Topological realizations of groups in Alexandroff spaces

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Abstract

Given a group G, we provide a constructive method to get infinitely many (non-homotopy-equivalent) Alexandroff spaces, such that the group of autohomeomorphisms, the group of homotopy classes of self-homotopy equivalences and the pointed version are isomorphic to G. As a result, any group G can be realized as the group of homotopy classes of self-homotopy equivalences of a topological space X, for which there exists a CW complex $\mathcal{K}(X)$ and a weak homotopy equivalence from $\mathcal{K}(X)$ to X.

1 Introduction

The algebraic topology of finite spaces is becoming a significant part of topology. It is mainly due to two relatively old papers, [15] and [17]. Approximately from the turn of the century, it was born a renewed interest on this subject and probably it will grow up as soon as researchers find better and better results on approximation of spaces and maps by means of finite data.

Up to our knowledge, there are two monographs focused on algebraic aspect of the topology of finite spaces, one is [4] which is essentially the Ph. D. thesis of Barmak (under the supervision of G. Minian). The other one is due to J.P. May [14], it is related to some REU programs developed by the author at the University of Chicago and it is probably one of the main reasons of the current interest on the subject. Another one reason is the introduction of finite spaces to deal with problems in computational topology mainly those related to Topological Data Analysis.

Some of the results given by McCord in [15] can be rephrased in the following way: *A group can be realized as the fundamental group of a compact polyhedron if and only if it can be realized as the fundamental group of a finite topological space satisfying the separation T_0 property.*

In fact a much stronger and much more general result was given in [15] involving weak homotopy equivalences, general simplicial complexes with the weak topology and a suitable extension of "finite topological space" introduced by Alexandroff in [1] by means of imposing that the intersection of arbitrary open sets is open.

The problem to realize groups as the homeomorphism group of a topological space has been widely studied. We are not going to list in our references all those we know. We only refer herein a few of them, those specially related to the subject of our paper. In particular [5] deals with the realization of finite groups as the full group of homeomorphisms of a finite topological space or, equivalently, the automorphism group of a finite partially ordered set. They focused on trying to give, for any finite group G, a finite topological space with the lower cardinality possible having G as the corresponding group of homeomorphisms.

This research is partially supported by Grant MTM2015-63612-P and Grant BES-2016-076669 from Ministerio de Economia y Competitividad (Spain).

Other groups associated with topological spaces are the groups of homotopy classes of homotopy equivalences of any topological space, both pointed and unpointed or free. Also the corresponding realizability problem is of interest in the literature when we restrict the class of spaces. This problem can be stated as follows:

Given a group G, is there a space X, with the same homotopy type of a CW-complex such that the group of homotopy classes of self-homotopy equivalences of X is (isomorphic to) G?

Alternatively, also a point $x \in X$ such that G is the group of self-homotopy equivalences of (X, x) in the pointed category? This problem has a long history. In $HPol$, the full subcategory of $HTop$ whose objects are all topological spaces having the homotopy type of a polyhedron, the problem of realizability has appeared in many papers for over fifty years [3],[10],[12],[16] and it has been placed as the first problem to solve in [2], a list of open problems about groups of self-homotopy equivalences. In this direction, a complete answer for the finite and pointed case was obtained by C. Costoya and A. Viruel [7].

Theorem 1.1 ([7]). *Every finite group G can be realized as the group of self-homotopy equivalences of infinitely many (non-homotopy-equivalent) rational elliptic spaces X.*

In a recent paper [8], the free case has been completely solved using tools of highly algebraic character and for Eilenberg-MacLane spaces.

In this paper, we want to show that Alexandroff spaces form a good environment for realizing groups as homeomorphisms groups of such spaces or as groups of homotopy classes of homotopy equivalences or even as groups of homotopy classes of pointed homotopy equivalences. Moreover, Alexandroff spaces are very close to CW-complexes if we look only at topological invariants such as the homotopy groups and singular homology groups because any Alexandroff space is the codomain of a weak homotopy equivalence from a CW-complex as proved by McCord in [15]. Furthermore, it can also be deduced the next: *Every group can be realized as the fundamental group of an Alexandroff space.*

The other foundational paper on finite spaces due to Stong [17] plays also a central role herein. There, it was accomplished an interesting study on the homotopy-type classification of finite spaces. Among the things Stong introduced, there is the important concept of Core or minimal finite space. These spaces have the important property that any homotopy equivalence of two of them is in fact a homeomorphism because what really happens is that any self-map homotopy equivalent to the identity in any of them is in fact the identity. We also have to mention that M. Kukiela [13] extended some of these concepts and results from finite spaces to general Alexandroff spaces, which is our framework.

As a summary, in this paper, we construct for any group G an Alexandroff space, with the property that any self-homotopy equivalence is a homeomorphism having G as the group of self-homeomorphism. Later, we construct another space closely related to the first one adding only a point $*$ in such a way that any autohomeomorphism must fix the new point $*$.

A weak homotopy equivalence is a map between topological spaces which induces isomorphisms on all homotopy groups. Furthermore, we say that two topological spaces X, Y are weak homotopy equivalent (or they have the same weak homotopy type) if there exists a sequence of spaces $X = X_0, X_1, ..., X_n = Y$ such that there are weak homotopy equivalences $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for every $0 \geq i \geq n-1$. Herein, we state the two main results of the paper.

Theorem 1. *Every group G can be realized as the group of self-homotopy equivalences of a topological space X, for which there exists a CW complex $\mathcal{K}(X)$ and a weak homotopy equivalence from $\mathcal{K}(X)$ to X.*

In fact, we can find infinite (non-homotopy-equivalent) Alexandroff spaces in the same weak homotopy type satisfying that result and the pointed version:

Corollary 1. *Every group can be realized as the group of pointed homotopy classes of pointed self-homotopy equivalences of infinitely many (non-homotopy-equivalent) Alexandroff spaces in the same weak homotopy type.*

To obtain that topological spaces we solve firstly the realizability problem in the topological category (Top). We generalize the construction for the finite case of Barmak and Minian in [5] to a more general setting.

Theorem 2. *Every group can be realized as the group of autohomeomorphisms of an Alexandroff space.*

Then, we propose some modifications of the Alexandroff space built in Theorem 2 so as to get rigidity in terms of homotopy. We prove the following.

Theorem 3. *Every group can be realized as the group of self-homotopy equivalences of infinitely many (non-homotopy-equivalent) Alexandroff spaces in the same weak homotopy type.*

The organization of the paper is as follows. In Section 2, we introduce the basic definitions and theorems from the literature that we will use in future sections. In section 3, given a group G, we provide a method to obtain three Alexandroff spaces X_G,$\overline{X}_G$,$\overline{X}^*_G$ that are the candidates to solve the problem of realizability for the category Top, $HTop$ and the pointed version, we also present some examples in detail. In Section 4, we show that $Aut(X_G)$ is isomorphic to G, solving the problem of realizability for the topological category. In section 5, we prove the main result of the paper, that is to say, every group can be realized as the group of self-homotopy equivalences of infinitely many (non-homotopy-equivalent) Alexandroff spaces in the same weak homotopy type and the pointed version. The proof is divided into two auxiliary lemmas and use the main result of Section 4. As a consequence, using the theory of McCord, it will be deduced that any group G can be realized as the group of self-homotopy equivalences of a topological X, for which there exists a CW complex $\mathcal{K}(X)$ and a weak homotopy equivalence from $\mathcal{K}(X)$ to X. In section 6, we study some properties of the space $\overline{X}_G$ and its McCord complex $\mathcal{K}(\overline{X}_G)$.

We also want to point out a result of independent interest showing that the unique continuous flow (or continuous dynamical system) in any Alexandroff space is the trivial one. Someone can think that this result is extremely trivial if one take, as example of Alexandroff spaces, the discrete ones. In this example, the triviality is due to the fact that in discrete spaces the unique paths are the constant ones. On the contrary, in general Alexandroff spaces one can have a lot of non-trivial paths or even of homotopy classes of paths because one can easily prove, using McCord's results and Eilenberg-McClane spaces, that any group G can be realized as the fundamental group of a path-connected Alexandroff space.

2 Preliminaries

We introduce some concepts that will be used in future sections. First of all we recall a result of Alexandroff [1].

Theorem 2.1 ([1])**.** *For a partially ordered set (poset) $(X, \geq)$ the family of upper (lower) sets of $\geq$ is a T_0 topology on X, that makes X a T_0 topological space with the property that* *the arbitrary intersection of open sets is open. For a T_0 topological space (X, τ) such that the arbitrary intersection of open sets is open the relation $x \leq_\tau y$ if and only if $U_x \subset U_y$ ($U_y \subset U_x$), where U_x is the intersection of all open sets containing $x \in X$, is a partial order on X. Moreover, the two associations relating T_0 topologies and partial orders are mutually inverse.*

Given $(X, \leq)$ a poset, an upper (lower) set S is a subset of X such that if $x \in S$ and $y \leq x$ ($x \leq y$) then $y \in S$. From now on, we will deal posets and Alexandroff spaces as the same object without explicit mention and all of them will be T_0, U_x will denote the open set that consist of the intersection of all open sets containing $x \in X$. Furthermore, the Alexandroff's theorem allows to express some topological notions using the partial order. For instance, let X, Y be Alexandroff spaces, $f : X \to Y$ is a continuous function if and only if f is order preserving. If X and Y are finite topological spaces, $f, g : X \to Y$ are homotopic if and only if there exists a sequence of continuous maps with $f(x) = f_0(x) \leq f_1(x) \geq f_2(x) \leq ...f_n(x) = g(x)$ for every x in X. In the case that X and Y are Alexandroff spaces and $f(x) \geq g(x)$ for every $x \in X$ we get that f is homotpic to g. In addition, a path for an Alexandroff space is a sequence of elements $(x_0, x_1..., x_n)$ such that x_i comparable to x_{i+1} for every $i = 1, ..., n$. A great introduction can be found in J. P. May's notes [14]. On the other hand, finite T_0 topological spaces are a specific case of Alexandroff spaces, a good reference for the finite case is [4].

Stong in [17] provided an ingenious method to classify finite spaces by their homotopy type.

Definition 2.1. *Let X be a finite space*

- *$x \in X$ is linear, or up beat point following modern notation, if there exists $y > x$ with the property that for every $z > x$ we have $z \geq y$.*
- *$x \in X$ is colinear, or down beat point following modern notation, if there exists $y < x$ with the property that for every $z < x$ we have $y \geq z$.*

Moreover, we say that a finite space X is a core if X is T_0 and has no linear or colinear points (beat points).

If we remove the colinear and linear points (beat points) of a finite space X, the homotopy type of X does not change, we can repeat that process until there are no colinear or linear points, then we would get what is called the core of the space X denoted by X^c.

Theorem 2.2 ([17]). *A homotopy equivalence between two cores is a homeomorphism.*

The notion of core introduced by Stong and some results for finite spaces were generalised for Alexandroff spaces by Kukiela in [13]. In concrete, Kukiela obtained an analogue of the Theorem 2.2 that we will use in Section 5 Lemma 5.2. We recall two definitions.

