School Choice with Transferable Student Characteristics

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Abstract

We consider a school choice problem where schools' priorities depend on transferable students' characteristics. A school choice algorithm selects for each profile of students' preferences over schools an assignment of students to schools and a final allocation of characteristics (an extended matching). We define the Student Exchange with Transferable Characteristics (SETC) class of algorithms. Each SETC always selects a constrained efficient extended matching. That is an extended matching that i) is stable according to the priorities generated by the final allocation of characteristics and ii) is not Pareto dominated by another stable extended matching. Every constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. When students' characteristics are fully transferable, a specific algorithm in the SETC family is equivalent to the application of the Top Trade Cycle Algorithm starting from the Student Optimal Stable Matching

Keywords: School Choice, Transferable Characteristics, Priorities, Constrained Effiency.

JEL Classification C78, D61, D78, I20

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School Choice with Transferable Student Characteristics*

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Abstract

We consider a school choice problem where school priorities depend on (transferable) student characteristics. We define the Student Exchange with Transferable Characteristics (SETC) class of algorithms. Each SETC algorithm generates a constrained efficient extended matching, that is, a matching and allocation of student characteristics such that the matching i) is stable according to the priorities generated by that allocation of characteristics and ii) is not Pareto dominated by another stable matching under any allocation of characteristics. Every constrained efficient extended matching that Pareto improves upon a stable extended matching is the outcome of an algorithm in the SETC class. When student characteristics are fully transferable, a specific algorithm in the SETC family selects the matching obtained with the Efficiency Adjusted Deferred Acceptance Mechanism.

Keywords: School Choice, Transferable Characteristics, Priorities, Constrained Efficiency.

JEL: C78, D61, D78, I20.

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1 Introduction

The School Choice problem studies the mechanisms employed by many school districts to assign students to public schools (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). The canonical formulation of a school choice problem considers a set of students, a set of schools, and the schools' quotas, which represent the capacity of each school. Each student submits a list of school preferences to a central placement authority such as a school district, and each school has a priority ranking that determines who receives a seat in case a school experiences excessive demand. The school district decides which students attend each school using an algorithm (or mechanism) that selects a matching of students to schools considering the students' reported preferences as well as the schools' priorities. A major concern regarding the design of school choice programs has been the ability to fairly match students to schools. That is, all students who obtain a seat at a given school should have a higher priority at that school than the students who preferred that school rather than the one they are matched to. During recent years, a vast majority of school districts have implemented school choice algorithms based on Gale and Shapley's Deferred Acceptance (DA) Algorithm (Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005; Pathak, 2016). The application of the student- proposing DA algorithm to prospective students always results in a stable matching, that is, a fair, individually rational, and non-wasteful matching.[1] However, the matching can be Pareto dominated by another matching that does not respect schools' priorities.

In this paper, we present a school choice problem with extended priorities. In the canonical school choice problem, priorities are a primitive aspect of the model, but school districts use several criteria to determine priority orders for schools, such as different characteristics of potential students or tie-breaker lotteries. For example, the Boston School district considers (lexicographically) the existence of an older sibling and walk zone proximity (Abdulkadiroğlu et al., 2005). In Spain, characteristics such as family income, number of siblings, siblings attending the school, and "legacy" points (if parents or older siblings attended that school) are considered. Students receive points for each of these characteristics, and the students are then prioritized depending on the number of points

[1] A matching is individually rational if no student is assigned to a school that she would rather not attend. A matching is non-wasteful if every school that a student prefers to the school she is assigned to has filled all its available seats.

that they receive.[2] Our model relies on student characteristics as primitives. In a school choice problem with extended priorities, students are initially endowed with characteristics specific to individual schools.[3] Each school priority is defined over pairs of students and the specific student characteristics of that school. Students can exchange the characteristics of different schools and thus affect their positions in the priority rankings of those schools. In this context, a matching that Pareto improves the initial matching but may not respect fairness according to the priority rankings of students generated by the initial distribution of characteristics may become fair after an exchange of the relevant characteristics among students.[4] Our aim is to explore the possibility of obtaining efficiency gains with respect to outcomes of the student-proposing DA algorithm that can be justified under schools' priority rankings after exchanges of characteristics among students.

We propose a class of school choice algorithms, namely, the Student Exchange with Transferable Characteristics (SETC) algorithms. Each algorithm in this class selects a *constrained efficient extended matching*. That is, the outcome of an SETC algorithm is a matching of students to schools and an allocation of specific transferable characteristics such that the matching i) is stable with respect to school priorities for new allocations of characteristics, and ii) is not Pareto dominated by another stable matching with respect to further admissible reallocations of characteristics. Any such constrained efficient extended matching can be obtained by an algorithm in the SETC class (Theorem 1).

Our general framework cannot be directly compared with previous analyses of the

[2]See Casalmiglia et al. (2020); Górtazar et al. (2020) for detailed descriptions of Barcelona and Madrid's priority systems.

[3]A similar formulation was independently proposed by Duddy (2019). This paper discusses the informational shortcomings of the current priority based model and proposes a formulation based on a "priority matrix".

[4]An obvious candidate for such a transferable characteristic would be the tie-breaking lottery draws used at different schools when they use different tie-breaking criteria, which can be exchanged. Multiple tie-breaking criteria are justified, since they reduce the chances that overdemanded schools will systematically reject a student who has a bad lottery draw (Arnosti, 2016). Amsterdam's school choice program reform in 2014 introduced a system based on the DA algorithm with multiple tie-breaking criteria. In 2015, this decision was challenged in court by families who wished to switch school seats, which could be justified by an exchange of the priorities established by multiple tie-breaking criteria. The issue is discussed in Ashlagi et al. (2019) and https://www.nemokennislink.nl/publicaties/schoolstrijd-in-amsterdam/(Schoolstrijd in Amsterdam) (Arnout Jaspers, Kennislink, July 1, 2015, accessed April 27th 2021).

possibility of relaxing the trade-off between stability and efficiency by dropping stability constraints when some students do not benefit from exercising their priority rights, as is done in Kesten (2010) and Alcalde and Romero-Medina (2017). For this reason, in Section 4, we study a case in which all characteristics are fully transferable. In this case, our extended framework can be reduced to the canonical school choice problem with priorities. In this setting, the set of outcomes produced by an SETC algorithm coincides with the set of $\alpha-$fair matchings defined in Alcalde and Romero-Medina (2017) (Corollary 2). Furthermore, when we can define the EADA-SETC algorithm, a particular member of the SETC class that selects an extended matching where the resulting matching coincides with the outcome of theEfficiency Adjusted Deferred Acceptance Mechanism (EADAM) defined by Kesten (2010) (Theorem 2).

Our approach can be useful for improving efficiency in situations where we can distinguish between allocative criteria (such as tie-breakers) and fairness constraints (such as the need for siblings to attend the same school) in the formation of priorities. For example, we can think about the allocation of medical resources. Consider a situation where public medical services are regionally managed, as is done in Spain. In this case, a patient may be eligible for different treatments or drugs depending on her characteristics (such as age, life expectancy or health) and on the assortment of treatments available from her regional government. If we view patients' characteristics as determinants of their access to treatment and their residences only as allocative criteria, the algorithms in the SETC class allow us to implement welfare-improving exchanges of treatments or drugs across regions while respecting fairness requirements.

1.1 Related Literature

The school choice problem was first presented by Balinski and Sönmez (1999), who introduce the idea of fairness to the context of allocating school seats to students. Abdulkadiroğlu and Sönmez (2003) analyze this problem from a mechanism design perspective. These authors show that a student-proposing DA algorithm always selects stable matches and is strategy-proof.[5] They also study an adaptation of Gale's Trop Trading Cycle mechanism (TTCM) (Shapley and Scarf, 1974) and show that it always selects Pareto-efficient matchings and is strategy-proof. Unfortunately, stable matchings are not efficient, and,

[5]A mechanism is strategy-proof if students have incentives to report their true preferences.

indeed can have severe levels of inefficiency (Dur and Morrill, 2017; Kesten, 2010; Abdulkadiroğlu et al., 2009).

There have been attempts to alleviate the trade-off between stability and efficiency by weakening the notion of fairness. Kesten (2010) proposes an EADAM that finds a constrained efficient matching by incorporating the possibility that students may consent to renounce their priorities in relation to schools where they cannot obtain a seat according to the student-proposing DA algorithm.[6] Alcalde and Romero-Medina (2017) propose an alternative weakening of fairness dubbed α-equitability. Ehlers and Morrill (2020) relax the fairness constraint and propose a stable set of legal matchings that are not dominated in terms of fairness by any other legal matching. Finally, Alva and Manjunath (2019) present the concept of stable domination.

