ALGEBRAIC MULTIPLICITY AND TOPOLOGICAL DEGREE FOR FREDHOLM OPERATORS

JULIÁN LÓPEZ-GÓMEZ, JUAN CARLOS SAMPEDRO

This paper is dedicated to Shair Ahmad
at the occasion of his 85th anniversary
with admiration for his mathematical work
and profound personal esteem

"Estos sabios te enseñarán el camino de la eternidad,
te elevarán a un lugar desde donde nadie podrá derribarte;
éste es el único medio de prolongar la vida mortal,
o, mejor, de convertirla en inmortal.
Los honores, los monumentos, todo lo que la ambición ha decretado
y se ha esforzado en levantar se desploma rápidamente,
nada existe que resista largo tiempo,
demoliendo con preferencia lo que ella misma ha consagrado."

Lucius Annaeus Seneca (Extractum ab "De brevitate vitae")

Abstract. This paper tries to establish a link between topological and algebraic methods in nonlinear analysis showing how the topological degree for Fredholm operators of index zero of Fitzpatrick, Pejsachowicz and Rabier [11] can be determined from the generalized algebraic multiplicity of Esquinas and López-Gómez [8], [7], [22], in the same vein as the Leray–Schauder degree can be calculated from the Schauder formula through the classical algebraic multiplicity.

1. Introduction

In 1991, assuming that X and Y are two Banach spaces and $\mathfrak{L}:[a,b]\to\mathcal{L}(X,Y)$ is a continuous path of linear Fredholm operators of index zero with invertible endpoints, Fitzpatrick and Pejsachowicz [9, 10] introduced an homotopy invariant of $\mathfrak{L}$, the parity of $\mathfrak{L}$ on $[a,b]$, denoted by $\sigma(\mathfrak{L},[a,b])$, which later played a pivotal role in the construction of a degree extending the Leray–Schauder degree to the class of Fredholm operators with index zero, [11]. In [9] it was shown that, generically, the parity counts, modulus 2, the number of transversal intersections of $\mathfrak{L}([a,b])$ with the set of singular operators between X and Y, $\mathcal{S}(X,Y)$, and that the local parity,

$$\sigma(\mathfrak{L},\lambda_0):=\lim_{\varepsilon\downarrow 0}\sigma(\mathfrak{L},[\lambda_0-\varepsilon,\lambda_0+\varepsilon]), \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47H11, 58C40.

Key words and phrases. Schauder formula, Fredholm paths, Leray–Schauder degree, degree of Fitzpatrick, Pejsachowicz and Rabier, generalized algebraic multiplicity.

The first author has been supported by the Research Grant PGC2018-097104-B-100 of the Spanish Ministry of Science, Technology and Universities and by the Institute of Interdisciplinar Mathematics of Complutense University. The second author has been supported by PhD Grant PRE2019_1_0220 of the Basque Country Government.

remains invariant under Lyapunov–Schmidt reductions if $\lambda_0 \in (a,b)$ is an isolated singular value of $\mathfrak{L}(\lambda)$. Finally, Fitzpatrick and Pejsachowicz tried to establish that

$$\sigma(\mathfrak{L},\lambda_0) = (-1)^{m(\lambda_0)} \tag{1.2}$$

for any of the existing concepts of algebraic multiplicities, $m(\lambda_0)$, in the context of local and global bifurcation theory; among them, those introduced by Magnus [26], Ize [15], Esquinas and López-Gómez [8], and Esquinas [7].

Unfortunately, in [9] it was proven less than claimed, because in 1991, it was completely unknown whether, or not, the algebraic multiplicities of [26] and [7] were well defined. Thus, (1.2) remained unproven in some pivotal cases from the point of view of the applications, though it was indeed established for the Ize multiplicity, [15]. However, since the multiplicity of Ize [15] had been introduced as the order at λ_0 of $\det \mathfrak{L}(\lambda)$ through a preliminary Lyapunov-Schmidt decomposition for analytic paths, the Ize multiplicity is far from being directly computable in terms of the original path of Fredholm operators, $\mathfrak{L}(\lambda)$. This explains why Fitzpatrick and Pejsachowicz [9] had to face the technical problem of the invariance of the local parity by Lyapunov–Schmidt reductions, which is rather artificial from a purely algebraic perspective, and makes harder than necessary using the abstract theory in many concrete applications.

It was not until 2001, that Chapters 4 and 5 of [22] characterized whether the algebraic multiplicities of [26], [8] and [7] were well defined through the new pivotal concept of *algebraic eigenvalue*, unknown in [9]. A singular value, λ_0, of a continuous path of Fredholm operators, $\mathfrak{L}(\lambda)$, is said to be *k-algebraic* if there exist $\varepsilon > 0$ and $C > 0$ such that $\mathfrak{L}(\lambda)$ is an isomorphism if $0 < |\lambda - \lambda_0| < \varepsilon$, with

$$\|\mathfrak{L}^{-1}(\lambda)\| < \frac{C}{|\lambda-\lambda_0|^k} \quad \text{if } \; 0 < |\lambda-\lambda_0| < \varepsilon, \tag{1.3}$$

and k is the least positive integer for which (1.3) holds. It turns out that the multiplicities of [26], [8] and [7] are well defined if, and only if, the path $\mathfrak{L}(\lambda)$ is of class $\mathcal{C}^r$ for some $r \geq 1$ and λ_0 is a k-algebraic eigenvalue of $\mathfrak{L}(\lambda)$ with $1 \leq k \leq r$. According to Theorems 4.4.1 and 4.4.4 of [22], when $\mathfrak{L}$ is analytic and $\mathfrak{L}([a,b])$ contains some invertible operator, then the set of singular values of $\mathfrak{L}$, denoted by $\Sigma(\mathfrak{L})$, is discrete and any singular value, $\lambda_0 \in \Sigma(\mathfrak{L})$, is an algebraic eigenvalue of $\mathfrak{L}$. Therefore, the algebraic multiplicities of Magnus [26] and López-Gómez and Esquinas [8], [7] are well defined at these singular values. But this problem remained open in [26] and [7]. Naturally, ten years before, Fitzpatrick and Pejsachowicz in [9], could only use these concepts of multiplicity heuristically, but not rigorously because it was completely unknown whether, or not, they were really defined. It was unknown even in the context of analytic Fredholm paths of index zero.

Short time later, in 2004, Mora-Corral [27] axiomatized the theory of algebraic multiplicities for $\mathcal{C}^\infty$-Fredholm paths by establishing that, modulus a normalization condition (see Theorem 3.3), the algebraic multiplicity of Esquinas and López-Gómez, [8], [7], [22], denoted by χ through this paper, is the unique map

$$\chi[\cdot,\lambda_0] : \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon),\Phi_0(X)) \longrightarrow [0,\infty]$$

satisfying the *product formula*

$$\chi[\mathfrak{L}\circ\mathfrak{M},\lambda_0] = \chi[\mathfrak{L},\lambda_0] + \chi[\mathfrak{M},\lambda_0]$$

for every $\mathfrak{L},\mathfrak{M} \in \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon),\Phi_0(X))$. In this paper, $\Phi_0(X)$ stands for the set of Fredholm operators with index zero in the Banach space X. These more recent findings, outside the general scope of [9], [10] and [11], were covered systematically in the monograph [25], where, in addition, the theory of Göhberg and Sigal [14] was substantially generalized to a non-analytic setting and the existence of the local Smith canonical form, through the length of all Jordan chains of $\mathfrak{L}(\lambda)$, was characterized by means of the concept of k-algebraic eigenvalue. These results also characterized whether the multiplicity of Rabier [30] is well defined.

The interest of all these algebraic multiplicities, which, as soon as they satisfy the product formula, must equal χ, relies upon the crucial fact that, in the special case when $I_X - \mathfrak{L}(\lambda)$ is compact, according to Theorem 5.6.2 of [22], $\chi[\mathfrak{L}, \lambda_0]$ is odd if, and only if, the Leray–Schauder degree, $\deg_{LS}(\mathfrak{L}(\lambda), B_R(0))$ changes as λ crosses λ_0. Throughout this paper, for any given $x_0 \in X$ and $R > 0$, $B_R(x_0)$ stands for the ball of radius R centered at x_0. Thus, for any nonlinear compact perturbation of $\mathfrak{L}(\lambda)$, say $\mathfrak{N}(\lambda, x)$, i.e. a nonlinear compact map such that $\mathfrak{N}(\lambda, 0) = 0$ and $\mathfrak{N}(\lambda, x) = o(\|x\|)$ as $x \to 0$, the set of nontrivial solutions of the equation

$$\mathfrak{L}(\lambda)x + \mathfrak{N}(\lambda, x) = 0 \tag{1.4}$$

possesses a component bifurcating from $x = 0$ at $\lambda = \lambda_0$ if and only if $\chi[\mathfrak{L}, \lambda_0]$ is odd. Moreover, this component satisfies the global alternative of Rabinowitz [31]. As the number of applications of this result is huge, its mathematical relevance is considerable. However, as in many applications $\mathfrak{L}(\lambda)$ is a Fredholm path which cannot be expressed as a compact perturbation of the identity map, the degree of Fitzpatrick, Pejsachowicz and Rabier [11], $\deg_{FPR}(\mathfrak{L}(\lambda), B_R(0))$, became a powerful device from the point of view of its applications (see [10]).

The main goal of this paper is calculating $\deg_{FPR}$ through the multiplicity χ, in a similar way as the Leray–Schauder degree of an invertible compact perturbation of the identity, $L = I_X - K$, is determined from the classical algebraic multiplicity, for any bounded open set Ω with $0 \notin \partial\Omega$, through the Schauder formula

$$\deg_{LS}(L, \Omega) = (-1)^{\sum_{i=1}^{q} \mathfrak{m}_{\mathrm{alg}}[I_X - L, \mu_i]} \tag{1.5}$$

where

$$\mathrm{Spec}(I_X - L) \cap (1, \infty) = \{\mu_1, \mu_2, ..., \mu_q\} \qquad \mu_i \neq \mu_j \quad \text{if} \quad i \neq j,$$

and $\mathfrak{m}_{\mathrm{alg}}[I_X - L, \mu_i]$ stands for the classical algebraic multiplicity of μ_i as an eigenvalue of $K := I_X - L$ and $\mathrm{Spec}(I_X - L)$ denotes its classical spectrum. Our extension of the Schauder formula is motivated by the fact that (1.5) can be equivalently expressed as

$$\deg_{LS}(L, \Omega) = (-1)^{\sum_{i=1}^{q} \chi[\mathfrak{L}, \lambda_i]} \tag{1.6}$$

(see Theorem 3.6 of Section 3) where $\mathfrak{L}(\lambda)$ is the analytic Fredholm path defined by

$$\mathfrak{L}(\lambda) := (1 - \lambda)I_X + \lambda L, \qquad \lambda \in [0, 1],$$

and $\{\lambda_1, \lambda_2, ..., \lambda_q\}$ is the set of singular values of $\mathfrak{L}(\lambda)$ in $(0, 1)$. Note that $\deg_{LS}(L, \Omega) = 1$ if this set is empty. Although the formulas (1.5) and (1.6) are equivalent, (1.6) is far more versatile than (1.5) from the point of view of the applications, as it will become apparent shortly. Adopting an algorithmic perspective, the Schauder formula (1.5) relies on the classical concept of eigenvalue and algebraic multiplicity. Thus, in order to detect through it any change of the degree of a linear mapping depending on a parameter μ, L_μ, as μ varies, one should face the, very hard, problem of determining all the classical eigenvalues of the operators L_μ, while, according to (1.6), it is unnecessary to determine the classical spectrum of the operator L_μ as μ varies, but simply catching the oddities of the (generalized) algebraic multiplicities of the path

$$\mathfrak{L}_\mu(\lambda) = (1 - \lambda)I_X + \lambda L_\mu, \quad \lambda \in [0, 1],$$

which is a substantially simpler task from a technical point of view, as it can be accomplished through a finite algorithm (see [22]). Here relies the advantage of expressing the Schauder formula in terms of the algebraic multiplicity χ. As a byproduct of (1.6), for any regular admissible pair, (f,Ω), with respect to the Leray–Schauder degree, it becomes apparent that

$$\deg_{LS}(f,\Omega) := \sum_{x\in f^{-1}(0)\cap\Omega} (-1)^{\sum_{i=1}^{q_x}\chi[\mathfrak{L}_x,\lambda_{x,i}]} \tag{1.7}$$

where, for every $\lambda\in[0,1]$ and $x\in f^{-1}(0)\cap\Omega$,

$$\mathfrak{L}_x(\lambda) := (1-\lambda)I_X + \lambda Df(x), \quad \lambda\in[0,1],$$

and, for every $x\in f^{-1}(0)\cap\Omega$,

$$\Sigma(\mathfrak{L}_x) = \{\lambda_{x,1},\lambda_{x,2},\dots,\lambda_{x,q_x}\}.$$

Therefore, the Leray–Schauder degree can be calculated by means of the algebraic multiplicity χ. The main result of this paper extends (1.7) to the context of the degree of Fitzpatrick, Pejsachowicz and Rabier [11] by establishing that also this degree for Fredholm maps can be determined from the multiplicity χ.