Definition 2.2. *Let X be an Alexandroff space (with the distinguished point p), $r : X \to X$ (keeping the distinguished point fixed) is a comparative retraction if r is a retraction in the usual sense and $r(x) \leq x$ or $r(x) \geq x$ for every $x \in X$. The class of all comparative retractions is denoted by $\mathcal{C}$. The space X ((X, p)) is called a $\mathcal{C}$-core if there is no other retraction $r : X \to X$ ($r : (X, p) \to (X, p)$) in $\mathcal{C}$ other than the identity id_X.*

Definition 2.3. *We say a $\mathcal{C}$-core X ((X, p)) is locally a core if for every $x \in X$ there exists a finite set $A_x \subset X$ containing x such that for every $y \in A_x$, then either $y = p$ or $|A_x \cap max(\{z \in X | z < y\}| \geq 2$ if y is not minimal in X and $|A_x \cap min(\{z \in X | z > y\})| \geq 2$ if y is not maximal in X, that is to say, y is not a beat point of A_x.*

Theorem 2.3 ([13]). *If X (X,p) is locally a core, then there is no map in $C(X,X)$ homotopic to id_X other than id_X, where $C(X,Y)$ denotes the space of continuous maps from X to Y equipped with the compact-open topology.*

Corollary 2.1 ([13]). *If X,Y are locally cores (with distinguished points p,q), then X (or (X,p)) is homotopy equivalent to Y (or (Y,q)) if and only if X is homeomorphic to Y ((X,p) is homeomorphic to (Y,q))*

The same year it is published the paper of Stong mentioned before, it is also published [15], where McCord studied the weak homotopy type of Alexandroff spaces using simplicial complexes.

The key to get the most important result in that paper relies in the next theorem, which is somehow an adaptation of a theorem by Dold and Thom [9].

Definition 2.4. *An open cover $\mathcal{U}$ of a space B will be called basis-like if whenever $x \in U \cap V$ and $U,V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $x \in W \subset U \cap V$.*

Theorem 2.4 ([15]). *Suppose p is a map of a space E into a space B for which there exists a basis-like open cover $\mathcal{U}$ of B satisfying the following condition: For each $U \in \mathcal{U}$, the restriction $p_{|p^{-1}(U)} : p^{-1}(U) \to U$ is a weak homotopy equivalence. Then p itself is a weak homotopy equivalence.*

Every Alexandroff space X admits a basis-like open cover $\mathcal{U} := \{U_x | x \in X\}$, where U_x denotes the intersection of every open set containing x; U_x can also be seen as the set of $y \in X$ with $y \leq x$. In addition, U_x is a contractible space.

Definition 2.5. *Let X be an Alexandroff space, that is to say, a poset using the order induced by the relation $x \geq y$ for $x,y \in X$ if and only if $U_x \subset U_y$. We can consider the ordered complex (or McCord complex) $\mathcal{K}(X)$, i.e. the vertices of the complex are the points of X and the simplices are the finite, totally ordered subsets of X. The geometric realization of $\mathcal{K}(X)$ will be denoted by $|\mathcal{K}(X)|$.*

If X is an Alexandroff space, for every $u \in |\mathcal{K}(X)|$ we have that u is contained in a unique open simplex $(x_0, ..., x_r)$, where $x_0 < ... < x_r$. Then, $f_X : |\mathcal{K}(X)| \to X$ is defined by $f_X(u) = x_0$. McCord showed that f_X is continuous and has the property that $|\mathcal{K}(U_x)|$ is a deformation retract of $f_X^{-1}(U_x)$ and contractible. From here,

Theorem 2.5 ([15]). *There exists a correspondence that assigns to each Alexandroff space X a simplicial complex $\mathcal{K}(X)$ and a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \to X$.*

We can also visualize an Alexandroff space X using a Hasse diagram $H(X)$, which is a directed graph, the vertices are the points of X. We have an edge between two vertices x and y if and only if $x < y$ (or $x > y$) and there is no $z \in X$ with $x < z < y$ (or $x > z > y$).

Finally, we revise some results from basic set theory. A complete introduction and description can be found in [11].

Definition 2.1. *A set T is transitive if every element of T is a subset of T. A set α is an ordinal number if α is transitive and α is well-ordered by $\in_\alpha$ ($A, B \in \alpha$, $A \in_\alpha B$ if and only if $A \in B$).*

In the next proposition we recollect some properties for ordinal numbers, where the order $<$ is the one given by $\alpha < \beta$ if and only if $\alpha \in \beta$ with α and β ordinal numbers.

Proposition 2.1. *Some properties that hold for ordinal numbers:*

- *If α is an ordinal number, $S(\alpha) := \alpha \cup \{\alpha\}$ is an ordinal number, we will denote $S(\alpha)$ by $\alpha + 1$, so $\alpha + n = S(\overset{n}{\cdots}(S(\alpha))$. An ordinal number α is called successor ordinal if $\alpha = S(\beta)$ for some β. Otherwise, is called a limit ordinal. $\mathbb{N}$ is a limit ordinal and will be denoted by ω.*
- *If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*
- *$\alpha < \beta$ and $\alpha > \beta$ cannot both hold.*
- *Either $\alpha < \beta$ or $\alpha = \beta$ or $\alpha > \beta$ holds.*
- *Every non-empty set of ordinal numbers has a $<$-least element. Consequently, every set of ordinal numbers is well-ordered by $<$.*
- *If α is an ordinal number $\alpha \notin \alpha$.*
- *Every element of an ordinal number is an ordinal number.*

The next theorem require the Axiom Schema of Replacement, which is an axiom in Zermelo–Fraenkel set theory (ZF) and corresponds to 3.1 Theorem in [11].

Theorem 2.6. *Every well-ordered set is isomorphic to a unique ordinal number, where isomorphism in this context means that there is a bijective function that preserves the order.*

Then, as a corollary we can obtain that every set is isomorphic to an ordinal number using the Axiom of Choice. Hence, we will work in Zermelo–Fraenkel set theory with the axiom of choice (ZFC).

3 Construction of X_G, $\overline{X}_G$ and $\overline{X}^*_G$

Let G be a group and S a set of non-trivial generators of G, i.e. the identity element is not a generator and we do not have repetitions of elements in S. By the Axiom of Choice, S is a well-ordered set and by Theorem 2.6 there is a bijective map order preserving between S and the ordinal number α'. Moreover, we can find an ordinal number α, where we denote by $-1 \in \alpha$ and $0 \in \alpha$ the first and the second elements of α respectively, such that $\alpha \setminus \{\{-1\}, \{0\}\}$ is in bijective correspondence with α'. We consider $X_G = G \times \alpha$ with the next relations:

- $(g, \beta) < (g, \gamma)$ if $-1 \leq \beta < \gamma$ where $g \in G$ and $-1, \beta, \gamma \in \alpha$.
- $(gh_\beta, -1) < (g, \gamma)$ if $0 < \beta \leq \gamma$ where $g \in G$ and $0, \beta, \gamma \in \alpha$.

where the set of generators is represented by $h_\beta \in S$ with $0 < \beta < \alpha$. If S is infinite, the ordinal number α will be consider as a limit ordinal (cardinality of S). It is trivial to check that X_G with the previous relations is a partial ordered set and therefore via Theorem 2.1 a T_0 Alexandroff space.

Remark 3.1. *If the set of non-trivial generators S is finite or countable, it is not necessary to use the ordinal number theory. For the finite case, that is to say, $|S| = r$, we have $X_G = G \times \{-1, 0, 1, ..., r\}$, if G is also finite, X_G is a finite T_0 topological space. In addition, for this case the construction is the same given in [5]. If S is countable we only need to consider $X_G = G \times (\mathbb{N} \cup \{-1\} \cup \{0\})$. The relations defined above for both cases remain the same. Therefore, for these cases it is not necessary to use the Axiom of Choice and the Axiom Schema of Replacement, the arguments that will be used in future sections will also hold true.*

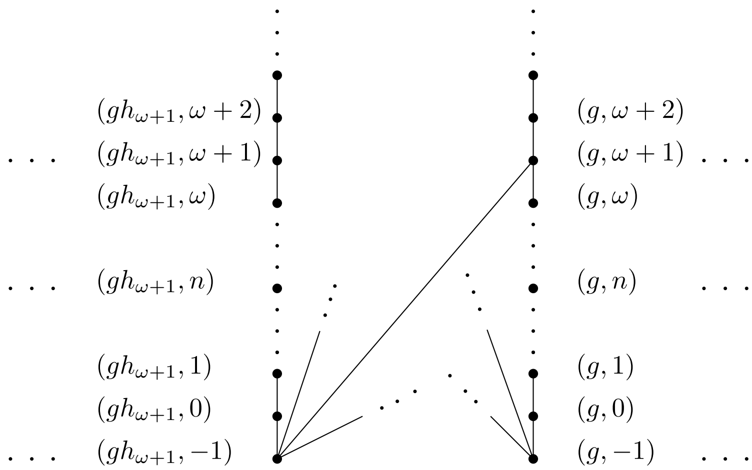

Figure 1: Schematic Hasse diagram of X_G

X_G is far from being a core for the finite case or a locally core for the non-finite case because every point of the form $(g,\beta) \in X_G$, with $\alpha > \beta > 0$, is clearly a beat point. Thus, we need to add points so as to get a good candidate ($\overline{X}_G$) to be a locally core or core. The previous property is crucial to obtain the main result of Section 5.

If S is infinite (finite), for every element $(g,\beta) \in X_G$ with $\alpha > \beta \geq 0$ (except for β such that $\beta + 1 = \alpha$), we consider $S_{(g,\beta)}$ and $T_{(g,\beta)}$ in the following way: $S_{(g,\beta)} := \{A_{(g,\beta)}, B_{(g,\beta)}, C_{(g,\beta)}, D_{(g,\beta)}\}$ with the relations $A_{(g,\beta)} > C_{(g,\beta)}, D_{(g,\beta)}; B_{(g,\beta)} > C_{(g,\beta)}, (g,\beta)$ and $(g,\beta) > D_{(g,\beta)}$; and $T_{(g,\beta)} := \{E_{(g,\beta)}, F_{(g,\beta)}, G_{(g,\beta)}, H_{(g,\beta)}, I_{(g,\beta)}, J_{(g,\beta)}\}$ with the relations $(g,\beta) > H_{(g,\beta)}; E_{(g,\beta)} > (g,\beta), I_{(g,\beta)}; F_{(g,\beta)} > H_{(g,\beta)}, J_{(g,\beta)}$ and $G_{(g,\beta)} > I_{(g,\beta)}, J_{(g,\beta)}$. Then, we define

$$\overline{X}_G = \bigcup_{\substack{(g,\beta)\in G\times\alpha \\ \beta\geq 0, \beta+1\neq\alpha}} (S_{(g,\beta)} \cup T_{(g,\beta)}) \cup X_G$$

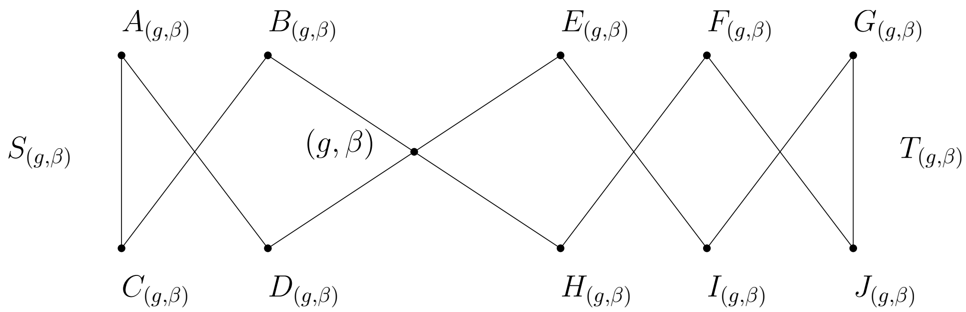

Figure 2: Hasse diagram of $S_{(g,\beta)} \cup T_{(g,\beta)} \cup (g,\beta)$

Remark 3.2. *If G is finite, it is not necessary to consider the spaces of the form $T_{(g,i)}$, i.e. $\overline{X}_G = \bigcup_{(g,i)\in G\times\{0,\ldots,r-1\}} S_{(g,i)} \cup X_G$, where $|S| = r$.*

Finally, we only need to add one extra point $\{*\}$ to $\overline{X}_G$ so as to obtain a pointed version for the Theorem 3. The point $\{*\}$ will play the role of a fixed point for every self-homotopy equivalence. We define $\overline{X}^*_G$ as the union of $\overline{X}_G$ and $\{*\}$, where $* > (g,-1)$ for every $g \in G$.

