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Article

Locally Quasi-Convex Compatible Topologies on a Topological Group

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Abstract: For a locally quasi-convex topological abelian group (G, τ), we study the poset $\mathscr{C}(G, \tau)$ of all locally quasi-convex topologies on G that are compatible with τ (*i.e.*, have the same dual as (G, τ)) ordered by inclusion. Obviously, this poset has always a bottom element, namely the weak topology $\sigma(G, \widehat{G})$. Whether it has also a top element is an open question. We study both quantitative aspects of this poset (its size) and its qualitative aspects, e.g., its chains and anti-chains. Since we are mostly interested in estimates "from below", our strategy consists of finding appropriate subgroups H of G that are easier to handle and show that $\mathscr{C}(H)$ and $\mathscr{C}(G/H)$ are large and embed, as a poset, in $\mathscr{C}(G, \tau)$. Important special results are: (i) if K is a compact subgroup of a locally quasi-convex group G, then $\mathscr{C}(G)$ and $\mathscr{C}(G/K)$ are quasi-isomorphic (3.15); (ii) if D is a discrete abelian group of infinite rank, then $\mathscr{C}(D)$ is quasi-isomorphic to the poset $\mathfrak{F}_D$ of filters on D (4.5). Combining both results, we prove that for an LCA (locally compact abelian) group G with an open subgroup of infinite co-rank (this class includes, among others, all non-σ-compact LCA groups), the poset $\mathscr{C}(G)$ is as big as the underlying topological structure of (G, τ) (and set theory) allows. For a metrizable connected compact group X, the group of null sequences $G = c_0(X)$ with the topology of uniform convergence is studied. We prove that $\mathscr{C}(G)$ is quasi-isomorphic to $\mathscr{P}(\mathbb{R})$ (6.9).

Keywords: locally quasi-convex topology; compatible topology; quasi-convex sequence; quasi-isomorphic posets; free filters; Mackey groups

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1. Introduction

All groups in this paper are abelian, and for a group G, we denote by $\mathcal{L}(G)$ (resp., $\mathcal{T}(G)$) the set of all (Hausdorff) group topologies on G.

Varopoulos posed the question of the description of the group topologies on an abelian group G having a given character group H and called them compatible topologies for the duality (G, H) [1]. As the author explains, the question is motivated by Mackey's theorem, which holds in the framework of locally convex spaces. He treated the question within the class of locally precompact abelian groups. Later on, this problem was set in a bigger generality in [2]; namely, within the class of locally quasi-convex groups. This is a class of abelian topological groups, which properly contains the class of locally convex spaces, a fact that makes the attempt to generalize the Mackey–Arens Theorem more natural. We denote by $\mathscr{C}(G, \tau)$, or simply by $\mathscr{C}(G)$ if no misunderstanding can arise, the set of all locally quasi-convex group topologies on G, which are compatible for $G^\wedge$, the character group of G.

The bottom element of $\mathscr{C}(G, \tau)$ is the weak topology $\sigma(G, G^\wedge)$. In [2], it was asked if the poset $\mathscr{C}(G, \tau)$ has a top element. We denote the supremum (in $\mathcal{L}(G)$) of $\mathscr{C}(G, \tau)$ by $S(G, \tau)$. Then, $S(G, \tau)$ is a locally quasi-convex topology; nevertheless, the question of whether $S(G, \tau) \in \mathscr{C}(G, \tau)$ is still open. In case $S(G, \tau) \in \mathscr{C}(G, \tau)$, it is called Mackey topology and denoted by $\mu(G, G^\wedge)$. Furthermore, if $\tau = \mu(G, G^\wedge)$, the group (G, τ) is called a Mackey group. This is a generalization of the notion of a Mackey locally convex space.

Theorem 1.1. ([2]) *If a locally quasi-convex group G is Čech complete (in particular, complete metrizable or locally compact abelian (LCA)), then G is a Mackey group.*

In particular, one has the following immediate corollary concerning the special case when $|\mathscr{C}(G, \tau)| = 1$ (*i.e.*, $\tau = \sigma(G, G^\wedge) = \mu(G, G^\wedge)$):

Corollary 1.2. *If G is an LCA group, then $|\mathscr{C}(G)| = 1$ if and only if G is compact.*

Further attention to topological groups G with $|\mathscr{C}(G)| = 1$ is paid in [3,4], where many examples are given, inspired by [5] (in particular, it is proven that this equality holds for pseudocompact abelian groups).

This paper offers a solution for the following questions from [3] in the case that G is a non-σ-compact LCA group:

Questions 1.3. Let G be a locally quasi-convex topological group.

(a) [3] (Question 8.92) Compute the cardinality of the poset $\mathscr{C}(G)$.
(b) [3] (Problem 8.93) Under which conditions on the group G is the poset $\mathscr{C}(G)$ a chain?

More precisely, in the light of Corollary 1.2, we show that if an LCA group is sufficiently far from being compact (e.g., non-σ-compact), then the poset $\mathscr{C}(G)$ is as big (and as far from being a chain) as possible (see Section 1.2 for details).

1.1. Measuring Posets of Group Topologies

In order to face Question 1.3, one needs a tool to measure the poset $\mathscr{C}(G)$ of group topologies.

The complete lattice $\mathcal{L}(G)$ for a group G and some of its subsets (e.g., $\mathcal{T}(G)$, the subset $\mathcal{B}(G)$ of precompact topologies, its subset $\mathcal{P}sc(G)$ of pseudocompact topologies, *etc.*) have been studied by many authors [6–11]. In particular, many cardinal invariants of the specific subposets of $\mathcal{T}(G)$ have been computed by using the simple idea of replacing the complicated posets $\mathcal{T}(G)$, $\mathcal{B}(G)$, $\mathcal{P}sc(G)$, *etc.*, by some naturally-defined simple posets of purely combinatorial nature (e.g., the powerset $\mathscr{P}(G)$ ordered by inclusion). Since $\mathscr{C}(G) \cap \mathcal{B}(G) = \{\sigma(G, \widehat{G})\}$ is a singleton for a locally quasi-convex group G, we cannot make use of the above-mentioned results where $\mathcal{B}(G)$ (and even $\mathcal{P}sc(G)$) were shown to be as big as possible.

Let us recall that a subset A of a partially-ordered set X is called anti-chain if its members are pairwise incomparable. The maximal size of an anti-chain in a partially-ordered set X is called the width of X and denoted by $\mathrm{width}(X)$.

In order to measure the size and width of the poset $\mathscr{C}(G)$, we introduce the following notion:

Definition 1.4. Two posets $(X, \leq)$ and $(Y, \leq)$ are:

- isomorphic (we write $X \cong Y$) if there exists a poset isomorphism $X \longrightarrow Y$;
- ([9–11]) quasi-isomorphic (we write $X \overset{q.i.}{\cong} Y$) if there exist poset embeddings:

$$(X, \leq) \hookrightarrow (Y, \leq) \quad \text{and} \quad (Y, \leq) \hookrightarrow (X, \leq)$$

Clearly, quasi-isomorphic posets share the same monotone cardinal invariants, e.g., cardinality, width, maximum size of chains, *etc*. As a "sample" poset of combinatorial nature will be used, the poset is defined in the following example.

Example 1.5. Let X be an infinite set, and let $\mathbf{Fil}_X$ be the set of all free filters (*i.e.*, filters $\mathscr{F}$ on X with $\bigcap \mathscr{F} = \emptyset$) ordered by inclusion. The bottom element φ_0 of $\mathbf{Fil}_X$ is known as the Fréchet filter; its elements are the complements of finite sets. A filter on X is free iff it contains φ_0. For the sake of completeness, we shall add to $\mathbf{Fil}_X$ also the power set $\mathscr{P}(X)$ of X to obtain the complete lattice:

$$\mathfrak{F}_X := \mathbf{Fil}_X \cup \{\mathscr{P}(X)\} \subseteq \mathscr{P}(\mathscr{P}(X))$$

having as top element $\mathscr{P}(X)$ and bottom element φ_0. We shall denote $\mathfrak{F}_X$ also by $\mathfrak{F}_\kappa$ where $\kappa = |X|$, if we do not need to indicate the specific set X and only care about its size κ. Then:

$$\mathrm{width}(\mathfrak{F}_X) = |\mathfrak{F}_X| = 2^{2^{|X|}} \tag{1}$$

since there are $2^{2^{|X|}}$ ultrafilters on X ([12]) that obviously form an anti-chain; the reverse inequalities $\mathrm{width}(\mathfrak{F}_X) \leq |\mathfrak{F}_X| \leq 2^{2^{|X|}}$ are obvious, as $\mathfrak{F}_X$ is contained in the power set $\mathscr{P}(\mathscr{P}(X))$ of $\mathscr{P}(X)$ having size $2^{2^{|X|}}$.

Our choice of $\mathfrak{F}_X$ as a sample poset is justified by the following simple fact:

Proposition 1.6. *For every infinite group G, there exists a poset embedding $\mathcal{T}(G) \to \mathfrak{F}_{|G|}$.*

Proof. Every Hausdorff group topology τ on G is completely determined by the filter $\mathcal{N}_{G,\tau}$ of all τ-neighborhoods of zero. Since τ is Hausdorff, zero is the only common point of all members of $\mathcal{N}_{G,\tau}$. Hence, by restricting this filter to the set $X = G \setminus \{0\}$, we obtain an element of $\mathfrak{F}_X$. In other words, we defined an injective monotone map $\mathcal{T}(G) \to \mathfrak{F}_X$. Note that the discrete topology on X is mapped to $\mathscr{P}(X)$. $\square$

Let us recall that a poset $(X, \leq)$ is called a dcpo (directedly complete poset) if all directed suprema in X exist ([13] (Definition 2.1.13)). Clearly, a dcpo with a bottom element is a complete lattice precisely when it is a lattice. The relevant fact that the poset $\mathscr{C}(G,\tau)$ is a dcpo was established in [2] (Proposition 1.14):

Fact 1.7. [2] $\mathscr{C}(G,\tau)$ is a dcpo, *i.e.*, stable under directed suprema taken in the complete lattice $\mathcal{L}(G)$.

According to the above fact, the poset $\mathscr{C}(G)$ has maximal elements (actually, every element of $\mathscr{C}(G)$ is contained in a maximal element of $\mathscr{C}(G)$), so it has a top element precisely when it is a lattice. In such a case, $\mathscr{C}(G)$ is a complete lattice, and G has a Mackey topology.