Subsequently, we will denote by $\mathscr{A}_F$ the set of admissible triples for the degree $\deg_{FPR}$, $\mathscr{R}_F$ stands for the subset of $\mathscr{A}_F$ consisting of all regular admissible triples (see Section 5), and $\mathscr{RV}_f$ is the set of regular values of f. Given a triple, (f,Ω,ε), ε stands for an orientation of the image $Df(\Omega)$. The main result of this paper can be stated as follows.

Theorem 1.1. *Let $(f,\Omega,\varepsilon)\in\mathscr{A}_F$ be a Fredholm admissible triple with $\Omega\neq\emptyset$. Then, for every $L\in Df(\Omega)\cap GL(X,Y)$,*

$$\deg_{FPR}(f,\Omega,\varepsilon) = \varepsilon(L)\cdot\sum_{x\in f^{-1}(y)\cap\Omega}(-1)^{\sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})}\chi[\mathfrak{L}_{\omega,x},\lambda_x]}$$

where $\mathfrak{L}_{\omega,x}\in\mathscr{C}^\omega([a,b],\Phi_0(X,Y))$ is an analytical curve $\mathscr{C}$-homotopic to some curve $\mathfrak{L}_x\in\mathcal{C}([a,b],Df(\Omega))$ connecting $Df(x)$ to L, and $y=0$ if $(f,\Omega,\varepsilon)\in\mathscr{R}_F$, whereas $y\in\mathscr{RV}_f$ is any regular value of f sufficiently close to 0 if $(f,\Omega,\varepsilon)\notin\mathscr{R}_F$.

These formulas express the degree of Fitzpatrick, Pejsachowicz and Rabier [11] in terms of the multiplicity χ. Thus, they allow to calculate $\deg_{FPR}$ algorithmically, liberating it of the topological artillery used in its definition. So, expressing it in a versatile way from the point of view of the applications.

The distribution of this paper is the following. As we would like to be as much self-contained as possible, to facilitate the reading of this technical paper Section 2 collects the main fundamentals of the Leray–Schauder degree delivering a geometrical perspective of it that can be easily extrapolated to the degree of [11]. Then, after reviewing the main properties of χ, Section 3 establishes (1.6) and (1.7). In Section 4 we calculate rigorously the parity of Fitzpatrick and Pejsachowicz [9] through the multiplicity χ and obtain some (new) stability results that are necessary to prove Theorem 1.1. Finally, in Section 5 we shortly review the degree of Fitzpatrick, Pejsachowicz and Rabier [11] and characterize the orientability by using the geometrical ideas already introduced and discussed in Section 2 in the context of the Leray–Schauder degree. The new perspective facilitates extraordinarily the proof of Theorem 1.1, which has been delivered in Section 6. This paper concludes by extracting some simple consequences from Theorem 1.1.

2. Leray–Schauder degree and Schauder's formula

In this section we shortly review the Leray–Schauder degree. It was introduced in [19] to get some pioneering existence results on Nonlinear Partial Differential Equations, and refined, very substantially, the finite-dimensional degree introduced by Brouwer [5] to obtain his celebrated fixed point theorem. Roughly spoken, the Leray–Schauder degree is a generalized (topological) counter of the number of zeros that a continuous map, f, linear or nonlinear, can have on an open bounded subset, Ω, of a real Banach space, X. To be defined, the map f must be a compact perturbation of the identity map. Although this always occurs in a finite-dimensional context, it fails to be true in many important applications.

Throughout this paper, for any given pair of real Banach spaces X, Y with $X \subset Y$, we denote by $\mathcal{L}_c(X,Y)$ the set of linear and continuous operators, $L \in \mathcal{L}(X,Y)$, which are a compact perturbation of the identity map, $L = I_X - K$. Then, the *linear group*, $GL(X,Y)$ is defined as the set of linear isomorphisms $L \in \mathcal{L}(X,Y)$. Similarly, the *compact linear group*, $GL_c(X,Y)$, is defined as $GL(X,Y) \cap \mathcal{L}_c(X,Y)$.

The fastest way to introduce the Leray–Schauder degree proceeds through the following axiomatizing theorem. Subsequently, for any pair of real Banach spaces X, Y such that $X \subset Y$, any open and bounded domain $\Omega \subset X$ and any map $f : \overline{\Omega} \subset X \to Y$, it is said that (f, Ω) is an *admissible pair* if:

i) $f \in \mathcal{C}(\overline{\Omega}, Y)$;
ii) f is a compact perturbation of the identity map I_X;
iii) $0 \notin f(\partial\Omega)$.

The class of admissible pairs will be denoted by $\mathscr{A}$. Note that $(I_X, \Omega) \in \mathscr{A}$ for every open and bounded subset $\Omega \subset X$ such that $0 \notin \partial\Omega$. Actually, $(I_X, \Omega) \in \mathscr{A}_{GL}$, where $\mathscr{A}_{GL}$ stands for the set of admissible pairs $(L, \Omega) \in \mathscr{A}$ such that $L \in GL_c(X,Y)$.

Theorem 2.1. *For any given pair of real Banach spaces, X, Y such that $X \subset Y$, there exists an unique integer valued map, $\deg_{LS} : \mathscr{A} \to \mathbb{Z}$, satisfying the following properties:*

(N) ***Normalization:*** $\deg_{LS}(I_X, \Omega) = 1$ *if* $0 \in \Omega$.

(A) ***Additivity:*** *For every $(f, \Omega) \in \mathscr{A}$ and any pair of open disjoint subsets, Ω_1 and Ω_2, of Ω such that $0 \notin f(\overline{\Omega} \backslash (\Omega_1 \uplus \Omega_2))$,*

$$\deg_{LS}(f, \Omega) = \deg_{LS}(f, \Omega_1) + \deg_{LS}(f, \Omega_2). \tag{2.1}$$

(H) ***Homotopy Invariance:*** *For every homotopy $H \in \mathcal{C}([0,1] \times \overline{\Omega}, X)$ such that $(H(t, \cdot), \Omega) \in \mathscr{A}$ for each $t \in [0,1]$,*

$$\deg_{LS}(H(0, \cdot), \Omega) = \deg_{LS}(H(1, \cdot), \Omega).$$

Moreover, for every $(L, \Omega) \in \mathscr{A}_{GL}$ with $0 \in \Omega$,

$$\deg_{LS}(L, \Omega) = (-1)^{\sum_{i=1}^{q} \mathfrak{m}_{\mathrm{alg}}[I_X - L, \mu_i]} \tag{2.2}$$

where

$$\mathrm{Spec}(I_X - L) \cap (1, \infty) = \{\mu_1, \mu_2, ..., \mu_q\} \qquad \mu_i \neq \mu_j \quad \textit{if } i \neq j.$$

The existence part of Theorem 2.1 goes back to Brouwer [5] in $\mathbb{R}^N$ and to Leray and Schauder [19] in arbitrary real Banach spaces. The uniqueness assertion is attributable to Führer [13], in $\mathbb{R}^N$, and to Amann and Weiss [1] in the infinite-dimensional setting. The map $\deg_{LS}$ is usually refereed to as the *Leray–Schauder degree*. Although it goes back to [19], the formula (2.2) is usually referred to as the *Schauder formula*. It should be recalled that, setting $K := I_X - L$, for any eigenvalue $\mu \in \mathrm{Spec}(K)$, the classical algebraic multiplicity of μ is defined by

$$\mathfrak{m}_{\mathrm{alg}}[K, \mu] = \dim \mathrm{Ker}[(\mu I_X - K)^{\nu(\mu)}],$$

where $\nu(\mu)$ is the *algebraic ascent* of μ, i.e. the minimal integer, $\nu \geq 1$, such that

$$\mathrm{Ker}[(\mu - K)^{\nu}] = \mathrm{Ker}[(\mu I_X - K)^{\nu+1}].$$

In Theorem 2.1, the axiom (N) is called the *normalization property* because, for every $n \in \mathbb{Z}$, the map $n \deg_{LS}$ satisfies the axioms (A) and (H), though not (N). Thus, the axiom (N) normalizes the degree so that, for the identity map, it provides us with its exact number of zeroes. The *degree* of a polynomial, in the plane, is nothing but its number of zeroes on a sufficiently large disc. The degree is a generalized topological counter of the number of zeroes that a continuous map has on an open and bounded subset of X. The axiom (A) packages three basic properties that any counter of zeros should satisfy. Indeed, by choosing $\Omega = \Omega_1 = \Omega_2 = \emptyset$, it becomes apparent that

$$\deg_{LS}(f, \emptyset) = 0, \tag{2.3}$$

so establishing that no continuous map can admit a zero in the empty set. Moreover, in the special case when $\Omega = \Omega_1 \uplus \Omega_2$, (2.1) establishes the *additivity property* of the counter of zeroes. Finally, in the special case when $\Omega_2 = \emptyset$, it follows from (2.1) and (2.3) that

$$\deg_{LS}(f, \Omega) = \deg_{LS}(f, \Omega_1),$$

which is usually refereed to as the *excision property* of the degree. If, in addition, also $\Omega_1 = \emptyset$, then

$$\deg_{LS}(f, \Omega) = 0 \quad \text{if } \; f^{-1}(0) \cap \overline{\Omega} = \emptyset.$$

Therefore, for every $(f, \Omega) \in \mathscr{A}$ such that $\deg_{LS}(f, \Omega) \neq 0$, the equation $f(x) = 0$ admits, at least, a solution in Ω. This key property of the degree is refereed to as the *fundamental or solution property* of the degree.

The axiom (H) establishes the *invariance by homotopy* of the degree. Besides it entails that the degree is something else than the exact number of zeroes of the map f on Ω, as, otherwise, it would not be satisfied, it endows the degree with the possibility of being computed in the practical situations of interest from the point of view of the applications. Nevertheless, when dealing with analytic maps in $\mathbb{C}$, it coincides with the exact number of zeroes of the map, counting orders (see, e.g., Chapter 11 of [21]).