Example 3.1. *Let us consider the dihedral group of four elements $D_4 = \{a, b | a^2 = b^2 = abab = e\}$, we take $S = \{a, b\}$ as a set of non-trivial generators, where a is associated to 1 and b to 2 in the construction of the finite spaces X_{D_4}, $\overline{X}_{D_4}$ and $\overline{X}^*_{D_4}$. The Hasse diagram of that spaces can be found in Figure 3, where we have in black the Hasse diagram of X_{D_4}, the Hasse diagram of $\overline{X}_{D_4}$ is in black, blue and red, the Hasse diagram of $\overline{X}^*_{D_4}$ corresponds to the entire diagram of the Figure 3, where we have in purple the new part respect to $\overline{X}_{D_4}$. If we have followed the Remark 3.2, then the Hasse diagram of $\overline{X}_{D_4}$ would be represented only by black and red.*

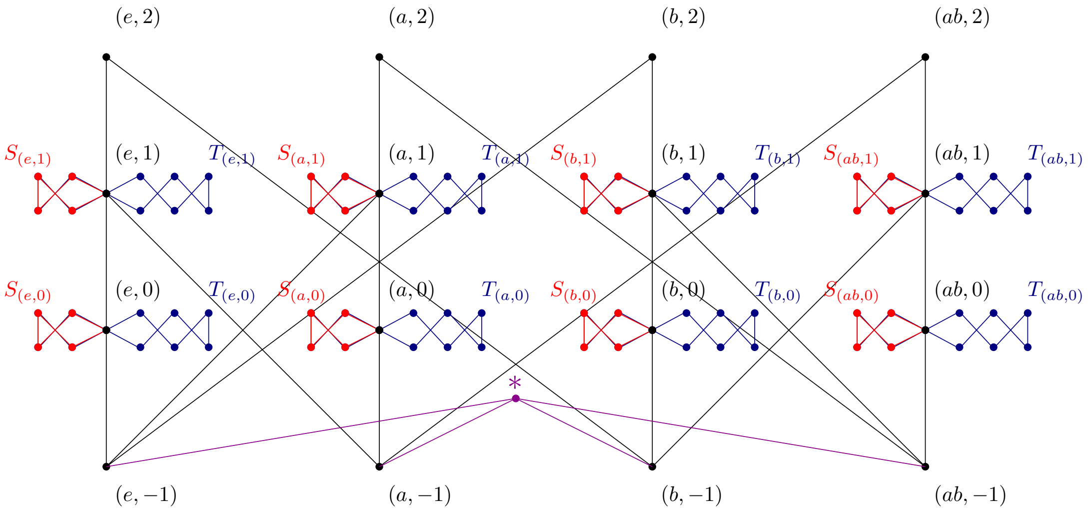

Figure 3: Hasse diagram of X_{D_4},$\overline{X}_{D_4}$ and $\overline{X}^*_{D_4}$

Example 3.2. *Let us consider the cyclic group of three elements $C_3 = \{e, a, a^2\}$, we consider $S = \{a\}$ as a set of non-trivial generators. We define the column associated to an element x of C_3 by $C_x := \{S_x, T_x, (x, -1), (x, 0), (x, 1)\}$.*

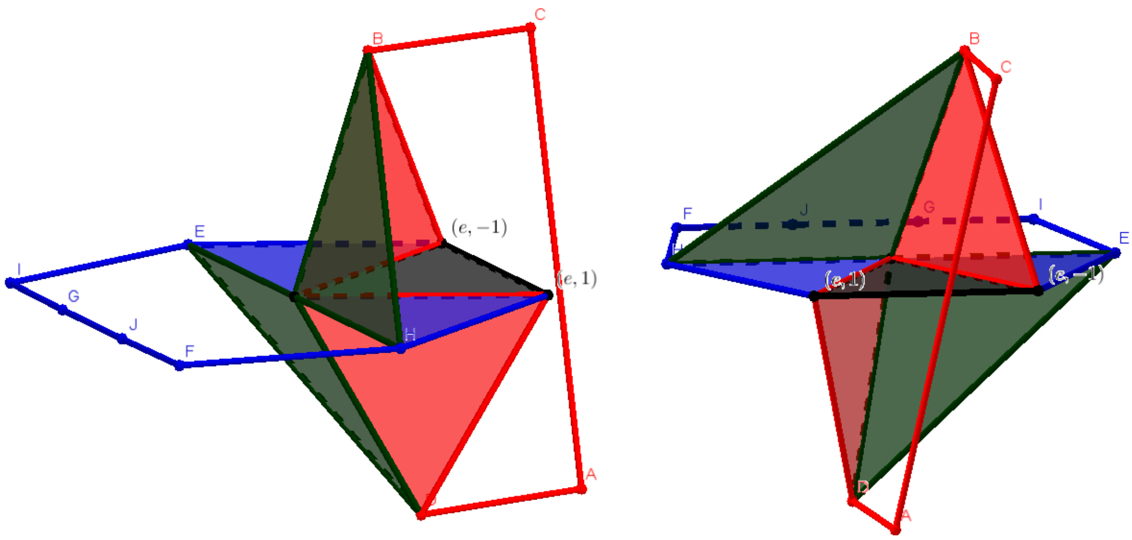

Figure 4: McCord complex of the column associated to e, C_e, from two different perspectives.

In Figure 4, we have the McCord complex of C_e, $\mathcal{K}(C_e)$, where we have in black the simplices from $\mathcal{K}(X_{C_3} \cap C_e)$, in red the simplices from $\mathcal{K}(S_{(e,0)} \cap C_e)$, in blue the simplices *from $\mathcal{K}(T_{(e,0)} \cap C_e)$, and finally in green the simplices that combine elements from $S_{(e,0)} \cap C_e$ and $T_{(e,0)} \cap C_e$, for example the simplex $< D_{(e,0)}, (e,0), E_{(e,0)} >$. We have omitted the subscript of the elements for simplicity.*

In Figure 5, we can observe the McCord Complexes from a top perspective and the Hasse diagrams of $X_{C_3}, \overline{X}_{C_3}$ and $\overline{X}^*_{C_3}$. The color black will be associated to X_{C_3}, black, red, blue and green are related to $\overline{X}_{C_3}$, finally the entire diagram and simplicial complex correspond to $\overline{X}^*_{C_3}$. It is quite easy to generalize this construction for the cyclic group of n elements C_n in order to obtain the Hasse diagrams and McCord Complexes for $X_{C_n}, \overline{X}_{C_n}$ and $\overline{X}^*_{C_n}$. Every column for an element x of C_n will have exactly the same McCord complex obtained in Figure 4. The McCord complex $\mathcal{K}(\overline{X}^*_{C_n})$ follows the same structure obtained in Figure 5, instead of 3 columns ($\mathcal{K}(C_x)$) we will have n columns around a kind of a circle or a polygon with n sides.

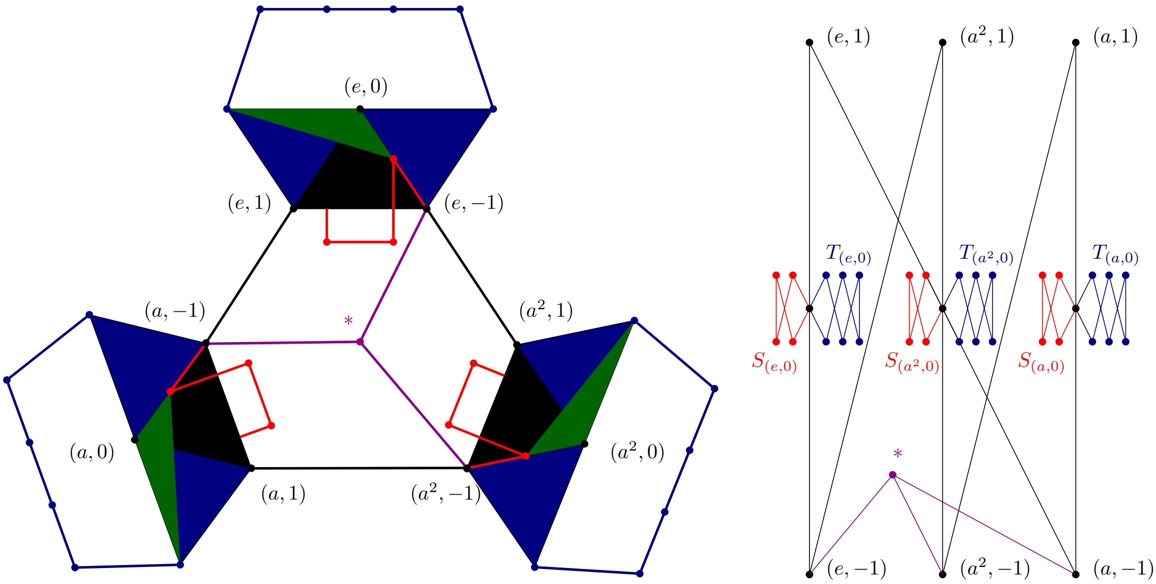

Figure 5: Top view of the McCord complex and Hasse diagram of $X_{C_3}, \overline{X}_{C_3}$ and $\overline{X}^*_{C_3}$

4 The group of autohomeomorphisms of X_G

In this section, we solve the realizability problem for groups in the topological category (Top), that is to say, Theorem 2, the proof is essentially the same given by Barmak and Minian in [5]. They proved that every finite group G can be realized as the group of autohomeomorphisms of a finite T_0 topological space (poset) with $n(r+2)$ points, where $|G| = n$ and $|S| = r$, improving the results obtained by Birkhoff [6] and Thornton [18], that used $n(n+1)$ and $n(2r+1)$ points respectively for the construction of the poset.

Before giving the proof of Theorem 2 we study a property of X_G that will be used also in Lemma 5.1.

Proposition 4.1. *If $f : X_G \to X_G$ is a homeomorphism, f preserves the levels, that is to say, $f(g,\beta) = (h,\beta)$ for every $(g,\beta) \in X_G$ and some $h \in G$.*

Proof. Let f be an autohomeomorphism of X_G, we argue by transfinite induction with respect to the index σ in $G \times \sigma$ with $\sigma < \alpha$, i.e. the levels or the second coordinate of the points of X_G. We show the result for the elements of the form $(g,-1)$ with $g \in G$, the first level. Suppose $f(g,-1) = (h,\beta)$ for some $h \in G$ and $\beta > -1$. Therefore, $f^{-1}(h,\beta) = (g,-1)$, but $(h,-1) < (h,\beta)$, so $f^{-1}(h,-1) < f^{-1}(h,\beta)$ and $f^{-1}(h,-1) = (g,-1)$ by the minimality of $(g,-1)$, which leads to a contradiction with the injectivity of f^{-1}. Then, we can deduce that f and f^{-1} preserve the level -1.

