Camanho and Dyson (2006) for measuring the technology gap to satisfy the circular relation also vanishes. Both results will make it easier for practitioners applying the two indexes in different economic sectors regardless of how many groups are being compared.

Keywords: Data Envelopment Analysis, group performance, Malmquist index.

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1. Introduction

In many sectors related to services supply, it is worth evaluating alternative approaches for carrying out the production process. The different groups of decision making units (DMUs) basically transform the same set of inputs into the same set of outputs. However, the internal organizational, activities and managerial techniques used to deliver the service may differ. Evaluating different production alternatives is especially important, for example, in the public sector, where it often has to be decided whether public services should be funded by public or private sources. This is a typical research issue in the educational sector (for a review see De Witte and López-Torres, 2015), where previous literature focuses on comparing the performance of different groups of schools (e.g. by school ownership, by region, by country).

To carry out this comparison among groups, Camanho and Dyson (2006) introduced a Malmquist-type index (hereinafter CDMI) to provide an average indicator of the relative performance of two or more groups of DMUs within a period of time. The main advantage of this approach compared with others (Charnes et al., 1981), is that it does not assume convex combinations of group specific frontiers to be feasible. This appealing property has allowed to the CDMI to be used successfully in empirical applications, to provide performance comparisons of groups of units in cross-sectional studies (see, for example, Vaz and Camanho, 2012, Ferreira and Marques, 2014 and Thanassoulis et al. 2015).

The CDMI has been recently extended by Aparicio et al. (2017) in the context where a panel data or a pseudo-panel data is available and the objective is to determine how the measured performance gap changes over time. A panel implies that the same group of DMUs is observed across all analyzed periods. However, in the educational sector, it is common to extract random waves of representative samples of schools at regional or country level, in order to analyze their performance over time (PISA, TIMSS and PIRLS are some international education databases belonging to this category). The production units contained in each wave vary from one period to another and are usually anonymous for researchers. This organization of the information is referred to as a pseudo-panel database and, in this case, the approach introduced in Aparicio et al. (2017) can be used to compare the performance of these representative groups of DMUs over time using a pseudo panel Malmquist-type index (PPMI).

However, there are two aspects that must be necessarily improved in the approach introduced by Aparicio et al. (2017) for endowing the PPMI approach with greater usefulness among practitioners. First, the PPMI is the ratio of two CDMIs and measures the relative performance gap change between two groups of DMUs within periods t and t+1. In this sense, the PPMI consists of the mix of two Malmquist-type indexes. Accordingly, one should expect that the PPMI could be interpreted in terms of the ratio of aggregated productivity changes in the two groups between periods t and t+1. However, this is only true under some conditions. In particular, a factor baptized as a divergence component (DC), which is difficult to interpret, needs to be equal to one (Proposition 1 in

Aparicio et al. 2017), something that does not always happen. Indeed, in the empirical application used by Aparicio et al. (2017) for showing how PPMI works, the two DCs were different from one (specifically, 0.9138 and 0.9304).

Second, the PPMI index inherits the non-circularity of the original CDMI. In our context, circularity (Frisch, 1936) implies that, in frameworks where it is relevant to compare the performance of more than two groups, the direct comparison between two groups is equivalent to their indirect comparison through a third group, whatever the third group selected for the assessment is. Unfortunately, despite the easy and interesting interpretation of the CDMI, this index does not satisfy circularity. This was the reason why Camanho and Dyson (2006) suggested an 'adjusted index' of their original index, following the EKS method (see Caves et al., 1982). In order to meet this remarkable property, Camanho and Dyson (2006) incorporate information on all the units in all the groups for determining the comparison between two specific groups and, consequently, making the interpretation of the 'circular' final index and its possible decomposition (not shown in Camanho and Dyson, 2006) more difficult.

In summary, given the difficulty of interpreting the DC in the PPMI approach and seekingto correct the non-circularity of the index in an easy way at the same time, in this paper, we define a new PPMI which fixes a baseline group. This allows both the PPMI index and its components to be adequately interpreted, thus making it easier for practitioners to apply. Moreover, the new index and its decomposition meet circularity. Finally, assuming a baseline group as reference technology allows the CDMI index to satisfy the circular relation property directly.