As a rather direct application of the excision property and the axiom (H) the next generalized version of the invariance by homotopy holds. For any given open subset $\mathcal{O} \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$, we are denoting

$$\mathcal{O}_t := \{x \in X \; : \; (t, x) \in \mathcal{O}\}.$$

Corollary 2.2. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and suppose that $\mathcal{O}$ is an open and bounded subset of $[a, b] \times X$. Let $H \in \mathcal{C}(\overline{\mathcal{O}}, Y)$ be such that $(H(t, \cdot,), \mathcal{O}_t) \in \mathscr{A}$ for each $t \in [a, b]$. Then,*

$$\deg_{LS}(H(t, \cdot), \mathcal{O}_t) = \deg_{LS}(H(a, \cdot), \mathcal{O}_a) \quad \textit{for all } \; t \in [a, b].$$

As a by-product of Corollary 2.2, it becomes apparent that, for every $L \in GL_c(X, Y)$, $x_0 \in X$, $R > 0$ and $t \in [0, 1]$,

$$\deg_{LS}(L(\cdot - x_0), B_R(x_0)) = \deg_{LS}(L(\cdot - tx_0), B_R(tx_0)) = \deg_{LS}(L, B_R), \tag{2.4}$$

where we are denoting $B_R := B_R(0)$.

Subsequently, we introduce the set of *regular admissible pairs*, $\mathscr{R}$, as the set of pairs $(f, \Omega) \in \mathscr{A}$ such that $f \in \mathcal{C}^1(\Omega, Y)$ and $Df(x) \in GL_c(X, Y)$ for all $x \in f^{-1}(0) \cap \Omega$. Combining the inverse function theorem with the fact that f is a compact perturbation of I_X and taking into account that f cannot admit zeroes on $\partial\Omega$, it is apparent that $f^{-1}(0) \cap \Omega$ is finite, possibly empty. Thus, in general,

$$f^{-1}(0) \cap \Omega = \{x_1, ..., x_n\} \quad \text{if } \; (f, \Omega) \in \mathscr{R}.$$

Now, choose a sufficiently small $\varepsilon > 0$ so that

$$\overline{B}_\varepsilon(x_i) \cap \overline{B}_\varepsilon(x_j) = \emptyset, \quad 1 \leq i < j \leq n.$$

Then, according to the axiom (A),

$$\deg_{LS}(f,\Omega)=\sum_{j=1}^{n}\deg_{LS}(f,B_\varepsilon(x_j)). \tag{2.5}$$

Moreover, thanks to the excision property, $\deg_{LS}(f,B_\varepsilon(x_j))$ is independent of the value of ε as soon as it is sufficiently small as to satisfy (2.5). Actually, for every $j\in\{1,...,n\}$, the map

$$H_j(t,x):=Df(x_j)(x-x_j)+t[f(x)-Df(x_j)(x-x_j)],\qquad (t,x)\in[0,1]\times\overline{B}_\varepsilon(x_j),$$

satisfies $(H_j(t,\cdot),B_\varepsilon(x_j))\in\mathscr{A}$ for all $t\in[0,1]$. Consequently, by the axiom (H), it follows from (2.5) that

$$\deg_{LS}(f,\Omega)=\sum_{j=1}^{n}\deg_{LS}(Df(x_j)(\cdot-x_j),B_\varepsilon(x_j)).$$

Therefore, by (2.4), we find that, for every $R>0$,

$$\deg_{LS}(f,\Omega)=\sum_{j=1}^{n}\deg_{LS}(Df(x_j),B_R).$$

Consequently, if

$$\mathrm{Spec}(I_X-Df(x_j))\cap(1,\infty)=\{\mu_{j,1},\mu_{j,2},...,\mu_{j,q_j}\},\quad 1\le j\le n,$$

then, it follows from the Schauder formula that

$$\deg_{LS}(f,\Omega)=\sum_{j=1}^{n}(-1)^{\sum_{i=1}^{q_j}\mathfrak{m}_{\mathrm{alg}}[I_X-Df(x_j),\mu_{j,i}]}. \tag{2.6}$$

Conversely, one might define the Leray–Schauder degree for regular pairs through the formula (2.6) and then use the Sard–Smale theorem, [32], [33], in order to extend it for general admissible pairs, as e.g. in the classical textbook of Lloyd [20].

Adopting a geometrical point of view, the construction of the Leray–Schauder degree can also be based upon the concept of orientation for $X=Y$. Let $H\in\mathcal{C}([0,1]\times\overline{\Omega},X)$ be a homotopy with $(H(t,\cdot),\Omega)\in\mathscr{A}_{GL}$ for each $t\in[0,1]$. Since H can be regarded as the continuous path $\mathfrak{L}\in\mathcal{C}([0,1],GL_c(X))$ defined by $\mathfrak{L}(t):=H(\cdot,t)$, $t\in[0,1]$, by the axiom (H), the integer $\deg_{LS}(\mathfrak{L}(t),\Omega)$ is constant for all $t\in[0,1]$. This introduces an equivalence relation between the operators of $GL_c(X)$. Indeed, for every pair of operators L_0, $L_1\in GL_c(X)$, it is said that $L_0\sim L_1$ if L_0 and L_1 are homotopic in $\mathscr{A}_{GL}$ in the sense that $L_0=\mathfrak{L}(0)$ and $L_1=\mathfrak{L}(1)$ for some path $\mathfrak{L}\in\mathcal{C}([0,1],GL_c(X))$. This equivalence relation divides the compact linear group into two path connected components, $GL_c^+(X)$ and $GL_c^-(X)$, separated away by $\mathcal{S}(X)\cap GL_c(X)$, where

$$\mathcal{S}(X):=\mathcal{L}(X)\setminus GL(X),$$

as illustrated in Figure 1.

Conversely, if $GL_c^+(X)$ stands for the path connected component of $GL_c(X)$ containing I_X, the fact that a given operator, $L\in GL_c(X)$ belongs to one component, or another, defines an *orientation* on L. This allows us to define a map,

$$\deg_{LS}(L,\Omega):=\begin{cases}1 & \text{if } L\in GL_c^+(X) \text{ and } 0\in\Omega,\\ -1 & \text{if } L\in GL_c^-(X) \text{ and } 0\in\Omega,\\ 0 & \text{if } L\in GL_c(X) \text{ and } 0\notin\Omega,\end{cases} \tag{2.7}$$

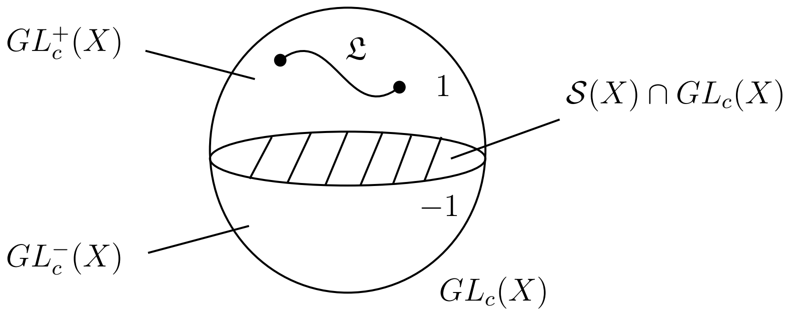

FIGURE 1. The two path connected components of $GL_c(X)$.

that verifies the three axioms of the Leray–Schauder degree within the class $\mathscr{A}_{GL}$ and, in particular, is homotopically invariant. Once defined the degree in $\mathscr{A}_{GL}$, one can extend this restricted concept of degree to the regular pairs $(f,\Omega) \in \mathscr{R}$ through the identity

$$\deg_{LS}(f,\Omega) = \sum_{x \in f^{-1}(0)\cap\Omega} \deg_{LS}(Df(x),\Omega).$$

Finally, according to the Sard–Smale theorem and the homotopy invariance property, it can be extended to be defined for general admissible pairs, $(f,\Omega) \in \mathscr{A}$. A crucial feature that facilitates this construction of the degree is the fact that the space $GL_c(X)$ consists of two path-connected components. Thus, it admits an orientation. This fails to be true in more general contexts, as it will be apparent later in Section 4.

3. The Schauder formula through the multiplicity χ

The classical spectral theory deals with straight lines $\mathfrak{L} \in \mathcal{C}([a,b],\mathcal{L}_c(X,Y))$ of the form $\lambda I_X - K$ for some compact operator K and their intersections with the space of singular operators $\mathcal{S}(X,Y)$. In this context, $\lambda_0 \in [a,b]$ is said to be an eigenvalue of the straight line $\mathfrak{L}(\lambda) = \lambda I_X - K$ if $\mathfrak{L}(\lambda_0) \in \mathcal{S}(X,Y)$. These linear paths, in particular, lie in the set of Fredholm operators of index zero, $\Phi_0(X,Y)$.

More generally, given two real Banach spaces, X and Y, nonlinear spectral theory deals with general continuous paths in $\Phi_0(X,Y)$, $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$, generalizing the classical theory not only because it deals with arbitrary continuous curves, not merely straight lines, but also because the paths can lie in $\Phi_0(X,Y)$; not only in

$$\mathcal{L}_c(X,Y) \subset \Phi_0(X,Y).$$

This section collects some of the most basic concepts and results in this field. Subsequently, by a *Fredholm path, or curve,* it is meant any map $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$. Given a Fredholm path, $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$, it is said that $\lambda \in [a,b]$ is a *generalized eigenvalue* of $\mathfrak{L}$ if $\mathfrak{L}(\lambda) \notin GL(X,Y)$. Then, the *generalized spectrum* of $\mathfrak{L}$, $\Sigma(\mathfrak{L})$, consists of the set of all these generalized eigenvalues, i.e.,

$$\Sigma(\mathfrak{L}) := \{\lambda \in [a,b] : \mathfrak{L}(\lambda) \notin GL(X,Y)\}.$$

According to Lemma 6.1.1 of [22], $\Sigma(\mathfrak{L})$ is a compact subset of $[a,b]$, though, in general, one cannot say anything more about it, because for any given compact subset of $[a,b]$, J, there exists a continuous function $\mathfrak{L} : [a,b] \to \mathbb{R}$ such that $J = \mathfrak{L}^{-1}(0)$.

The following concept is pivotal in nonlinear spectral theory. It was introduced in [22] to characterize whether, or not, the algebraic multiplicity of [8] and [7] is well defined.

Definition 3.1. *Let $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ and $k \in \mathbb{N}$. A generalized eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be a k-**algebraic eigenvalue** if there exits $\varepsilon > 0$ such that*

(a) $\mathfrak{L}(\lambda) \in GL(X,Y)$ *if* $0<|\lambda-\lambda_0|<\varepsilon$*;*
(b) *There exits* $C>0$ *such that*

$$\|\mathfrak{L}^{-1}(\lambda)\| < \frac{C}{|\lambda-\lambda_0|^k} \quad \text{if } 0<|\lambda-\lambda_0|<\varepsilon; \tag{3.1}$$

(c) k *is the least positive integer for which* (3.1) *holds.*

The set of algebraic eigenvalues of $\mathfrak{L}$ or order k will be denoted by $\mathrm{Alg}_k(\mathfrak{L})$. Thus, the set of *algebraic eigenvalues* can be defined by

$$\mathrm{Alg}(\mathfrak{L}) := \biguplus_{k\in\mathbb{N}} \mathrm{Alg}_k(\mathfrak{L}).$$

According to Theorems 4.4.1 and 4.4.4 of [22], when $\mathfrak{L}(\lambda)$ is real analytic in $[a,b]$, i.e., $\mathfrak{L}\in\mathcal{C}^\omega([a,b],\Phi_0(X,Y))$, either $\Sigma(\mathfrak{L})=[a,b]$, or $\Sigma(\mathfrak{L})$ is finite and $\Sigma(\mathfrak{L})\subset\mathrm{Alg}(\mathfrak{L})$.