The closest paper to ours is that of Dur et al. (2019), which proposes an alternative weakening of stability called partial stability. Under partial stability, certain priorities of certain students at certain schools are ignored. Then, the welfare gains that can be captured by applying the improvement cycles approach proposed by Erdil and Ergin (2008) for school choice problems with weak priorities and arbitrary tie-breakers are explored. Similar to Dur et al. (2019) our paper uses improvement cycles. However, beyond this point, the two papers have considerable differences. First, the primitives in our model are not school priorities but the individual student characteristics on which those priorities are based. Second, in our case, the resulting extended matching is an allocation of both school seats and student characteristics. Third, the possible welfare gains that we capture are derived from exchanges of characteristics. That is, the resulting extended matching of our model is justified by the final allocation of transferable characteristics. Fourth, the SETC algorithms consider exchanges of characteristics and, contrary to the stable improvement cycle algorithm in Erdil and Ergin (2008), some of the students who participate in these cycles only exchange characteristics and facilitate other exchanges, and they are weakly better off. Finally, there is a technical difference between these papers. Our framework does not require that additional conditions be imposed on the set of priorities that may be ignored.[7] Our results only require that school priorities are complete and monotonic in terms of student characteristics.

[6]See also Tang and Yu (2014) for an alternative algorithm for the EADAM.

[7]Assumption 1 in Dur et al. (2019). See Remark 1 in Section 2.

The rest of the paper is organized as follows. In Section 2, we introduce the model and notation utilized. In Section 3, we present our main results. In Section 4, we relate our framework of transferable characteristics to that of school choice with consent proposed by Kesten (2010). In Section 5, we conclude this paper. In Section 6, we provide the proofs of our work.

2 Notation and Definitions

We present the elements of the canonical school choice problem and introduce a school choice problem with extended priorities and transferable characteristics.

Let I be a finite set of students and S be a finite set of schools where the students must be allocated. Each student i has a strict preference of P_i over $S \cup \{\varnothing\}$,[8] where $\{\varnothing\}$ denotes the option of being unassigned. We use R_i to signify the weak preference relation associated with P_i, which is defined in the standard way. Each school s has a limited number of seats available q_s.

A ***matching*** is a function $\mu : I \to S \cup \{\varnothing\}$ such that (i) for each $i \in I$, $\mu(i) \in S \cup \{\varnothing\}$ and (ii) for each $s \in S$, $\mu^{-1}(s) \leq q_s$.[9] A matching μ' ***Pareto dominates*** the matching μ if for each $i \in I$ $\mu'(i)\ R_i\ \mu(i)$ and for some $j \in I$ $\mu'(j)\ P_j\ \mu(j)$.

The final component of the canonical school choice problem is school priorities. Each school ranks its prospective students according to a priority order. Our contribution is to explore the structures of such priority orders. We consider that school priorities may depend on different student characteristics. Some of these characteristics are intrinsic to individual students, but others can be exchanged among students. The relevant priorities for schools depend on the final allocation of such characteristics.

For each student i, let $\omega(i) = (\omega^s(i))_{s \in S}$ be the initial endowment of the transferable characteristics that influence the position of student i at each school s. For each school s, let $\Omega^s \equiv \cup_{i \in I} \omega^s(i)$. A permutation of the transferable characteristics for school s, $\lambda^s : I \to \Omega^s$, is a bijection from I to Ω^s.[10] Let $\mathcal{L}^s$ be a set of all the permutations of

[8] A strict preference is a complete, antisymmetric, and transitive binary relation.

[9] For any set A, $\#A$ stands for the cardinality of the set A.

[10] For each $i \in I$ and $s \in S$, there is $j \in I$ with $\lambda^s(i) = \omega^s(j)$, and for each $j, j' \in I$, $\lambda^s(j) \neq \lambda^s(j')$.

transferable characteristics for s. We call $\lambda = (\lambda_s)_{s\in S}$ an ***allocation of transferable characteristics***. Finally, for each student i and each allocation λ, $\lambda(i) \equiv (\lambda^s(i))_{s\in S}$. We use ω to denote the initial endowment allocation of transferable characteristics.

When characteristics are transferable, their assignment is relevant. Note that each allocation of transferable characteristics λ can be obtained via the characteristic exchange cycles between the students. An ***extended matching*** is a pair (μ, λ) where μ is a matching and λ is an allocation of transferable characteristics. Abusing notation, we say that the extended matching (μ, λ) Pareto dominates the extended matching (μ', λ') if μ Pareto dominates μ'.

In a school choice problem with extended priorities, school priorities not only compare students but also pairs of students and the allocations of transferable characteristics that they present to the school choice process. Hence, each school has a complete, transitive, and antisymmetric binary relation $\succ_s$ with $I \times \Omega^s$. We use the notation $\succsim_s$ to refer to the weak priority relation associated with $\succ_s$.

Throughout this paper, we assume that transferable characteristics are monotonous in the sense that they affect all students in the same direction.

Monotonous Priorities For each $i, j \in I$, $s \in S$, and for each $l, l' \in \Omega^s$, $(i, l) \succ_s (i, l')$ if and only if $(j, l) \succ_s (j, l')$.

Under monotonous priorities, for each s, the set Ω^s is naturally ordered; abusing notation, for each $L^s \subseteq \Omega^s$, we define

$$\max\{L^s\} \equiv \{l \in L^s| \text{ for each } i \in I, l' \in L^s,\ (i, l) \succsim_s (i, l')\}.$$

Remark 1. *Under monotonous priorities, for each school s, λ^s, $i_0, i_1, i_2, i_3 \in I$, and extended priority $\succsim$, if*

$$(i_1, \lambda^s(i_1)) \succ_s (i_2, \lambda^s(i_2)) \succ_s (i_3, \lambda^s(i_3)), \quad \textit{and}$$
$$(i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_1, \lambda^s(i_1))$$

then $(i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_2, \lambda^s(i_2))$.

A school choice problem with extended priorities is a 6-tuple $(I, S, (R_i)_{i\in I}, \omega, (\succ_s)_{s\in S}, (q_s)_{s\in S})$. Henceforth, we consider an arbitrary school choice problem with extended priorities and do not refer to the specific problem whenever it does not induce confusion.

Finally, we present a stability notion that takes into account the fact that school priorities depend on the identities of students and some transferable characteristics.

An extended matching (μ, λ) is **(ex-post) stable** if:

- μ is ***fair under*** λ: For each $i, j \in I$, $\mu(j), P_i\ \mu(i)$ implies $(j, \lambda^s(j)) \succ_{\mu(j)} (i, \lambda^s(i))$.
- μ is ***individually rational***: For each $i \in I$, $\mu(i)\ R_i\ \{\varnothing\}$
- μ is ***non wasteful***: For no $i \in N$ or $s \in S$, $s\ P_i\ \mu(i)$ and $\#\mu^{-1}(s) < q_s$.

The definition of *(ex-post) stable* coincides with the natural notion of stability. An *(ex-post) stable* extended matching does not elicit complaints from students who would like to change the school that they are assigned to. Extended matching respects the priorities of students under the final allocation of transferable characteristics λ.

It is worth noting that our notion of *(ex-post) stable* is parallel to the concept of *partial stability* in Dur et al. (2019) but we provide a rationale and structure for the initial priorities (under the initial endowment of transferable characteristics) that are not necessarily respected in the final assignment of students to schools. In light of Remark 1, since schools have monotonous extended priorities, we do not need to introduce additional restrictions on students' priorities that will not necessarily be respected.[11]

We are interested in obtaining *(ex-post) stable* extended matchings that are not Pareto dominated by other *(ex-post) stable* extended matchings. When there is no possibility of exchanging transferable characteristics, the student-proposing DA algorithm selects an *(ex-post) stable* extended matching based on the initial endowment of transferable characteristics. In the extended matching framework, the student-proposing DA algorithm runs as follows: In the first step, all students apply to their favorite schools. Then, each school matches each applicant according to her respective transferable characteristics in order of extended priority, placing them on waitlists until the corresponding quota is reached; then, all remaining applicants are rejected. In each subsequent step, all the

[11]See Assumption 1 in Dur et al. (2019).

students rejected in the previous step apply to their favorite schools among those who have not yet rejected them. Then, each school organizes its applicants (both new and waitlisted) in order of extended priority, on a new waitlist until their quotas are reached, and they reject all the remaining applicants. The algorithm stops when no students are rejected. Each student is then assigned to the school at which she is currently waitlisted or remains unassigned to a school if she has been rejected at every school that she prefers to $\varnothing$. Gale and Shapley (1962) prove that the selected matching Pareto dominates all other *(ex-post) stable* extended matchings under an initial endowment of transferable characteristics. We can apply the same logic to any allocation of transferable characteristics λ, and we define μ_λ^{SO} as the matching obtained by the student-proposing DA algorithm under λ. We call $(\mu_\omega^{SO}, \omega)$ the ***student optimal stable extended matching (SOSEM)***.[12]

If students may exchange their transferable characteristics, we are able to find *(ex-post) stable* extended matchings (μ, λ) such that μ Pareto dominates μ_ω^{SO}. We focus on extended matchings that can be obtained through limited exchanges of transferable characteristics that justify changes in the students' matches.

Given an extended matching (μ, λ), we say that $(\bar{\mu}, \bar{\lambda})$ is a ***reshuffling of*** (μ, λ) if for each $i \in I$ and for each $s \notin \{\mu(i), \bar{\mu}(i)\}$, $\lambda^s(i) = \bar{\lambda}^s(i)$.

We now present a notion that captures the idea of obtaining efficient matchings that satisfy the requirements of *fairness* and *stability* when transferable characteristics can be exchanged.