1.2. Main Results

We give for a large class of LCA groups concrete descriptions of the set of compatible locally quasi-convex group topologies. More precisely, we first establish the following.

(a) If G is a locally quasi-convex abelian group and K is a compact subgroup of G, then $\mathscr{C}(G) \cong \mathscr{C}(G/K)$ (Theorem 3.15).

(b) If H is an open subgroup of G, then there exist poset embeddings $\mathscr{C}(H) \overset{\Psi}{\hookrightarrow} \mathscr{C}(G) \overset{\Theta}{\hookleftarrow} \mathscr{C}(G/H)$ (Theorem 3.6 and Corollary 3.11).

(c) For every discrete group D of infinite rank, the set $\mathscr{C}(D)$ is quasi-isomorphic to the set of filters on D (Lemma 4.5).

Item (a) above gives a precise and useful form of the intuitive understanding (based on Corollary 1.2) that compact subgroups are "negligible" when $\mathscr{C}(G)$ is computed for a locally quasi-convex group G. In particular, this holds for an LCA group G and a compact subgroup K. This allows us to reduce the computation of $\mathscr{C}(G)$ to the case of much simpler (e.g., discrete) groups G.

Items (b) and (c) yield the following:

Theorem A. If H is an open subgroup of a locally quasi-convex group $(G, \mathcal{T})$ with $r(G/H) \geq \omega$, then there is a poset embedding:

$$\mathfrak{F}_{|G/H|} \hookrightarrow \mathscr{C}(G) \tag{2}$$

so that $|\mathscr{C}(G)| \geq \mathrm{width}(\mathscr{C}(G)) \geq 2^{2^{|G/H|}}$.

Corollary 1.8. *If a locally quasi-convex group $(G, \mathcal{T})$ has an open subgroup of infinite co-rank, then:*

$$|\mathscr{C}(G)| \geq \mathrm{width}(\mathscr{C}(G)) \geq 2^{\mathfrak{c}}$$

so $\mathscr{C}(G)$ is not a chain.

This answers Problem 8.93 of [3] (*i.e.*, Question 1.3(b)) in the case of groups that have a discrete quotient of infinite rank.

Item (b) and Theorem A suggest considering open subgroups H of G with the highest possible co-rank. Motivated by this, we introduce the following notion.

Definition 1.9. A topological abelian group G is called *r-disconnected* if G has an open subgroup H of infinite co-rank.

For example, a non-σ-compact LCA group is r-disconnected (see Section 2 for a proof, as well as for further details). According to (2), r-disconnected groups G have a sufficiently large poset $\mathscr{C}(G)$. The term r-disconnected is suggested by the fact that an r-disconnected group G has open subgroups H of infinite co-rank, so its degree of disconnectedness of G "is measured" by the rank of G/H. Now, we introduce a cardinal invariant to carry out the measuring of r-disconnectedness in a natural way:

Definition 1.10. For a topological group G define the discrete rank (d-rank) of G by:

$$\varrho(G) = \sup\{r(G/H) : H \text{ open subgroup of } G\} \tag{3}$$

Clearly, r-disconnected groups G have infinite $\varrho(G)$, but a group with $\varrho(G) = \omega$ need not be r-disconnected unless G is LCA. For the properties of this cardinal invariant and its connection to the compact covering number $k(G)$, see Section 2.

In the case of LCA groups, it is possible to provide an embedding in the opposite direction of the embedding (2). In fact, we prove the following theorem (see Section 5 for the proof of Theorems A and B).

Theorem B. For every r-disconnected LCA group G, there exist poset embeddings:

$$\mathfrak{F}_{\varrho(G)} \hookrightarrow \mathscr{C}(G) \hookrightarrow \mathfrak{F}_{\mathfrak{c}\cdot\varrho(G)} \tag{4}$$

in particular,

$$2^{2^{\varrho(G)}} \leq |\mathscr{C}(G)| \leq 2^{2^{\mathfrak{c}\cdot\varrho(G)}} \quad \text{and} \quad 2^{2^{\varrho(G)}} \leq \mathrm{width}(\mathscr{C}(G)) \leq 2^{2^{\mathfrak{c}\cdot\varrho(G)}}$$

If G is totally disconnected, then $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{\varrho(G)}$.

It turns out that the inclusions (4) hold also under a slightly stronger condition of purely topological flavor

Corollary C. Let G be a non σ-compact LCA group, then $\varrho(G) > \omega$; so, the inclusions (4) hold.

The proof of this corollary will be given in Section 5.

Since both sides of the concluding inequality in Theorem B coincide when $2^{\varrho(G)} \geq 2^{\mathfrak{c}}$, we obtain the equality $|\mathscr{C}(G)| = 2^{2^{\varrho(G)}}$. One can say something more precise under a stronger assumption:

Corollary 1.11. *If an LCA group G has $\varrho(G) \geq \mathfrak{c}$, then $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{\varrho(G)}$.*

Since $\varrho(G) \geq \mathfrak{c}$ implies that the group G is r-disconnected, Theorem B applies. Now, the assertion follows from the inclusions (4).

From Corollary C and Corollary 1.11, one can immediately deduce:

Corollary 1.12. *Under the assumption of CH, $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{\varrho(G)}$ holds for every non-σ-compact LCA group.*

The last section is dedicated to a natural generalization of the metrizable LCA groups, namely the complete metrizable locally quasi-convex (so, necessarily Mackey) groups. In Theorem 6.1, we prove that for every non-trivial compact connected metrizable group X, the group $c_0(X)$ of null sequences in X carries a Polish Mackey topology τ, and $\mathscr{C}(G)$ is quasi-isomorphic to $\mathscr{P}(\mathbb{N})$, so it contains exactly $\mathfrak{c}$ many connected separable metrizable locally quasi-convex non-Mackey topologies compatible with $\mathfrak{p}_0$, the topology induced by the product topology $c_0(X) \hookrightarrow X^{\mathbb{N}}$.

The paper is organized as follows. Properties of the d-rank and its connection to the compact covering number are exposed in Section 2. In Section 3, we give general properties of the compatible topologies and of the poset $\mathscr{C}(G)$ (mainly invariance properties w.r.t. passage to products, subgroups and quotient groups). They enable us to prove in Section 4 that $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{|G|}$ for every discrete group G of infinite rank. Using this fact, we prove Theorems A and B and Corollary C in Section 5. Finally, in Section 6, we investigate $\mathscr{C}(G)$ for $G = c_0(X)$, where X is a non-trivial, compact connected metrizable group, and we show that $\mathscr{C}(G)$ is quasi-isomorphic to $\mathscr{P}(\mathbb{R})$. In Section 7, we collect final remarks and open questions.

The main results of the paper were exposed in talks of the second named author at the Prague TopoSym 2011 and at the Seventh Italian Spanish Conference on General Topology in Badajoz in September 2010, as well as at a Colloquim talk at Complutense University of Madrid in 2010.

Notation and preliminaries. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the natural numbers, by $\mathbb{P}$ the prime numbers, by $\mathbb{Z}$ the group of integers, by $\mathbb{Q}$ the group of rational numbers, by $\mathbb{R}$ the group of real numbers and by $\mathbb{Z}_m$ the cyclic group of order m. Let $\mathbb{T}$ denote the quotient group $\mathbb{R}/\mathbb{Z}$. We shall identify it with the interval $(-1/2, 1/2]$ with addition modulo one. It is isomorphic to the unit circle in the complex plane, with the ordinary product of complex numbers. Let $\mathbb{T}_+ =: [-1/4, 1/4] \subseteq \mathbb{T}$. For a topological abelian group G, the character group or dual group $G^\wedge$ is the set of all continuous homomorphisms from G to $\mathbb{T}$, with addition defined pointwise.

For an abelian topological group (G, τ) and its dual $G^\wedge$, we shall denote the weak topology $\sigma(G, G^\wedge)$ also by τ^+. An abelian topological group (G, τ) is said to be maximally almost periodic (MAP), if the continuous characters of G separate the points of G (*i.e.*, if τ^+ is Hausdorff).

In the beginning of the 1950s, Vilenkin defined the notions of locally quasi-convex subsets and locally quasi-convex groups for abelian Hausdorff groups. These settings generalize the terms convexity and local convexity in topological vector spaces. A subset M of an abelian topological group G is said to be quasi-convex if every element in $G \setminus M$ can be separated from M by means of a continuous character. More precisely: for any $z \notin M$, there exists $\xi \in G^\wedge$, such that $\xi(M) \subseteq \mathbb{T}_+$ and $\xi(z) \notin \mathbb{T}_+$. An abelian topological group is called locally quasi-convex if it has a neighborhood basis of zero consisting of quasi-convex sets. The most prominent examples of locally quasi-convex groups are $\mathbb{T}$, $\mathbb{R}$ and discrete groups and, more generally, locally compact abelian groups and locally convex vector spaces (see [14] for an account of the properties of quasi-convex groups).

For an abelian group G, we shall denote by $\mathcal{LQC}(G)$ the poset of all Hausdorff locally quasi-convex group topologies on G. Then, $\mathcal{B}(G) \subseteq \mathcal{LQC}(G) \subseteq \mathcal{T}(G)$.

We use frequently the fact that the supremum of locally quasi-convex topologies is again locally quasi-convex.

Consider the mapping:

$$\mathcal{LQC}(G) \longrightarrow \mathrm{Hom}(G, \mathbb{T}),\ \tau \longmapsto (G, \tau)^\wedge$$

Its restriction to the set $\mathcal{B}(G)$ of all precompact topologies is injective. The image of this mapping is the set of all dense subgroups of the dual group of the discrete group G, and the fiber of $(G, \tau)^\wedge$ is precisely $\mathscr{C}(G, \tau)$. Therefore, $\{\mathscr{C}(G, \tau) : \tau \in \mathcal{LQC}(G)\}$ forms a partition of $\mathcal{LQC}(G)$. Since $(G, \tau)^\wedge = (G, \tau^+)^\wedge$ and τ^+ is a precompact, we obtain:

$$\mathcal{LQC}(G) = \bigcup_{\tau \in \mathcal{LQC}(G)} \mathscr{C}(G, \tau) = \bigcup_{\tau \in \mathcal{LQC}(G)} \mathscr{C}(G, \tau^+) = \biguplus_{\tau \in \mathcal{B}(G)} \mathscr{C}(G, \tau) \tag{5}$$

One of the aims of this paper is to show that each member of the partition (5) containing a non-σ-compact LCA group topology has the same size as the whole $\mathcal{LQC}(G)$ and actually the same size as $\mathcal{T}(G)$.