According to the theory developed in Chapter 7 of [25], $\lambda_0\in\mathrm{Alg}(\mathfrak{L})$ if, and only if, the lengths of all Jordan chains of $\mathfrak{L}$ at λ_0 are uniformly bounded above. This allowed to characterize whether, or not, $\mathfrak{L}(\lambda)$ admits a local Smith form at λ_0.

The next concept, going back to [8], is pivotal in nonlinear spectral theory as it allows to introduce a generalized algebraic multiplicity, $\chi[\mathfrak{L},\lambda_0]$, in a rather natural manner. Subsequently, we will denote

$$\mathfrak{L}_j := \frac{1}{j!}\mathfrak{L}^{(j)}(\lambda_0), \quad 1\le j\le r,$$

if these derivatives exist.

Definition 3.2. *Let* $\mathfrak{L}\in\mathcal{C}^r([a,b],\Phi_0(X,Y))$ *and* $1\le k\le r$*. Then, a given eigenvalue* $\lambda_0\in\Sigma(\mathfrak{L})$ *is said to be a* k***-transversal eigenvalue*** *of* $\mathfrak{L}$ *if*

$$\bigoplus_{j=1}^{k}\mathfrak{L}_j\left(\bigcap_{i=0}^{j-1}\mathrm{Ker}(\mathfrak{L}_i)\right)\oplus R(\mathfrak{L}_0)=Y \quad \text{with} \quad \mathfrak{L}_k\left(\bigcap_{i=0}^{k-1}\mathrm{Ker}(\mathfrak{L}_i)\right)\neq\{0\}.$$

For these eigenvalues, the *algebraic multiplicity of* $\mathfrak{L}$ *at* λ_0, $\chi[\mathfrak{L},\lambda_0]$, was introduced in [8] through

$$\chi[\mathfrak{L},\lambda_0] := \sum_{j=1}^{k} j\cdot\dim\mathfrak{L}_j\left(\bigcap_{i=0}^{j-1}\mathrm{Ker}(\mathfrak{L}_i)\right). \tag{3.2}$$

According to Theorems 4.3.2 and 5.3.3 of [22], for every $\mathfrak{L}\in\mathcal{C}^r([a,b],\Phi_0(X,Y))$, $k\in\{1,2,...,r\}$ and $\lambda_0\in\mathrm{Alg}_k(\mathfrak{L})$, there exists a polynomial $\Phi:\mathbb{R}\to\mathcal{L}(X)$ with $\Phi(\lambda_0)=I_X$ such that λ_0 is a k-transversal eigenvalue of the path

$$\mathfrak{L}^\Phi := \mathfrak{L}\circ\Phi\in\mathcal{C}^r([a,b],\Phi_0(X,Y)). \tag{3.3}$$

Moreover, $\chi[\mathfrak{L}^\Phi,\lambda_0]$ is independent of the curve of *trasversalizing local isomorphisms* Φ chosen to transversalize $\mathfrak{L}$ at λ_0 through (3.3), regardless Φ is a polynomial or not. Therefore, the next generalized concept of algebraic multiplicity is consistent

$$\chi[\mathfrak{L},\lambda_0] := \chi[\mathfrak{L}^\Phi,\lambda_0].$$

This concept of algebraic multiplicity can be easily extended by setting

$$\chi[\mathfrak{L},\lambda_0]=0 \quad \text{if} \quad \lambda_0\notin\Sigma(\mathfrak{L})$$

and

$$\chi[\mathfrak{L},\lambda_0]=+\infty \quad \text{if} \quad \lambda_0\in\Sigma(\mathfrak{L})\setminus\mathrm{Alg}(\mathfrak{L}) \ \text{ and } \ r=+\infty.$$

Thus, $\chi[\mathfrak{L},\lambda]$ is well defined for all $\lambda\in[a,b]$ of any smooth path $\mathfrak{L}\in\mathcal{C}^\infty([a,b],\Phi_0(X,Y))$ and, in particular, for any analytical curve $\mathfrak{L}\in\mathcal{C}^\omega([a,b],\Phi_0(X,Y))$. The next uniqueness result, going back to [27], axiomatizes it (see also Chapter 6 of [25]), much like Theorem 2.1 axiomatizes the Leray–Schauder degree.

Theorem 3.3. *For every $\varepsilon > 0$, the algebraic multiplicity χ is the unique map*

$$\chi[\cdot, \lambda_0] : \mathcal{C}^\infty((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \Phi_0(X)) \longrightarrow [0, \infty]$$

such that

(PF) *For every pair $\mathfrak{L}, \mathfrak{M} \in \mathcal{C}^\infty((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \Phi_0(X))$,*

$$\chi[\mathfrak{L} \circ \mathfrak{M}, \lambda_0] = \chi[\mathfrak{L}, \lambda_0] + \chi[\mathfrak{M}, \lambda_0].$$

(NP) *There exits a rank one projection $P_0 \in L(X)$ such that*

$$\chi[(\lambda - \lambda_0)P_0 + I_X - P_0, \lambda_0] = 1.$$

The axiom (PF) is the *product formula* and (NP) is a *normalization property* for establishing the uniqueness of the algebraic multiplicity. From these axioms one can derive all the remaining properties of the generalized algebraic multiplicity χ. Among them, that it equals the classical algebraic multiplicity when

$$\mathfrak{L}(\lambda) = \lambda I_X - K$$

for some compact operator K. Indeed, according to [25], for every smooth path $\mathfrak{L} \in \mathcal{C}^\infty((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \Phi_0(X))$, the following properties are satisfied:

- $\chi[\mathfrak{L}, \lambda_0] \in \mathbb{N} \uplus \{+\infty\}$;
- $\chi[\mathfrak{L}, \lambda_0] = 0$ if, and only if, $\mathfrak{L}(\lambda_0) \in GL(X)$;
- $\chi[\mathfrak{L}, \lambda_0] < \infty$ if, and only if, $\lambda_0 \in \mathrm{Alg}(\mathfrak{L})$.
- If $X = \mathbb{R}^N$, then, in any basis,
$$\chi[\mathfrak{L}, \lambda_0] = \mathrm{ord}_{\lambda_0} \det \mathfrak{L}(\lambda).$$
- Let $L \in \mathcal{L}(X)$ be such that $\lambda I_X - L \in \Phi_0(X)$. Then, for every $\lambda_0 \in \mathrm{Spec}(L)$, there exists $k \geq 1$ such that

$$\begin{aligned} \chi[\lambda I_X - L, \lambda_0] &= \sup_{n \in \mathbb{N}} \dim \mathrm{Ker}[(\lambda_0 I_X - L)^n] \\ &= \dim \mathrm{Ker}[(\lambda_0 I_X - L)^k] = \mathfrak{m}_{\mathrm{alg}}[L, \lambda_0]. \end{aligned} \tag{3.4}$$

Therefore, χ extends, very substantially, the classical concept of algebraic multiplicity. Among the most useful properties of χ counts the product formula established by the next result, going back to Theorem 5.6.1 of [25], where we are denoting by $\kappa[\mathfrak{L}, \lambda_0]$ the *algebraic ascent* of λ_0, defined, for every $\mathfrak{L} \in \mathcal{C}^r([a, b], \Phi_0(X, Y))$, by

$$\kappa[\mathfrak{L}, \lambda_0] := \begin{cases} 0 & \text{if } \lambda_0 \notin \Sigma(\mathfrak{L}), \\ k & \text{if } \lambda_0 \in \mathrm{Alg}_k(\mathfrak{L}), \quad k \in \{1, ..., r\}, \\ +\infty & \text{if } \lambda_0 \in \Sigma(\mathfrak{L}) \setminus \mathrm{Alg}(\mathfrak{L}). \end{cases}$$

Theorem 3.4. *Let $\mathfrak{L} \in \mathcal{C}^r([a, b], \Phi_0(X, Y))$, $\mathfrak{M} \in \mathcal{C}^s([a, b], \Phi_0(Y, Z))$, for some $r, s \in \mathbb{N} \uplus \{+\infty\}$ such that $\chi[\mathfrak{L}, \lambda_0]$ and $\chi[\mathfrak{M}, \lambda_0]$ are well defined and*

$$\min\{r, s\} \geq \kappa[\mathfrak{L}, \lambda_0] + \kappa[\mathfrak{M}, \lambda_0].$$

Then, $\chi[\mathfrak{M} \circ \mathfrak{L}, \lambda_0]$ is well defined and the next ***product formula*** *holds*

$$\chi[\mathfrak{M} \circ \mathfrak{L}, \lambda_0] = \chi[\mathfrak{M}, \lambda_0] + \chi[\mathfrak{L}, \lambda_0].$$

In particular, when $r = s = +\infty$,

$$\chi[\mathfrak{M} \circ \mathfrak{L}, \lambda] = \chi[\mathfrak{M}, \lambda] + \chi[\mathfrak{L}, \lambda] \quad \text{for all } \lambda \in [a, b].$$

The next result, going back to Theorem 5.6.1 of [22], shows that χ detects all changes of the Leray–Schauder degree.

Theorem 3.5. *Let $\mathfrak{L}\in\mathcal{C}^r([a,b],\mathcal{L}(X))$ for some $r\in\mathbb{N}\uplus\{+\infty\}$ such that $\mathfrak{L}(\lambda)\in GL_c(X)$ for every $\lambda\in[a,b]\backslash\{\lambda_0\}$ and $\lambda_0\in\mathrm{Alg}_k(\mathfrak{L})$ for some $k\in\{1,2,...,r\}$. Then, for every $R>0$,*

$$\deg_{LS}(\mathfrak{L}(\lambda_a),B_R)\cdot\deg_{LS}(\mathfrak{L}(\lambda_b),B_R)=(-1)^{\chi[\mathfrak{L},\lambda_0]}$$

for all $\lambda_a\in[a,\lambda_0)$ and $\lambda_b\in(\lambda_0,b]$. Therefore, $\deg_{LS}(\mathfrak{L}(\lambda),B_R)$ changes as λ crosses λ_0 if, and only if, $\chi[\mathfrak{L},\lambda_0]$ is odd.

As a byproduct of Theorem 3.5 and (2.7), if $\chi[\mathfrak{L},\lambda_0]\in 2\mathbb{N}+1$, then the operators of the path $\mathfrak{L}(\lambda)$ change of orientation when λ crosses λ_0.

To conclude this section, we will show how the Schauder formula (2.2) can be reformulated, equivalently, in terms of the generalized algebraic multiplicity χ of the canonical homotopy between the original linear map and the identity map.