An extended matching (μ, λ) is ***constrained efficient*** if it is *(ex-post) stable* and if μ' does not Pareto dominate μ for any *(ex-post) stable* reshuffling (μ', λ').

Example 1. *Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_{s_x} = 1$ for $x = 1, 2, 3$. The students' preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3
$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$

[12] Note that if there is a matching $\nu \neq \mu_\omega^{SO}$ such that (ν, ω) is *(ex-post) stable*, then μ_ω^{SO} Pareto dominates ν.

Each school uses a cardinal raking based on two criteria to create their priority rankings. They consider scores related to nontransferable characteristics intrinsic to each student as well as transferable characteristics, for example, students' legacy points if their parents are alumni at the school.

Assume that legacy is a transferable characteristic. Student i_1 has legacy points for s_1, student i_2 has legacy points for s_2, while student i_3 has no legacy points. Hence, we can write the endowment allocation of the initial transferable characteristics as:

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), \omega^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (0,1,0) \\ (0,0,0) \end{pmatrix}.$$

Additionally, the school priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_2, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_2, 0)$	$(i_1, 0)$	$(i_1, 0)$

Finally, $\mu_\omega^{SO} = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$.

When students $\{i_1, i_2\}$ exchange their transferable characteristics, the allocation of these exchangeable characteristics is:

$$\begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix}.$$

Additionally, the extended school priorities under λ are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_2, 1)$	$(i_1, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_1, 0)$	$(i_2, 0)$	$(i_1, 0)$

The application of the student-proposing DA algorithm under the allocation of transferable characteristics λ results in the matching $\mu_\lambda^{SO} = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$. Hence, $(\mu_\lambda^{SO}, \lambda)$ is (ex-post) stable. *Clearly, μ_λ^{SO} Pareto dominates μ_ω^{SO}. Actually, there is no matching μ' that Pareto dominates μ_λ^{SO}. Hence, the extended matching $(\mu_\lambda^{SO}, \lambda)$ is* constrained efficient.

The following example extends Example 1 and illustrates the difficulties that may arise when characteristics are not school specific. It may be the case that some students can exchange seats at their initially assigned schools and that with exchanges of transferable characteristics, all the remaining student priorities regarding those schools can be respected. However, if transferable characteristics are not school specific, their exchange may trigger a chain of fairness violations in other schools.[13]

Example 2. *Let $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, and $q_{s_x} = 1$ for $x = 1, \ldots, 5$. The students' relevant preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
s_4	s_1	s_1	s_4	s_5
s_2	s_2	s_2	s_5	s_4
s_1	s_3	s_3	$\{\varnothing\}$	$\{\varnothing\}$
s_3	$\{\varnothing\}$	$\{\varnothing\}$		
$\{\varnothing\}$				

The student characteristics in this example exhibit the same distribution as those in Example 1, but the transferable characteristics of school s_2 are also relevant for school s_4. Hence, the initial endowment allocation of transferable characteristics is:

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega(i_3) \\ \omega(i_4) \\ \omega(i_5) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1), \omega^{s_4}(i_1), \omega^{s_5}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2), \omega^{s_4}(i_2), \omega^{s_5}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), \omega^{s_3}(i_3), \omega^{s_4}(i_3), \omega^{s_5}(i_3)) \\ (\omega^{s_1}(i_4), \omega^{s_2}(i_4), \omega^{s_3}(i_4), \omega^{s_4}(i_4), \omega^{s_5}(i_4)) \\ (\omega^{s_1}(i_5), \omega^{s_2}(i_5), \omega^{s_3}(i_5), \omega^{s_4}(i_5), \omega^{s_5}(i_5)) \end{pmatrix} = \begin{pmatrix} (1_1, 0, 0, 0, 0) \\ (0, 1_2, 0, 0, 0) \\ (0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0) \end{pmatrix}.$$

[13] This example can be readily tailored to illustrate the problems that arise if we hold that the reallocation of transferable characteristics are not organized by reshufflings.

Additionally, the school priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
$(i_1, 1_1)$	$(i_1, 1_2)$	$(i_3, 0)$	$(i_5, 0)$	$(i_4, 0)$
$(i_4, 0)$	$(i_2, 1_2)$	$(i_2, 0)$	$(i_1, 1_2)$	$(i_5, 0)$
$(i_2, 1_1)$	$(i_3, 0)$	$(i_1, 0)$	$(i_4, 0)$	
$(i_3, 0)$	$(i_2, 0)$		$(i_1, 0)$	
$(i_2, 0)$				

and $\mu^{SO}_{\omega} = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5)\}$.

When students $\{i_1, i_2\}$ *exchange their transferable characteristics, the allocation of exchangeable characteristics is* λ'.

$$\lambda' = \begin{pmatrix} (0, 1_2, 0, 0, 0) \\ (1_1, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0) \end{pmatrix}.$$

Through exchanges of transferable characteristics, students i_1 *and* i_2 *can improve their school matches by exchanging their respective seats at* s_1 *and* s_2. *This exchange of transferable characteristics helps avoid any legitimate claim of student* i_3 *on* s_1, *and the resulting matching* $\mu' = \{(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5)\}$ *represents a Pareto improvement with respect to* μ. *However, the extended matching* (μ', λ') *is not* (ex-post) stable. *After obtaining* 1_2, *student* i_1 *could claim student* i_4*'s position at school* s_4. *In fact,* $\mu^{SO}_{\lambda'} = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_5), (i_5, s_4)\}$. *That is, the initial Pareto improvement achieved by the aforementioned exchange of transferable characteristics initiates a chain reaction that leads to an extended matching where the students initiating exchanges do not benefit.*

3 Improvement Cycles for Extended Matchings

In this section, we propose a systematic method to obtain *constrained efficient* extended matchings. Our approach follows those of Erdil and Ergin (2008) and Dur et al. (2019), who propose a method for finding fair Pareto-improving trade cycles based on the outcome of the student-proposing DA algorithm for coarse priorities with arbitrary tie-breakers and partially non-enforceable priorities, respectively. In both papers, the logic behind improving the cycles is related to the idea of vacancy chains introduced by Blum et al. (1997). For an initial stable matching involving a position at some school, if the student assigned to that position finds another position, her seat can be occupied by another student such that no other student with higher priority at that school prefers that seat to the position at the initial matching.

The following concepts extend the graph–theoretical approach presented by Dur et al. (2019) to the extended priorities framework. We use a similar logic when relevant school priorities depend on students' transferable characteristics. The difference between our model and that of Dur et al. (2019) is that in our model, students may be willing to move to a position at a desirable school but, depending on the student who exchange the transferable characteristics, some violation of fairness may appear. Therefore, Pareto improvements involving two students may require the participation of additional students who only exchange transferable characteristics without becoming involved in a school exchange.

Given an *(ex-post) stable* extended matching (μ, λ), for each school $j \in I$, let:

- $D_{(\mu,\lambda)}(j) = \{i \in I : \mu(j)\ R_i\ \mu(i)\}$ and $\tilde{D}_{(\mu,\lambda)}(j) = \{i \in I : \mu(j)\ P_i\ \mu(i)\}$.
- $X_{(\mu,\lambda)}(j) = \{i \in D_{(\mu,\lambda)}(j) : \forall k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\}, (i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (k, \lambda^s(k))\}$.

The set $\tilde{D}_{(\mu,\lambda)}(j)$ contains all the students who prefer the match of student j over their own matches. The set $D_{(\mu,\lambda)}(j)$ also includes all the students who are matched to $\mu(j)$. The set $X_{(\mu,\lambda)}(j)$ includes all the students who would be willing to occupy j's position at $\mu(j)$ and therefore would not be envious if they were matched to $\mu(j)$ should j leave her position. The members of $X_{(\mu,\lambda)}(j)$ are the students in $D_{(\mu,\lambda)}(j)$ who are ranked above the remaining members of $\tilde{D}_{(\mu,\lambda)}(j)$ and have either obtained $\lambda^{\mu(j)}(j)$ or maintained $\lambda^{\mu(j)}(j)$.

Hence, if j moves to a preferred school and a member of $X_{(\mu,\lambda)}(j)$ obtains j's position at $\mu(j)$, no one could argue that this change violates her priority for $\mu(j)$.

Let $G = (V; E)$ be a (directed) application graph with the set of vertices V and the set of directed edges E, which is a set of ordered pairs from V .

For each extended matching (μ, λ), $G(\mu, \lambda) = (I; E(\mu, \lambda))$ is the ***(directed) application graph associated with*** (μ, λ) where the set of directed edges $E(\mu, \lambda) \subseteq I \times I$ is defined by $ij \in E(\mu, \lambda)$ (that is, i points to j) if and only $i \in X_{(\mu,\lambda)}(j)$. A set of edges $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a path if the related vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct, and it is a cycle if the vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct and $i_1 = i_{n+1}$. Student i is involved in cycle ϕ if there is a student j such that $ij \in \phi$. A cycle $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is solved when for each $ij \in \phi$, student i is assigned to $\mu(j)$ and a new matching is obtained. Formally, we denote the solution of a cycle by the operation $\circ$, that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, \ldots, i_n\}$, $\eta(i') = \mu(i')$. A cycle ϕ is an ***improvement cycle*** for $G(\mu, \lambda)$ if there is an $ij \in \phi$ such that $i \in \tilde{D}_{(\mu,\lambda)}(j)$.