For an abelian group G, we denote by $r_0(G)$ the free-rank of G, and for a prime p, we denote by $r_p(G)$ the p-rank of G (namely, $\dim_{\mathbb{Z}_p} \{x \in G : px = 0\}$). Finally, the rank of G is defined by:

$$r(G) = r_0(G) + \sup\{r_p(G) : p \in \mathbb{P}\}$$

For a subgroup H of G, the co-rank of H in G is defined to be the rank $r(G/H)$ of the quotient group.

2. The Connections between the Compact Covering Number and the d-Rank of Topological Groups

For a topological group G, we denote by $k(G)$ the compact covering number of G, *i.e.*, the minimum number of compact sets whose union covers G. Clearly, either $k(G) = 1$ (precisely when G is compact) or $k(G) \geq \omega$. If $k(G) \leq \omega$, the group is called σ-compact. We shall see below (Theorem 2.7) that there is a close connection between the cardinal invariants $k(G)$ and $\varrho(G)$.

Clearly, $\varrho(G) = 0$ precisely when G has no proper open subgroups. This class of groups (in different, but equivalent terms) was introduced by Enflo [15] under the name locally generated groups (connected groups are obviously the leading example of locally generated groups). Therefore, the non-r-disconnected groups G (in particular, the groups with $\varrho(G) < \omega$) can be considered as a natural generalization of the locally generated groups introduced by Enflo.

Furthermore, $\varrho(G) \leq \omega$ when G is either σ-compact or separable, since both conditions imply that all discrete quotients of G are countable (see Theorem 2.7 for the connection between σ-compactness and countability of the d-rank ϱ).

As the next example shows, the supremum in Equation (3) need not be attained by any specific open subgroup of G. We shall see in Theorem 2.7 that for r-disconnected LCA groups, this supremum is always a maximum.

Example 2.1. Consider the group $G = \bigoplus_n G_n$, where $G_n = \bigoplus_{\aleph_n} \mathbb{Q}$ is discrete and G carries the product topology. A base of neighborhoods of zero is formed by the open subgroups $W_m = \bigoplus_{n>m} G_n$, so that G is r-disconnected as $r(G/W_m) = \aleph_m$ for every $m \in \mathbb{N}$. Moreover, $\varrho(G) = |G| = \aleph_\omega$. On the other hand, every open subgroup H of G contains some W_m, so $r(G/H) \leq r(G/W_m) = \aleph_m < \varrho(G)$.

Generalizing the well-known fact that connected locally compact groups are σ-compact, we characterize below the LCA groups that are not r-disconnected, showing that they are σ-compact of a very special form (see Example 2.6). They will be the object of study in [16], whereas in this paper, we are interested in r-disconnected groups that may well be σ-compact (an example to this effect is any discrete countable group of infinite rank).

Let us start with the description of the groups of finite rank.

Fact 2.2. Let G be an infinite abelian group. Then, the following statements are equivalent:

(a) $r(G) < \infty$;
(b) G is isomorphic to a subgroup of a group of the form $\mathbb{Q}^m \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$, where $m, k \in \mathbb{N}$ and $p_1, \ldots, p_k$ are not necessarily distinct primes;
(c) $G \cong L \times F \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$, where $k \in \mathbb{N}$, F is a finite abelian group, L is a subgroup of $\mathbb{Q}^m$ ($m \in \mathbb{N}$) and $p_1, \ldots, p_k$ are not necessarily distinct primes;
(d) G contains no infinite direct sum of non-trivial subgroups;
(e) G contains no subgroup H, which is a direct sum of $|G|$-many non-trivial subgroups.

Proof. (a) $\to$ (b). Let $D(G)$ be the divisible hull of G. Then, $r(D(G)) = r(G)$, since $r_0(D(G)) = r_0(G)$ and $r_p(D(G)) = r_p(G)$ by the fact that G is essential in $D(G)$ (see [17] (Lemma 24.3)). Therefore, (a) implies that $r(D(G)) < \infty$. Now, (b) follows from the structure theorem for divisible abelian groups.

(b) $\to$ (c) Assume wlogthat G is a subgroup of $H = \mathbb{Q}^m \times D$, where $D = \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$. Then, the torsion subgroup $t(G)$ of G is a subgroup of D; hence, $t(G) = F \times T$ for subgroup $T \cong \prod_{s=1}^t \mathbb{Z}(p_{i_s}^\infty)$ of G and for appropriate $1 \leq i_1 < \ldots < i_t \leq k$ and a finite subgroup F of $\prod_{i=1}^k \mathbb{Z}(p_i^\infty)$. Since T is a divisible subgroup of G, it splits in G. Therefore, there exists a subgroup G_1 (containing F) of G, such that $G = G_1 \times T$. Note that $t(G_1) \cong t(G/T) = t(G)/T \cong F$, since T is a torsion group. By a theorem of Kulikov [17] (Theorem 27.5), the torsion part of an abelian group splits when it is finite, so we can write $G_1 = t(G_1) \times G_2$, where G_2 is a torsion-free subgroup of G_1 (isomorphic to $G_1/t(G_1) \cong G/t(G)$). Since $r_0(G_2) \leq r_0(G) \leq r_0(H) = m < \infty$, one has $D(G_2) = \mathbb{Q}^{m_2}$ for some $m_2 \leq m$, so G_2 is isomorphic to a subgroup of $\mathbb{Q}^m$.

(c) $\to$ (d) is obvious and (d) $\to$ (e) is trivial.

To prove the implication of (e) $\to$ (a), assume for a contradiction that $r(G)$ is infinite. Then, G contains a direct sum of $r(G)$-many non-trivial subgroups (it will be isomorphic to $(\bigoplus_{r_0(G)} \mathbb{Z}) \oplus (\bigoplus_p \bigoplus_{r_p(G)} \mathbb{Z}_p)$). To end the proof, it suffices to note that $|G| = r(G)$ whenever the latter cardinal is infinite [17]. □

It is important to note that while the d-rank is obviously monotone with respect to taking quotients, the rank is not (*i.e.*, a quotient of a group may have a bigger rank that the group itself, as we shall see in a while).

A group G of finite rank may have infinite $\varrho(G)$ (e.g., $r(\mathbb{Q}) = 1$, while $\varrho(\mathbb{Q}) = \omega$, witnessed by $r(\mathbb{Q}/\mathbb{Z}) = \omega$). In order to understand better the properties of the d-rank of rational groups, we introduce some specific rational groups.

Let $\mathbb{Q}_p$ be the subgroup of $\mathbb{Q}$ formed by all rational numbers having only powers of a given fixed prime number p in the denominator. For a set π of prime numbers, denote by $\mathbb{Q}_\pi$ the subgroup $\sum_{p\in\pi}\mathbb{Q}_p$ of $\mathbb{Q}$, *i.e.*, $\mathbb{Q}_\pi$ consists of those rational numbers for which the primes from $\mathbb{P}\setminus\pi$ are excluded from denominators. For completeness, let $\mathbb{Q}_\emptyset = \mathbb{Z}$.

Example 2.3. For $\pi \subseteq \mathbb{P}$, one has $r(\mathbb{Q}_\pi) = 1$ and $\varrho(\mathbb{Q}_\pi) = |\pi|$. In particular, $\varrho(\mathbb{Q}_\pi^n) < \infty$ (for $n \in \mathbb{N}$) if and only if π is finite. Actually, a torsion-free group G has finite d-rank if and only if G is isomorphic to a subgroup of $\mathbb{Q}_\pi^n$ for some $n \in \mathbb{N}$ and some finite $\pi \subseteq \mathbb{P}$ ([18] (Lemma 10.8)).

One can describe the non-r-disconnected discrete groups as follows:

Fact 2.4. ([18] (Lemma 10.12)) A discrete abelian group G is non-r-disconnected (*i.e.*, has $\varrho(G) < \infty$) if and only if $G \cong L \times F \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$, where L is a torsion-free non-r-disconnected group (*i.e.*, isomorphic to a subgroup of $\mathbb{Q}_\pi^n$ for some finite $\pi \subseteq \mathbb{P}$ and $n \in \mathbb{N}$), F is a finite abelian group, $k \in \mathbb{N}$ and $p_1, \dots, p_k$ are not necessarily distinct primes.

In order to describe also the non-r-disconnected LCA groups, we need a fundamental fact from the structure theory of LCA groups.

Fact 2.5. According to the structure theory, an LCA group G is topologically isomorphic to $\mathbb{R}^n \times H$, where $n \in \mathbb{N}$ and the group H has a compact open subgroup K ([19]). Therefore, the quotient group $D = H/K$ is discrete.

Example 2.6. Let G be a non-r-disconnected LCA group.

(a) G has a compact subgroup K, such that $G/K \cong \mathbb{R}^n \times L \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$ for some $n, k \in \mathbb{N}$, a torsion-free discrete non-r-disconnected group L, and not necessarily distinct primes p_i.

(b) The quotient $\mathbb{R}^n \times L \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$ does not depend on the choice of the compact group K up to isomorphism in the following sense. If K_1 is another compact subgroup of G, such that $G/K_1 \cong \mathbb{R}^{n_1} \times L_1 \times \prod_{j=1}^{k_1} \mathbb{Z}(q_j^\infty)$ with $n_1, k_1 \in \mathbb{N}$, a torsion-free discrete non-r-disconnected group L_1, and not necessarily distinct primes q_j, then $G/K \cong G/K_1$ (so $n_1 = n$, $L_1 \cong L$, $k_1 = k$ and $p_i = q_j$ for all $i = 1, 2, \dots, k$ and an appropriate permutation of the primes q_j).

Proof. (a) Indeed, as we saw above, there exists a closed subgroup H of G with a compact open subgroup K, such that $G = \mathbb{R}^n \times H$, so $N = \mathbb{R}^n \times K$ is an open subgroup of G. By our hypothesis, $H/K \cong G/N$ has finite d-rank. By Fact 2.4, $H/K \cong L \times F \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$, where L is a a torsion-free discrete non-r-disconnected group, F is a finite group and $p_1, \dots, p_k$, with $k \in \mathbb{N}$, are not necessarily distinct primes. Choosing K a bit larger, we can assume without loss of generality that $F = 0$. Since $G/K \cong \mathbb{R}^n \times H/K$, we are done.