Theorem 3.6. *For every $(L,\Omega)\in\mathscr{A}_{GL}$ with $0\in\Omega$,*

$$\deg_{LS}(L,\Omega)=(-1)^{\sum_{i=1}^{q}\chi[\mathfrak{L},\lambda_i]}\tag{3.5}$$

where $\mathfrak{L}\in\mathcal{C}^\omega([0,1],\mathcal{L}_c(X,Y))$ is the analytic (linear) path defined by

$$\mathfrak{L}(\lambda):=(1-\lambda)I_X+\lambda L,\qquad \lambda\in[0,1],$$

and

$$\Sigma(\mathfrak{L})=\{\lambda_1,\lambda_2,...,\lambda_q\}\subset(0,1).\tag{3.6}$$

Proof. As already commented above, owing to Theorems 4.4.1 and 4.4.4 of [22], $\Sigma(\mathfrak{L})$ is finite and $\Sigma(\mathfrak{L})\subset\mathrm{Alg}(\mathfrak{L})$, because $\mathfrak{L}(0)=I_X$ and $\mathfrak{L}(1)=L$ are invertible. The result is obvious if $\Sigma(\mathfrak{L})=\emptyset$, since in this case, I_X and L are in the same path connected component. Thus, by (2.7), $\deg_{LS}(L,\Omega)=1$. Suppose that (3.6) holds for some $q\geq 1$. First, note that, by the definition of the segment $\mathfrak{L}(\lambda)$,

$$\mathrm{Spec}(I_X-L)\cap(1,\infty)=\{\lambda_1^{-1},\lambda_2^{-1},...,\lambda_q^{-1}\}.$$

Thus, by (2.2),

$$\deg_{LS}(L,\Omega)=(-1)^{\sum_{i=1}^{q}\mathfrak{m}_{\mathrm{alg}}[I_X-L,\lambda_i^{-1}]}.\tag{3.7}$$

Moreover,

$$\begin{aligned}\chi[\mathfrak{L},\lambda_i]&=\chi[(1-\lambda)I_X+\lambda L,\lambda_i]=\chi[I_X-\lambda(I_X-L),\lambda_i]\\&=\chi[\lambda(\lambda^{-1}I_X-(I_X-L)),\lambda_i]=\chi[\lambda^{-1}I_X-(I_X-L),\lambda_i].\end{aligned}$$

The last step follows from the product formula, because

$$\begin{aligned}\chi[\lambda(\lambda^{-1}I_X-(I_X-L)),\lambda_i]&=\chi[\lambda I_X\circ(\lambda^{-1}I_X-(I_X-L)),\lambda_i]\\&=\chi[\lambda I_X,\lambda_i]+\chi[\lambda^{-1}I_X-(I_X-L),\lambda_i]\\&=\chi[\lambda^{-1}I_X-(I_X-L),\lambda_i].\end{aligned}$$

On the other hand, by (3.4), it is apparent that

$$\chi[\lambda^{-1}I_X-(I_X-L),\lambda_i]=\mathfrak{m}_{\mathrm{alg}}[I_X-L,\lambda_i^{-1}].$$

Therefore, for every $i\in\{1,...,q\}$, we find that

$$\chi[\mathfrak{L},\lambda_i]=\mathfrak{m}_{\mathrm{alg}}[I_X-L,\lambda_i^{-1}].$$

Consequently, substituting these identities in (3.7) yields

$$\deg_{LS}(L,\Omega)=(-1)^{\sum_{i=1}^{q}\chi[\mathfrak{L},\lambda_i]}.$$

So, completing the proof. □

As a by-product of Theorem 3.6 and the degree formula for regular admissible pairs, the next result holds.

Corollary 3.7. *Let* $(f,\Omega)\in\mathscr{R}$ *be a regular admissible pair with* $0\in\Omega$. *Then,*

$$\deg_{LS}(f,\Omega) := \sum_{x\in f^{-1}(0)\cap\Omega} (-1)^{\sum_{i=1}^{q_x}\chi[\mathfrak{L}_x,\lambda_{x,i}]} \tag{3.8}$$

where, for every $\lambda\in[0,1]$ *and* $x\in f^{-1}(0)\cap\Omega$,

$$\mathfrak{L}_x(\lambda) := (1-\lambda)I_X + \lambda Df(x) \in \mathcal{C}^\omega([0,1],\mathcal{L}_c(X,Y))$$

and, for every $x\in f^{-1}(0)\cap\Omega$,

$$\Sigma(\mathfrak{L}_x) = \{\lambda_{x,1},\lambda_{x,2},...,\lambda_{x,q_x}\}.$$

In the particular case when $X=Y$, the formula (3.5) establishes a linking between the operator L and the identity map I_X, through the Fredholm path $\mathfrak{L}(\lambda)$ and the generalized algebraic multiplicity χ, which allows us to ascertain weather, or not, I_X and L belong to the same connected component of $GL_c(X)$, $GL_c^{\pm}(X)$ as we can appreciate in Figure 2. Thus, from a geometric perspective, (3.5) is substantially sharper than (2.2).

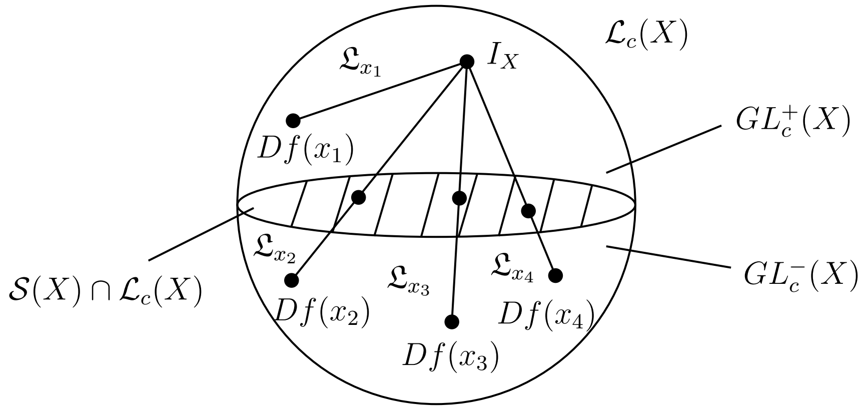

FIGURE 2. Graphic illustration of Corollary 3.7 when $X=Y$.

Adopting another, more algorithmic, point of view, the Schauder formula (2.2) relies on the classical concept of eigenvalue and algebraic multiplicity. Thus, in order to detect through it any change of the degree of a linear map depending on a parameter μ, L_μ, as μ changes, one should face the, very hard, problem of determining all the classical eigenvalues of the operators L_μ, while, according to (3.5), determining the classical spectrum of the operator L_μ as μ varies is unnecessary, but simply ascertaining the oddities of the generalized algebraic multiplicities of

$$\mathfrak{L}_\mu(\lambda) = (1-\lambda)I_X + \lambda L_\mu, \quad \lambda\in[0,1],$$

which is a substantially simpler task from a technical point of view.

4. Calculating the parity of Fredholm maps

We begin by studying the structure of the space of linear Fredholm operators of index zero, $\Phi_0(X,Y)$. The space $\Phi_0(X,Y)$ is an open path connected subset of $\mathcal{L}(X,Y)$. Recall that, in general, $\Phi_0(X,Y)$ is not linear and whenever $GL(X)$ is contractible $\pi_1(\Phi_0(X))\simeq\mathbb{Z}_2$, where π_1 stands for the first fundamental group (see [10]). For every $n\in\mathbb{N}$, the set of *singular operators of order* n is defined by

$$\mathcal{S}_n(X,Y) := \{L\in\Phi_0(X,Y): \ \dim \mathrm{Ker}(L) = n\}.$$

Thus, the set of *singular operators* can be defined through

$$\mathcal{S}(X,Y) := \Phi_0(X,Y)\backslash GL(X,Y) = \biguplus_{n\in\mathbb{N}} \mathcal{S}_n(X,Y).$$

According to [9], for every $n \in \mathbb{N}$, $\mathcal{S}_n(X,Y)$ is a Banach submanifold of $\Phi_0(X,Y)$ of codimension n^2. This feature allows us to regard $\mathcal{S}(X,Y)$ as an hypersurface of $\Phi_0(X,Y)$. On the other hand, by Theorem 1 of Kuiper's article [16], the space of isomorphisms of any separable infinite dimensional Hilbert space, H, denoted by $GL(H)$, is path connected. Thus, defining an orientation in $GL(X,Y)$ for general Banach spaces X,Y is not possible since in general, $GL(X,Y)$ is path connected. This fact reveals a fundamental difference between finite and infinite dimensional vector normed spaces, because, for every $N \in \mathbb{N}$, the space $GL_c(\mathbb{R}^N) = GL(\mathbb{R}^N)$ is divided into two path connected components, $GL^{\pm}(\mathbb{R}^N)$. This situation is illustrated in Figure 3.

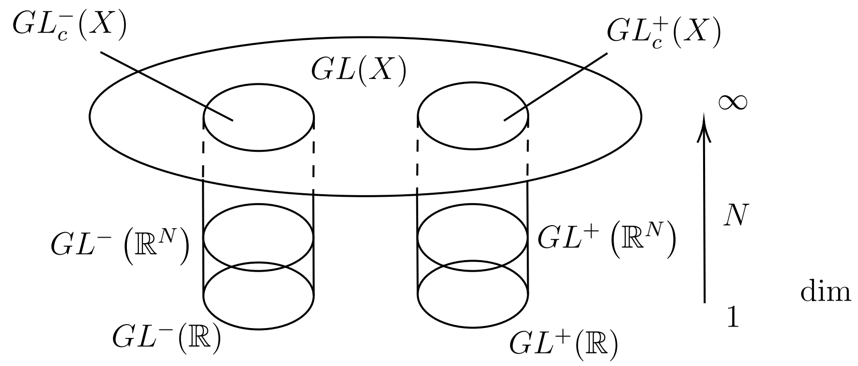

FIGURE 3. The evolution of $GL(X)$ in terms of the dimension.

A key technical tool to overcome this difficulty in order to define the degree for Fredholm operators of index zero is provided by the concept of *parity* introduced by Fitzpatrick and Pejsachowicz in [10]. Essentially, the parity is a generalized local detector of the change of orientability of a given *admissible path*. A Fredholm path $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ is said to be *admissible* if $\mathfrak{L}(a), \mathfrak{L}(b) \in GL(X,Y)$. Subsequently, the set of all admissible paths will be denoted by $\mathscr{C}([a,b],\Phi_0(X,Y))$, and, for every $r \in \mathbb{N} \uplus \{+\infty, \omega\}$, we will set

$$\mathscr{C}^r([a,b],\Phi_0(X,Y)) := \mathcal{C}^r([a,b],\Phi_0(X,Y)) \cap \mathscr{C}([a,b],\Phi_0(X,Y)).$$

The fastest way to introduce the notion of parity consists in defining it for $\mathscr{C}$-transversal paths, which are going to be introduced next, and then for general admissible curves through the density of $\mathscr{C}$-transversal paths in $\mathscr{C}([a,b],\Phi_0(X,Y))$. The density of $\mathscr{C}$-transversal curves is proven in [9]. This simplifies the scheme of Fitzpatrick and Pejsachowicz [10] through the systematic use of a *parametrix*, whose existence requires the technicalities of the theory of fibre bundles.

A given Fredholm path, $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$, is said to be *$\mathscr{C}$-transversal* if

i) $\mathfrak{L} \in \mathscr{C}^1([a,b],\Phi_0(X,Y))$;
ii) $\mathfrak{L}([a,b]) \cap \mathcal{S}(X,Y) \subset \mathcal{S}_1(X,Y)$ and it is finite;
iii) $\mathfrak{L}$ is transversal to $\mathcal{S}_1(X,Y)$ at each point of $\mathfrak{L}([a,b]) \cap \mathcal{S}(X,Y)$.

The curve $\mathfrak{L} \in \mathcal{C}^1([a,b],\Phi_0(X,Y))$ is said to be transversal to $\mathcal{S}_1(X,Y)$ at λ_0 if

$$\mathfrak{L}'(\lambda_0) + T_{\mathfrak{L}(\lambda_0)}\mathcal{S}_1(X,Y) = \mathcal{L}(X,Y),$$

where $T_{\mathfrak{L}(\lambda_0)}\mathcal{S}_1(X,Y)$ stands for the tangent space to the manifold $\mathcal{S}_1(X,Y)$ at $\mathfrak{L}(\lambda_0)$.