The following algorithm is built on an *(ex-post) stable* extended matching and is defined by solving cycles iteratively.

Student Exchange with Transferable Characteristics (SETC) Algorithm:

Step 0: Let (μ_0, λ_0) be an *(ex-post) stable* extended matching.

Step $k \geq 1$: Given the extended matching $(\mu_{k-1}, \lambda_{k-1})$,

(k.1) if there is an improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, solve any one of such cycles, for example, ϕ_k; let $\mu_k = \phi_k \circ \mu_{k-1}$, and define λ_k as follows. For each $i \in I$, let $s_k = \mu_k(i)$ and $s_0 = \mu_0(i)$.

- For each $s \notin \{s_0, s_k\}$, $\lambda_k^s(i) = \lambda_0^s(i)$.
- If there is no i' such that $ii' \in \phi_k$, then $\lambda_k^{s_k}(i) = \lambda_{k-1}^{s_k}(i)$.
- If there is an i' such that $ii' \in \phi_k$, then $\lambda_k^{s_k}(i) = \max\{\lambda_{k-1}^{s_k}(i), \lambda_{k-1}^{s_k}(i')\}$.
- If there is a j such that $\lambda_0^{s_0}(i) = \lambda_k^{s_0}(j)$, then $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(j)$; otherwise, $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(i)$.

Next, move to Step $k+1$.

$(k.1)$ if there is no improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, then the algorithm stops and $(\mu_{k-1}, \lambda_{k-1})$ is the obtained extended matching.

Since sets of schools and students are finite, the algorithm stops after a finite number of steps. Actually, at most, there are $\frac{1}{2}\#I(\#I-1)$ possible Pareto improvements. Note that the definition of the SETC algorithm entails a class of algorithms, as there may be several incompatible Pareto improvements and the order in which cycles are solved may lead to different final outcomes.

Example 3 shows the relevance of constructing improvement cycles for students who do not strictly benefit from exchanging their transferable characteristics.

Example 3. *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, $q_{s_x} = 1$ for $x = 1, 3$; and $q_{s_2} = 2$. The students' preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_2	s_1	s_1	s_2
s_1	s_2	s_2	s_3
s_3	s_3	s_3	s_1
$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$	$\{\varnothing\}$

The initial endowment of transferable characteristics w is defined by:

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \\ \omega(i_4) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), \omega^{s_3}(i_3)) \\ (\omega^{s_1}(i_4), \omega^{s_2}(i_4), \omega^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (1,0,0) \\ (0,0,0) \\ (0,1,0) \end{pmatrix}.$$

Additionally, the school priorities are as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_4, 1)$	$(i_4, 0)$
$(i_3, 1)$	$(i_3, 0)$	$(i_3, 0)$
$(i_2, 1)$	$(i_1, 1)$	$(i_2, 0)$
$(i_4, 0)$	$(i_2, 0)$	$(i_1, 0)$
$(i_3, 0)$	$(i_1, 0)$	

The application of the student-proposing DA algorithm produces the matching $\mu_\omega^{SO} = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_2)\}$. *The assignment* $\mu' = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_2)\}$ *Pareto dominates* μ, *but if* i_1 *and* i_3 *exchange their transferable characteristics, then the resulting extended matching does not respect student* i_2*'s priority at school* s_2 *because* $\omega^{s_2}(i_3) = 0$ *and* $(i_2, 0) \succ_{s_2} (i_1, 0)$. *However, if student* i_4 *participates in this exchange of characteristics, we obtain the following reshuffling of transferable characteristics:*

$$\lambda = \begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \\ \lambda(i_4) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \\ (\lambda^{s_1}(i_4), \lambda^{s_2}(i_4), \lambda^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix}.$$

Indeed, $\mu' = \mu_\lambda^{SO}$ *and* (μ', λ) *are* (ex-post) stable. *Moreover, since students* i_1, i_3, *and* i_4 *obtain seats at their most preferred schools, there is no matching* ν *that Pareto dominates* μ'. *Hence,* (μ', λ) *is* constrained efficient.

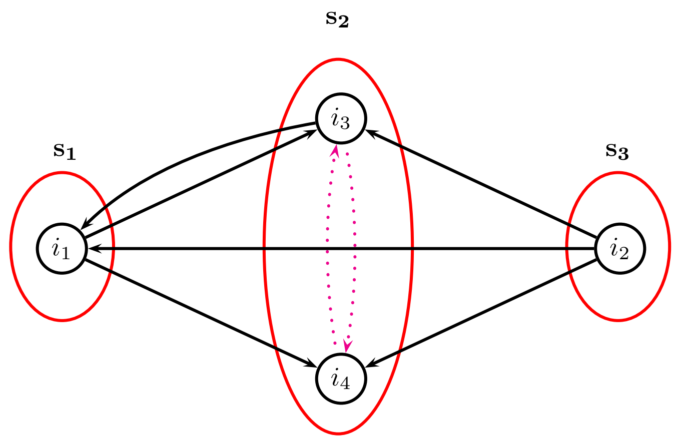

Figure 1: Example 3. Student i_x points student i_y if $i_x \in D_{(\mu,\omega)}(i_y)$. Solid lines: i_x points at i_y if $i_x \in \tilde{D}_{(\mu,\omega)}(i_y)$. Dotted Lines: i_x points at i_y if $i_x \in D_{(\mu,\omega)}(i_y)$, $\mu(i_x) = \mu(i_y)$.

Starting from the SOSEM, Figure 1 presents a graph where each student points to the positions of the students whose positions they would like to occupy (including indifference relations). Figure 2 shows the strict improvements that would not elicit justified envy, and it can be observed that no cycle can be constructed. Note that i_1 *does not point at*

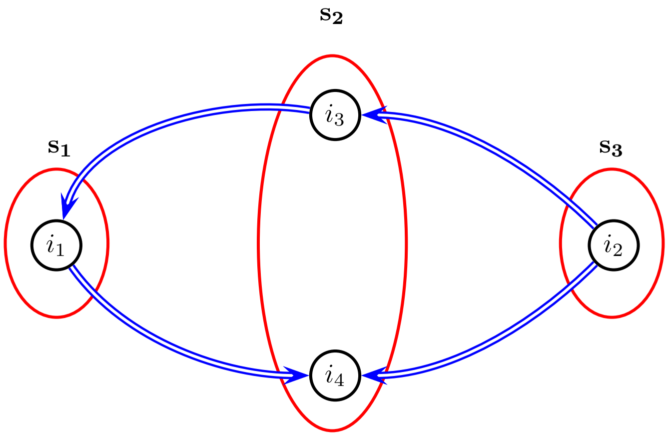

Figure 2: Example 3. Graph associated with (μ, ω). Student i_x points at student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$ and $\mu(i_x) \neq \mu(i_y)$.

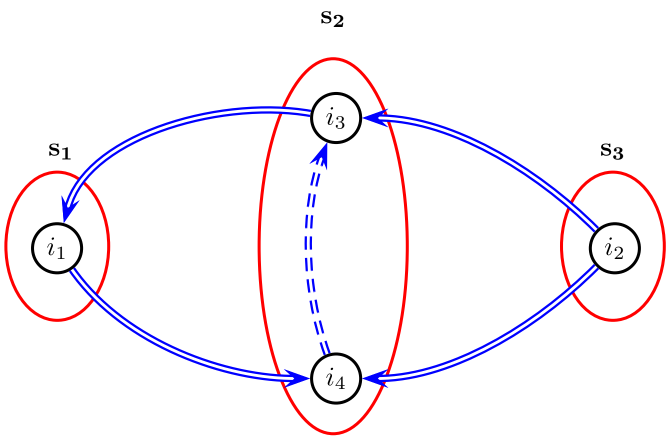

Figure 3: Example 3. $G(\mu, \omega)$. Student i_x points at student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

i_3 *because* $(i_2, \omega_{i_2}^{s_2}) \succ_{s_2} (i_1, \max\{\omega_{i_1}^{s_2}, \omega_{i_3}^{s_2}\})$. *Figure 3 presents the graph associated with* (μ, ω). *We observe the existence of a unique cycle, namely,* $\gamma = i_1 i_4 i_3 i_1$. *Solving* γ *generates the extended matching* (μ', λ). *Finally, Figure 4 presents the graph* $G(\mu', \lambda)$. *This graph contains no improvement cycles, and indeed, the extended matching* (μ', λ) *is* (ex-post) stable *and* constrained efficient*.*

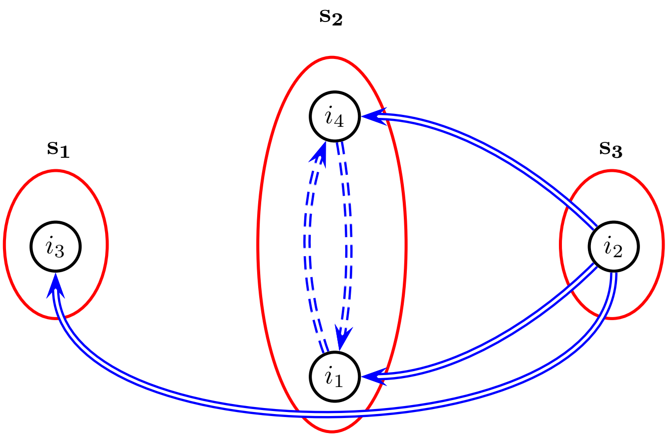

Figure 4: Example 3. $G(\mu', \lambda)$. Student i_x points at student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

Remark 2. *The school priorities presented in Example 3 are consistent with point-system based priorities. Point systems generate additively separable extended priorities. That is, for each school* s*, for each pair of students* i_x, i_y*, and for each* $\lambda^s, \bar{\lambda}^s \in \Omega^s$*,* $(i_x, \lambda^s) \succ_s (i_y, \lambda^s)$ *if and only if* $(i_x, \bar{\lambda}^s) \succ_s (i_y, \bar{\lambda}^s)$*.*

Next, we present our main results. Starting from any *(ex-post) stable* extended matching, the application of an algorithm of the SETC class always yields a *constrained efficient* and *(ex-post) stable* extended matching. Moreover, any *constrained efficient* extended matching can be obtained with an SETC algorithm by starting with the SOSEM. Hence, SETC-class algorithms identify all improvement cycles that yield *(ex-post) stable* extended matchings.