(b) Since $K + K_1$ is a compact subgroup of G containing K and K_1 as open subgroups, both indexes $[(K + K_1) : K]$ and $[(K + K_1) : K_1]$ are finite. Therefore, $F = (K + K_1)/K$ is a finite subgroup of G/K, and similarly, $F_1 = (K + K_1)/K_1$ is a finite subgroup of G/K_1. Moreover,

$$(G/K)/F \cong G/(K + K_1) \cong (G/K_1)/F_1 \tag{6}$$

Since F is a finite subgroup of $t(G/K) \cong \prod_{i=1}^{k} \mathbb{Z}(p_i^\infty)$, it is easy to see that $t(G/K)/F \cong t(G/K)$. Similarly, $t(G/K_1)/F_1 \cong t(G/K_1)$. This yields the isomorphisms $(G/K)/F \cong G/K$ and $(G/K_1)/F_1 \cong G/K_1$. Now, the isomorphisms (6) yield the desired isomorphism $G/K \cong G/K_1$. These isomorphisms take the connected component $c(G/K) \cong \mathbb{R}^n$ of G/K to the the connected component $c(G/K_1) \cong \mathbb{R}^{n_1}$ of G/K_1; in particular, $n_1 = n$. Moreover, these isomorphisms takes the torsion subgroup $t(G/K) \cong \prod_{i=1}^{k} \mathbb{Z}(p_i^\infty)$ of G/K to the torsion subgroup $t(G/K_1) \cong \prod_{j=1}^{k_1} \mathbb{Z}(q_j^\infty)$ of G/K_1. Hence, $k_1 = k$, $p_i = q_j$ for all $i = 1, 2, \ldots, k$ and an appropriate permutation of the primes q_j. Finally, the isomorphism induces an isomorphism of the quotients $(G/K)/(c(G/K) \times t(G/K)) \cong L$ and $(G/K_1)/(c(G/K_1) \times t(G/K_1)) \cong L_1$. Therefore, we deduce $L_1 \cong L$. □

We need a property of the discrete rank $\varrho(G)$, when the group G is LCA and r-disconnected. Since it is related to the rank of discrete quotients of G, it has also a connection to the compact covering number $k(G)$, as Theorem 2.7 shows.

Theorem 2.7. *Let G be an r-disconnected group. Then, $k(G) \geq \varrho(G)$. If G is LCA, then:*

(a) G has an open σ-compact subgroup L with $r(G/L) = \varrho(G)$.
(b) every discrete quotient of G has a size at most $\varrho(G)$.
(c) there exists a compact subgroup N of G, such that $G/N \cong \mathbb{R}^n \times D$ for some discrete abelian group D with $|D| = \varrho(G)$ and $n \in \mathbb{N}$.
(d) $k(G) = \varrho(G)$.

Proof. Assume that $G = \bigcup_{i \in I} K_i$, where each K_i is compact. Then, for every open subgroup H of G with infinite G/H, the quotient map $q : G \to G/H$ takes each K_i to a finite subset of G/H. Since G/H is infinite, one has $|G/H| \leq |I|$. This proves that $\varrho(G) \leq k(G)$.

Now, suppose that G is LCA. We may assume without loss of generality that $G = \mathbb{R}^n \times H$ with n, H and K, as in Fact 2.5. Then, the subgroup $L = \mathbb{R}^n \times K$ of G is open and σ-compact. Moreover, $G/L \cong H/K$ is discrete. Every open subgroup of G contains $\mathbb{R}^n \times \{0\}$, so having the form $\mathbb{R}^n \times O$, for some open subgroup O of H. Hence, our hypothesis of r-disconnectedness on G implies that H is r-disconnected, as well, and $\varrho(G) = \varrho(H) \geq r(H/K) = |H/K|$.

(a) To see that $\varrho(G) = \varrho(H) \leq r(H/K)$, take any open subgroup N of H. We can assume wlog that N is contained in K. Then, the quotient K/N is both discrete and compact, hence finite. Therefore, $r(H/N) = r(H/K)$ as the quotient K/N is finite, while $r(H/N)$ and $r(H/K)$ are both infinite.

(b) Let $O \leq G$ be an open subgroup. If $r(G/O) < \infty$, then $|G/O| \leq \omega \leq \rho(G)$; otherwise, if $r(G/O) = \infty$, then $|G/O| = r(G/O) \leq \rho(G)$.

(c) follows from the above argument.

(d) We only have to check that $k(G) \leq \varrho(G)$. Since $\mathbb{R}^n$ is σ-compact and K is compact, one has $k(G) \leq \omega \cdot |D| = \omega \cdot \varrho(G) = \varrho(G)$. □

3. Some General Properties of Compatible Topologies

Let (G, τ) be a topological group. Another group topology ν on G is said to be compatible for (G, τ) if it has the same character group as the original one, that is if $(G, \nu)^\wedge = (G, \tau)^\wedge$.

We start with the following general facts providing easy upper bounds for $|\mathscr{C}(G, \tau)|$:

Proposition 3.1. *If (G, τ) is Mackey, then:*
(a) $|\mathscr{C}(G, \tau)| \leq 2^{|\tau|}$; so $|\mathscr{C}(G, \tau)| \leq 2^{2^{\mathrm{w}(G)}}$, where $\mathrm{w}(G)$ denotes the weight of G;
(b) if $\mathscr{C}_\kappa(G)$ denotes the subset of the compatible topologies of local weight $\leq \kappa$, then $|\mathscr{C}_\kappa(G)| \leq |\tau|^\kappa$. In particular, the cardinality of the set of metrizable compatible topologies is not greater than $|\tau|^\omega$.

Proof. (a) For the first inequality, it suffices to note that any compatible group topology is a subset of τ. For the second assertion, note that $|\tau| \leq 2^{\mathrm{w}(G)}$.

(b) follows from the fact that a group topology with local weight $\leq \kappa$ is completely determined by the assignment of a family of size $\leq \kappa$ that forms its local base at zero. □

In order to understand the structure (in particular, the size) of the poset $\mathscr{C}(G)$ for a topological group G, we relate $\mathscr{C}(G)$ to the corresponding posets $\mathscr{C}(H)$ and $\mathscr{C}(G/H)$ for convenient subgroups $H < G$.

Proposition 3.2. *Let $G = H_1 \times H_2$ be a locally quasi-convex group. Then,*

$$\mathscr{C}(H_1) \times \mathscr{C}(H_2) \longrightarrow \mathscr{C}(G),\ (\nu_1, \nu_2) \longmapsto \nu_1 \times \nu_2$$

is a poset embedding.

Proof. This mapping is injective and preserves the order. It remains to show that it is well defined. Therefore, let τ be the original topology on G and τ_1, τ_2 the induced topologies on H_1 and H_2 respectively. Take $\nu_1 \in \mathscr{C}(H_1, \tau_1)$ and $\nu_2 \in \mathscr{C}(H_2, \tau_2)$. Then, $\nu_1 \times \nu_2$ is again a locally quasi-convex topology on $G = H_1 \times H_2$. Clearly,

$$(H_1 \times H_2, \nu_1 \times \nu_2)^\wedge = (H_1, \nu_1)^\wedge \times (H_2, \nu_2)^\wedge = (H_1, \tau_1)^\wedge \times (H_2, \tau_2)^\wedge = (G, \tau)^\wedge$$

so $\nu_1 \times \nu_2$ is compatible. The assertion follows. □

A subgroup H of a topological abelian group G is called:

- dually closed if for every $x \in G \setminus H$, there exists $\chi \in G^\wedge$, such that $\chi(H) = \{0\}$ and $\chi(x) \neq 0$;
- dually embedded if each continuous character of H can be extended to a continuous character of G.

Remark 3.3. It is well known that if H is an open subgroup, it is dually closed and dually embedded, but in general, a closed subgroup need not have these properties.

(a) It is easy to see that a subgroup H of a topological group (G, τ) is dually closed if and only if H is τ^+-closed.
(b) It is straightforward to prove that if $G = H_1 \oplus H_2$ is equipped with the product topology, then H_1 and H_2 are dually embedded in G. Moreover, H_1 and H_2 are dually closed in G precisely when H_1 and H_2 are maximally almost periodic.

(c) It follows from Item (a) that a pair of compatible topologies shares the same dually-closed subgroups.

Lemma 3.4. *Let (G, τ) be a topological abelian group and H a dually-closed and dually-embedded subgroup. If $M \subseteq H$ is quasi-convex in H, it is also quasi-convex in G.*

Proof. We must check that every $x \in G \setminus M$ can be separated from M by means of a continuous character. Consider two cases:

(a) $x \in H \setminus M$. Then, there exists $\chi \in H^\wedge$, such that $\chi(M) \subseteq \mathbb{T}_+$ and $\chi(x) \notin \mathbb{T}_+$. Now, any extension of χ, say $\tilde{\chi} \in G^\wedge$, does the job.

(b) $x \in G \setminus H$. Since H is dually closed, there exists $\xi \in G^\wedge$, such that $\xi(H) = \{0\}$ and $\xi(x) \neq 0$. A suitable multiple of ξ gives a character that separates x from M. □

Proposition 3.5. *[20] (1.4) Every compact subgroup of a maximally almost periodic group is dually embedded and dually closed.*

Proposition 3.6. *Let (G, τ) be a locally quasi-convex group and H an open subgroup. Then, there exists a canonical poset embedding:*

$$\Psi : \mathscr{C}(H, \tau|_H) \longrightarrow \mathscr{C}(G, \tau)$$

Proof. Fix $(H, \nu) \in \mathscr{C}(H, \tau|_H)$, and let $\mathcal{N}_{H,\nu}(0)$ be a basis of quasi-convex zero neighborhoods for ν. If $\mathcal{N}_{H,\nu}(0)$ is considered as a basis of zero neighborhoods in G, we obtain a new group topology $\Psi(\nu)$ on G, for which H is an open subgroup. According to 3.4, $\Psi(\nu)$ is locally quasi-convex.