Suppose that we have proved the result for every $\beta < \gamma < \alpha$, i.e. for every $(g,\beta) \in X_G$ with $\beta < \gamma$, $f(g,\beta) = (h,\beta)$ for some $h \in G$, then we can deduce that for every $g \in G$ and $\beta < \gamma$ we get $f^{-1}(g,\beta) = (h,\beta)$ for some $h \in G$ because f is a homeomorphism. If $f(g,\gamma) = (h,\tau)$, where $\tau < \gamma$, then $f^{-1}(h,\tau) = (g,\gamma)$, which leads to a contradiction with our hypothesis. If $f(g,\gamma) = (h,\tau)$, where $\tau > \gamma$, $f^{-1}(h,\tau) = (g,\gamma)$. On the other hand, $(h,\gamma) < (h,\tau)$ so $f^{-1}(h,\gamma) < f^{-1}(h,\tau) = (g,\gamma)$ by the continuity of f^{-1}, which leads to a contradiction with our hypothesis because $f^{-1}(h,\gamma)$ is an element in a lower level than γ but $f(f^{-1}(h,\gamma)) = (h,\gamma)$. □

Remark 4.1. *If we consider an autohomeomorphism $f : X_G \setminus \{(g,-1) \in X_G | g \in G\} \to X_G \setminus \{(g,-1) \in X_G | g \in G\}$. Then, f preserves the levels, i.e. $f(g,\beta) = (h,\beta)$ for some $h \in G$ and every $\beta \geq 0$. It is necessary to show the result for the first level, in this case the level 0, and then apply transfinite induction to conclude as we did in the proof of Proposition 4.1. Suppose $f(g,0) = (h,\beta)$ for some $h \in G$ and $\beta > 0$. Therefore, $f^{-1}(h,\beta) = (g,0)$ and $f^{-1}(h,0) = (g,0)$, by the continuity of f^{-1} and minimality of $(g,0)$, which leads to a contradiction with the injectivity of f^{-1}. Then, we can deduce that f and f^{-1} preserve the first level 0. We only need to repeat the same argument used in the previous proposition so as to obtain the result.*

Proof of Theorem 2. Given a group G, we consider the Alexandroff space X_G. We define $\varphi : G \to Aut(X_G)$ by $\varphi(g)(s,\beta) = (gs,\beta)$, where $(s,\beta) \in X_G$. We only need to show that φ is an isomorphism of groups. First of all, we check that φ is well defined, i.e. if $g \in G$ we have $\varphi(g) \in Aut(X)$. $\varphi(g) : X_G \to X_G$ is clearly continuous because preserves the order. By construction, $\varphi(g)$ is also bijective. The inverse of $\varphi(g)$ is $\varphi(g^{-1})$, which is also continuous. It is straightforward to check that φ is a homomorphism of groups.

We prove that φ is a monomorphism of groups. Suppose that $\varphi(g) = Id$, where $Id : X_G \to X_G$ denotes the identity, then $(ge,-1) = \varphi(g)(e,-1) = (e,-1)$, where e denotes the identity element of the group G, so $g = e$.

Now we verify that φ is an epimorphism of groups. Let us take $f \in Aut(X_G)$. By Proposition 4.1, $f(e,-1) = (h,-1)$ for some $h \in G$, we also have that $\varphi(h)(e,-1) = (h,-1)$. We consider $Y := \{x \in X_G | f(x) = \varphi(h)(x)\}$. We will show that Y is open, closed and X_G is a connected space. Thus, $Y = X_G$ because Y is non-empty since $(e,-1) \in Y$.

Y is open, let us take $(g,\beta) = x \in Y$, $f(x) = \varphi(h)(x)$ and $f_{|U_x}, \varphi(h)_{|U_x} : U_x \to U_{f(x)}$. By Proposition 4.1, $f(x) = \varphi(h)(x) = (s,\beta)$ for some $s \in G$. On the other hand, there is only one element for each level γ with $0 \leq \gamma \leq \beta$ in U_x and $U_{f(x)}$, (g,γ) and (s,γ) respectively. In concrete, U_x consists of (g,γ) with $-1 \leq \gamma \leq \beta$ and elements of the form $(gh_\gamma,-1)$ with $\gamma \leq \beta$, where $h_\gamma \in S$, the description of $U_{f(x)}$ is similar. Hence, by Proposition 4.1 we can deduce $f(g,\gamma) = \varphi(h)(g,\gamma) = (s,\gamma)$ with $0 \leq \gamma \leq \beta$ and therefore by the continuity of f and $\varphi(g)$ we get $f(y) = \varphi(g)(y)$ for every $y \in U_x$. Thus, $U_x \subset Y$.

Y is closed, let us take $(k,\beta) = x \in X_G \setminus Y$. By Proposition 4.1, $f(k,\beta) = (g,\beta) = \varphi(gk^{-1})(k,\beta)$ for some $g \in G$. Furthermore, $g \neq hk$ because otherwise we would get

$$f(k,\beta) = (g,\beta) = (hk,\beta) = \varphi(hkk^{-1})(k,\beta) = \varphi(h)(k,\beta)$$

which leads to contradiction with $x \notin X_G \setminus Y$. We can repeat the same argument used before to get that $f_{|U_x} = \varphi(gk^{-1})_{|U_x}$, but $gk^{-1} \neq h$, so $f(y) = \varphi(gk^{-1})(y) \neq \varphi(h)(y)$ for every $y \in U_x$ and $U_x \cap Y = \emptyset$.

X_G is a connected space, we will show that X_G is path connected. Then, we need to show that for every $x, y \in X_G$ there is a path from x to y, that is to say, a sequence $(x = x_0, x_1 ..., x_\alpha = y)$ with x_i comparable to x_{i+1}. It is only necessary to check the situation for points of the form $(g, -1), (h, -1)$ with $g, h \in G$ and $g \neq h$, the reason is the first relation of the partial ordered given in X_G. We have that $g = kk^{-1}g$, without loss of generality we can assume that $k^{-1} = h_1, ..., h_n$ and $g = h_{n+1}, ..., h_m$, where $h_i \in S$ and $i = 1, ..., m \in \alpha$. Thus,

$$\begin{aligned}(g, -1) = (kk^{-1}h_{n+1}, ..., h_m, -1) < (kk^{-1}h_{n+1}, ..., h_{m-1}, m) > (kk^{-1}h_{n+1}, ..., h_{m-1}, -1) < \\ < (kk^{-1}h_{n+1}, ..., h_{m-2}, m-1) > (kk^{-1}h_{n+1}, ..., h_{m-2}, -1) < ... > (kh_1, ..., h_n, -1) < \\ < (kh_1, ..., h_{n-1}, n) > (kh_1, ..., h_{n-1}, -1) < ... > (k, -1)\end{aligned}$$

□

Remark 4.2. *From the proof of Theorem 2, it can be deduced the next property: if $f, g \in Aut(X_G)$ such that there exists $x \in X_G$ satisfying $f(x) = g(x)$, then $f = g$.*

Alternatively, the isomorphism of groups φ from the proof of Theorem 2 can be seen as a group action $\varphi : X_G \times G \to X_G$. By remark 4.2, the action of the group is free. On the other hand,

Proposition 4.2. *If A is an Alexandroff space, the only continuous flow map $\varphi : A \times \mathbb{R} \to A$ is the trivial one, i.e. $\varphi_t = Id$ for every $t \in \mathbb{R}$.*

Proof. We will treat A as a poset $(A, \leq)$ with the opposite order, that is to say, in Theorem 2.1 we consider the lower sets. We argue by contradiction, suppose that φ is not trivial. Then, there exists $x \in A$ with $\varphi^s(x) \neq x$ for some $s \in \mathbb{R}$. On the other hand, φ is continuous at $\varphi(x, 0) = x$, so there exists $\epsilon > 0$ such that $\varphi(F_x \times (-\epsilon, \epsilon)) \subset F_x$, where $F_x = \{y \in A | y \geq x\}$ is the minimal open set containing x.

Firstly, we will show that s can be considered in $(-\epsilon, \epsilon)$. If there is $s \in (-\epsilon, \epsilon)$ with $\varphi(x, s) \neq x$, we have finished. If there is no $s \in (-\epsilon, \epsilon)$ with $\varphi(x, s) \neq x$, we argue by contradiction. Suppose $s \notin (-\epsilon, \epsilon)$ and $s > 0$, the case when $s < 0$ is analogue. We can take $0 < \tau < \frac{\epsilon}{2}$, $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (n\tau, (n+2)\tau)$. Therefore, $s \in (n\tau, (n+2)\tau)$ for some $n \in \mathbb{N}$, $s - n\tau \in (0, 2\tau) \subset (0, \epsilon)$. By hypothesis, $\varphi(x, s - n\tau) = x$. Then,

$$\begin{aligned}x \neq \varphi(x, s) = \varphi(x, s - n\tau + n\tau) = \varphi(\varphi(x, \tau), (n-1)\tau + s - n\tau) = \varphi(x, (n-1)\tau + s - n\tau) = \\ = \varphi(\varphi(x, \tau), (n-2)\tau + s - n\tau) = \varphi(x, (n-2)\tau + s - n\tau) = ... = \varphi(x, s - n\tau) = x\end{aligned}$$

Hence, we can assume $s \in (-\epsilon, \epsilon)$ such that $\varphi(x, s) = y \in F_x$ with $y \neq x$ $(y > x)$. $\varphi^s : F_x \to F_x$ is a homeomorphism because $\varphi^t \in Aut(A)$ for every $t \in \mathbb{R}$. Thus, there exists $z \in F_x$ $(z > x)$ such that $\varphi^s(z) = x$. φ^s should preserve the order, so $x = \varphi^s(z) > \varphi^s(x) = y$ but $y > x$. □

5 The group of self-homotopy equivalences of $\overline{X}_G$ and $\overline{X}^*_G$

In this section, we show that every group G can be realized as the group of self-homotopy equivalences of infinitely many (non-homotopy-equivalent) Alexandroff spaces in the same weak homotopy type, Theorem 3, and the pointed version, Corollary 1. Firstly, we will show the result for $\overline{X}_G$, we divide the proof into two lemmas that show the rigidity of $\overline{X}_G$ in terms of autohomeomorphisms.

Lemma 5.1. *Given a group G, we get that $Aut(X_G)$ is isomorphic to $Aut(\overline{X}_G)$.*

Proof. Firstly, we will show that every $f \in Aut(\overline{X}_G)$ satisfies $f(X_G) = X_G$. Therefore, we would get $f_{|X_G} \in Aut(X_G)$. We study some properties by cases:

If $x = (g, \beta)$ with $\alpha > \beta \geq 0$, we get $f(g, \beta) \in X_G$. We argue by contradiction.

- The set of generators of G is infinite or $x = (g, \beta)$ with $\beta + 1 \neq \alpha$ for the finite case. $f(x) = y$, where y is $A_{(h,\gamma)}, B_{(h,\gamma)}, E_{(h,\gamma)}, F_{(h,\gamma)}$ or $G_{(h,\gamma)}$ for some $(h, \gamma) \in X_G$. But, $(g, \beta + 1) > (g, \beta)$, which implies $f(g, \beta + 1) = y = f(x)$ by the maximality of y and then a contradiction.