When $\mathfrak{L}$ is $\mathscr{C}$-transversal, the *parity* of $\mathfrak{L}$ in $[a,b]$ is defined by

$$\sigma(\mathfrak{L},[a,b]) := (-1)^k,$$

where $k\in\mathbb{N}$ equals the cardinal of $\mathfrak{L}([a,b])\cap\mathcal{S}(X,Y)$. Thus, the parity of a $\mathscr{C}$-transversal path, $\mathfrak{L}(\lambda)$, is the number of times, mod 2, that $\mathfrak{L}(\lambda)$ intersects transversally the singular hypersurface $\mathcal{S}(X,Y)$.

The fact that the $\mathscr{C}$-transversal paths are dense in the set of all admissible curves, together with the next stability property: for any $\mathscr{C}$-transversal path $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$, there exists $\varepsilon>0$ such that

$$\sigma(\mathfrak{L},[a,b]) = \sigma(\tilde{\mathfrak{L}},[a,b])$$

for all $\mathscr{C}$-transversal path $\tilde{\mathfrak{L}}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ such that $\|\mathfrak{L}-\tilde{\mathfrak{L}}\|_\infty<\varepsilon$ (see [9]), allows us to define the parity for a general admissible path $\mathfrak{L}\in\mathscr{C}([a,b],\Phi_0(X,Y))$ through

$$\sigma(\mathfrak{L},[a,b]) := \sigma(\tilde{\mathfrak{L}},[a,b]),$$

where $\tilde{\mathfrak{L}}$ is any $\mathscr{C}$-transversal curve satisfying $\|\mathfrak{L}-\tilde{\mathfrak{L}}\|_\infty<\varepsilon$ for sufficiently small $\varepsilon>0$.

Originally, this concept was introduced by Fitzpatrick and Pejsachowicz via the Leray–Schauder degree through the intermediate notion of *parametrix*, assigning to every Fredholm path $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ another path $\mathfrak{L}_c\in\mathcal{C}([a,b],\mathcal{L}_c(X))$ with values in the space of the compact perturbations of the identity map, $\mathcal{L}_c(X)$. Precisely, for any given Fredholm path, $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$, a *parametrix* of $\mathfrak{L}(\lambda)$ is any curve

$$\mathfrak{P}\in\mathcal{C}([a,b],GL(Y,X))$$

such that

$$\mathfrak{P}(\lambda)\circ\mathfrak{L}(\lambda)\in\mathcal{L}_c(X) \quad \text{for all } \lambda\in[a,b].$$

According to [10], any Fredholm path $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ admits a parametrix $\mathfrak{P}\in\mathcal{C}([a,b],GL(Y,X))$. However, the proof of this fact relies upon some nontrivial technicalities of the theory of fibre bundles. Once established the existence of a parametrix for every Fredholm path, the next result establishes the precise relation between the parity and the Leray–Schauder degree.

Theorem 4.1. *Let $\mathfrak{L}\in\mathscr{C}([a,b],\Phi_0(X,Y))$ be an admissible Fredholm curve. Then, for every parametrix $\mathfrak{P}\in\mathcal{C}([a,b],GL(Y,X))$ of $\mathfrak{L}(\lambda)$ and any $R>0$,*

$$\sigma(\mathfrak{L},[a,b]) = \deg_{LS}(\mathfrak{P}(a)\circ\mathfrak{L}(a),B_R)\cdot\deg_{LS}(\mathfrak{P}(b)\circ\mathfrak{L}(b),B_R). \tag{4.1}$$

Actually, Fitzpatrick and Pejsachowicz [10] introduced the parity of a given Fredholm path through the formula (4.1). Then, they established in [9] that this concept equals the previous one. It should be noted that, although [10] was published in 1993, two years later than [9], the reference [10] had been already included in the list of references of [9].

Subsequently, it is said that a homotopy $H\in\mathcal{C}([0,1]\times[a,b],\Phi_0(X,Y))$ is *admissible* if $H([0,1]\times\{a,b\})\subset GL(X,Y)$, and two paths, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, are said to be *$\mathscr{C}$-homotopic* if they are homotopic through an admissible homotopy. A fundamental property of the parity is its invariance under admissible $\mathscr{C}$-homotopic paths (see [10]).

Our next result establishes that, as soon as the Fredholm path $\mathfrak{L}(\lambda)$ is defined in $\mathcal{L}_c(X)$, every transversal intersection with $\mathcal{S}(X)$ induces a change of orientation, i.e., a change of path-connected component. This fact, motivates the geometrical interpretation of the parity as a local detector of change of orientation of the operators of a Fredholm path.

Theorem 4.2. *Let $\mathfrak{L}\in\mathscr{C}([a,b],\mathcal{L}_c(X))$ be an admissible curve with values in $\mathcal{L}_c(X)$. Then, $\sigma(\mathfrak{L},[a,b])=-1$ if, and only if, $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ lie in different path-connected components of $GL_c(X)$.*

Proof. Suppose that $\sigma(\mathfrak{L},[a,b])=-1$. Then, by Theorem 4.1, for every $R>0$ and each parametrix $\mathfrak{P}\in\mathcal{C}([a,b],GL(X))$ of $\mathfrak{L}(\lambda)$,

$$-1=\sigma(\mathfrak{L},[a,b])=\deg_{LS}(\mathfrak{P}(a)\circ\mathfrak{L}(a),B_R)\cdot\deg_{LS}(\mathfrak{P}(b)\circ\mathfrak{L}(b),B_R).$$

On the other hand, since $\mathfrak{L}([a,b])\subset\mathcal{L}_c(X)$, the map $\mathfrak{P}\in\mathcal{C}([a,b],GL(X))$ defined by $\mathfrak{P}(\lambda)=I_X$ for all $\lambda\in[a,b]$ is a parametrix of $\mathfrak{L}(\lambda)$. Thus,

$$\deg_{LS}(\mathfrak{L}(a),B_R)\cdot\deg_{LS}(\mathfrak{L}(b),B_R)=-1.$$

Therefore, $\deg_{LS}(\mathfrak{L}(a),B_R)$ and $\deg_{LS}(\mathfrak{L}(b),B_R)$ have different signs within $\{-1,1\}$. Thus, by (2.7), $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ lie in different path-connected components.

Conversely, if $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ are in different path-connected components, then, again by (2.7), we find that

$$\deg_{LS}(\mathfrak{L}(a),B_R)\cdot\deg_{LS}(\mathfrak{L}(b),B_R)=-1$$

for all $R>0$. Therefore, since $\mathfrak{P}(\lambda)=I_X$, $\lambda\in[a,b]$, is a parametrix of $\mathfrak{L}(\lambda)$, we can conclude that $\sigma(\mathfrak{L},[a,b])=-1$. This ends the proof. □

As illustrated by Figure 4, each transversal intersection of the path $\mathfrak{L}(\lambda)$ with the singular hypersurface $\mathcal{S}(X)$ can be viewed as a change of path-connected component.

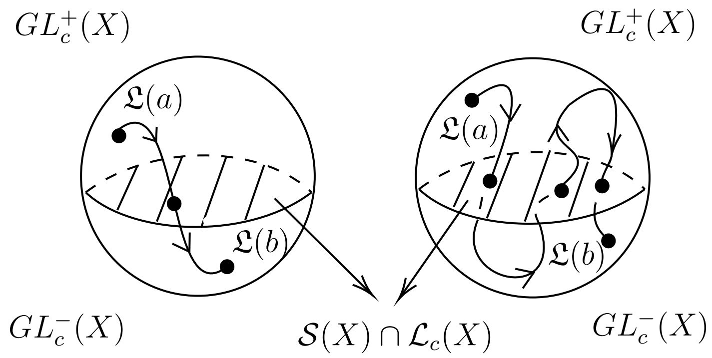

FIGURE 4. Geometrical interpretation of the parity on $\mathcal{L}_c(X)$.

As the parity is a topological invariant, it is difficult to compute it even in the simplest cases. Our next result shows how the parity of any analytic Fredholm path

$$\mathfrak{L}\in\mathcal{C}^\omega([a,b],\Phi_0(X,Y))$$

can be computed though the multiplicity χ. Since χ is an algebraic invariant, easily computable in practice (see [22]), this result is important from the point of view of the applications.

Theorem 4.3. *Let $\mathfrak{L}\in\mathscr{C}^\omega([a,b],\Phi_0(X,Y))$ be an analytical admissible curve. Then,*

$$\sigma(\mathfrak{L},[a,b])=(-1)^{\sum_{i=1}^n\chi[\mathfrak{L},\lambda_i]},$$

where $\Sigma(\mathfrak{L})=\{\lambda_1,\lambda_2,...,\lambda_n\}$.

Proof. By hypothesis, $\mathfrak{L}(a),\mathfrak{L}(b)\in GL(X,Y)$. Thus, by Theorems 4.4.1 and 4.4.4 of [22], $\Sigma(\mathfrak{L})$ is finite and $\lambda\in\mathrm{Alg}(\mathfrak{L})$ for all $\lambda\in\Sigma(\mathfrak{L})$. Since $\Sigma(\mathfrak{L})$ is finite, $[a,b]$ can be divided into intervals of the form

$$[a,b]=\bigcup_{i=1}^n[a_i,b_i]$$

with $a_1 = a$, $b_n = b$ and $\lambda_i \in (a_i, b_i)$ for each $i \in \{1, 2, ..., n\}$. Hence, by the additivity property of the parity (see [9] and [10]), it becomes apparent that

$$\sigma(\mathfrak{L}, [a, b]) = \prod_{i=1}^{n} \sigma(\mathfrak{L}, [a_i, b_i]). \tag{4.2}$$

As $\mathfrak{L}(a_i), \mathfrak{L}(b_i) \in GL(X, Y)$ for every $i \in \{1, 2, ..., n\}$, the parities $\sigma(\mathfrak{L}, [a_i, b_i])$ are well defined. Let $\mathfrak{P}_i \in \mathcal{C}([a_i, b_i], GL(Y, X))$ be a parametrix of $\mathfrak{L}|_{[a_i,b_i]}$ for each $i \in \{1, 2, ..., n\}$. Then, according to (4.1), we have that

$$\sigma(\mathfrak{L}, [a_i, b_i]) = \deg_{LS}(\mathfrak{P}_i(a_i) \circ \mathfrak{L}(a_i), B_R) \cdot \deg_{LS}(\mathfrak{P}_i(b_i) \circ \mathfrak{L}(b_i), B_R), \tag{4.3}$$

regardless the size of $R > 0$. Since $\mathfrak{P}_i(\lambda) \in GL(Y, X)$ for each $\lambda \in [a_i, b_i]$, the multiplicity $\chi[\mathfrak{P}_i, \lambda_i]$ is well defined and $\chi[\mathfrak{P}_i, \lambda_i] = 0$. Thus, by Theorem 3.4, $\chi[\mathfrak{P}_i \circ \mathfrak{L}, \lambda_i]$ is well defined and

$$\chi[\mathfrak{P}_i \circ \mathfrak{L}, \lambda_i] = \chi[\mathfrak{P}_i, \lambda_i] + \chi[\mathfrak{L}, \lambda_i] = \chi[\mathfrak{L}, \lambda_i]. \tag{4.4}$$

Hence, by Theorem 3.5, it follows from (4.4) that, for every $i \in \{1, ..., n\}$,

$$\deg_{LS}(\mathfrak{P}_i(a_i) \circ \mathfrak{L}(a_i), B_R) \cdot \deg_{LS}(\mathfrak{P}_i(b_i) \circ \mathfrak{L}(b_i), B_R) = (-1)^{\chi[\mathfrak{P}_i \circ \mathfrak{L}, \lambda_i]} = (-1)^{\chi[\mathfrak{L}, \lambda_i]}. \tag{4.5}$$

Therefore, substituting (4.5) and (4.3) into (4.2) yields

$$\sigma(\mathfrak{L}, [a_i, b_i]) = \prod_{i=1}^{n} \sigma(\mathfrak{L}, [a_i, b_i]) = (-1)^{\sum_{i=1}^{n} \chi[\mathfrak{L}, \lambda_i]},$$

which ends the proof. □

More generally, the next result holds.