Since H is an open subgroup of G both w.r.t. to τ and to $\Psi(\nu)$ and since a homomorphism $\chi : G \to \mathbb{T}$ is continuous iff its restriction to an open subgroup is continuous, it is sufficient to prove that $(H, \Psi(\nu)|_H)$ is compatible with $(H, \tau|_H)$. Obviously, $\Psi(\nu)|_H = \nu \in \mathscr{C}(H, \tau|_H)$; hence, the assertion follows. □

Remark 3.7. In the sequel, we denote by $Q(\tau)$ the quotient topology of a topology τ (the quotient group will always be quite clear from the context).

Let H be a dually-closed subgroup of the Hausdorff abelian group G. By Remark 3.3(c), H is τ closed for every topology $\tau \in \mathscr{C}(G)$. Hence, for $\tau \in \mathscr{C}(G)$, the group topology $Q(\tau)$ is Hausdorff and:

$$Q : \mathscr{C}(G) \longrightarrow \mathcal{T}(G/H),\ \tau \longmapsto Q(\tau)$$

is an order-preserving mapping.

Of course, Q is in general not injective. The situation improves when K is a compact subgroup and G is a locally quasi-convex Hausdorff group (see Theorem 3.15).

The next fact is probably known, but hard to find in the literature; hence, we prefer to give a proof for the reader's convenience.

Lemma 3.8. *Let (G, τ) be a MAP abelian group and H a dually-closed subgroup. Then, for the quotient G/H, one has $Q(\tau)^+ = Q(\tau^+)$, i.e., the Bohr topology of the quotient coincides with the quotient topology of the Bohr topology τ^+.*

Proof. Since both $Q(\tau)^+$ and $Q(\tau^+)$ are precompact group topologies on G/H, it is sufficient to check that they are compatible. To this end, note that $Q(\tau)^+ \geq Q(\tau^+)$, since $Q(\tau) \geq Q(\tau^+)$ and $Q(\tau^+)$ is precompact. To see that they are compatible, take a $Q(\tau)^+$-continuous character $\chi : G/H \to \mathbb{T}$. Then, it is also $Q(\tau)$-continuous. Hence, the composition with the canonical projection $q : G \to G/H$ produces a τ-continuous character $\xi = \chi \circ q$ of G. Since ξ is τ^+-continuous, as well, from the factorization $\xi = \chi \circ q$, we deduce that χ is also $Q(\tau^+)$-continuous. Hence, $Q(\tau)^+ \leq Q(\tau^+)$. □

Next, we are interested in embedding $\mathscr{C}(G/H)$ into $\mathscr{C}(G)$. The following notation will be used in the sequel:

Notation 3.9. *Let H be a closed subgroup of the topological abelian group (G, τ). Denote by $q : G \to G/H$ the canonical projection. Further, for a group topology $\theta \in \mathcal{T}(G/H)$, we denote by $q^{-1}(\theta)$ the initial topology, namely the group topology $\{q^{-1}(O) : O \in \theta\}$. It is straightforward to prove that whenever θ is locally quasi-convex, then $q^{-1}(\theta)$ is locally quasi-convex, as well.*

Theorem 3.10. *Let H be a dually-closed subgroup of the locally quasi-convex Hausdorff group (G, τ). The mapping:*

$$\Theta : \mathscr{C}(G/H, Q(\tau)) \longrightarrow \mathscr{C}(G, \tau),\ \theta \longrightarrow q^{-1}(\theta) \vee \tau^+$$

is a poset embedding with left inverse Q.

Proof. According to 3.9, $\Theta(\theta)$ is a locally quasi-convex group topology on G and finer than τ^+, hence Hausdorff.

Before proving that $\Theta(\theta)$ is compatible, we show that $Q \circ \Theta(\theta) = \theta$. This will imply (once it is shown that Θ is well defined) that Q is a left inverse for Θ, and hence, Θ is injective. A neighborhood basis of zero in $(G, \Theta(\theta))$ is given by sets of the form $V = q^{-1}(U) \cap W$ where W is a neighborhood of zero in (G, τ^+) and U a neighborhood of zero in $(G/H, \theta)$. Observe that $q(V) = U \cap q(W)$. This implies the first equality in the following chain of equalities:

$$Q(\Theta(\theta))) = \theta \vee Q(\tau^+) \stackrel{3.8}{=} \theta \vee Q(\tau)^+ = \theta$$

(the last equality follows from $\theta \geq Q(\tau)^+$).

Let us show now that $\Theta(\theta)$ is compatible. Since $\Theta(\theta) \geq \tau^+$, we only need to show that if $\chi \in (G, \Theta(\theta))^\wedge$, then χ is also continuous with respect to τ. It is easy to check that $\Theta(\theta)|_H = \tau^+|_H$. Hence, we obtain that $\chi|_H : (H, \tau^+|_H) \to \mathbb{T}$ is continuous. Thus, $\chi|_H$ is a continuous character of the precompact group $(H, \tau^+|_H)$, which is dually embedded in (G, τ^+). Hence, there exists a continuous character $\chi_1 \in (G, \tau^+)^\wedge$, which extends χ. Since $\chi_1 \in (G, \tau^+)^\wedge = (G, \tau)^\wedge$, it is sufficient to show that $\chi - \chi_1 \in (G, \tau)^\wedge$ or, equivalently, we may suppose that $\chi \in H^\perp$.

Hence, $\overline{\chi} : (G/H, \underbrace{Q(\Theta(\theta))}_{=\theta}) \to \mathbb{T},\ x + H \mapsto \chi(x)$ is well defined and continuous. Since θ is compatible with $Q(\tau)$, we deduce that $\overline{\chi} : (G/H, Q(\tau)) \to \mathbb{T}$ is continuous, and hence, $\chi = \overline{\chi} \circ q : (G, \tau) \to \mathbb{T}$ is continuous.

This completes the proof. □

Since open or compact subgroups are dually closed, the above theorem gives:

Corollary 3.11. *Let G be a locally quasi-convex group, and let H be an open or a compact subgroup of G. Then, $\Theta : \mathscr{C}(G/H) \to \mathscr{C}(G)$ is a poset embedding.*

Remark 3.12. Let (G, τ) be a locally quasi-convex group, and let H be an open subgroup of G. The images $\Theta(\mathscr{C}(G/H))$ and $\Psi(\mathscr{C}(H))$ in $\mathscr{C}(G)$ of both embeddings, obtained in Theorem 3.6 and Corollary 3.11, meet in a singleton, namely:

$$\Theta(\mathscr{C}(G/H)) \cap \Psi(\mathscr{C}(H)) = \{\mathcal{T}_0 \vee \tau^+\}$$

where $\mathcal{T}_0$ is the topology on G with neighborhood basis $\{H\}$.

Indeed, one can see first that if $\mathcal{T} \in \Psi(\mathscr{C}(H))$, then $\mathcal{T} \geq \mathcal{T}_0 \vee \tau^+$. On the other hand, if $\mathcal{T} \in \Theta(\mathscr{C}(G/H))$, then $\mathcal{T} = \Theta(\theta) \leq \Theta(\delta_{G/H}) = \mathcal{T}_0 \vee \tau^+$ for some $\theta \in \mathscr{C}(G/H)$, where $\delta_{G/H}$ denotes the discrete topology on G/H. Combining both inclusions, we obtain $\mathcal{T} = \mathcal{T}_0 \vee \tau^+$. On the other hand, to see that the topology $\mathcal{T}_0 \vee \tau^+$ is an element of the intersection, it suffices to realize that $\mathcal{T}_0 \vee \tau^+ = \Psi(\tau_+|_H) = \Theta(\delta_{G/H})$.

In the proof of the next result, namely that for a locally quasi-convex group G and a compact subgroup K of G, the posets $\mathscr{C}(G)$ and $\mathscr{C}(G/K)$ are isomorphic, we need the following results from [21]:

Theorem 3.13. *Let (G, τ) be a locally quasi-convex Hausdorff group, and let K be a subgroup of G.*

(a) [21] (Theorem (3.5)) If $(K, \tau^+|_K)$ is compact, then $\tau|_K = \tau^+|_K$, in particular $(K, \tau|_K)$ is compact.
(b) [21] (Theorem (2.7)) If K is compact, then G/K is a locally quasi-convex Hausdorff group.

Further, we need:

Lemma 3.14 (Merzon)**.** *Let $\tau_1 \leq \tau_2$ be group topologies on a group G, and let H be a subgroup of G. If the subspace topologies $\tau_1|_H = \tau_2|_H$ and the quotient topologies $Q(\tau_1) = Q(\tau_2)$ coincide, then $\tau_1 = \tau_2$.*

Theorem 3.15. *Let (G, τ) be a locally quasi-convex Hausdorff group and K a compact subgroup of G. Then:*

$$\Theta : \mathscr{C}(G/K) \longrightarrow \mathscr{C}(G) \quad \textit{and} \quad Q : \mathscr{C}(G) \longrightarrow \mathscr{C}(G/K)$$

are mutually inverse poset isomorphisms.

Proof. Let us show first that $Q : \mathscr{C}(G) \to \mathscr{C}(G/K), \theta \mapsto Q(\theta)$ is well defined. According to 3.13(b), $Q(\theta)$ is a locally quasi-convex Hausdorff group topology. In order to show that $Q(\theta)$ is compatible for $\theta \in \mathscr{C}(G)$, we fix a continuous character $\chi : (G/K, Q(\theta)) \to \mathbb{T}$. Let $q : G \to G/K$ be the quotient homomorphism. Then, $\chi \circ q : (G, \theta) \to \mathbb{T}$ is continuous. Since θ is compatible for (G, τ), also $\chi \circ q : (G, \tau) \to \mathbb{T}$ is continuous, and hence, $\chi : (G, Q(\tau)) \to \mathbb{T}$ is continuous. On the other hand, $\theta \geq \tau^+$, and hence, $Q(\theta) \geq Q(\tau^+) = Q(\tau)^+$. Combining both conclusions, we obtain that $Q(\theta)$ is compatible for $(G, Q(\tau))$.

Taking into account 3.10, it is sufficient to prove that $\Theta \circ Q = \mathrm{id}_{\mathscr{C}(G)}$.