2. Background

First of all, let us introduce some notation. Let us assume that we have observed n^{A^s} DMUs in group A in period s, $s = t, t+1$, which produce output $y^{A^s} \in R_+^q$ from input $x^{A^s} \in R_+^k$ and that we have also observed n^{B^s} DMUs in group B in period s, $s = t, t+1$, which produce output $y^{B^s} \in R_+^q$ from input $x^{B^s} \in R_+^k$. The DMUs operating in group A in period s are represented by their input-output vector as $\left(x_j^{A^s}, y_j^{A^s}\right)$, $j = 1, ..., n^{A^s}$. In the same way, $\left(x_j^{B^s}, y_j^{B^s}\right)$ denotes the input-output vector of DMU j, $j = 1, ..., n^{B^s}$, belonging to group B in period s. $D^{A^s}\left(x_j^{B^h}, y_j^{B^h}\right) = \inf\left\{\theta : \left(\left(x_j^{B^h}, y_j^{B^h} / \theta\right)\right) \in T^{A^s}\right\}$ represents the Shephard output distance function from observation $\left(x_j^{B^h}, y_j^{B^h}\right)$ in group B in period h, $h = t, t+1$, to the frontier of the technology of group A in period s, $s = t, t+1$, T^{A^s}. A similar notation is used for the distance for a unit in A with respect to the technology of group B, and for the distance from a unit that belongs to the same group as the technology of reference.

The CDMI is an adaptation of the Malmquist index to provide a cross-sectional comparison of the performance of DMUs operating under different conditions rather than a measurement of the productivity change between two periods. The CDMI for comparing the performance of two groups of DMUs, A and B, associated with different programs, ownerships or practices in one time period s is defined as follows:

$$\mathrm{CDMI}_s^{\mathrm{AB}} = \left[\frac{\left(\prod_{j=1}^{n^{A^s}} D^{A^s}(x_j^{A^s}, y_j^{A^s}) \right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{A^s}(x_i^{B^s}, y_i^{B^s}) \right)^{1/n^{B^s}}} \cdot \frac{\left(\prod_{i=j}^{n^{A^s}} D^{B^s}(x_j^{A^s}, y_j^{A^s}) \right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{B^s}(x_i^{B^s}, y_i^{B^s}) \right)^{1/n^{B^s}}} \right]^{1/2} . \quad (1)$$

Additionally, the relative performance gap measured in (1) may be decomposed into the following terms:

$$\mathrm{CDMI}_s^{\mathrm{AB}} = \underbrace{\frac{\left(\prod_{j=1}^{n^{A^s}} D^{A^s}(x_j^{A^s}, y_j^{A^s}) \right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{B^s}(x_i^{B^s}, y_i^{B^s}) \right)^{1/n^{B^s}}}}_{EG_s^{AB}} \cdot \underbrace{\left[\frac{\left(\prod_{i=j}^{n^{A^s}} D^{B^s}(x_j^{A^s}, y_j^{A^s}) \right)^{1/n^{A^s}}}{\left(\prod_{j=1}^{n^{A^s}} D^{A^s}(x_j^{A^s}, y_j^{A^s}) \right)^{1/n^{A^s}}} \cdot \frac{\left(\prod_{i=1}^{n^{B^s}} D^{B^s}(x_i^{B^s}, y_i^{B^s}) \right)^{1/n^{B^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{A^s}(x_i^{B^s}, y_i^{B^s}) \right)^{1/n^{B^s}}} \right]^{1/2}}_{TG_s^{AB}} . \quad (2)$$

The ratio EG_s^{AB} compares within-group efficiency spreads, measuring the technical efficiency gap between both groups, while the ratio TG_s^{AB} evaluates the productivity gap between the frontiers of the two analyzed groups, A and B.

With the aim of analyzing the evolution of the performance gap between groups A and B over two time periods, t and $t+1$, Aparicio et al. (2017) proposed making the ratio of two CDMIs calculated for the two considered periods. The application of this strategy allows the evolution of the different groups of DMUs over time to be checked, thus providing policy makers with valuable information. The relative performance gap change between A and B within t and $t+1$ is defined as the pseudo-panel Malmquist index (PPMI) as follows:

$$PPMI_{t,t+1}^{AB} = \frac{CDMI_{t+1}^{AB}}{CDMI_t^{AB}} . \quad (3)$$

The PPMI can also be decomposed into efficiency gap change (EGC) and technological gap change (TGC), as Equation 4 shows:

$$PPMI_{t,t+1}^{AB} = \frac{CDMI_{t+1}^{AB}}{CDMI_t^{AB}} = \frac{EG_{t+1}^{AB}}{EG_t^{AB}} \cdot \frac{TG_{t+1}^{AB}}{TG_t^{AB}} = EGC_{t,t+1}^{AB} \cdot TGC_{t,t+1}^{AB} \cdot . \quad (4)$$