- The set of generators of G is finite and $x = (g, \beta)$ with $\beta + 1 = \alpha$, $f(x)$ can only be of the form (h, β), $B_{(h,\beta-1)}$ or $E_{(h,\beta-1)}$ for some $h \in G$ because f is a homeomorphism so it preserves the maximal chains of length $|\alpha|$ (maximal length), where we denote by $\beta - 1$ the ordinal number γ with $S(\gamma) = \beta$ because β is not a limit ordinal and $\beta \geq 1$. Suppose that $f(g, \beta) = B_{(h,\beta-1)}$ (the case $f(g, \beta) = E_{(h,\beta-1)}$ is similar), so $f^{-1}(B_{(h,\beta-1)}) = (g, \beta)$, by continuity $f^{-1}(C_{(h,\beta-1)}) < (g, \beta)$. There are three options for $f^{-1}(C_{(h,\beta-1)})$ due to the minimality of $C_{(h,\beta-1)}$:
  (1) $f^{-1}(C_{(h,\beta-1)})$ is equal to $(g, -1)$ or $(gh_i, -1)$ for some $h_i \in S$, therefore we obtain a contradiction studying the image of $(g, -1) < (g, 0) < (g, 1)$ or $(gh_i, -1) < (gh_i, 0) < (gh_i, 1)$ by f.
  (2) If $f^{-1}(C_{(h,\beta-1)}) = D_{(g,l)}$ with $l < \beta$, we get a contradiction studying the image of $D_{(g,l)} < (g, l) < (g, l + 1)$ by f.
  (3) $f^{-1}(C_{(h,\beta-1)}) = H_{(g,l)}$ we only need to repeat a similar argument used in (2).

- $f(x) = y$, where y is $C_{(h,\gamma)}, D_{(h,\gamma)}, H_{(h,\gamma)}, I_{(h,\gamma)}, J_{(h,\gamma)}$ for some $(h, \gamma) \in X_G$ or $(h, -1)$ for some $h \in G$. We have $(g, -1) < (g, \beta)$, we deduce by the minimality of y that $f(g, -1) = y = f(x)$ and then the contradiction.

Thus, we have that $f' := f_{|X_G \setminus \{(g,-1) \in X_G | g \in G\}} : X_G \setminus \{(g, -1) \in X_G | g \in G\} \to X_G \setminus \{(g, -1) \in X_G | g \in G\}$ is a homeomorphism. By Remark 4.1, we know that for every $g \in G$ and $\beta \geq 0$ we get $f'(g, \beta) = (h, \beta)$ for some $h \in G$. It only remains to show that if $x = (g, -1)$, we get $f(g, -1) \in X_G$. Again, we argue by contradiction.

- $f(x) = D_{(h,0)}$ for some $h \in G$. We have $(g, -1) < (g, 0)$, so $D_{(h,0)} = f(g, -1) < f(g, 0) = (h, 0)$ by the previous property and continuity of f. Moreover, $D_{(g,0)}, H_{(g,0)} < (g, 0)$. Then $f(g, 0) = (h, 0) > f(D_{(g,0)}), f(H_{(g,0)})$, there are two options:
  (1) $f(D_{(g,0)}) = (h, -1)$ and $f(H_{(g,0)}) = H_{(h,0)}$, but $A_{(g,0)} > D_{(g,0)}$, so $f(A_{(g,0)}) > f(D_{(g,0)}) = (h, -1)$. Thus,

$$f(A_{(g,0)}) = \begin{cases} (h, \gamma) & \gamma > 0 \\ B_{(h,\gamma)} & \gamma \geq 0 \\ E_{(h,\gamma)} & \gamma \geq 0 \\ y \geq (k, \delta) & \delta \geq 1 \end{cases}$$

  where $g = kh_\delta$ and h_δ is some generator. For all cases, the maximal chain containing $f(A_{(g,0)})$ at the top has at least length 3, while the maximal chain containing $A_{(g,0)}$ at the top has length 2. Therefore, we will obtain a contradiction with the injectivity studying f^{-1}.
  (2) $f(D_{(g,0)}) = H_{(h,0)}$ and $f(H_{(g,0)}) = (h, -1)$. Then, we can repeat the same argument used in (1) to $f(H_{(g,0)}) = (h, -1)$ so as to get the contradiction.

- $f(x) = H_{(h,0)}$ for some $h \in G$. We can repeat the same argument used for $f(x) = D_{(h,0)}$.

- $f(x) = y$, where y is $D_{(h,\gamma)}$ or $H_{(h,\gamma)}$ for some $h \in G$ and $\gamma > 0$. We have $(g, 0) > (g, -1)$, so $f(g, 0) > f(g, -1) = y$. Then, $f(g, 0) = (h, \delta)$ with $\delta \geq \gamma$ or $f(g, 0) = z$ with $z = B_{(h,\delta)}, E_{(h,\delta)}$ and $\delta \geq \gamma$, both cases leads to contradiction, the first case with the property obtained using Remark 4.1 and the second case with the property proved at the beginning of the proof.

- The rest of the cases are trivial using the fact that $(g, -1)$ is a minimal element and its part of an infinite chain given by $(g, -1) < (g, 0) < (g, 1) < ...$ or a maximal chain of length $|\alpha|$ in the case of a group with a finite set of generators.

From here, it is routine to deduce that $f(S_{(g,\beta)} \cup T_{(g,\beta)}) = S_{f(g,\beta)} \cup T_{f(g,\beta)}$. In concrete, $f(w_{(g,\beta)}) = w_{f(g,\beta)}$, where $w_{(g,\beta)} \in S_{(g,\beta)} \cup T_{(g,\beta)}$.

Now, we define $\phi : Aut(X_G) \to Aut(\overline{X}_G)$ given by $\phi(f) = \overline{f}$, where $\overline{f}$ is the natural extension, i.e. $f_{|X_G} = \overline{f}_{|X_G}$ and $\overline{f}(S_{(g,\beta)}) = S_{f(g,\beta)}$, $\overline{f}(T_{(g,\beta)}) = T_{f(g,\beta)}$. ϕ is clearly a well defined homomorphism of groups. If $f, s \in Aut(X_G)$ with $f \neq s$, it is immediate that $\phi(f) \neq \phi(s)$. In addition, if $f \in Aut(\overline{X}_G)$ we get $f_{|X_G} \in Aut(X_G)$ and $\phi(f_{|X_G}) = f$. Therefore, ϕ is an isomorphism of groups. □

Lemma 5.2. *Given a group G, we have that $\mathcal{E}(\overline{X}_G)$ is isomorphic to $Aut(\overline{X}_G)$.*

Proof. We need to prove that $\overline{X}_G$ is locally a core and then apply Corollary 2.1. Firstly, we show that $\overline{X}_G$ is a $\mathcal{C}$-core. Hence, we need to verify that the only comparative retraction of X is the identity. We argue by contradiction. Suppose that there exists a comparative retraction $r \neq Id$. Then, there exists $x \in X$ with $r(x) < x$ or $r(x) > x$. We study all possible cases.

Suppose that $r(x) > x$:

- x is a maximal element, that is to say, $x = A_{(g,\beta)}, B_{(g,\beta)}, E_{(g,\beta)}, F_{(g,\beta)}, G_{(g,\beta)}$, where $g \in G$ and $\beta \geq 0$, or $x = (g, \beta)$ with $\beta + 1 = \alpha$. We get $r(x) = x$ which leads to a contradiction.

- x is of the form (g, β) with $g \in G$ and $\beta \geq 0$ such that $\beta + 1 \neq \alpha$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} (g, \gamma) & \gamma > \beta \quad (1) \\ B_{(g,\gamma)} & \gamma \geq \beta \quad (2) \\ E_{(g,\gamma)} & \gamma \geq \beta \quad (3) \end{cases}$$

(1) $r(x) = (g, \gamma)$, we have $B_{(g,\beta)} > (g, \beta)$, so $r(B_{(g,\beta)}) \geq r(g, \beta) = (g, \gamma)$, but $B_{(g,\beta)}$ is not comparable to $r(B_{(g,\beta)})$ because $B_{(g,\beta)}$ is a maximal element and $r(B_{(g,\beta)}) \not< B_{(g,\beta)}$.
(2) $r(x) = B_{(g,\gamma)}$, we have $E_{(g,\beta)} > (g, \beta)$, so $r(E_{(g,\beta)}) = B_{(g,\gamma)}$ by the maximality of $B_{(g,\gamma)}$ and continuity of r, but it is clear that $E_{(g,\beta)}$ is not comparable to $B_{(g,\gamma)} = r(E_{(g,\beta)})$.
(3) $r(x) = E_{(g,\gamma)}$, we have $B_{(g,\beta)} > (g, \beta)$, so $r(B_{(g,\beta)}) = E_{(g,\gamma)}$ by the maximality of $E_{(g,\gamma)}$ and continuity of r, but it is clear that $B_{(g,\beta)}$ is not comparable to $E_{(g,\gamma)} = r(B_{(g,\beta)})$.

- x is of the form $(g,-1)$ with $g \in G$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} y \geq (g,0) & y \in \overline{X}_G \;\; (1) \\ y \geq (k,\beta) & y \in \overline{X}_G \;\; (2) \end{cases}$$

  where $\beta \geq 1$, $g = kh_\beta$ for some $k \in G$ and some generator $h_\beta \in S$.
  (1) $r(x) = y \geq (g,0)$, we have $(k,\beta) > x$, so $r(k,\beta) \geq r(x) \geq (g,0)$. Clearly, (k,β) is not comparable to $r(k,\beta)$.
  (2) $r(x) = y \geq (k,\beta)$, we have $(g,0) > (g,-1)$, so $r(g,0) \geq r(x) \geq (k,\beta)$. Again, it is clear that $(g,0)$ is not comparable to $r(g,0)$.

- x is of the form $C_{(g,\beta)}$ for some $(g,\beta) \in \overline{X}_G$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} A_{(g,\beta)} & (1) \\ B_{(g,\beta)} & (2) \end{cases}$$

  (1) $r(x) = A_{(g,\beta)}$, we have $C_{(g,\beta)} < B_{(g,\beta)}$, so $r(C_{(g,\beta)}) = A_{(g,\beta)} \leq r(B_{(g,\beta)})$. Then, $r(B_{(g,\beta)}) = A_{(g,\beta)}$ and $B_{(g,\beta)}$ is not comparable to $r(B_{(g,\beta)})$.
  (2) $r(x) = B_{(g,\beta)}$, we have $C_{(g,\beta)} < A_{(g,\beta)}$, so $r(C_{(g,\beta)}) = B_{(g,\beta)} \leq r(A_{(g,\beta)})$. Then, $r(A_{(g,\beta)}) = B_{(g,\beta)}$ and $A_{(g,\beta)}$ is not comparable to $r(A_{(g,\beta)})$.

- x is of the form $I_{(g,\beta)}$ or $J_{(g,\beta)}$ for some $(g,\beta) \in \overline{X}_G$. We only need to repeat the previous argument ($x = C_{(g,\beta)}$) to get a contradiction.