Theorem 1. *For each problem, an extended matching is* constrained efficient *and Pareto dominates the SOSEM if and only if it is obtained with an algorithm of the SETC class by starting with the SOSEM.*

The proof of Theorem 1 follows similar arguments to those related to the proof of Dur et al. (2019) but the extended model entails important intricacies. Transferable characteristics differ between students, and only exchanges involving specific students in a school may be mutually viable. Moreover, improvement cycles may need to involve students who do not strictly benefit from these exchanges but are needed to facilitate reassignment through transferable characteristic trades.

Certain immediate consequences follow from Theorem 1. Since the result of an SETC algorithm is *constrained efficient* with respect to its final allocation of transferable characteristics, then it is equal to the result of the SOSEM for the final allocation of transferable characteristics.

Corollary 1. *For each problem, each* (ex-post) stable *matching* (μ_0, ω)*, and each SETC algorithm, if the extended matching* (μ, λ) *is an outcome of an SETC algorithm then* $(\mu, \lambda) = (\mu_\lambda^{SO}, \lambda)$.

We conclude this section by analyzing the incentives of students to reveal their true preferences when an allocation of school seats is determined by an SETC algorithm. For that purpose, we need to introduce further notation related to the outcomes of different problems and defined for different profiles of student preferences.

Let $\mathcal{P}$ denote the complete set of student preference profiles and $\mathcal{M}$ be a set of all the extended matchings. A mechanism is a mapping $\Psi : \mathcal{P} \to \mathcal{M}$.

The application of an SETC algorithm starting with the SOSEM that corresponds to each preference profile defines a mechanism that always selects an *(ex-post) stable* and *constrained efficient* extended matching. We call this class of mechanisms the ***students' optimal transferable characteristics (SOTC)*** class of mechanisms.

Strategy-proofness A mechanism Ψ satisfies ***strategy-proofness*** if for each $i \in N$, each $P, P' \in \mathcal{P}$, such that for each $j \neq i$, $P_j = P'_j$, $\Psi(P) = (\mu, \lambda)$ and $\Psi(P') = (\mu', \lambda')$, $\mu(i)\ R_i\ \mu'(i)$.

Strategy-proofness implies that no student has the capacity or incentive to manipulate the results of this mechanism by misreporting her preferences regarding schools. According to the results in the work of Abdulkadiroğlu et al. (2009); Kesten (2010); Alva and Manjunath (2019); Kesten and Kurino (2019), since the matching selected by any SETC algorithm that starts with the SOSM matching Pareto dominates the SOSM matching for the initial endowment of characteristics and is not Pareto dominated by any other matching, any mechanism in the SOTC class is manipulable in relation to some profile of student preferences.

Proposition 1. *There is no mechanism in the SOTC class that satisfies* strategy-proofness*.*

4 Fully Transferable Characteristics

In the previous sections, we analyze a new setting where trade-offs between stability and efficiency can be attenuated. In a school choice problem with extended priorities, some violations of initial priorities can be justified after exchanges of transferable characteristics. The introduction of this new component to the canonical school choice problem does not allow us to make an immediate comparison to previous works that consider dropping stability constraints when some students do not benefit from exercising their priority rights.

In particular, the concepts of α-stability in Alcalde and Romero-Medina (2017) and of students consenting to drop their initial priorities in Kesten (2010) imply that proposed matchings are met with no objections although the initial priorities of some students are not respected by these matchings. In both cases, the exertion of some priorities by some students at some schools that block the assignment of seats to other students may not ultimately lead to a placement improvement for the blocking student. Therefore, students are either not allowed to exert their priority rights (Alcalde and Romero-Medina (2017)) or encouraged not to claim a seat if doing so would be ineffective(Kesten (2010)).

Although the proposals of Alcalde and Romero-Medina (2017) and Kesten (2010) are presented in terms of the canonical school choice problem, preventing the immediate comparison of this work with our results, there is an extreme class of extended priorities that allows us to view both of these proposals as particular cases of extended matchings obtained by SETC algorithms. This can be done in the domain of *fully transferable extended priorities*, which are completely defined by transferable characteristics and which allow us to address both concepts of priority renouncement.

Fully Transferable Extended Priorities. For each $i, i', j, j' \in I$, $s \in S$, and for each $\lambda^s, \bar{\lambda}^s \in \mathcal{L}^s$, $(i, \lambda^s) \succ_s (i', \bar{\lambda}^s)$ if and only if $(j, \lambda^s) \succ_s (j', \bar{\lambda}^s)$.

In cases where transferable characteristics entirely determine school priorities, if a student participates in an improvement cycle and receives the transferable characteristic that initially secured his or her seat, then the initial priority that another student may have had for that seat is no longer relevant.

In the context of fully transferable priorities, the analyses of the SETC algorithms are simpler, since any student that desires the position of another student can obtain it with an exchange of transferable characteristics.

Proposition 2. *Let* (μ, λ) *be an* (ex-post) stable *extended matching and* $G(\mu, \lambda)$ *the (directed) application graph associated with* (μ, λ)*. If school extended priorities are fully transferable and* $i \in \tilde{D}_{(\mu,\lambda)}(j)$*, then* $ij \in G(\mu, \lambda)$*.*

In the context of fully transferable priorities, students who exchange their characteristics but remain assigned to the same school do not need to participate in improvement cycles. Moreover, since any Pareto improvement of a matching can be achieved by forming disjointed exchange cycles among students and because such cycles correspond to the cycle in graph $G(\mu^{SO}_{\omega}, \omega)$ of Theorem 1 and Proposition 2, we immediately derive the following result.

Proposition 3. *Let school extended priorities be fully transferable. If* μ *is a matching that Pareto dominates* μ^{SO}_{ω} *and* μ *is not Pareto dominated by any matching* ν*, then there is a reshuffling* λ *such that* (μ, λ) *is the result of the application of an SETC algorithm that starts with the SOSEM.*

Alcalde and Romero-Medina (2017, Theorem 1) proves that the set of efficient matchings that are Pareto improvements over the initial optimal student matching coincides with an ideal set of matchings such that under the initial priorities, no student can pose an admissible objection. That is, whenever a student proposes (objects) an alternative matching where she would obtain a preferred seat for which she has a priority right, another student could rightfully object to that alternative matching (α-*fair* matchings). Hence, the set of matchings produced by an SETC algorithm coincides with the set of α-*fair* matchings produced under the assumption of fully transferable extended priorities.

Corollary 2. *Let school extended priorities be fully transferable. A matching* μ *is an* α-fair *matching if and only if there is a reshuffling* λ *such that the extended matching* (μ, λ) *is the result of the application of an SETC algorithm that starts with the SOSEM*

Kesten (2010) occupies a central position in the extant analysis of Pareto-efficient matching in the context of school choice. This study proposes the idea of consent. Students can consent to withdraw their claims to seats that they will not accept. This idea leads to a modification of the student-proposing DA algorithm that yields a Pareto-efficient matching with "minimal" violations of initial priorities, the ***Efficiency Adjusted Deferred Acceptance Algorithm (EADA)***. Tang and Yu (2014) present a simpler algorithm with the same outcome using the concept of underdemanded schools that we present now.[14]

(Simplified) Efficiency Adjusted Deferred Acceptance Algorithm (EADA): Given a matching μ, a school s is ***underdemanded in relation to*** μ if no student prefers s to the school to which they are assigned by μ. The simplified EADA algorithm works by executing the student-proposing DA algorithm iteratively after sequentially altering the preferences of students assigned to underdemanded schools. Starting with the SOSEM, as a first step, the student-proposing DA algorithm is executed a second time with the students previously assigned to underdemanded schools listing those schools as their top choices. Therefore, in this second stage, students at underdemanded schools retain their seats, and their potential priorities at schools where they cannot obtain seats become ineffective. This process is repeated iteratively until there are no underdemanded schools.