The CDMI, despite being a Malmquist-type index, does not measure productivity change but relative performance between two groups of units. As a consequence, the 'derived' PPMI does not generally have a direct relationship with productivity change over time of units in groups A and B. However, Aparicio et al. (2017) identified under which conditions the PPMI is related to the ratio between an aggregated measure of productivity changes, from t to $t+1$, of the units corresponding to group A and an aggregated measure of productivity changes, from t to $t+1$, of the units belonging to group B, as follows.

Proposition 1 (Aparicio et al., 2017).

$$PPMI_{t,t+1}^{AB} = \frac{\left[\frac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{A^{t+1}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t}}} D^{A^{t+1}}(x_j^{A^{t}}, y_j^{A^{t}})\right)^{1/n^{A^{t}}}} \cdot \frac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{A^{t}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t}}} D^{A^{t}}(x_j^{A^{t}}, y_j^{A^{t}})\right)^{1/n^{A^{t}}}} \right]^{1/2}}{\left[\frac{\left(\prod_{j=1}^{n^{B^{t+1}}} D^{B^{t+1}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{j=1}^{n^{B^{t}}} D^{B^{t+1}}(x_j^{B^{t}}, y_j^{B^{t}})\right)^{1/n^{B^{t}}}} \cdot \frac{\left(\prod_{j=1}^{n^{B^{t+1}}} D^{B^{t}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{j=1}^{n^{B^{t}}} D^{B^{t}}(x_j^{B^{t}}, y_j^{B^{t}})\right)^{1/n^{B^{t}}}} \right]^{1/2}} \tag{5}$$

if and only if

$$DC_{t,t+1}^{AB} = \frac{\left[\frac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{B^{t+1}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{B^{t}}} D^{B^{t+1}}(x_j^{B^{t}}, y_j^{B^{t}})\right)^{1/n^{B^{t}}}} \cdot \frac{\left(\prod_{j=1}^{n^{B^{t+1}}} D^{B^{t}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t}}} D^{B^{t}}(x_j^{A^{t}}, y_j^{A^{t}})\right)^{1/n^{A^{t}}}} \right]^{1/2}}{\left[\frac{\left(\prod_{j=1}^{n^{B^{t+1}}} D^{A^{t+1}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t}}} D^{A^{t+1}}(x_j^{A^{t}}, y_j^{A^{t}})\right)^{1/n^{A^{t}}}} \cdot \frac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{A^{t}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{B^{t}}} D^{A^{t}}(x_j^{B^{t}}, y_j^{B^{t}})\right)^{1/n^{B^{t}}}} \right]^{1/2}} = 1. \tag{6}$$

In the above proposition, the term $DC_{t,t+1}^{AB}$ is introduced, baptized as the 'Divergence Component' of the PPMI. It indicates how far the PPMI is from the right-hand side of (5). However, the interpretation of $DC_{t,t+1}^{AB}$ in (6) is cumbersome.

Regarding the satisfaction of circularity, it is well-known that the CDMI does not meet this property for more than two groups (Camanho and Dyson, 2006). Let us assume that we have observed units belonging to three groups instead of two: groups A, B and C. Then, for a fixed period of time s, in general, $\mathrm{CDMI}_s^{\mathrm{AC}} \neq \mathrm{CDMI}_s^{\mathrm{AB}} \cdot \mathrm{CDMI}_s^{\mathrm{BC}}$. This weakness is also inherited for the PPMI, what ("which") means that $PPMI_{t,t+1}^{AC} \neq PPMI_{t,t+1}^{AB} \cdot PPMI_{t,t+1}^{BC}$.

3. New results

Most national statistical offices make use of a fixed-base index, mainly because reference weights will remain the same for all the fixed-base index computations. This type of indexes has a number of interesting properties. In particular, and with respect to productivity measurement, Berg et al. (1992) showed that the base-period Malmquist index fulfills the circular relation, but the adjacent period index does not. In this section, we define a new base-group CDMI and a new base-group base-period PPMI, showing also that the DC vanishes while circularity holds.