- x is of the form $D_{(g,\beta)}$ for some $(g,\beta) \in \overline{X}_G$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} A_{(g,\beta)} & (1) \\ y \geq (g,\beta) & y \in \overline{X}_G \;\; (2) \end{cases}$$

  (1) $r(x) = A_{(g,\beta)}$, we have $(g,\beta) > D_{(g,\beta)}$, so $r(g,\beta) \geq r(D_{(g,\beta)}) = A_{(g,\beta)}$. Then, $r(g,\beta) = A_{(g,\beta)}$ and (g,β) is not comparable to $r(g,\beta)$.
  (2) $r(x) = y \geq (g,\beta)$, we have $A_{(g,\beta)} > D_{(g,\beta)}$, so $r(A_{(g,\beta)}) \geq r(D_{(g,\beta)}) = y \geq (g,\beta)$. From here, it is easy to deduce that $A_{(g,\beta)}$ is not comparable to $r(A_{(g,\beta)})$.

- x is of the form $H_{(g,\beta)}$ for some $(g,\beta) \in \overline{X}_G$. We only need to repeat the previous argument ($x = D_{(g,\beta)}$) to get a contradiction.

Suppose that $r(x) < x$:

- x is a minimal element, that is to say, $x = (g,-1), C_{(g,\beta)}, D_{(g,\beta)}, H_{(g,\beta)}, I_{(g,\beta)}, J_{(g,\beta)}$, where $g \in G$ and $\alpha > \beta \geq 0$. We get $r(x) = x$ which leads to a contradiction.

- x is of the form (g,β) with $\alpha > \beta \geq 0$ and $\beta + 1 \neq \alpha$ (for the finite case of generators). We have the next options for $r(x)$.

$$r(x) = \begin{cases} (g,\gamma) & \gamma < \beta \;\; (1) \\ (gh_\delta,-1) & \delta \leq \beta \;\; (2) \\ D_{(g,\gamma)} & \gamma \leq \beta \;\; (3) \\ H_{(g,\gamma)} & \gamma \leq \beta \;\; (4) \end{cases}$$

  where h_δ is some generator in S.
  (1) $r(x) = (g,\gamma)$, we have $D_{(g,\beta)} < (g,\beta)$, so $r(D_{(g,\beta)}) \leq r(g,\beta) = (g,\gamma)$. Clearly, $D_{(g,\beta)}$ is not comparable to $r(D_{(g,\beta)})$.
  (2) $r(x) = (gh_\delta,-1)$, we have $D_{(g,\beta)} < (g,\beta)$, so $r(D_{(g,\beta)}) \leq r(g,\beta) = (gh_\delta,-1)$.

Then, $r(D_{(g,\beta)}) = (gh_\delta, -1)$ and $D_{(g,\beta)}$ is not comparable to $r(D_{(g,\beta)})$.
(3) $r(x) = D_{(g,\gamma)}$, we have $H_{(g,\beta)} < (g, \beta)$, so $r(H_{(g,\beta)}) \leq r(x) = D_{(g,\gamma)}$. Then, $r(H_{(g,\beta)}) = D_{(g,\gamma)}$ and $H_{(g,\beta)}$ is not comparable to $r(H_{(g,\beta)})$.
(4) $r(x) = H_{(g,\gamma)}$, we only need to repeat the previous argument of the point (3).

- x is of the form (g, β), where $\beta + 1 = \alpha$, i.e. the set of generators S of G is finite. We only need to verify that (g, β) is not a down beat point and argue as we did in the previous point. We know that $(g, \beta) > (g, \beta - 1), (gh_\beta, -1)$, where we denote by $\beta - 1$ the ordinal number γ with $S(\gamma) = \beta$ because β is not a limit ordinal and $\beta \geq 1$, we show that there is no $z \in \overline{X}_G$ such that $(g, \beta) > z > (gh_\beta, -1)$. We argue by contradiction, if there exists z with that property, z is of the form (g, i) for $0 < i \leq \beta - 1$. We have $(g, \beta) > (g, i) > (gh_\beta, -1)$, so $h_\beta = e$ (identity element) or there exists $0 < j \leq i$, $\beta \neq j$ with $h_\beta = h_j$ and then we get the contradiction for both cases because S is a non-trivial set of generators. Hence, $r(x)$ can only be of the form:

$$r(x) = \begin{cases} y \leq (g, \gamma) & \gamma < \beta, \ \ y \in \overline{X}_G \ \ (1) \\ (gh_\beta, -1) & \qquad (2) \end{cases}$$

(1) $r(x) = y \leq (g, \gamma)$, we have that $(gh_\beta, -1) < (g, \beta)$, so $r(gh_\beta, -1) \leq r(x)$ and $r(gh_\beta, -1)$ is not comparable to $(gh_\beta, -1)$ due to the non-triviality of S (no repetition of generators).
(2) $r(x) = (gh_\beta, -1)$, we have $(g, \beta - 1) < x$, so $r(g, \beta - 1) = (gh_\beta, -1)$ and $(gh_\beta, -1)$ is not comparable to $r(gh_\beta, -1)$ again due to the not triviality of S ($e \notin S$).

- x is of the form $A_{(g,\beta)}$ for some $(g, \beta) \in \overline{X}_G$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} C_{(g,\beta)} & (1) \\ D_{(g,\beta)} & (2) \end{cases}$$

(1) $r(x) = C_{(g,\beta)}$, we have $D_{(g,\beta)} < A_{(g,\beta)}$, we deduce $r(D_{(g,\beta)}) = C_{(g,\beta)}$. Then, $D_{(g,\beta)}$ is not comparable to $r(D_{(g,\beta)})$.
(2) $r(x) = D_{(g,\beta)}$. We only need to argue as we did in (1).

- x is of the form $F_{(g,\beta)}$ or $G_{(g,\beta)}$ for some $(g, \beta) \in \overline{X}_G$. The argument to get the contradiction for both cases is the same used before when $x = A_{(g,\beta)}$.

- x is of the form $B_{(g,\beta)}$ for some $(g, \beta) \in \overline{X}_G$. We have the next options for $r(x)$.

$$r(x) = \begin{cases} C_{(g,\beta)} & \qquad (1) \\ y \leq (g, \beta) & y \in \overline{X}_G \ \ (2) \end{cases}$$

(1) $r(x) = C_{(g,\beta)}$, we have $(g, \beta) < B_{(g,\beta)}$. From here, we deduce $r(g, \beta) = C_{(g,\beta)}$ and then (g, β) is not comparable to $r(g, \beta)$.
(2) $r(x) = y \leq (g, \beta)$, we have $C_{(g,\beta)} < B_{(g,\beta)}$, so $r(C_{(g,\beta)}) \leq r(B_{(g,\beta)}) = y \leq (g, \beta)$. Thus, $C_{(g,\beta)}$ is not comparable to $r(C_{(g,\beta)})$.

- x is of the form $E_{(g,\beta)}$ for some $(g, \beta) \in \overline{X}_G$. We can adapt the argument used before when $x = B_{(g,\beta)}$.

Hence, we have shown that $\overline{X}_G$ is a $\mathcal{C}$-core. It remains to prove that $\overline{X}_G$ is locally a core. For every $x \in \overline{X}_G$ of the form (g, β) or $x \in S_{(g,\beta)}, T_{(g,\beta)}$, where $\beta \geq 0$, $\beta + 1 \neq \alpha$ and $g \in G$, we consider $A_x = S_x \cup T_x \cup x$. If $x = (g, -1)$, where $g = kh_\gamma$ for some $k \in G$ and generator h_γ, we consider $A_x = S_{(g,0)} \cup T_{(g,0)} \cup S_{(k,\gamma)} \cup T_{(k,\gamma)} \cup (g, -1) \cup (k, \gamma) \cup (g, 0)$. If $x = (g, \beta)$ with $\beta + 1 = \alpha$, we consider $A_x = S_{(g,\beta-1)} \cup T_{(g,\beta-1)} \cup S_{(gh_\beta,0)} \cup T_{(gh_\beta,0)} \cup x \cup (gh_\beta, -1) \cup (gh_\beta, 0) \cup (g, \beta - 1)$, where h_β is a generator. It is immediate to show that A_x satisfies the property asked in Definition 2.3.

□

Remark 5.1. *If G is a finite group, it is not necessary to use the general results of Kukiela for Alexandroff spaces [13]. As we mentioned in Remark 3.1, if G is finite, X_G is a finite T_0 topological space. Therefore, $\overline{X}_G$ is also a finite T_0 topological space with $n(r+2)+10nr$ points, where $|G| = n$ and $|S| = r$. Thus, we only need to verify that there are no beat points to get $\mathcal{E}(\overline{X}_G) \simeq Aut(X_G)$, that is to say, apply Theorem 2.2.*

In general, for an arbitrary Alexandroff space we can not expect to obtain an isomorphism of groups between the group of autohomeomorphisms and the group of homotopy classes of self-homotopy equivalences.

Example 5.1. *Let $A = \{a, b, c, d, e\}$, we use the topology associated to the next partial order via Theorem 2.1: $a, b < d, c, e$ and $c < e$. Firstly, we study $Aut(A)$. An autohomeomorphism preserves the order and therefore should send maximal chains to maximal chains, in A, there are two maximal chains, $a < c < e$ and $b < c < e$. From here, it is easy to deduce that e, d and c are fixed points for every autohomeomorphism and then $Aut(A) \simeq \mathbb{Z}_2$. On the other hand, $A^c = \{a, b, c, d\}$ is the core of A because e is clearly a down beat point and A^c does not contain beat points. Hence, $\mathcal{E}(A) \simeq \mathcal{E}(A^c)$. In addition, A^c is a core, by Theorem 2.2, we get $Aut(A^c) \simeq \mathcal{E}(A^c)$. From here, it is immediate that $\mathcal{E}(A^c)$ is the Klein four-group. We describe the two generators f and g of $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathcal{E}(A)$. f is given by $f(a) = b, f(b) = a, f(c) = c, f(d) = d$ and g is given by $g(c) = d, g(d) = c, g(a) = a, g(b) = b$. A schematic situation in the Hasse diagrams can be seen in Figure 6.*

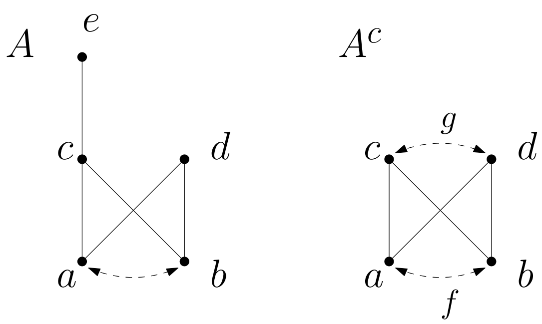

Figure 6: Hasse diagram of A and A^c

Theorem 5.1. *Every group can be realized as the group of self-homotopy equivalences of an Alexandroff space.*

Proof. Given a group G, we consider X_G and $\overline{X}_G$. By Theorem 2, $G \simeq Aut(X_G)$. In addition, by Lemma 5.1 and Lemma 5.2 we get $\mathcal{E}(\overline{X}_G) \simeq Aut(X_G)$. □

Remark 5.2. *By Remark 4.2, Lemma 5.1 and Lemma 5.2, it can be deduced that the set $L_{x,y} = \{f : (\overline{X}_G, x) \to (\overline{X}_G, y) | f(x) = y$ and $f \in \mathcal{E}(\overline{X}_G)\}$ has cardinality 1, if there exists $f \in \mathcal{E}(\overline{X}_G)$ with $f(x) = y$, or cardinality 0 if there is no $f \in \mathcal{E}(\overline{X}_G)$ with $f(x) = y$.*

A slight modification of the construction made in Section 3 can provide us with infinite Alexandroff spaces satisfying the Theorem 5.1. We only need to change $T_{(g,\beta)}$ by $T^n_{(g,\beta)}$ for every $(g,\beta) \in \overline{X}_G$, with $\alpha > \beta \geq 0$, $\beta + 1 \neq \alpha$ and $n \in \mathbb{N}$, so as to get $\overline{X}^n_G$. $T^n_{(g,\beta)}$ consists of $2n+4$ points, in concrete, $T^n_{(g,\beta)} = \{x_1, ..., x_{2+n}, y_1, ..., y_{2+n}\}$, where x_i denotes the maximal elements and y_i denotes the minimal elements for $i = 1, ..., 2+n$. The relations are given by

$$(g,i) < x_1 > y_2 < x_3 > ... < x_{1+n} > y_{2+n} < x_{2+n} > y_{1+n} < x_n > ... < x_2 > y_1 < (g,i)$$

An example of the Hasse diagrams can be seen in Figure 7.