Next, we propose the EADA-SETC algorithm, a specific SETC algorithm that under transferable priorities selects the matching obtained by the (simplified) EADA algorithm. The successive selection of cycles utilized by this algorithm involves identifying the students assigned to underdemanded schools, dropping the potential cycles involving those students, and of the remaining cycles, solving those that would satisfy the extended

[14]The resulting matching is a unique efficient matching μ^* such that there is no other matching ν that can improve the situation of any student whose priority is violated in μ unless ν violates the priority of a student whose situation is worsened (Reny, 2021).

priorities under the initial endowments of the students who are not assigned to underdemanded schools first. This process is equivalent to running the student-proposing DA algorithm when students who are assigned to underdemanded schools report that those underdemanded schools are their most preferred alternative. Running this process iteratively until an extended matching where no school is underdemanded is obtained leads to a *constrained efficient* extended matching through the matching process of the EADA algorithm.

Let us consider a matching μ and a subset of students $S \subseteq N$; let the graph $G^*(\mu, S)$ be such that $ij \in G^*(\mu, S)$ if and only if $i, j \in S$ and $\mu(j)\ P_i\ \mu(i)$ and for each $l \in S \setminus \{i\}$ with $\mu(j)\ P_l\ \mu(l)$ and $(i, \omega^{\mu(j)}(i)) \succ_i (l, \omega^{\mu(j)}(l))$.

EADA-SETC Algorithm:

Step *0*. Let (μ_0, λ_0) be the *student optimal extended matching* $I_0 = N$ and U_0 the set of underdemanded schools at μ_0.

Step *k*≥*1*. Given $(\mu_{k-1}, \lambda_{k-1})$:

Stage *k.0*. If $U_{k-1} = \varnothing$, stop; $(\mu_{k-1}, \lambda_{k-1})$ is the selected extended matching. Otherwise, let $(\mu_k^0, \lambda_k^0) = (\mu_{k-1}, \lambda_{k-1})$, $I_k = I_{k-1} \setminus \{l \in I_{k-1}, \mu_k(l) \in U_{k-1} \cup \{l\}\}$ and $S_k = S_{k-1} \setminus U_{k-1}$, and move to stage $k.1$.

Stage *k.t* (*t*≥*1*). Let $G_k^t = G^*(\mu_k^{t-1}, I_k)$:

- If there is one or more cycles at G_k^t, solve one of the cycles at G_k^t, for example, ϕ; let $\mu_k^t = \phi \circ \mu_k^{t-1}$, λ_k^t as in Step $(k.1)$ of the SETC algorithm and move to Stage $k.(t+1)$.
- If there is no cycle at G_k^t, let $(\mu_k, \lambda_k) = (\mu_k^{t-1}, \lambda_k^{t-1})$ and let U_k be the set of underdemanded schools at $(\mu_k^{t-1}, \lambda_k^{t-1})$, and move to Step $k+1$.

Theorem 2. *Under fully transferable characteristics, the EADA-SETC algorithm selects the extended matching (μ, λ), where μ is the EADA matching.*

5 Conclusions

In this paper, we generalize the school choice problem by defining school priorities in terms of (transferable) student characteristics. We define a family of algorithms – the Student Exchange with Transferable Characteristics (SETC) class– which start with an *(ex-post) stable* extended matching and produce *(ex-post) stable* extended matchings that are not Pareto dominated by other *(ex-post) stable* extended matchings. Moreover, any constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. Finally, we show that an algorithm in the SETC class produces a matching that is consistent with the matching of the Kesten (2010) EADA algorithm when school extended priorities are fully transferable.

Although the focus of this work has been on school choice, our model applies to the allocation of any object under priorities as long as those priorities are based on (multiple) individual characteristics. Recent research on the allocation of medical resources in the context of triage has proven the possibilities of integrating ethical values into the fair allocation of a single scarce resource by reserving part of the resource capacity for certain groups of individuals (Pathak et al., 2020). Our work provides techniques that facilitate Pareto improvements on fair allocations when there is more than one type of object, ethical considerations may be relaxed, and transfers of characteristics are allowed.

6 Proofs

We present separately the proofs of necessity and sufficiency sides of Theorem 1, and then the proofs of the remaining results.

6.1 Proof of Theorem 1 "if" part

Although Theorem 1 refers specifically to the application of SETC algorithms to the SOSEM, the analysis can be carried out from any arbitrary *(ex-post) stable* extended matching.

For a given problem $(I, S, (R_i)_{i\in I}, \omega, (\succ_s)_{s\in S}, (q_s)_{s\in S})$ and a n*(ex-post) stable* extended matching (μ_0, λ_0) consider an algorithm in the SETC class. Let K be the last step of the algorithm starting by (μ_0, λ_0), and for each $k \in \{1, \dots, K\}$, let (μ_k, λ_k) be the extended matching selected by the algorithm at step k. A cycle is solved at each step of the algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving the cycle. Thus, for each $k \geq 1$, μ_k *Pareto dominates* $\mu_k - 1$. Moreover, if student j is not involved in any improvement cycle at Step k, then $\tilde{D}_{(\mu_k,\lambda_k)}(j) \subseteq \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Hence, if i points to j in $G(\mu_{k-1}, \lambda_{k-1})$ and both students are not involved in an improvement cycle at Step k then i points to j in $G(\mu_k, \lambda_k)$.

Lemma 1. *Each extended matching obtained by a SETC algorithm is stable.*

Proof. Let $(\mu_k, \lambda_k\}$ be the extended matching obtained at Step $k \in \{0, \dots, K-1\}$. We prove the result by induction on k. The initial extended matching (μ_0, λ_0) is stable.

Fairness. Assume that $(\mu_{k-1}, \lambda_{k-1})$ is *fair under* λ_{k-1}. Take any pair of students (i, j) such that $\mu_k(j)\ P_i\ \mu_k(i)$. At each step of the algorithm, each student is either better off (she is in a solved cycle) or she is assigned to the same school as in the previous step. Let ϕ_k denote the improvement cycle solved in step k. Assume first that j is not involved in the cycle ϕ_k. Since $\mu_k(j)\ P_i\ \mu_k(i)$, $\mu_{k-1}(j)\ P_i\ \mu_{k-1}(i)$ and $i \in \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Then, by fairness of $(\mu_{k-1}, \lambda_{k-1})$, $(j, \lambda^{\mu_{k-1}(j)}(j)) \succ_{\mu_{k-1}} (i, \lambda^{\mu_{k-1}(j)}(i))$. Since j is not involved in ϕ_k, $\lambda^{\mu_{k-1}(j)}(j) = \lambda^{\mu_k(j)}(j)$. Since $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j)$, $\lambda^{\mu_{k-1}(j)}(i) = \lambda^{\mu_k(j)}(i)$. Therefore $(j, \lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)} (i, \lambda^{\mu_k(j)}(i))$. Assume now that j is involved in ϕ_k. Let $j' \in I$ be such that $j'j \in \phi_k$. Hence, $\mu_{k-1}(j')\ P_i\ \mu_{k-1}(i)$, $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j')$, and $\lambda^{\mu_{k-1}(j')}(i) = \lambda^{\mu_k(j')}(i)$. Since $j'j \in \phi_k$, $(j, \max\{\lambda_{k-1}^{\mu_{k-1}(j')}(j'), \lambda_{k-1}^{\mu_{k-1}(j)} \succ_{\mu_{k-1}(j')} (i, \lambda^{\mu_{k-1}(j')}(i))$, and $(j, \lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)} (i, \lambda^{\mu_k(j)}(i))$. Since i, j are arbitrary, (μ_k, λ_k) is *fair under* λ_k.

Individual Rationality. Since μ_0 is *individually rational*, and each student is never worse off after each step of the algorithm, the μ_K is *individually rational.*

Non-Wastefulness. The initial match μ_0 is *non-wasteful.* At each step students are assigned to better schools swapping their positions at schools, hence $\#\mu_k^{-1}(s)$ remains constant at any step of the algorithm. Assume school s has an empty slot at step k, then the school s has an empty slot at step 0. Since μ_0 is *non-wasteful* and *individually rational,* for each student i with $\mu_0(i) \neq s$, $\mu_0(i)\ P_i\ s$. Since for each i, $\mu_k(i)\ R_i\ \mu_0(i)$, $\mu_k(i)\ R_i\ s$, and (μ_k, λ_k) is *non-wasteful.* □

Lemma 2. *For each stable extended matching* (μ, λ) *and* $j \in I$*,* $X_{(\mu,\lambda)}(j) \subseteq \mu(\mu^{-1}(j))\backslash\{j\}$ *if and only if* $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$*.*

Proof. If $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$, since $D_{(\mu,\lambda)}(j) = \mu(\mu^{-1}(j))$ and $X_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$, the result is immediate. On the other hand, if $\tilde{D}_{(\mu,\lambda)}(j) \neq \{\varnothing\}$, then by completeness and transitivity of school priorities there is $i \in \tilde{D}_{(\mu,\lambda)}(j)$ such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$. By monotonicity of priorities, $(i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(i)\}) \succ_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Therefore, $\mu(i) \neq \mu(j)$ and $i \in X_{(\mu,\lambda)}(j)$. $\square$