Let us assume that we have observed the DMUs of a so-called reference group R. Then, the base-group CDMI for comparing the performance of A and B in one time period s is defined as follows:

$$\mathrm{CDMI}_s^{\mathrm{AB}}\left(R^s\right) = \frac{\left(\prod_{j=1}^{n^{A^s}} D^{R^s}(x_j^{A^s}, y_j^{A^s})\right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{R^s}(x_i^{B^s}, y_i^{B^s})\right)^{1/n^{B^s}}}. \tag{7}$$

Notice that in (7) it is not necessary to resort to the geometric mean as in (1) to aggregate the set of distances, since the technology of the group R is adopted as reference technology regardless of the evaluated units. Moreover, (7) can be decomposed into two subcomponents:

$$\mathrm{CDMI}_s^{\mathrm{AB}}\left(R^s\right) = \underbrace{\frac{\left(\prod_{j=1}^{n^{A^s}} D^{A^s}(x_j^{A^s}, y_j^{A^s})\right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{B^s}(x_i^{B^s}, y_i^{B^s})\right)^{1/n^{B^s}}}}_{EG_s^{AB}} \cdot \underbrace{\left[\frac{\left(\prod_{i=j}^{n^{A^s}} D^{R^s}(x_j^{A^s}, y_j^{A^s})\right)^{1/n^{A^s}}}{\left(\prod_{j=1}^{n^{A^s}} D^{A^s}(x_j^{A^s}, y_j^{A^s})\right)^{1/n^{A^s}}} \cdot \frac{\left(\prod_{i=1}^{n^{B^s}} D^{B^s}(x_i^{B^s}, y_i^{B^s})\right)^{1/n^{B^s}}}{\left(\prod_{i=1}^{n^{B^s}} D^{R^s}(x_i^{B^s}, y_i^{B^s})\right)^{1/n^{B^s}}}\right]^{1/2}}_{TG_s^{AB}\left(R^s\right)}. \tag{8}$$

To deal with panels and pseudo panels, we suggest to use the base-group base-period PPMI for measuring the relative performance gap change between A and B within t and $t+1$, defined as:

$$PPMI_{t,t+1}^{AB}\left(R^h\right) = \frac{CDMI_{t+1}^{AB}\left(R^h\right)}{CDMI_t^{AB}\left(R^h\right)}. \tag{9}$$

Notice that in (9) we fix the reference group and the period of time for evaluating both the CDMI in t and in $t+1$. In other words, we do not make the baseline group dependent on the period of time associated with the corresponding CDMI. As we will show later, this will allow us to determine a technical gap change component that is related to the ratio of the aggregated technical change in group A to the aggregated technical change in group B.

Then, substituting (7) in (9), we have that

$$PPMI_{t,t+1}^{AB}\left(R^h\right)=\frac{\dfrac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{R^h}(x_j^{A^{t+1}},y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{i=1}^{n^{B^{t+1}}} D^{R^h}(x_i^{B^{t+1}},y_i^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}}{\dfrac{\left(\prod_{j=1}^{n^{A^{t}}} D^{R^h}(x_j^{A^{t}},y_j^{A^{t}})\right)^{1/n^{A^{t}}}}{\left(\prod_{i=1}^{n^{B^{t}}} D^{R^h}(x_i^{B^{t}},y_i^{B^{t}})\right)^{1/n^{B^{s}}}}}=\frac{\dfrac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{R^h}(x_j^{A^{t+1}},y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t}}} D^{R^h}(x_j^{A^{t}},y_j^{A^{t}})\right)^{1/n^{A^{t}}}}}{\dfrac{\left(\prod_{i=1}^{n^{B^{t+1}}} D^{R^h}(x_i^{B^{t+1}},y_i^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{i=1}^{n^{B^{t}}} D^{R^h}(x_i^{B^{t}},y_i^{B^{t}})\right)^{1/n^{B^{s}}}}}. \tag{10}$$