Theorem 4.4. *Let $\mathfrak{L} \in \mathscr{C}^r([a, b], \Phi_0(X, Y))$ be an admissible path such that*

$$\Sigma(\mathfrak{L}) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$$

with $\lambda_i \in \mathrm{Alg}_{k_i}(\mathfrak{L})$ and $k_i \leq r$ for all $i \in \{1, ..., n\}$. Then,

$$\sigma(\mathfrak{L}, [a, b]) = (-1)^{\sum_{i=1}^{n} \chi[\mathfrak{L}, \lambda_i]}. \tag{4.6}$$

According to [22], the assumption that $\lambda_i \in \mathrm{Alg}_{k_i}(\mathfrak{L})$ with $k_i \leq r$ for all $i \in \{1, ..., n\}$ guarantees that $\chi[\mathfrak{L}, \lambda_i]$ is well defined and finite. The remaining technical details of the proof of Theorem 4.4 can be adapted *mutatis mutandis* from the proof of Theorem 4.3.

Our next result extends Theorem 4.4 to determine the parity of a general continuous path of Fredholm operators via homotopy techniques.

Theorem 4.5. *Any continuous path $\mathfrak{L} \in \mathscr{C}([a, b], \Phi_0(X, Y))$ is $\mathscr{C}$-homotopic to an analytic curve $\mathfrak{L}_\omega \in \mathscr{C}^\omega([a, b], \Phi_0(X, Y))$. Moreover, for any of these analytic paths,*

$$\sigma(\mathfrak{L}, [a, b]) = (-1)^{\sum_{i=1}^{n} \chi[\mathfrak{L}_\omega, \lambda_i]},$$

where

$$\Sigma(\mathfrak{L}_\omega) = \{\lambda_1, \lambda_2, ..., \lambda_n\}.$$

Proof. First, we will show the existence of $\mathfrak{L}_\omega$. Since $\mathfrak{L}$ is continuous and $[a, b]$ compact, $\mathfrak{L}([a, b])$ is compact. Let $\{B_{\varepsilon_i}(x_i)\}_{i \in \mathcal{I}}$ be a covering of $\mathfrak{L}([a, b])$ consisting of open balls $B_{\varepsilon_i}(x_i)$ with $x_i \in \mathfrak{L}([a, b])$ and $\varepsilon_i > 0$ sufficiently small so that $B_{\varepsilon_i}(x_i) \subset \Phi_0(X, Y)$. Observe that this ε_i exists since $\mathfrak{L}([a, b]) \subset \Phi_0(X, Y)$ and $\Phi_0(X, Y)$ is open. Then, there exists a finite subset $\{i_1, i_2, ..., i_n\} \subset \mathcal{I}$ such that

$$\mathfrak{L}([a, b]) \subset \bigcup_{j=1}^{n} B_{\varepsilon_{i_j}}(x_{i_j}).$$

By smoothing any piecewise linear approximation of $\mathfrak{L} \in \bigcup_{j=1}^{n} B_{\varepsilon_{i_j}}(x_{i_j})$ with extremal points $\mathfrak{L}(a), \mathfrak{L}(b)$, we can get the existence of an analytic path

$$\mathfrak{L}_\omega \in \mathcal{C}^\omega([a,b], \Phi_0(X,Y))$$

such that $\mathfrak{L}_\omega(a) = \mathfrak{L}(a)$, $\mathfrak{L}_\omega(b) = \mathfrak{L}(b)$, and

$$\|\mathfrak{L}_\omega - \mathfrak{L}\|_\infty < \varepsilon := \min\{\varepsilon_{i_1}, \varepsilon_{i_2}, ..., \varepsilon_{i_n}\}.$$

Then, the map

$$\begin{array}{rccl} H: & [0,1]\times[a,b] & \longrightarrow & \Phi_0(X,Y) \\ & (\mu,\lambda) & \mapsto & \mu\mathfrak{L}_\omega(\lambda) + (1-\mu)\mathfrak{L}(\lambda) \end{array}$$

defines an admissible homotopy between $\mathfrak{L}$ and $\mathfrak{L}_\omega$ in $\Phi_0(X,Y)$. Indeed, since

$$\|\mu\mathfrak{L}_\omega + (1-\mu)\mathfrak{L} - \mathfrak{L}\|_\infty = \|\mu\mathfrak{L}_\omega - \mu\mathfrak{L}\|_\infty = |\mu|\|\mathfrak{L}_\omega - \mathfrak{L}\|_\infty \le \|\mathfrak{L}_\omega - \mathfrak{L}\|_\infty < \varepsilon,$$

for each $\mu \in [0,1]$, it becomes apparent that

$$H([0,1]\times[a,b]) \subset \bigcup_{j=1}^{n} B_{\varepsilon_{i_j}}(x_{i_j}) \subset \Phi_0(X,Y).$$

Moreover, $H(\cdot,a) = \mathfrak{L}(a)$ and $H(\cdot,b) = \mathfrak{L}(b)$. Thus, the existence of $\mathfrak{L}_\omega$ gets shown. It is fundamental to observe that the mayor difficulty for defining the homotopy is to guarantee that

$$H([0,1]\times[a,b]) \subset \Phi_0(X,Y),$$

which is not obvious since $\Phi_0(X,Y)$ is not linear. Thus, we needed to cover $\mathfrak{L}([a,b])$ with finitely many balls, to make it sure.

By the invariance by admissible homotopies of the parity, for any of these paths $\mathfrak{L}_\omega$,

$$\sigma(\mathfrak{L},[a,b]) = \sigma(\mathfrak{L}_\omega,[a,b]).$$

Therefore, by Theorem 4.3, we conclude that

$$\sigma(\mathfrak{L},[a,b]) = \sigma(\mathfrak{L}_\omega,[a,b]) = (-1)^{\sum_{i=1}^{n}\chi[\mathfrak{L}_\omega,\lambda_i]}.$$

The proof is complete. ☐

Let $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ and $\lambda_0 \in \Sigma(\mathfrak{L})$ an isolated eigenvalue. Then, the *localized parity* of $\mathfrak{L}$ at λ_0 is defined through

$$\sigma(\mathfrak{L},\lambda_0) := \lim_{\varepsilon\downarrow 0}\sigma(\mathfrak{L},[\lambda_0-\varepsilon,\lambda_0+\varepsilon]).$$

Given $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ and $\delta > 0$, an isolated eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be *δ-isolated* if

$$\Sigma(\mathfrak{L}) \cap [\lambda_0-\delta,\lambda_0+\delta] = \{\lambda_0\}.$$

Corollary 4.6. *Let $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ and $\lambda_0 \in \Sigma(\mathfrak{L})$ be a δ-isolated eigenvalue. Then, for every $\mathfrak{L}_\omega \in \mathscr{C}^\omega([\lambda_0-\delta,\lambda_0+\delta],\Phi_0(X,Y))$ $\mathscr{C}$-homotopic to $\mathfrak{L}|_{[\lambda_0-\delta,\lambda_0+\delta]}$,*

$$\sigma(\mathfrak{L},\lambda_0) = (-1)^{\sum_{\lambda\in\Sigma(\mathfrak{L}_\omega)}\chi[\mathfrak{L}_\omega,\lambda]}. \tag{4.7}$$

Moreover, if $\mathfrak{L} \in \mathcal{C}^r([a,b],\Phi_0(X,Y))$ with $r \in \mathbb{N} \uplus \{\infty,\omega\}$ and $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$ for some $1 \le k \le r$, then

$$\sigma(\mathfrak{L},\lambda_0) = (-1)^{\chi[\mathfrak{L},\lambda_0]}.$$

Proof. For the first statement, just take $[a,b] = [\lambda_0-\delta,\lambda_0+\delta]$ in Theorem 4.5. The second statement follows from Theorem 4.4 for $[a,b] = [\lambda_0-\delta,\lambda_0+\delta]$ and the definition of δ-isolated eigenvalue. ☐

The identity (4.7) establishes the precise relationship between the topological notion of parity and the algebraic concept of multiplicity. As it will become apparent in the next section, the importance of Corollary 4.6 relies on the fact that, since the localized parity detects any change of orientation, (4.7) makes intrinsic to the concept of algebraic multiplicity any change of the local degree. This optimizes the detections of the changes of the degree from a computational point of view. Figure 5 illustrates Corollary 4.6. The superior pair of plots shows a continuous path $\mathfrak{L}$ with parity 1 at λ_0, together with a close analytic path $\mathfrak{L}_\omega$ with $\chi[\mathfrak{L}_\omega, \lambda_0]$ even. In the inferior one, $\sigma(\mathfrak{L}, \lambda_0) = -1$ and $\chi[\mathfrak{L}_\omega, \lambda_0]$ is odd.

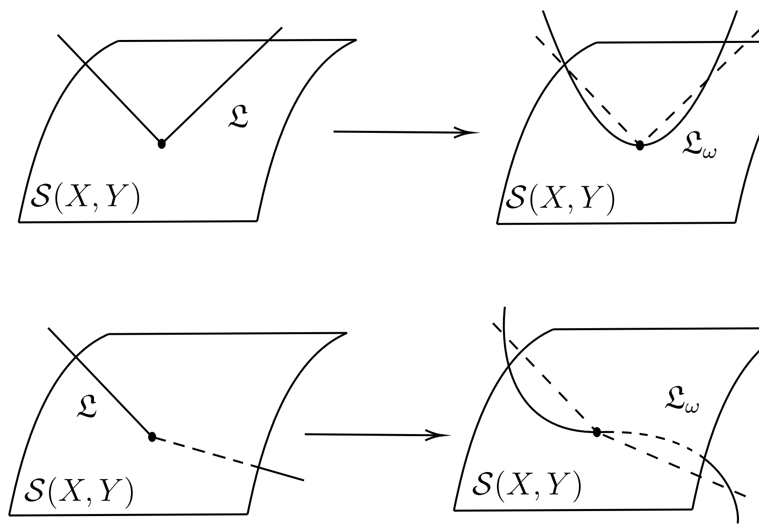

FIGURE 5. Graphic illustration of Corollary 4.6.

Our last result establishes that the crossing number of any $\mathscr{C}$-transversal path, $\mathfrak{L}$, sufficiently close to a given analytic curve, $\mathfrak{L}_\omega$, is an invariant which is congruent (mod 2) to $\chi[\mathfrak{L}_\omega, \lambda_0]$. This evidences the relationship between the multiplicity and the number of intersections of a perturbed path $\mathfrak{L}$ with the singular manifold. Thus, it establishes a sort of geometrical counterpart of the algebraic concept of multiplicity. After the local result, we will obtain the global one. Delivering the result in this way, facilitates the understanding of its most geometrical aspects.