Therefore, fix $\theta \in \mathscr{C}(G)$. The topology $\Theta(Q(\theta)) = q^{-1}(Q(\theta)) \vee \tau^+$ is coarser than θ, so applying Merzon's Lemma, it is sufficient to show that θ and $q^{-1}(Q(\theta)) \vee \tau^+$ coincide on K and on G/K. Since

$\theta^+ = \tau^+$ and since $(H, \tau^+|_K)$ is compact, we obtain by Theorem 3.13(a) that $(K, \theta|_K) = (K, \theta^+|_K)$ is compact. Hence: $\theta|_K \geq \Theta(Q(\theta))|_K \geq \tau^+|_K = \theta^+|_K = \theta|_K$ imply that $\Theta(Q(\theta))|_K = \theta|_K$. For the quotient topologies, the following holds by Theorem 3.10: $Q(\Theta(Q(\theta))) = (Q \circ \Theta)(Q(\theta)) = Q(\theta)$. Combining the partial results, the theorem is proven. □

Since every locally compact abelian group is locally quasi-convex, from Theorem 3.15, we immediately obtain:

Corollary 3.16. *Let G be a locally compact abelian group and K a compact subgroup of G. Then, $\mathscr{C}(G) \cong \mathscr{C}(G/K)$.*

Corollary 3.17. *Let G be a σ-compact group with an open compact subgroup. Then, there is a poset embedding $\mathscr{C}(G) \hookrightarrow \mathfrak{F}_\omega$; in particular, $|\mathscr{C}(G)| \leq 2^{\mathfrak{c}}$.*

Proof. Let K be an open compact subgroup of G. Without loss of generality, we may assume that G is not compact, and hence, G/K is infinite.

As a consequence of the above theorem, $Q : \mathscr{C}(G) \to \mathscr{C}(G/K)$ is a poset isomorphism. Since G/K is countable and discrete, 1.6 establishes a poset embedding $\mathscr{C}(G/K) \hookrightarrow \mathcal{T}(G/K) \hookrightarrow \mathfrak{F}_\omega$. □

Note that a group G as in the above corollary is necessarily locally compact. We shall see in the sequel (see Example 2.6) that unless G has a very special structure, one has actually a quasi-isomorphism between $\mathscr{C}(G)$ and $\mathfrak{F}_\omega$, in particular $|\mathscr{C}(G)| = 2^{\mathfrak{c}}$.

4. Compatible Topologies for Discrete Abelian Groups

We intend to reduce the study of the poset $\mathscr{C}(G)$ for a LCA group G to the case of infinite direct sums of countable groups. This is why, throughout the first part of this section, γ will denote an infinite cardinal and C_α will be a non-zero countable group for every $\alpha < \gamma$. Our intention is to show that for the discrete group $G = \bigoplus_{\alpha<\gamma} C_\alpha$, the poset $\mathscr{C}(G)$ is quasi-isomorphic to $\mathfrak{F}_\gamma$ (so, it contains an anti-chain of the maximal possible size 2^{2^γ} (note that $\gamma = |G|$), in particular $\mathscr{C}(G)$ has width and size 2^{2^γ}).

In order to get (many) group topologies on G, we need a frequently-used standard construction based on filters on γ.

Notation 4.1. *Every free filter φ on γ defines a topology τ_φ on G with a base $\{W_B : B \in \varphi\}$ of neighborhoods of zero, where:*

$$W_B = \bigoplus_{\alpha \in B} C_\alpha$$

(here, we are identifying the direct sum defining W_B with a subgroup of G in the obvious way, by adding zeros in the coordinates $\alpha \notin B$). Using the fact that each W_B is a subgroup of G and $W_{B_1} \cap W_{B_2} = W_{B_1 \cap B_2}$ for $B_1, B_2 \in \varphi$, one can easily prove that τ_φ is a Hausdorff group topology on G. Moreover, τ_φ is locally quasi-convex, since every basic open neighborhood W_B is an open subgroup, hence quasi-convex.

Finally, let us mention the fact (although it will not be used here) that, according to [22], τ_φ is complete when φ is an ultrafilter.

Lemma 4.2. *Let* δ *denote the discrete topology on* G*. The mapping:*

$$\Xi : \mathfrak{F}_\gamma \longrightarrow \mathscr{C}(G),\ \varphi \longmapsto \tau_\varphi \vee \delta^+$$

is a poset embedding.

Proof. Obviously $\delta^+ \leq \tau_\varphi \vee \delta^+ \leq \delta$, so the topology $\tau_\varphi \vee \delta^+$ is compatible. Since δ^+ is precompact, δ^+ is also locally quasi-convex. Moreover, $\tau_\varphi \vee \delta^+$ is locally quasi-convex, as proven in Notation 4.1. This shows that Ξ is well defined.

It is obvious that Ξ preserves the order. Let us show that Ξ is injective: To that end we fix $\varphi, \psi \in \mathfrak{F}_\gamma$ and assume that $\tau_\varphi \vee \delta^+ = \tau_\psi \vee \delta^+$, in particular $\tau_\varphi \vee \delta^+ \leq \tau_\psi \vee \delta^+$, and hence, $\tau_\varphi \leq \tau_\psi \vee \delta^+$. Let $B \in \varphi$, then $W_B \in \tau_\varphi$, so $W_B \in \tau_\psi \vee \delta^+$ by our hypothesis. Then, there exist $B' \in \psi$ and a finite set $F \subseteq G^\wedge$, such that $U := W_{B'} \cap F^\triangleleft \subseteq W_B$. Since $U \subseteq W_{B'}$, as well, we conclude that $U \subseteq W_B \cap W_{B'} = W_{B\cap B'}$. Since U is a neighborhood of zero in the Bohr topology of the subgroup $W_{B'}$, this proves that also $W_{B\cap B'}$ is a neighborhood of zero in the Bohr topology of $W_{B'}$. This means that $W_{B\cap B'}$ is an open subgroup of $W_{B'}$ equipped with the Bohr topology. Since every open subgroup in a precompact topology must have finite index, we deduce that $W_{B\cap B'}$ has finite index in $W_{B'}$. Therefore, $W_{B'}/W_{B\cap B'} \cong \bigoplus_{\alpha \in B' \setminus B} C_\alpha$ is finite. Consequently, also the set $S = B' \setminus B$ is finite. Since $\gamma \setminus S \in \psi$, we conclude that also $B'' := B' \setminus S = B' \cap (\gamma \setminus S) \in \psi$. On the other hand, $B'' \setminus B = \emptyset$, *i.e.*, $B'' \subseteq B$. This proves that $B \in \psi$. Therefore, $\varphi \subseteq \psi$. The other inclusion is proven analogously. □

Remark 4.3. Observe that $\Xi(\mathcal{P}(\gamma)) = \delta$.

Corollary 4.4. *For* G *as above, the sets* $\mathscr{C}(G)$ *and* $\mathfrak{F}_\gamma$ *are quasi-isomorphic, in particular* $\mathrm{width}\,(\mathscr{C}(G)) = |\mathscr{C}(G)| = 2^{2^{|G|}}$.

Further, $\mathscr{C}(G)$ *has chains of size (at least)* γ*.*

Proof. According to 4.2 and 1.6, the sets $\mathscr{C}(G)$ and $\mathfrak{F}_\gamma$ are quasi-isomorphic. Since $|G| = \gamma$ and since there are 2^{2^γ} different free ultrafilters in $\mathfrak{F}_\gamma$, the first assertion follows.

$\mathfrak{F}_\gamma$ has chains of length γ. Indeed, one can easily produce a chain of length γ in $\mathfrak{F}_\gamma$ by using any partition of γ in γ pairwise disjoint sets of size γ each. □

Theorem 4.5. *Let* G *be a discrete abelian group of infinite rank. Then,* $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{|G|}$ *holds. In particular,* G *admits* $2^{2^{|G|}}$ *pairwise incomparable compatible group topologies and* $|\mathscr{C}(G)| = 2^{2^{|G|}}$.

Proof. Let $\gamma = |G|$. According to 1.6, there are poset embeddings:

$$\mathscr{C}(G) \longrightarrow \mathcal{T}(G) \longrightarrow \mathfrak{F}_\gamma.$$

It is easy to see that G has a subgroup H isomorphic to a direct sum $\bigoplus_{\alpha<\gamma} C_\alpha$, where each C_α is a non-trivial countable group (actually, it can be taken to be a cyclic group). According to Lemma 4.2 and Proposition 3.6, there are poset embeddings $\Xi : \mathfrak{F}_\gamma \to \mathscr{C}(H)$ and $\Psi : \mathscr{C}(H) \to \mathscr{C}(G)$. This proves that $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{|G|}$. The second assertion follows from (1). □

Corollary 4.6. *Suppose that* H *is a discrete abelian group for which* $\mathfrak{F}_\omega$ *does not embed in* $\mathscr{C}(H)$*. Then,* H *has finite* d*-rank; in particular,* H *is countable.*

Proof. If H has infinite d-rank, then 4.5 implies the existence of a poset embedding $\mathfrak{F}_{|H|} \to \mathscr{C}(H)$. Hence, the hypothesis implies that H has finite rank. □

The case of finite rank abelian groups will be considered separately in [16].

5. Proofs of Theorem A, Theorem B and Corollary C

We intend to apply the results obtained so far to describe the poset of all compatible group topologies for r-disconnected groups (in particular, for non-σ-compact LCA groups).

Here comes the proof of Theorem A. It shows that for an LCA group G with large discrete rank $\varrho(G)$, the poset $\mathscr{C}(G)$ is quite large (in particular, $|\mathscr{C}(G)| < 2^{2^{\varrho(G)}}$ implies, among other things, that G is σ-compact).

Proof of Theorem A. Let H be an open subgroup of the locally quasi-convex group $(G, \mathcal{T})$, such that G/H has infinite rank. According to Corollary 3.11 and Theorem 4.5, there is a poset embedding $\mathfrak{F}_{|G/H|} \hookrightarrow \mathscr{C}(G)$. The assertion follows. □

This theorem gives as a by-product a description of the LCA groups G, such that $\mathfrak{F}_\omega$ does not embed in $\mathscr{C}(G)$.

Corollary 5.1. *If G is an LCA group, such that $\mathfrak{F}_\omega$ does not embed in $\mathscr{C}(G)$, then G is non-r-disconnected (so G contains a compact subgroup K, such that $G/K \cong \mathbb{R}^n \times L \times \prod_{i=1}^{k} \mathbb{Z}(p_i^\infty)$ for some $n, m, k \in \mathbb{N}$, a subgroup L of $\mathbb{Q}^m$ and not necessarily distinct primes p_i).*

Proof. This follows from Example 2.6 and Theorem A. □

Proof of Theorem B. First, we have to prove that if G is an r-disconnected LCA group, then there exist poset embeddings as in formula (4). Since the existence of the first one was already established in Theorem A, it remains to produce the embedding $\mathscr{C}(G) \hookrightarrow \mathfrak{F}_{\mathfrak{c}\cdot\varrho(G)}$. To this end, we use the isomorphism $\mathscr{C}(G) \cong \mathscr{C}(\mathbb{R}^n \times D)$ provided by the above fact. Since $|\mathbb{R}^n \times D| = \mathfrak{c} \cdot \varrho(G)$, Proposition 1.6 applies.