So, the $PPMI_{t,t+1}^{AB}\left(R^h\right)$ coincides with the ratio between an aggregated measure of productivity changes, from t to t+1, of the units corresponding to group A and an aggregated measure of productivity changes, from t to t+1, of the units belonging to B. Additionally, if we are working with a real panel data, $n^{A^t}=n^{A^{t+1}}=n^A$ and $n^{B^t}=n^{B^{t+1}}=n^B$, and the $PPMI_{t,t+1}^{AB}\left(R^h\right)$ can be rewritten as:

$$PPMI_{t,t+1}^{AB}\left(R^h\right)=\frac{\left(\prod_{j=1}^{n^A} M^{R^h}\left(x_j^{A^{t+1}},y_j^{A^{t+1}},x_j^{A^t},y_j^{A^t}\right)\right)^{1/n^A}}{\left(\prod_{i=1}^{n^B} M^{R^h}\left(x_i^{B^{t+1}},y_i^{B^{t+1}},x_i^{B^t},y_i^{B^t}\right)\right)^{1/n^B}}, \tag{11}$$

where $M^{R^h}\left(x_j^{A^{t+1}},y_j^{A^{t+1}},x_j^{A^t},y_j^{A^t}\right)$ denotes the traditional Malmquist index for units of group A when it is used the technology of R in period h as reference. A similar notation is used for group B.

As for the components of $PPMI_{t,t+1}^{AB}\left(R^h\right)$, it can also be decomposed into efficiency gap change (EGC) and technological gap change (TGC), as (12) shows:

$$PPMI_{t,t+1}^{AB}\left(R^h\right)=\frac{CDMI_{t+1}^{AB}\left(R^h\right)}{CDMI_{t}^{AB}\left(R^h\right)}=\frac{EG_{t+1}^{AB}}{EG_{t}^{AB}}\cdot\frac{TG_{t+1}^{AB}\left(R^h\right)}{TG_{t}^{AB}\left(R^h\right)}=EGC_{t,t+1}^{AB}\cdot TGC_{t,t+1}^{AB}\left(R^h\right). \tag{12}$$

Notice that $EGC_{t,t+1}^{AB}$ does not depend on R^h and, additionally, matches with the efficiency gap change in the original decomposition of PPMI (see expression (4)). Moreover, it is easy to see, from (8) and (12), that $EGC_{t,t+1}^{AB}$ coincides with the ratio between an aggregated measure of technical efficiency changes, from t to t+1, of the units in A and an aggregated measure of technical efficiency changes, from t to t+1, of the units belonging to B.

$$EGC_{t,t+1}^{AB} = \frac{\dfrac{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{A^{t+1}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{A^t}} D^{A^t}(x_j^{A^t}, y_j^{A^t})\right)^{1/n^{A^t}}}}{\dfrac{\left(\prod_{j=1}^{n^{B^{t+1}}} D^{B^{t+1}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{j=1}^{n^{B^t}} D^{B^t}(x_j^{B^t}, y_j^{B^t})\right)^{1/n^{B^t}}}}. \tag{13}$$

Therefore, $EGC_{t,t+1}^{AB}$ can always be interpreted as we were seeking.

Regarding $TGC_{t,t+1}^{AB}\left(R^h\right)$, we have that

$$TGC_{t,t+1}^{AB}\left(R^h\right) = \frac{TG_{t+1}^{AB}\left(R^h\right)}{TG_t^{AB}\left(R^h\right)} = \frac{\left[\dfrac{\left(\prod_{i=1}^{n^{A^{t+1}}} D^{R^h}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}}{\left(\prod_{j=1}^{n^{A^{t+1}}} D^{A^{t+1}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})\right)^{1/n^{A^{t+1}}}} \cdot \dfrac{\left(\prod_{i=1}^{n^{B^{t+1}}} D^{B^{t+1}}(x_i^{B^{t+1}}, y_i^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}{\left(\prod_{i=1}^{n^{B^{t+1}}} D^{R^h}(x_i^{B^{t+1}}, y_i^{B^{t+1}})\right)^{1/n^{B^{t+1}}}}\right]^{1/2}}{\left[\dfrac{\left(\prod_{i=1}^{n^{A^t}} D^{R^h}(x_j^{A^t}, y_j^{A^{t+1}})\right)^{1/n^{A^t}}}{\left(\prod_{j=1}^{n^{A^t}} D^{A^t}(x_j^{A^t}, y_j^{A^t})\right)^{1/n^{A^t}}} \cdot \dfrac{\left(\prod_{i=1}^{n^{B^t}} D^{B^t}(x_i^{B^t}, y_i^{B^t})\right)^{1/n^{B^t}}}{\left(\prod_{i=1}^{n^{B^t}} D^{R^h}(x_i^{B^t}, y_i^{B^t})\right)^{1/n^{B^t}}}\right]^{1/2}} =$$