Lemma 3. *Let* (μ, λ) *and* (η, λ') *be* (ex-post) stable *extended matchings such that* μ *Pareto dominates* η*. For each* $s \in S$*,* $\#\mu^{-1}(s) = \#\eta^{-1}(s)$*.*

Proof. Let $N = \{i \in I : \mu(i)\ P_i\ \eta(i)\}$. Since μ Pareto dominates η, for each $j \in I \setminus N$, $\mu(j) = \eta(j)$. Consider school s and assume that $\#(N \cap \mu^{-1}(s)) > \#(N \cap \eta^{-1}(s))$. This implies that $\#\eta^{-1}(s) < q_s$. For each $i \in N \cap \mu^{-1}(s)$, $\mu(i) = s\ P_i\ \eta(i)$, which contradicts η non-wastefulness. Hence, $\#(N \cap \mu^{-1}(s)) \leq \#(N \cap \eta^{-1}(s))$. Finally, assume to the contrary there is s such that the strict inequality holds. Summing up the inequalities across schools, the number of students in N who are assigned to some school in matching η is larger than the number of students in N that are assigned to some school in matching μ. Hence there is a student $i \in N$ such that $\eta(i) \in S$, and $\mu(i) = \{i\}$. Since η is a individually rational matching, $\eta(i)\ P_i\ \mu(i)$ which contradicts the definition of N. $\square$

Lemma 4. *An extended matching obtained by an SETC algorithm is* constrained efficient*.*

Proof. Let (μ, λ) be an extended matching obtained by an SETC algorithm. By Lemma 1, (μ, λ) is *(ex-post) stable*. We show that there is no *(ex-post) stable* extended matching (ν, λ') such that ν Pareto dominates μ. Assume to the contrary, that (ν, λ') is an *(ex-post) stable* extended matching and ν dominates μ. By the definition of the SETC algorithms, there is no improvement cycle in the graph $G(\mu, \lambda)$. There are two cases:

Case 1. For each $j \in I$ $\tilde{D}_{(\mu,\lambda)} = \{\varnothing\}$. Then for each $\in I$, $X_{(\mu,\lambda)}(j) \subseteq \mu^{-1}(j) \setminus \{j\}$. Thus each student is assigned to her best school at μ and ν does not Pareto dominate μ

Case 2. There are chains in $G(\mu, \lambda)$ involving students who would like to change her assigned school, but there is no cycle. This implies that there are students who are only pointed by the students assigned to the same school.

Assume we are in Case 2. Since there is no improvement cycle, there is a set of students who are not pointed by any other student in $G(\mu, \lambda)$. Let $I_1 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) = \varnothing\}$. Let $i_1 \in I_1$ and $s_1 = \mu(i_1)$. Note that for each $j \in \mu(s_1)$, $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ and $\mu(s_1) \subseteq I_1$ Since ν Pareto dominates μ, there does not exist any $j' \in I$, such that $\mu(j') \neq s_1$ and $\nu(j') = s_1$. Thus $\nu^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 3, $\#\mu^{-1}(s_1) = \#\nu^{-1}(s_1)$ and we get $\mu^{-1}(s_1) = \nu^{-1}(s_1)$. Since i_1 was arbitrary, this holds for each s such that $\mu^{-1}(s) \cap I_1 \neq \varnothing$.

Next, since there is no improvement cycle in $G(\mu, \lambda)$, then there is at least a student in $I \setminus I_1$ such that she is only pointed by students in I_1. Otherwise, there would be an improvement cycle or no improvement chains (Case 1). Let $I_2 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) \subseteq I_1\} \setminus I_1$ be the set of such students. Let $i_2 \in I_2$ and $s_2 = \mu(i_2)$. We first show that there is no j with $\mu(j) \neq s_2$ and $\nu(j) = s_2$. Assume to the contrary and since ν Pareto dominates μ, $s_2 \; P_j \; \mu(j)$ and thus, $j \in \tilde{D}_{(\mu,\lambda)}(i_2)$. Nevertheless, by definition i_2 is only pointed by students in I_1. By the previous paragraph, for each $j \in I_1$, $\mu(j) = \nu(j)$. Hence, $\nu^{-1}(s_2) \subseteq \mu^{-1}(s_2)$. By Lemma 3, $\#\mu^{-1}(s_2) = \#\nu^{-1}(s_2)$, and therefore $\mu^{-1}(s_2) = \nu^{-1}(s_2)$.

We can continue applying the same argument iteratively, to conclude that all students in any improving chain in $G(\mu, \lambda)$ have the same assignment under μ and ν. The students who are not in a chain in $G(\mu, \lambda)$, are contained in I_1 and have the same assignment in both μ and ν. We conclude that $\mu = \nu$ and ν does not Pareto dominate μ. $\square$

6.2 Proof of Theorem 1 "only if" part

Let (μ_0, λ_0) a partially stable extended matching. We prove that each *constrained efficient* matching that Pareto dominates (μ_0, λ_0) can be obtained by an algorithm in the SETC class.

We use again the notion of improvement cycle without making reference to the (directed) application graph associated with (μ, λ), $G(\mu, \lambda)$. The following lemma is a crucial first step for the construction of improvement cycles.

Lemma 5. *Let (μ, λ) and $(\nu, \bar{\lambda})$ be stable extended matchings such that ν Pareto dominates μ. Then there exists a set of disjointed improvement cycles $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ such that $\nu = \gamma_k \circ \ldots \circ \gamma_1 \circ \mu$, and there is λ'' obtained as in the definition of SETC such that (ν, λ'') is stable extended matching.*

Proof. Let $N \subseteq I$ be the set of students who strictly prefer their assignment under ν to the assignment under μ or such that $\lambda(i) \neq \lambda'(i)$. Partition the set N in three disjointed sets $N = N_1 \cup \ N_2 \cup N_3$. Define

$$\begin{aligned} N_1 &\equiv \{i \in N \mid \mu(i) = \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\}, \\ N_2 &\equiv \{i \in M \mid \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\}, \\ N_3 &\equiv \{i \in N \mid \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) = \lambda^{\nu(i)}(i)\}. \end{aligned}$$

Let $m = \#N$ and index the students in N in such that for each $j, j', j'' \in \{1, \ldots, m\}$ $i_j \in N_1$, $i_{j'} \in N_2$, $i_{j''} \in N_3$ if and only if $j < j' < j''$. Let $\tilde{G}[(\mu, \lambda), (\nu, \lambda')] = (N, E)$ be a directed graph such that the edges $E \subseteq N \times N$ are constructed in the following way:

- For each $i_j \in N_1$, i_j points l if and only if $\bar{\lambda}^{\mu(i_j)}(i_j) = \lambda^{\mu(i_j)}(l)$.
- For each $i_j \in N_2$, i_j points l if and only if $\bar{\lambda}^{\nu(i_j)}(i_j) = \lambda^{\nu(i_j)}(l)$.
- For each $i_j \in N_3$, i_j points an arbitrary student in $l \in N$ such that l has not been pointed by any $i_{j'}$ with $j' < j$ and $\mu(l) = \nu(i_j)$.[15]

In the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$, each student is pointed by a unique student and points to a unique student in N. Since N is finite, each student is in a cycle and no two cycles intersect. By construction, each of those cycles is an improvement cycle over μ and the extended matching $(\nu, \bar{\lambda})$ is obtained solving these cycles in any order. □

Lemma 6. *Let* (μ, λ) *be an* (ex-post) stable *and* $(\nu, \bar{\lambda})$ *a* (ex-post) stable *reshuffling of* (μ, λ) *such that* ν *Pareto dominates* μ*, then there exists a sequence of cycles* $(\gamma_1, \ldots, \gamma_k)$ *such that:*

- γ_1 *appears in* $G(\mu, \lambda)$*.*
- *For each* $k' \in \{2, \ldots, k\}$*,* $\gamma_{k'}$ *in* $G(\gamma_{k'-1} \circ \ldots \circ \gamma_1 \circ (\mu, \lambda))$*.*
- $\gamma_k \circ \gamma_{k-1} \circ \ldots \circ \gamma_1 \circ (\mu, \lambda)$*.*

Proof. By Lemma 5, we can construct a set of improvement cycles $\Phi = \{\phi_1, \ldots, \phi_q\}$. The result is trivial in the case where all the cycles in Φ appear in $G(\mu, \lambda)$: it follows that there are disjointed cycles in $G(\mu, \lambda)$ and solving them in any order leads to ν and to some λ'

[15]Note that since $(\nu, \bar{\lambda})$ is a reshuffling of (μ, λ) such a student l exists for each $i_j \in N_3$.

such that (ν, λ') is an *(ex-post) stable* reshuffling of (μ, λ). To prove the alternative case, we assume that none of the cycles in ϕ appears in $G(\mu, \lambda)$. This assumption is without loss of generality because of the following observation. If a cycle $\phi \in \Phi$ appears in $G(\mu, \lambda)$, then this cycle is solved first and $\mu' = \phi \circ \mu$ is obtained. If another cycle $\phi' \in \Phi$ also appears in $G(\mu', \lambda^*)$, by the fact that all the cycles in Φ are disjointed and that if there are two students forming a link in $G(\mu, \lambda)$, and those students do not belong to ϕ, then the link also appears in $G(\mu', \lambda^*)$. Following this logic, whenever a subset of cycles Φ appear in $G(\mu, \lambda)$, these cycles are solved first, and we focus on the case where none of the improvement cycles appear in $G(\mu, \lambda)$.