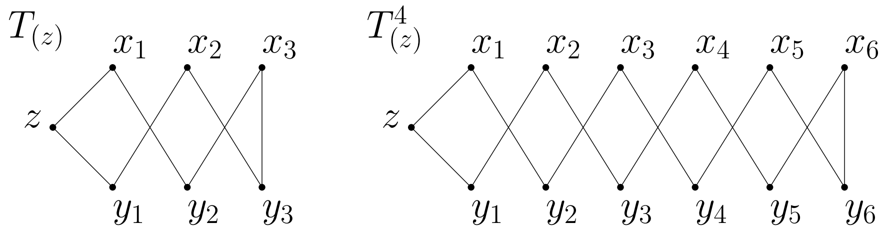

Figure 7: Hasse diagram of $T_{(z)}$ and $T^4_{(z)}$

It is clear that $T^1_{(g,\beta)} = T_{(g,\beta)}$ and $\overline{X}^1_G = \overline{X}_G$. We could argue in a similar way varying $S_{(g,\beta)}$ or $S_{(g,\beta)}$ and $T_{(g,\beta)}$ at the same time, for every $(g,\beta) \in \overline{X}_G$ with $\beta \geq 0$ and $\beta + 1 \neq \alpha$. The only condition that we need to keep is an asymmetry between the variation of $S_{(g,\beta)}$ and $T_{(g,\beta)}$, i.e. if $S^*_{(g,\beta)}$ and $T^*_{(g,\beta)}$ are variations of $S_{(g,\beta)}$ and $T_{(g,\beta)}$, then $T^*_{(g,\beta)}$ is not homeomorphic to $S^*_{(g,\beta)}$. The main reason for the previous condition is to keep invariant the number of autohomeomorphisms of $\overline{X}'_G$, where $\overline{X}'_G$ denotes one possible variation of $\overline{X}_G$. If $T^*_{(g,\beta)}$ is homeomorphic to $S^*_{(g,\beta)}$, we have introduced an infinite number of autohomeomorphisms given by the symmetry through (g,β) between $S^*_{(g,\beta)}$ and $T^*_{(g,\beta)}$ and keeping the rest of the points fixed. Therefore, we can not expect to obtain $Aut(\overline{X}'_G) \simeq Aut(\overline{X}_G)$ and $\mathcal{E}(\overline{X}'_G) \simeq \mathcal{E}(\overline{X}_G)$.

We only need to combine the previous results and constructions to obtain the proof of the Theorem 3.

Proof of Theorem 3. Given a group G we only need to consider $\{\overline{X}^n_G\}_{n\in\mathbb{N}}$. We can prove that $Aut(\overline{X}^n_G) \simeq Aut(X_G)$ using similar arguments from Lemma 5.1. Repeating the same proof of Lemma 5.2 we can obtain $\mathcal{E}(\overline{X}^n_G) \simeq Aut(\overline{X}^n_G)$. By Theorem 2, we get $Aut(X_G) \simeq G$.

If $\overline{X}^n_G$ homotopic to $\overline{X}^m_G$, where $m \neq n$, there exist $f : \overline{X}^n_G \to \overline{X}^m_G$ and $g : \overline{X}^m_G \to \overline{X}^n_G$, such that $f \circ g \simeq Id_{\overline{X}^m_G}$ and $g \circ f \simeq Id_{\overline{X}^n_G}$. By Corollary 2.1, $f \circ g = Id_{\overline{X}^m_G}$ and $g \circ f = Id_{\overline{X}^n_G}$. Then, f and g are bijective and continuous. But, it is immediate to check that $\overline{X}^n_G$ and $\overline{X}^m_G$ are not homeomorphic; a homeomorphism should preserve the chains of the form $(g,-1) < (g,0) < ...$, that is to say, $f(g,\beta) = (h,\beta)$ for some $h \in G$ and every $\beta < \alpha$. Then, we get the contradiction studying the image of $T^n_{(g,\beta)}$ by f. If G is a finite group, the result is immediate looking at the cardinality of $\overline{X}^m_G$ and $\overline{X}^n_G$.

To prove the last part, we define a candidate to be a weak homotopy equivalence $f^n : \overline{X}^n_G \to \overline{X}_G$. We define f^n by parts,

$$f^n(x) = \begin{cases} x & \text{if} \quad x \in \overline{X}^n_G \setminus (\bigcup_{x \in X^n_G} T^n_{(x)}) \\ x_i \in T_{(z)} & \text{if} \quad x = x_i \in T^n_{(z)} \quad i = 1,2,3 \quad \text{for some} \quad z = (g,\beta) \in X_G, \quad \beta \geq 0 \\ y_i \in T_{(z)} & \text{if} \quad x = y_i \in T^n_{(z)} \quad i = 1,2,3 \quad \text{for some} \quad z = (g,\beta) \in X_G, \quad \beta \geq 0 \\ x_3 \in T_{(z)} & \text{if} \quad x = x_i \in T^n_{(z)} \quad i = 4,...,2+n \quad \text{for some} \quad z = (g,\beta) \in X_G, \quad \beta \geq 0 \\ y_3 \in T_{(z)} & \text{if} \quad x = y_i \in T^n_{(z)} \quad i = 4,...,2+n \quad \text{for some} \quad z = (g,\beta) \in X_G, \quad \beta \geq 0 \end{cases}$$

f^n collapses $T^n_{(z)}$ to $T_{(z)}$ and keeps the rest of the points fixed. It is easy to show that $|\mathcal{K}(T_{(z)})|$ and $|\mathcal{K}(T^n_{(z)})|$ are homotopic to S^1 for every $z = (g,\beta) \in X_G$ with $\beta \geq 0$. From here, it can be deduced that $\mathcal{K}(f^n)$ is a homotopy equivalence. Then, by the 2-out-of-3 property for weak homotopy equivalences, that is to say, if f and g are two composable maps and 2 of the 3 maps f, g, fg are weak homotopy equivalences, then so is the third, we can deduce that f^n is a weak homotopy equivalence. Hence, $\overline{X}^n_G$ and $\overline{X}_G$ have the same weak homotopy type for every $n \in \mathbb{N}$.

□

Remark 5.3. *We can deduce from the proof of Theorem 3 an analogue of that theorem for the group of autohomeomorphism, i.e. every group G can be realized by the group of autohomeomorphism of infinitely many (non-homeomorphic) Alexandroff spaces.*

Proof of Corollary 1. The spaces used in Theorem 3 are far from satisfy that their group of pointed homotopy classes of pointed self-homotopy equivalences is isomorphic to G due to Remark 4.2. Nevertheless, it will be only necessary to add one extra point to $\overline{X}^n_G$ so as to obtain the desired result. We define $\overline{X}^{n*}_G = \overline{X}^n_G \cup *$, where $* > (g,-1)$ for every $g \in G$. Firstly, we will show that $\overline{X}^{n*}_G$ is a $\mathcal{C}$-core. Then, we need to verify that the only comparative retraction is the identity. We argue by contradiction, that is to say, there exists a comparative retraction r with $r(x) \neq x$ for some $x \in \overline{X}^{n*}_G$, from the proof of Theorem 3 it only remains to study the points of the form $(g,-1)$ and $*$. Suppose that $* > r(*) = (g,-1)$ ($r(*) > *$ is not possible by the maximality of $*$) for some $g \in G$, $(h,-1) < *$ for some $g \neq h \in G$, by the continuity of r we get that $r(h,-1) \leq r(*) = (g,-1)$ so $r(h,-1) = (g,-1)$ and $(h,-1)$ is not comparable to $r(h,-1)$. Let us consider $x = (g,-1)$, $* = r(x) > x$ (the other cases were studied in the proof of Theorem 3), $(g,0) > (g,-1)$, by the continuity of r, $r(g,-1) = * \leq r(g,0)$ so $* = r(g,0)$, but it is clear that $(g,0)$ is not comparable to $* = r(g,0)$.

It is immediate to show that $\overline{X}^{n*}_G$ is locally a core, we only need to define a finite set A_* satisfying the property asked in Definition 2.3. We take $x = (g,-1)$ and $y = (h,-1)$ for some $g,h \in G$. We consider the set A_x and A_y defined in the proof of Lemma 5.2. Therefore, $A_* = * \cup A_x \cup A_y$ trivially satisfies the property asked. Thus, $\overline{X}^{n*}_G$ is locally a core, so $\mathcal{E}(\overline{X}^{n*}_G) \simeq Aut(\overline{X}^{n*}_G)$.

Lastly, we need to show that $G \simeq Aut(\overline{X}^n_G) \simeq Aut(\overline{X}^{n*}_G)$. To do that, we will show that $*$ is a fixed point for every $f \in Aut(\overline{X}^{n*}_G)$. We argue by contradiction, suppose $*$ is not a fixed point, we study cases. If $f(*) = y$ with y a minimal element or an element of the form (g,γ) or $B_{(g,\gamma)}$ or $E_{(g,\gamma)}$ for some $g \in G$ and $-1 \leq \gamma < \alpha$, then we get the contradiction studying f^{-1}. If $f(*) = A_{(g,\gamma)}, F_{(g,\gamma)}, G_{(g,\gamma)}$ and $|G| > 2$ we obtain a contradiction with the bijectivity of f because there are at least 3 different elements smaller than $*$, but the image of $*$ by f only have 2 elements smaller than $f(*)$; if $|G| = 2$ ($G = \mathbb{Z}_2$), studying the image of $* > (g,-1) < (g,0) < (g,1)$ by f we obtain the contradiction. Then, it is trivial that $Aut(\overline{X}^n_G) \simeq Aut(\overline{X}^{n*}_G)$.

We consider the group of pointed self-homotopy equivalences of the pointed space $(\overline{X}^{n*}_G, *)$, from the previous arguments we can deduce that the previous group is isomorphic to G. Using the proof of Theorem 3, it is trivial to check that $\overline{X}_G^{n*}$ is homotopic to $\overline{X}_G^{m*}$ if and only if $m = n$. □

Finally, we get from the results obtained previously a direct proof for Theorem 1.

Proof of Theorem 1. It is an immediate consequence of Theorem 5.1 and the theory of McCord (Theorem 2.5). □

6 Some properties of $\overline{X}_G$

We study the weak homotopy type of the space $\overline{X}_G$. To do that we will obtained a good representation in terms of homotopy of the McCord complex $\mathcal{K}(\overline{X}_G)$ related to $\overline{X}_G$.