$$= \frac{\left[\left(\prod_{j=1}^{n^{A^{t+1}}} \dfrac{D^{R^h}(x_j^{A^{t+1}}, y_j^{A^{t+1}})}{D^{A^{t+1}}(x_j^{A^{t+1}}, y_j^{A^{t+1}})}\right)^{1/n^{A^{t+1}}} \cdot \left(\prod_{j=1}^{n^{A^t}} \dfrac{D^{A^t}(x_j^{A^t}, y_j^{A^t})}{D^{R^h}(x_j^{A^t}, y_j^{A^t})}\right)^{1/n^{A^t}}\right]^{1/2}}{\left[\left(\prod_{j=1}^{n^{B^{t+1}}} \dfrac{D^{R^h}(x_j^{B^{t+1}}, y_j^{B^{t+1}})}{D^{B^{t+1}}(x_j^{B^{t+1}}, y_j^{B^{t+1}})}\right)^{1/n^{B^{t+1}}} \cdot \left(\prod_{j=1}^{n^{B^t}} \dfrac{D^{B^t}(x_j^{B^t}, y_j^{B^t})}{D^{R^h}(x_j^{B^t}, y_j^{B^t})}\right)^{1/n^{B^t}}\right]^{1/2}}. \tag{14}$$

The numerator in (14) measures the frontier shift over time in group A, with respect to a reference technology, that corresponds to group R in period h. The same happens for the denominator for units belonging to B. In this way, $TGC_{t,t+1}^{AB}\left(R^h\right)$ coincides with the ratio of aggregated measures of technological changes for the units in the groups A and B. Notice that when one resorts to the base-group base-period PPMI, the divergence component (DC) that appeared in the approach by Aparicio et al. (2017) vanishes, simplifying the interpretation of the index and its components and, therefore, making it easier to be applied").

However, this is not the unique advantage of assuming a reference group. This approach also has interesting implications regarding the satisfaction of the circularity property. In particular, as happens with the base-period Malmquist index that fulfills the circular test (Berg et al. 1992), we next show that the base-period CDMI also meets this property.

$$\mathrm{CDMI}_s^{\mathrm{AB}}\left(R^s\right)\cdot\mathrm{CDMI}_s^{\mathrm{BC}}\left(R^s\right)=\frac{\left(\prod_{j=1}^{n^{A^s}}D^{R^s}(x_j^{A^s},y_j^{A^s})\right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{B^s}}D^{R^s}(x_i^{B^s},y_i^{B^s})\right)^{1/n^{B^s}}}\cdot\frac{\left(\prod_{j=1}^{n^{B^s}}D^{R^s}(x_j^{B^s},y_j^{B^s})\right)^{1/n^{B^s}}}{\left(\prod_{i=1}^{n^{C^s}}D^{R^s}(x_i^{C^s},y_i^{C^s})\right)^{1/n^{C^s}}}=$$
$$=\frac{\left(\prod_{j=1}^{n^{A^s}}D^{R^s}(x_j^{A^s},y_j^{A^s})\right)^{1/n^{A^s}}}{\left(\prod_{i=1}^{n^{C^s}}D^{R^s}(x_i^{C^s},y_i^{C^s})\right)^{1/n^{C^s}}}=\mathrm{CDMI}_s^{\mathrm{AC}}\left(R^s\right). \quad (15)$$

In this way, the base-period CDMI is circular without drastically adjusting the index introducing the observations of all the groups considered, as happened with the solution suggested by Camanho and Dyson (2006) when dealing with more than two groups of units.