Now, assume that G is totally disconnected. We have to prove that $\mathscr{C}(G) \overset{q.i.}{\cong} \mathfrak{F}_{\varrho(G)}$. According to Fact 2.5, the LCA group G is topologically isomorphic to $\mathbb{R}^n \times H$, where $n \in \mathbb{N}$ and the group H has a compact open subgroup K. Since the quotient group $D = H/K$ is discrete and $G/K \cong \mathbb{R}^n \times D$, Corollary 3.16 gives $\mathscr{C}(G) \cong \mathscr{C}(\mathbb{R}^n \times D)$. According to our hypothesis, G is totally disconnected, so G cannot contain subgroups of the form $\mathbb{R}^n$ with $n > 0$. Therefore, $G/K = D$ is discrete. Now, Corollary 3.16 implies that $\mathscr{C}(G) \cong \mathscr{C}(G/K)$. Moreover, $r(G/K)$ is infinite, since G is r-disconnected. Thus, $\mathscr{C}(G/K) \overset{q.i.}{\cong} \mathfrak{F}_{|G/K|} = \mathfrak{F}_{\varrho(G)}$ by Theorem 4.5. □

Proof of Corollary C. We have to prove that if G is non-σ-compact, then $\varrho(G) > \omega$, and there exists a poset embedding $\mathfrak{F}_{\varrho(G)} \hookrightarrow \mathscr{C}(G)$. By Theorem 2.7(d), we have $\varrho(G) = k(G) > \omega$; so, G is r-disconnected, and Theorem A applies. □

6. Metrizable Separable Mackey Groups with Many Compatible Topologies

For an abelian topological group X, we denote by $c_0(X)$ the subgroup of $X^{\mathbb{N}}$ consisting of all $(x_n) \in X^{\mathbb{N}}$, such that $x_n \to 0$ in X. We denote by $\mathfrak{p}$ and $\mathfrak{u}$ the product topology and the uniform topology of $X^{\mathbb{N}}$, respectively. We use $\mathfrak{p}_0$ and $\mathfrak{u}_0$ to denote the topologies induced by $\mathfrak{p}$ and $\mathfrak{u}$, respectively, on $c_0(X)$.

Throughout this section, let X be a non-trivial compact connected metrizable group:

Theorem 6.1. *[23] The group $G = (c_0(X), \mathfrak{u}_0)$ is a non-compact Polish connected Mackey group with $\mathfrak{u}_0^+ = \mathfrak{p}_0$.*

Observe that both $\mathfrak{u}_0$ and $\mathfrak{p}_0$ are metrizable and non-compact. In order to find more compatible topologies on G, we use the following construction.

Notation 6.2. *For a subset B of $\mathbb{N}$ and a neighborhood U of 0 in X, let:*

$$P(B,U) = c_0(X) \cap \left(U^B \times X^{\mathbb{N}\setminus B}\right)$$

The following properties of these sets will be frequently used in the sequel:

(a) $\bigcap_{i\in I} P(B_i,U) = P(\bigcup_{i\in I} B_i, U)$ for every subset $\{B_i : i \in I\} \subseteq \mathscr{P}(\mathbb{N})$;
(b) if $U \neq X$ and $P(B_1,U_1) \subseteq P(B,U)$, then $B \subseteq B_1$.

This follow directly from $X^{(\mathbb{N})} \cap (U_1^{B_1} \times X^{(\mathbb{N}\setminus B_1)}) \subseteq U^B \times X^{(\mathbb{N}\setminus B)}$.

Definition 6.3. For any $A \subseteq \mathbb{N}$, define a group topology $\mathfrak{t}_A$ on $c_0(X)$ having as a neighborhood basis at zero the family of sets $(P(B,U))$, where U runs through all neighborhoods of zero in X and B through all elements of $\mathscr{P}(\mathbb{N})$ with finite $A\Delta B$.

Obviously, $\mathfrak{t}_\emptyset = \mathfrak{p}_0$ and $\mathfrak{t}_\mathbb{N} = \mathfrak{u}_0$; more precisely, $\mathfrak{t}_A = \mathfrak{p}_0$ ($\mathfrak{t}_A = \mathfrak{u}_0$) if and only if A is finite (resp., co-finite).

Lemma 6.4. *If $P(A,U) \in \mathfrak{t}_B$ with $U \neq X$, then $A \setminus B$ is finite. Consequently, $A \setminus B$ is finite whenever $\mathfrak{t}_A \leq \mathfrak{t}_B$.*

Proof. By our hypothesis, there exists a subset B' of $\mathbb{N}$, such that $B'\Delta B$ is finite and $P(B',U') \subseteq P(A,U)$ for some neighborhood U' of 0 in X. Then, by Item (b) of Notation 6.2, $A \subseteq B'$. Since $|B'\Delta B| < \infty$, this proves that $A \setminus B$ is finite, as well.

Now, assume that $\mathfrak{t}_A \leq \mathfrak{t}_B$. Then, $P(A,U) \in \mathfrak{t}_A$, so $P(A,U) \in \mathfrak{t}_B$, as well. Hence, $A \setminus B$ is finite by the fist part of the argument. □

Proposition 6.5. *For the topological group $(G, \mathfrak{u}_0) = (c_0(X), \mathfrak{u}_0)$, the mapping:*

$$\mathscr{P}(\mathbb{N}) \longrightarrow \mathscr{C}(G),\ A \longmapsto \mathfrak{t}_A \tag{7}$$

is order preserving. Moreover, $\mathfrak{t}_A = \mathfrak{t}_B$ for $A, B \in \mathscr{P}(\mathbb{N})$ if and only if the symmetric difference $A\Delta B$ is finite.

Proof. The mapping (7) is well defined, since for any $A \subseteq \mathbb{N}$, we have $\mathfrak{p}_0 \subseteq \mathfrak{t}_A \subseteq \mathfrak{u}_0$. Since all of the topologies $\mathfrak{t}_A$ are also locally quasi-convex, they belong to $\mathscr{C}(G)$.

Assume that $\mathfrak{t}_A = \mathfrak{t}_B$. Applying twice the second assertion in Lemma 6.4, we conclude that $A\Delta B$ is finite. $\square$

Define an equivalence relation on $\mathscr{P}(\mathbb{N})$ by letting $A \sim B$ for $A, B \in \mathscr{P}(\mathbb{N})$ whenever $|A\Delta B| < \infty$. Denote the set of equivalence classes by $\mathscr{P}(\mathbb{N})_*$ and its elements by A_*; moreover, write $A \subseteq^* B$ ($A =^* B$) for subsets of $\mathbb{N}$ when $|A \setminus B| < \infty$ (resp., $A\Delta B$) is finite. In these terms, Proposition 6.5 can be reformulated as follows:

Corollary 6.6. *The mapping* $\mathscr{P}(\mathbb{N})_* \to \mathscr{C}(G),\ A_* \mapsto \mathfrak{t}_A$ *is a poset embedding.*

Remark 6.7. The above Proposition 6.5 implies that $\sup\{\mathfrak{t}_A, \mathfrak{t}_B\} \leq \mathfrak{t}_{A\cup B}$. However, one can easily check with Notation 6.2(a) that actually, $\sup\{\mathfrak{t}_A, \mathfrak{t}_B\} = \mathfrak{t}_{A\cup B}$ holds true. In particular, if $P(A, U) \in \sup\{\mathfrak{t}_{B_1}, \ldots, \mathfrak{t}_{B_n}\} = \mathfrak{t}_{B_1\cup\ldots\cup B_n}$ with $U \neq X$, then $A \subseteq^* \bigcup_i B_i$ by Lemma 6.4.

The structure of $(\mathscr{P}(\mathbb{N})_*, \subseteq)$ is very rich, as the following example shows.

Example 6.8. There exists an anti-chain of size $\mathfrak{c}$ in $(\mathscr{P}(\mathbb{N})_*, \subseteq)$. Although this fact is well known (see, for example, [24]), we give a brief argument for the reader's convenience.

As the poset $\mathscr{P}(\mathbb{N})_*$ is isomorphic to $\mathscr{P}(\mathbb{Q})_*$, it is enough to check that $\mathrm{width}(\mathscr{P}(\mathbb{Q})_*) = \mathfrak{c}$. For every $\rho \in \mathbb{R}$, pick a one-to-one sequence of rational numbers (r_n^ρ) converging to ρ. Then, the sets $A_\rho := \{r_n^\rho : n \in \mathbb{N}\}$, $\rho \in \mathbb{R}$, form an almost disjoint family witnessing $\mathrm{width}(\mathscr{P}(\mathbb{Q})_*) = \mathfrak{c}$.

All topologies of the form $\mathfrak{t}_A$ produced so far, including the top and the bottom element ($\mathfrak{u}_0$ and $\mathfrak{p}_0$, resp.) of $\mathscr{C}(G)$, are metrizable (so, second countable, since G is separable by (3.4) in [23]). This makes it natural to ask whether all compatible topologies are metrizable. The next theorem and Example 6.10 answer this question negatively in the strongest possible way.

Theorem 6.9. *The poset* $\mathscr{C}(G)$ *is quasi-isomorphic to* $\mathscr{P}(\mathbb{R})$*, so* $|\mathscr{C}(G)| = 2^{\mathfrak{c}}$ *and* $\mathscr{C}(G)$ *has chains of length* $\mathfrak{c}^+$.

Proof. According to Proposition 3.1, $\mathscr{C}(G)$ embeds into $\mathscr{P}(\mathbb{R})$ as $(G, \mathfrak{u}_0)$ is Mackey and $|\mathfrak{u}_0| = \mathfrak{c}$. Therefore, it suffices to show that $\mathscr{P}(\mathbb{R})$ embeds into $\mathscr{C}(G)$. It is enough to produce an embedding of $\mathscr{P}(\mathbb{R}) \setminus \{\emptyset\}$ into $\mathscr{C}(G) \setminus \{\mathfrak{p}_0\}$. To this end, we shall replace $\mathbb{R}$ by an almost disjoint family $\mathfrak{A}$ of size $\mathfrak{c}$ of infinite subsets of $\mathbb{N}$ (see Example 6.8).