We define the undirected graph $H_u(\overline{X}_G)$ given by the Hasse diagram of $\overline{X}_G$, the set of vertices are the points of $\overline{X}_G$ and there is an edge between two vertices x and y if and only if $x < y$ ($x > y$) and there is no $z \in \overline{X}_G$ with $x < z < y$ ($x > z > y$). It is clear that we have a well defined continuous inclusion $i : H_u(\overline{X}_G) \to |\mathcal{K}(\overline{X}_G)|$, which it is indeed a deformation retract (proof of Proposition 6.1).

Proposition 6.1. *If G is an infinite group, $|\mathcal{K}(\overline{X}_G)| \simeq \bigvee_{\aleph_\alpha} S^1$, where $\aleph_\alpha$ denotes the cardinal of α. If G is a finite group, $|\mathcal{K}(\overline{X}_G)|$ is homotopy equivalent to the wedge sum of $3nr - n + 1$ copies of S^1, where $|G| = n$ and $|S| = r$.*

Proof. Firstly, we will show that $H_u(\overline{X}_G)$ and $|\mathcal{K}(\overline{X}_G)|$ have the same homotopy type. The idea of the proof will be to show that i is a weak homotopy equivalence between two CW-complexes. Then, by a well known theorem of Whithead [19], we would get that $H_u(\overline{X}_G)$ is a deformation retract of $|\mathcal{K}(\overline{X}_G)|$.

By Theorem 2.5, we know that there is a weak homotopy equivalence $f : |\mathcal{K}(\overline{X}_G)| \to \overline{X}_G$, in concrete, $f^{-1}(U_x)$ is an open neighborhood of $|\mathcal{K}(U_x)|$ and homotopic to $|\mathcal{K}(U_x)|$. On the other hand, $|\mathcal{K}(U_x)|$ and U_x are contractible, where U_x is the open given by the intersection of every open set that contains $x \in \overline{X}_G$, $\mathcal{U}$ will denote the basis-like open cover given by $\{U_x\}_{x\in\overline{X}_n}$. It is straightforward to check that $f^{-1}(\mathcal{U}) = \{f^{-1}(U_x)|U_x \in \mathcal{U}\}$ is an open cover basis-like for $|\mathcal{K}(\overline{X}_G)|$.

In addition, $i^{-1}(f^{-1}(U_x))$ corresponds in homotopy to the undirected Hasse diagram of U_x, that is to say, $i^{-1}(f^{-1}(U_x)) \simeq H_u(U_x) \subset H_u(\overline{X}_G)$. But $H_u(U_x)$ is contractible for every $x \in \overline{X}_G$ since $H_u(U_x)$ is a tree, the vertices of $H(U_x)$ are x and $y < x$ with $y \in \overline{X}_G$. Thus, the inclusion i is a weak homotopy equivalence by Theorem 2.4.

For the infinite case, each column $C_g := \{(g,\beta), S_{(g,\beta)}, T_{(g,\beta)}|\beta < \alpha\}$ of the graph contains $2\aleph_\alpha = \aleph_\alpha$ copies of S^1, $H_u(T_{(x)}) \simeq H_u(S_{(x)}) \simeq S^1$, hence we can consider that $H_u(T_{(x)} \cup S_{(x)})$ is $S^1 \bigvee S^1$ with one vertex and two edges. We can collapse C_g to $(g,-1)$ for every $g \in G$, Figure 8.

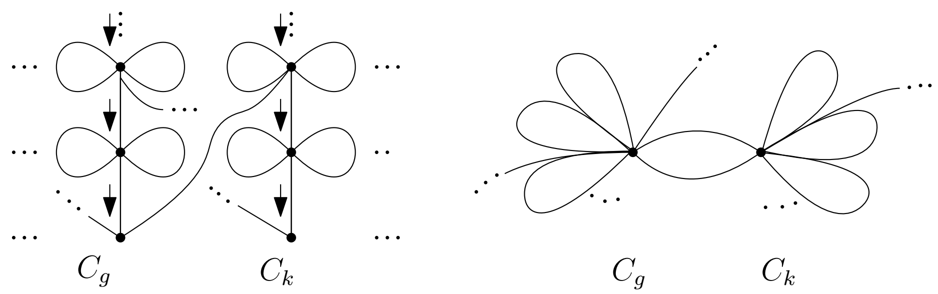

Figure 8: The collapse of the columns

Furthermore, from each element of the form $(g,-1)$, there is just a finite number of edges to other vertices ($g = h_\gamma ... h_\sigma$, where $h_\epsilon \in S$); and from each element of the form (g,β), there are $\aleph_\sigma < \aleph_\alpha$ number of edges that go to vertices of the form $(gh_\rho,-1)$, where $\rho < \beta$, i.e. in both cases there are at most $\aleph_\sigma$ number of copies of S^1, so $\aleph_\sigma\aleph_\alpha = \aleph_\alpha$. There are $\aleph_\alpha$ elements in G, then $\aleph_\alpha\aleph_\alpha = \aleph_\alpha$ copies of S^1 by the previous observations. To conclude, we only need to collapse horizontally to some $(g,-1)$ for $g \in G$.

For the finite case $|G| = n$ and $|S| = r$, it is easy to show that X_G has the weak homotopy type of a wedge sum of $n(r-1)+1$ copies of S^1. In concrete, $H_u(X_G)$ is homotopy equivalent to $\bigvee_{n(r-1)+1} S^1$. In $H_u(\overline{X}_G)$, we have added $2nr$ circles ($H_u(T_{(x)})$ and $H_u(S_{(x)})$) to $H_u(X_G)$. Therefore, $H_u(\overline{X}_G)$ has the homotopy type of a wedge sum of $3nr-n+1$ copies of S^1.

□

From the proof of the Proposition 6.1 and the fact that $H_u(T^n_{(x)}) \simeq H_u(T_{(x)}) \simeq S^1$, it can also be deduced that the elements of the sequence $\{\overline{X}^n_G\}_{n\in\mathbb{N}}$ built in Section 5 have the weak homotopy type of $\overline{X}_G$ for every $n \in \mathbb{N}$.

Remark 6.1. *If G is a finite group and we follow the Remark 3.2, that is to say, we do not use $T_{(g,i)}$ in the construction of $\overline{X}_G$, we would get the wedge sum of $2nr-n+1$ copies of S^1.*

Proposition 6.2. *Given a group G, the McCord functor induces a natural monomorphism of groups $\mathcal{K} : \mathcal{E}(\overline{X}_G) \to \mathcal{E}(\mathcal{K}(\overline{X}_G))$.*

Proof. If $f \in \mathcal{E}(\overline{X}_G) \simeq Aut(\overline{X}_G)$, then $\mathcal{K}(f) \in Aut(\mathcal{K}(\overline{X}_G))$ and $\mathcal{K}(f) \in \mathcal{E}(\mathcal{K}(\overline{X}_G))$. It is trivial to check that $\mathcal{K} : \mathcal{E}(\overline{X}_G) \to \mathcal{E}(\mathcal{K}(\overline{X}_G))$ is a well defined homomorphism of groups. Let us check the injectivity, if $f, g \in \mathcal{E}(\overline{X}_G)$ with $f \neq g$ there exists $x = (h,\beta) \in \overline{X}_G$ with $f(x) \neq g(x)$. Therefore, $f(S_x) \neq g(S_x)$ and $\mathcal{K}(f)(\mathcal{K}(S_x)) \neq \mathcal{K}(g)(\mathcal{K}(S_x))$. On the other hand, $|\mathcal{K}(S_x)| \simeq S^1$. Then, $\mathcal{K}(f)$ and $\mathcal{K}(g)$ send the same copy of S^1 to different copies of S^1 in $|\mathcal{K}(\overline{X}_G)|$. Using the Proposition 6.1, it can be deduced that $\mathcal{K}(f)$ is not homotopic to $\mathcal{K}(g)$. □

Remark 6.2. *It is not difficult to check that the monomorphism of groups from Proposition 6.2 is not an isomorphism. For instance, we only need to consider the continuous function that exchange $H_u(S_{(x)})$ with $H_u(T_{(x)})$ in $H_u(\overline{X}_G)$ for some $x = (g,\beta)$, i.e. the symmetry through x of $H_u(S_{(x)})$ and $H_u(T_{(x)})$ in $H_u(\overline{X}_G)$ that fixes the rest of the points.*

For a general Alexandroff space A, the McCord functor $\mathcal{K} : \mathcal{E}(A) \to \mathcal{E}(\mathcal{K}(A))$ is not necessarily a monomorphism of groups.

Example 6.1. *Let us consider the Alexandroff space A^c considered in Example 5.1. The McCord complex $\mathcal{K}(A^c)$ of A^c is a triangulation of S^1. Then, $\mathcal{E}(\mathcal{K}(A^c)) \simeq \mathcal{E}(S^1) \simeq \mathbb{Z}_2$, while $\mathcal{E}(A^c) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Corollary 6.1. *$\mathcal{E}(\bigvee_{\mathbb{N}} S^1)$ is not countable.*

Proof. By Proposition 6.2 and Proposition 6.1, every countable group can be embedded in $\mathcal{E}(\bigvee_{\mathbb{N}} S^1)$. But there is not a countable group containing all countable groups. □

Remark 6.3. *The image of the monomorphism of Proposition 6.2 is not a normal subgroup of $\mathcal{E}(\mathcal{K}(\overline{X}_G))$ in general. We consider the Example 3.2, i.e. $G = C_3$. $f \in G \simeq \mathcal{E}(\overline{X}_G)$ with $f \neq Id$. We take $\rho \in \mathcal{E}(\mathcal{K}(\overline{X}_G)) \simeq \mathcal{E}(\bigvee_{i=1}^7 S^1)$ the counterclockwise rotation of the copies of S^1. Then, we get that $\rho f \rho^- 1 \neq Id, a, a^2$.*

Example 6.2. *Let $G = \mathbb{Z}$, the group of integer numbers with the addition, we consider the set of two generator $S = \{1, -1\}$. The Hasse diagram of $\overline{X}_{\mathbb{Z}}$ can be seen in Figure 9. From the Hasse diagram, it can be deduced that $\mathbb{Z} \simeq Aut(\overline{X}_{\mathbb{Z}}) \simeq \mathcal{E}(\overline{X}_{\mathbb{Z}})$ is generated by the translation to the right and to the left of the columns of the Hasse diagram.*

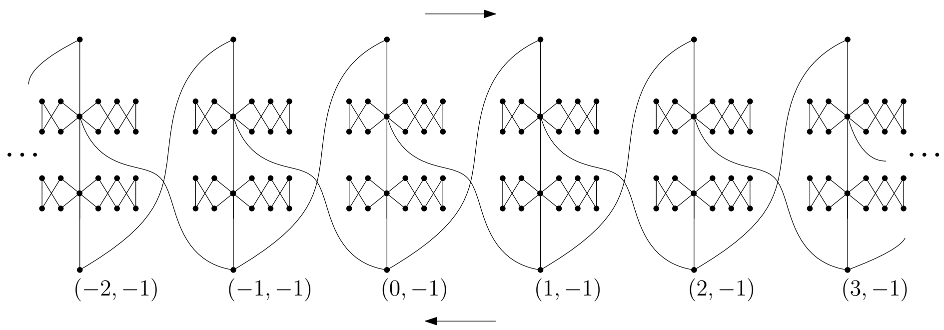

Figure 9: Hasse diagram of $\overline{X}_{\mathbb{Z}}$

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