To show the existence of a cycle in $G(\mu, \lambda)$ first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu, \lambda)$ and $lk \in \phi'$ for some $l \in I$ and $\phi' \in \Phi$. Consider an arbitrary $\phi \in \Phi$ and $ij \in \phi$.

- If $i \in X_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$ by construction. Moreover, i is a part of ϕ, which implies there exists $l \in I$ with $li \in \phi$.
- If $i \notin X_{(\mu,\lambda)}(j)$, there exists a student i' such that $i' \in \tilde{D}_{(\mu,\lambda)}(j)$ and $(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Let k be, between those students, one such that $(k, \max\{\lambda^{\mu(j)}(k), \lambda^{\mu(j)}(j)\}) \succ_{\mu(j)} (k', \max\{\lambda^{\mu(j)}(k'), \lambda^{\mu(j)}(j)\})$ for each $k' \in D_{\mu,\lambda)}(j)$.[16] Note that $k \in X_{(\mu,\lambda)}(j)$, and therefore $kj \in G(\mu, \lambda)$. Finally, we check that k is in an improvement cycle in Φ. That is, there is $\phi' \in \Phi$ such that $lk \in \phi'$ for some $l \in I$. Assume to the contrary that $\mu(k) = \nu(k)$, and $\mu(j)\ P_k\ \mu(k) = \nu(k)$. Note that $k \in X_{(\mu,\lambda)}(j)$, $i \notin X_{(\mu,\lambda)}(j)$, $\nu(i) = \mu(j)$, and $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$). Since $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, this is a contradiction, since $(\nu, \bar{\lambda})$ is *(ex-post) stable*. Thus, $\nu(k)\ P_k\ \mu(k)$, which implies that k is in an improvement cycle in Φ.

Thus, for any student j who is in an improvement cycle $\varphi \in \Phi$, there exists another student k such that $kj \in G(\mu, \lambda)$ and k is in an improvement cycle $\phi' \in \Phi$. Since the set of students in improvement cycles is finite, and each student is pointed at least by another student in N, and there exists a cycle γ_1 in $G(\mu, \lambda)$. Note that for each $ij \in \phi$ such that $ij \notin \gamma_1$, then $ij \notin G(\mu, \lambda)$, and $i \notin X_{(\mu,\lambda)}(j)$.

[16] By our definition of extended priorities the existence of such a student k is ensured. See Remark 1.

We next show that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and it is weakly Pareto dominated by ν. Since $\gamma_1 \circ \mu$ solves a cycle in $G(\mu, \lambda)$ clearly $\gamma_1 \circ \mu$ Pareto dominates μ. Hence, we focus on proving that ν (weakly) Pareto dominates $\gamma_1 \circ \mu$. For any $kj \in \gamma_1$ such that $(\gamma_1 \circ \mu)(k) \neq \mu(k)$ note that $(\gamma_1 \circ \mu)(k) = \mu(j)$.

- If $kj \in \phi$ for some $\phi \in \Phi$, then $\nu(k) = \mu(j)$.
- If $kj \notin \phi$ for any $\phi \in \Phi$, we claim that $\nu(k)\ R_k\ \mu(j)$. Suppose that $\mu(j)\ P_k\ \nu(k)$, that is, $k \in \tilde{D}_{(\nu,\bar{\lambda})}(j)$. Consider the student $i \in I$ such that $ij \in \phi$ for some $\phi \in \Phi$, so $\nu(i) = \mu(j)$. By the definition of γ_1, $ij \notin G(\mu, \lambda)$. implies $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$. Since $kj \in \gamma_1$, $kj \in G(\mu, \lambda)$ and $ij \notin G(\mu, \lambda)$, $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(k)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, which is a contradiction because $(\nu, \bar{\lambda})$ is *(ex-post) stable*.

Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than under the matching μ and worse off than under the matching ν. Each remaining student is assigned to the same school to which she's assigned under μ which implies that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and is weakly Pareto dominated by ν. Let λ_1 be the allocation of characteristics obtained by solving the cycle γ_1 according to the definition of the SETC algorithm. By the arguments in Lemma 1, $(\gamma_1 \circ \mu, \lambda_1)$ is *(ex-post) stable*. If the extended matching $(\gamma_1 \circ \mu)$ is equivalent to ν the proof is complete. If not we can use the same argument inductively. By Lemma 6, there is a set of distinct improvement cycles, such that the matching ν is obtained by solving these cycles over $\gamma_1 \circ \mu$ solving at each stage a cycle that appears in the graph defined by the SETC algorithm. □

6.3 Proof of the Remaining Results

Proof of Proposition 1. Let A be an algorithm in the SETC, define the SOTC mechanism Ψ that for each profile of students' preferences selects the matching obtained through the application of A at that preference profile. By Theorem 1, for each preference profile the extended matching selected by Ψ is *(ex-post) stable* and *constrained efficient*. For each $P \in \mathcal{P}$, Ψ selects an extended matching (μ, λ) such that μ Pareto dominates μ^S_ω. By Abdulkadiroğlu et al. (2009), the SOSEM is in the Pareto frontier of the set of mechanisms that satisfy *strategy-proofness*. Hence, Ψ violates *strategy-proofness*. □

Proof of Proposition 2. Let $s = \mu(j)$. Since $\in \tilde{D}_{(\mu,\lambda)}(j)$, $s\ P_i\ \mu(i)$. Since (μ, λ) is *(ex-post) stable*, for each $j' \neq i$ such that $s\ P_{j'}\ \mu(j')$, $(j, \lambda^s(j)) \succ_s (j', \lambda^s(j'))$. Therefore, since s extended priorities are fully transferable, $(i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (j', \lambda^s(j'))$ and $i \in X_{(\mu,\lambda)}(j)$. □

Proof of Theorem 2. Note first that by Proposition 2, for each k and t, $\hat{G}^t_k \subseteq G(\mu, \lambda)$. Thus, since (μ_0, λ_0) is (ex-post) stable and by the arguments in Lemma 1, (μ^t_k, λ^t_k) is also *(ex-post) stable.*

Note also that since for each k, t, and $j \in I_k$, there is at most another student i such that $ij \in \hat{G}^t_k$. This fact implies that if for some k and t there are two cycles those cycles are disjointed ($\phi, \phi' \in \hat{G}^t_k$, $\phi \cap \phi' = \varnothing$).

If $U_0 = \{\varnothing\}$ the algorithm stops immediately, $(\mu^{SO}_\omega, \omega)$ is *constrained efficient*, and μ^{SO}_ω coincides with the outcome of the EADA algorithm. If $U_0 \neq \{\varnothing\}$, note that (μ_0, λ_0) is *(ex-post) stable* but it is not *constrained efficient.* We prove the result by comparing the graph $\hat{G}^0_1$ defined at Step 1 of the algorithm with the (directed) application graph associated with (μ^0, λ^0) obtained for a school choice problem with alternative students' preferences and extended priorities.

Consider the school choice problem $(I, S, (R^*_i)_{i\in I}, \omega, (\succ^*_s)_{s\in S}, (q_s)_{s\in S})$. such that for each $i \in I_1$, $R^*_i = R_i$ and for each $j \notin R^*_j$ is such that for each $s \in S \setminus \{\mu^0(j)\}$, $\mu^0(j)\ P_j\ s$, and for each $i, j \in I$ and each allocation of transferable characteristics λ, $(i, \lambda^s(i)) \succsim^*_s (j, \lambda^s(j))$ if and only if $(i, \omega^s(i)) \succsim^*_s (j, \omega^s(j))$. That is, students assigned to underdemanded schools under μ^0 consider that school as the best possible alternative, and transferable characteristics are irrelevant for school priorities. For each extended matching (μ, λ), let's denote by $G^*(\mu, \lambda)$ the (directed) application graph associated with (μ, λ) for the problem $(I, S, (R^*_i)_{i\in I}, \omega, (\succ^*_s)_{s\in S}, (q_s)_{s\in S})$ Note that $G^*(\mu^0, \lambda^0)$ coincides with $\hat{G}^0_1$. By Theorem 1, starting with an *(ex-post) stable* extended matching, the SETC algorithm obtains a *constrained efficient* extended matching. Note that under the new priorities and preferences, since the transferable characteristics are irrelevant, the student-proposing DA algorithm obtains the unique *constrained efficient* matching (Gale and Shapley, 1962). This fact also implies that the order in which the cycles are solved at any stage $1.t$ is irrelevant and a unique extended matching (μ_1, λ_1) is obtained, and μ_1 coincides with the outcome of the student-proposing DA algorithm under the preferences $(R^*_i)_{i\in I}$. Hence, for each $t \geq 1$, $G^*(\mu^t, \lambda^t)$ also coincides with $\hat{G}_1^t$. We can repeat the argument for the subsequent steps, until for some $k \geq 0$, $U_k = \varnothing$, which completes the proof. □

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