For $\emptyset \neq \mathcal{B} \subseteq \mathfrak{A}$, let $\mathcal{T}_\mathcal{B} = \sup\{\mathfrak{t}_A :\ A \in \mathcal{B}\}$. It suffices to prove that the mapping:

$$\mathscr{P}(\mathfrak{A}) \setminus \{\emptyset\} \longrightarrow \mathscr{C}(G),\ \mathcal{B} \longmapsto \mathcal{T}_\mathcal{B}$$

is injective. To this end, it suffices to show that $\mathcal{T}_\mathcal{B} \leq \mathcal{T}_{\mathcal{B}'}$ implies $\mathcal{B} \subseteq \mathcal{B}'$ for $\emptyset \neq \mathcal{B}, \mathcal{B}' \in \mathscr{P}(\mathfrak{A})$. Pick $A \in \mathcal{B}$ and fix a neighborhood $U \subsetneq X$ of zero. As $P(A, U) \in \mathfrak{t}_A \subseteq \mathcal{T}_\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}'}$, there exists a finite subset $\mathcal{B}'_0 \subseteq \mathcal{B}'$, such that $P(A, U) \in \mathcal{T}_{\mathcal{B}'_0} = \mathfrak{t}_{\bigcup \mathcal{B}'_0}$. Then, $A \subseteq^* \bigcup \mathcal{B}'_0$ by Remark 6.7. Since the finite set $\{A\} \cup \mathcal{B}'_0$ is a subfamily of the almost disjoint family $\mathfrak{A}$, this may occur only if $A = B$ for some member $B \in \mathcal{B}'_0$, *i.e.*, $A \in \mathcal{B}'_0 \subseteq \mathcal{B}'$.

For the last assertion, we need the following notation. For infinite cardinals κ, λ, we shall write $C(\kappa, \lambda)$ if, for a set of size κ, the poset $\mathscr{P}(\kappa)$ has a chain of size λ (for more details, see [7,25]). In these terms, Sierpinksi has proven that $C(2^{<\lambda}, 2^\lambda)$ holds true for every infinite cardinal λ, where $2^{<\lambda} = \sup\{2^\mu : \mu < \lambda\}$. Now, let λ be the minimal cardinal, such that $2^\lambda > \mathfrak{c}$. Then, obviously, $2^{<\lambda} = \mathfrak{c}$, since $\lambda > \omega$. As $2^\lambda \geq \mathfrak{c}^+$, $C(2^{<\lambda}, 2^\lambda)$ implies that $C(\mathfrak{c}, \mathfrak{c}^+)$ holds true (see also [7] (Corollary 1.6) for the proof of $C(\kappa, \kappa^+)$ for arbitrary κ). In other words, $\mathscr{P}(\mathbb{R})$ admits chains of length $\mathfrak{c}^+$. $\square$

By Proposition 3.1, the set of metrizable group topologies in $\mathscr{C}(G)$ has cardinality $\leq \mathfrak{c}^\omega = \mathfrak{c}$. Hence, the above theorem provides $2^\mathfrak{c}$ many non-metrizable topologies in $\mathscr{C}(G)$. For the sake of completeness, we provide a short intrinsic proof of this fact.

Example 6.10. Let $\mathfrak{A}$ be as in the above proof. We show that $\mathcal{T}_\mathcal{B}$ is not metrizable whenever $\mathcal{B}$ is an uncountable subset of $\mathfrak{A}$. Since there are $2^\mathfrak{c}$ many such sets, this provides $2^\mathfrak{c}$ many non-metrizable topologies in $\mathscr{C}(G)$.

Assume that $\mathcal{T}_\mathcal{B}$ is metrizable, and let $(W_k)_{k\in\mathbb{N}}$ be a countable neighborhood basis at zero in $\mathcal{T}_\mathcal{B}$. For every k, there exists a finite subset $\mathcal{B}_k$ of $\mathcal{B}$, such that $W_k \in \mathcal{T}_{\mathcal{B}_k}$. Hence, there exist a neighborhood U_k of zero in X and $C_k \subseteq \mathbb{N}$ with:

$$C_k =^* \bigcup \mathcal{B}_k \quad \text{and} \quad P(C_k, U_k) \subseteq W_k. \tag{8}$$

Since $\mathcal{B}$ is uncountable, we can choose $A_0 \in \mathcal{B} \setminus \bigcup_{k\in\mathbb{N}} \mathcal{B}_k$. Let $U \subsetneq X$ be a neighborhood of zero. Then, $P(A_0, U) \in \mathfrak{t}_{A_0} \subseteq \mathcal{T}_\mathcal{B}$. Therefore, $W_k \subseteq P(A_0, U)$ for some $k \in \mathbb{N}$. Now, the inclusion in (8) yields $P(C_k, U_k) \subseteq P(A_0, U)$. Hence, Notation 6.2(b) implies $A_0 \subseteq C_k =^* \bigcup \mathcal{B}_k$. Since these are finitely many members of an almost disjoint family, we conclude that $A_0 \in \mathcal{B}_k$, a contradiction.

7. Final Comments and Open Questions

The embedding $\mathfrak{F}_\omega \hookrightarrow \mathscr{C}(G)$ remains true for many non-compact σ-compact LCA groups G. Actually, the σ-compact LCA groups G for which our approach does not ensure an embedding of $\mathfrak{F}_\omega$ in $\mathscr{C}(G)$ must be non-r-disconnected. Hence, they must have a very special structure (namely, contain a compact subgroup K, such that $G/K \cong \mathbb{R}^n \times L \times \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$ for some $n, m, k \in \mathbb{N}$, a discrete torsion-free non-r-discrete group L and not necessarily distinct primes p_i, see Example 2.6). Still simpler is this structure in the case when G is supposed additionally to have a compact open subgroup (this eliminates the vector subgroup $\mathbb{R}^n$). Indeed, in this case, G must contain an open subgroup of the form $K \times \mathbb{Z}^m$, for some $m \in \mathbb{N}$ and a compact group K, so that $G/(K \times \mathbb{Z}^m) \cong \prod_{i=1}^k \mathbb{Z}(p_i^\infty)$ for some $k \in \mathbb{N}$ and not necessarily distinct primes p_i.

In the general case, we have proved in [16] that $|\mathscr{C}(G)| \geq 3$ for every non-compact LCA group G. In particular, $|\mathscr{C}(\mathbb{R})| \geq 3$, $|\mathscr{C}(\mathbb{Z})| \geq 3$ and $|\mathscr{C}(\mathbb{Z}((p^\infty))| \geq 3$ for every prime p. Further progress in this direction depends pretty much on the following:

Questions 7.1. (a) Compute $|\mathscr{C}(\mathbb{Z})|$. Is it infinite? Is it countable? Is it at most $\mathfrak{c}$? At least $\mathfrak{c}$?

(b) Compute $|\mathscr{C}(\mathbb{R})|$. Is it infinite? Is it countable? Is it at most $\mathfrak{c}$? At least $\mathfrak{c}$?

(c) Compute $|\mathscr{C}(\mathbb{Z}(p^\infty))|$, where p is a prime. Is it infinite? Is it countable? Is it at most $\mathfrak{c}$? At least $\mathfrak{c}$?

Note that in (a) and (c), the group in question is countable, so that $2^{\mathfrak{c}}$ is an obvious upper bound in both cases.

Any information in the direction of Item (c) will throw light on the poset $\mathscr{C}(\mathbb{Q}_\pi)$ for all π containing p (as $\mathbb{Q}_\pi$ has a quotient isomorphic to $\mathbb{Z}(p^\infty)$, so the poset $\mathscr{C}(\mathbb{Z}(p^\infty)$ embeds into the poset $\mathscr{C}(\mathbb{Q}_\pi)$ by Theorem 3.10).

Clearly, CH is needed in Corollary 1.12 only to eliminate those groups G that have $\omega < \varrho(G) < \mathfrak{c}$. We do not know if the assertion from this corollary remains true without the assumption of CH. In particular, the following question remains open:

Question 7.2. Is $\mathscr{C}(\mathbb{R} \oplus \bigoplus_{\omega_1} \mathbb{Z}_2) \overset{q.i.}{\cong} \mathfrak{F}_{\omega_1}$?

Question 7.3. Let G be a non-precompact second countable Mackey group. Is it true that $|\mathscr{C}(G)| \geq \mathfrak{c}$?

Here comes a somewhat general question:

Problem 7.4. *Find sufficient conditions for a metrizable precompact group G to be Mackey (i.e., have $|\mathscr{C}(G)| = 1$).*

Such a sufficient condition was pointed out in [3]: all bounded metrizable precompact groups are Mackey.

The next question is related to Question 7.3:

Question 7.5. If for a group G, one has $|\mathscr{C}(G)| > 1$, can $\mathscr{C}(G)$ be finite? In particular, can one have a Mackey group G with $|\mathscr{C}(G)| = 2$?

The following conjecture will positively answer Question 7.3 (as well as Question 7.1) and negatively answer Question 7.5:

Conjecture 7.6. *[Mackey dichotomy] For a locally quasi-convex group G, one has either $|\mathscr{C}(G)| = 1$ or $|\mathscr{C}(G)| \geq \mathfrak{c}$.*

Positive evidence in the case of bounded groups can be obtained from the recent results in [26].

Let λ be a non-discrete linear topology on $\mathbb{Z}$. It is easy to see that $(\mathbb{Z}, \lambda)$ is metrizable and precompact. It is shown in [27] that these groups are not Mackey, *i.e.*, $|\mathscr{C}(\mathbb{Z}, \lambda)| > 1$.

Question 7.7. How large can $\mathscr{C}(\mathbb{Z}, \lambda)$ be?

Let (G, τ) be infinite torsion subgroup of $\mathbb{T}$ equipped with the (precompact metrizable) topology τ induced by $\mathbb{T}$. It was proven in [28] that (G, τ) is not a Mackey group, *i.e.*, $|\mathscr{C}(G, \tau)| > 1$.

Question 7.8. How large can be $\mathscr{C}(G, \tau)$ in this case?

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Author Contributions

Lydia Außenhofer, Dikran Dikranjan and Elena Martín-Peinador contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

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