Additionally, by (15), we get that

$$PPMI_{t,t+1}^{AB}\left(R^h\right)\cdot PPMI_{t,t+1}^{BC}\left(R^h\right)=\frac{CDMI_{t+1}^{AB}\left(R^h\right)}{CDMI_t^{AB}\left(R^h\right)}\cdot\frac{CDMI_{t+1}^{BC}\left(R^h\right)}{CDMI_t^{BC}\left(R^h\right)}=$$
$$=\frac{CDMI_{t+1}^{AC}\left(R^h\right)}{CDMI_t^{AC}\left(R^h\right)}=PPMI_{t,t+1}^{AC}\left(R^h\right). \quad (16)$$

Consequently, the base-group base-period PPMI is also circular. The same steps can be carried out to prove that the technological gap change component $TGC_{t,t+1}^{AB}\left(R^h\right)$ also satisfies circularity. As for the efficiency gap change $EGC_{t,t+1}^{AB}$, it is circular since it coincides with the ratio $EG_{t+1}^{AB}\big/EG_t^{AB}$, where numerator and denominator are circular (see Camanho and Dyson, 2006, p. 39).

4. Conclusions

In this note, we have extended and enhanced the Aparicio et al. (2017) approach by introducing a new base-group base-period PPMI for measuring the performance gap evolution between two or more groups of units. This approach endows the index and its components with a clear and straightforward interpretation in terms of the comparison of productivity change of units in A and B. Moreover, we have shown that the new index and its components satisfy the circular test.

We are aware that with the base-group base-period PPMI, in contrast to the original PPMI, we gain interpretability and fulfillment of the circular test but pay with base period dependency and with an underlying reference to a fixed technology. Although in some cases this could become a problem, we think that in most empirical situations it will be easy to find this baseline group and period technology to act as reference. For example, as mentioned in the introduction, international education databases provide different groups that could fulfill this role. For example, following Aparicio et al. (2017), a practitioner interested in monitoring; public, private government dependent schools and private schools performance over time within a region, could select a group of, let´s say, public schools in an adjacent region. If the target were to analyze groups of schools belonging to different regions of a same country, the reference could be a best practice region or country such as Finland and so on. For other sectors, the researcher should pursue a similar wise strategy.

Acknowledgements. J. Aparicio appreciates the financial support from the Spanish Ministry for Economy and Competitiveness (Ministerio de Economía, Industria y Competitividad), the State Research Agency (Agencia Estatal de Investigacion) and the European Regional Development Fund (Fondo Europeo de DEsarrollo Regional) under grant MTM2016-79765-P (AEI/FEDER, UE). D. Santin acknowledges the financial support of the Spanish Ministry of Economy and Competitiveness (ECO2014-53072-P).

References

Aparicio, J., Crespo-Cebada, E., Pedraja-Chaparro, F. and Santin, D. (2017) Comparing school ownership performance using a pseudo-panel database: A Malmquist-type index approach. European Journal of Operational Research, 256: 533-542.

Berg, S.A., Førsund, F.R. and Jansen, E.S. (1992) Malmquist Indices of Productivity Growth during the Deregulation of Norwegian Banking, 1980–89. Scandinavian Journal of Economics (Supplement), 211-228.

Camanho, A.S. and Dyson, R.G. (2006) Data envelopment analysis and Malmquist indices for measuring group performance. Journal of Productivity Analysis, 26: 35-49.

Caves, D.W., Christensen, L.R. and Diewert, W.E. (1982) Multilateral comparisons of output, input and productivity using superlative index numbers. Economic Journal, 92: 73-86.

Charnes, A.; Cooper, W. W. and Rhodes, E. (1981). Evaluating program and managerial efficiency: an application of data envelopment analysis to Program Follow Through, Management Science, 27, 668-697.

De Witte, K. and López-Torres, L. (2015). Efficiency in education: a review of literature and a way forward. Journal of the Operational Research Society, doi:10.1057/jors.2015.92.

Ferreira, D. and Marques, R.C. (2015) Did the corporatization of Portuguese hospitals significantly change their productivity? European Journal of Health Economics, 16 (3): 289-303.

Frisch R (1936) Annual survey of general economic theory: the problem of index numbers. Econometrica, 4: 1-38.

Thanassoulis, E., Shiraz, R.K. and Maniadakis, N. (2015) A cost Malmquist productivity index capturing group performance. European Journal of Operational Research, 241 (3): 796-805.

Vaz, C.B. and Camanho, A.S. (2012) Performance comparison of retailing stores using a Malmquist-type index. Journal of the Operational Research Society, 63(5): 631-645.

Westergaard, H. (1890) Die Grundgftge der Theorie der Statistik. Jena: Gustav Fischer.