Theorem 4.7. *Suppose that $\mathfrak{L}_\omega \in \mathcal{C}^\omega([a,b], \Phi_0(X,Y))$ and that $\lambda_0 \in \Sigma(\mathfrak{L}_\omega) \cap (a,b)$ is a δ-isolated eigenvalue. Then, there exists $\varepsilon > 0$ such that, for any $\mathscr{C}$-transversal path $\mathfrak{L} \in \mathcal{C}([\lambda_0 - \delta, \lambda_0 + \delta], \Phi_0(X,Y))$ with*

$$\|\mathfrak{L}_\omega - \mathfrak{L}\|_{\infty,[\lambda_0-\delta,\lambda_0+\delta]} < \varepsilon, \tag{4.8}$$

the next identity holds

$$\mathrm{Card}(\Sigma(\mathfrak{L})) \equiv \chi[\mathfrak{L}_\omega, \lambda_0] \pmod{2}. \tag{4.9}$$

Proof. By the analysis already done in [9], for any given compact interval J, the $\mathscr{C}$-transversal paths are dense in $\mathscr{C}(J, \Phi_0(X,Y))$. Thus, there always exists a $\mathscr{C}$-transversal curve, $\mathfrak{L} \in \mathcal{C}(J, \Phi_0(X,Y))$, such that

$$\|\mathfrak{L}_\omega - \mathfrak{L}\|_{\infty,J} < \varepsilon.$$

In particular, this holds true with $J = [\lambda_0 - \delta, \lambda_0 + \delta]$.

Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, where ε_1, $\varepsilon_2 > 0$ satisfy

$$B_{\varepsilon_1}(\mathfrak{L}_\omega(\lambda_0 - \delta)) \cup B_{\varepsilon_2}(\mathfrak{L}_\omega(\lambda_0 + \delta)) \subset GL(X, Y).$$

This ε exists because $\mathfrak{L}(\lambda_0 \pm \delta) \in GL(X,Y)$ and $GL(X,Y)$ is open. Let

$$\mathfrak{L} \in \mathcal{C}([\lambda_0 - \delta, \lambda_0 + \delta], \Phi_0(X,Y))$$

be a $\mathscr{C}$-transversal path satisfying (4.8). Then,

$$\begin{array}{rccl} H: & [0,1] \times [\lambda_0 - \delta, \lambda_0 + \delta] & \longrightarrow & \Phi_0(X,Y) \\ & (\mu, \lambda) & \mapsto & \mu\mathfrak{L}(\lambda) + (1-\mu)\mathfrak{L}_\omega(\lambda) \end{array}$$

provides us with an admissible homotopy, because, for each $\gamma \in \{\lambda_0 - \delta, \lambda_0 + \delta\}$,

$$H(\mu, \gamma) = \mathfrak{L}_\omega(\gamma) + \mu(\mathfrak{L}(\gamma) - \mathfrak{L}_\omega(\gamma)) \in B_\varepsilon(\mathfrak{L}_\omega(\gamma)) \subset GL(X,Y)$$

for all $\mu \in [0,1]$. Moreover, taking $\varepsilon > 0$ sufficiently small, one can guarantee that

$$H([0,1] \times [\lambda_0 - \delta, \lambda_0 + \delta]) \subset \Phi_0(X,Y),$$

much like in the proof of Theorem 4.5. Thus, by the invariance by admissible homotopies of the parity, it follows from Theorem 4.3 that

$$\sigma(\mathfrak{L}, [\lambda_0 - \delta, \lambda_0 + \delta]) = \sigma(\mathfrak{L}_\omega, [\lambda_0 - \delta, \lambda_0 + \delta]) = (-1)^{\chi[\mathfrak{L}_\omega, \lambda_0]}.$$

On the other hand, since $\Sigma(\mathfrak{L})$ is finite and each eigenvalue is transversal (which implies that $\chi[\mathfrak{L}, \lambda] = 1$ for $\lambda \in \Sigma(\mathfrak{L})$), it follows from Theorem 4.4 that

$$\sigma(\mathfrak{L}, [\lambda_0 - \delta, \lambda_0 + \delta]) = (-1)^{\sum_{\lambda \in \Sigma(\mathfrak{L})} \chi[\mathfrak{L}, \lambda]} = (-1)^{\sum_{\lambda \in \Sigma(\mathfrak{L})} 1} = (-1)^{\mathrm{Card}(\Sigma(\mathfrak{L}))}.$$

Therefore,

$$(-1)^{\chi[\mathfrak{L}_\omega, \lambda_0]} = (-1)^{\mathrm{Card}(\Sigma(\mathfrak{L}))},$$

which ends the proof. □

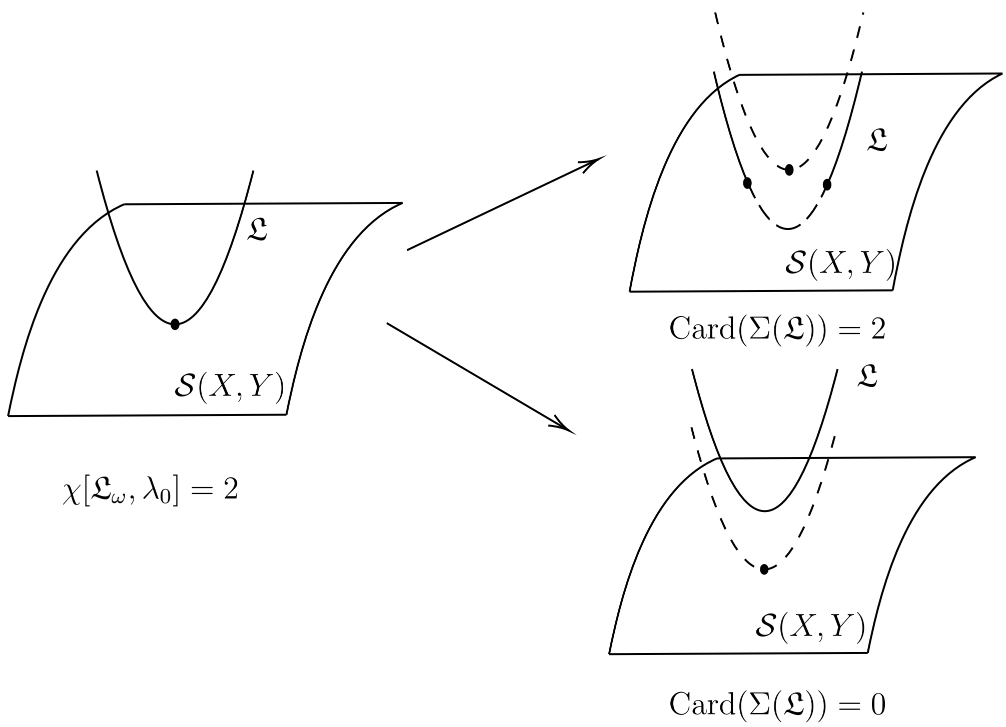

Figure 6. Graphical explanation of Theorem 4.7.

Figure 6 illustrates graphically the result of Theorem 4.7. The left plot shows a genuine situation where $\chi[\mathfrak{L}_\omega, \lambda_0] = 2$ with $\mathfrak{L}_\omega(\lambda)$ tangent at λ_0 to the singular manifold $\mathcal{S}(X,Y)$. The two plots on the right show two admissible perturbations from the original situation sketched on the left one. Therefore, Theorem 4.7 is optimal, in the sense that one cannot expect, in general, to have

$$\mathrm{Card}(\Sigma(\mathfrak{L})) = \chi[\mathfrak{L}_\omega, \lambda_0].$$

More generally, the next result holds.

Theorem 4.8. *For any given $\mathfrak{L}_\omega \in \mathscr{C}^\omega([a,b], \Phi_0(X,Y))$ there exists $\varepsilon > 0$ such that, for every $\mathscr{C}$-transversal path $\mathfrak{L} \in \mathcal{C}([a,b], \Phi_0(X,Y))$ with $\|\mathfrak{L}_\omega - \mathfrak{L}\|_\infty < \varepsilon$,*

$$\mathrm{Card}(\Sigma(\mathfrak{L})) \equiv \sum_{\lambda \in \Sigma(\mathfrak{L}_\omega)} \chi[\mathfrak{L}_\omega, \lambda] \ \ (\mathrm{mod.}\ 2).$$

Proof. The existence of $\mathscr{C}$-transversal curves satisfying the requirements of the theorem is guaranteed by the first remark of the proof of Theorem 4.7. Now, choose $\varepsilon > 0$ and the admissible homotopy H as in the previous proof, though defined in $[0,1] \times [a,b]$. Then, by the invariance by admissible homotopies of the parity, Theorem 4.3 implies that

$$\sigma(\mathfrak{L}, [a,b]) = \sigma(\mathfrak{L}_\omega, [a,b]) = (-1)^{\sum_{\lambda \in \Sigma(\mathfrak{L}_\omega)} \chi[\mathfrak{L}_\omega, \lambda]}.$$

On the other hand, since $\Sigma(\mathfrak{L})$ is finite and each eigenvalue is 1-transversal, it follows from Theorem 4.3 that

$$\sigma(\mathfrak{L}, [a,b]) = (-1)^{\sum_{\lambda \in \Sigma(\mathfrak{L})} \chi[\mathfrak{L}, \lambda]} = (-1)^{\sum_{\lambda \in \Sigma(\mathfrak{L})} 1} = (-1)^{\mathrm{Card}(\Sigma(\mathfrak{L}))}.$$

This ends the proof. □

5. Orientability. Fitzpatrick–Pejsachowicz–Rabier's degree

In this section, we will collect the most basic fundamentals of the topological degree for Fredholm operators of index zero introduced by Fitzpatrick, Pejsachowicz and Rabier in [10] and [11]. There is another degree for Fredholm operators of index zero due to Benevieri and Furi, [2, 3, 4], but this notion is not so intuitive geometrically as the one introduced in [11]. Moreover, the formulation of [11] is far more suitable for establishing the relationships between the degree and the algebraic multiplicity, though this task was accomplished partially in [24]. A coincidence degree for a class of particular linear paths of the form

$$\mathfrak{L}(\lambda) = L - \lambda A, \qquad \lambda \in \mathbb{C},$$

with L Fredholm of index zero, not necessarily continuous, and a suitable linear operator A, was introduced by Laloux and Mawhin [17, 18]. But this formulation remains outside the main scope of this paper.

As the principal difficulty to introduce a topological degree for Fredholm operators of index zero is the absence of orientations in the space of linear isomorphisms $GL(X,Y) \subset \Phi_0(X,Y)$, we will restrict our space of operators to subsets of $\Phi_0(X,Y)$ where is possible to introduce a notion of *orientability.* This is accomplished through the notion introduced in the next definition, going back to Fitzpatrick, Pejsachowicz and Rabier [11].

Definition 5.1. *A path connected subset $\mathcal{O} \subset \Phi_0(X,Y)$ such that $\mathcal{O} \cap GL(X,Y) \neq \emptyset$, is called* ***orientable*** *if there exists a map $\varepsilon : \mathcal{O} \cap GL(X,Y) \to \mathbb{Z}_2$, called orientation, such that*

$$\sigma(\mathfrak{L}, [a,b]) = \varepsilon(\mathfrak{L}(a)) \cdot \varepsilon(\mathfrak{L}(b)) \quad \text{for all } \mathfrak{L} \in \mathscr{C}([a,b], \mathcal{O}). \tag{5.1}$$

Actually, rather than orienting subsets of $\Phi_0(X,Y)$, in [11] were oriented, instead, maps $h:\Lambda\to\Phi_0(X,Y)$, where Λ is a path-connected topological space. However, if $h(\Lambda)\subset\Phi_0(X,Y)$ is orientable in the sense of Definition 5.1 with orientation ε, necessarily $h:\Lambda\to\Phi_0(X,Y)$ is orientable in the sense of [11] with orientation $\varepsilon(h(\cdot))$. It is worth mentioning that Benevieri and Furi proved in [3] that if the condition

$$\mathcal{O}\cap GL(X,Y)\neq\emptyset. \tag{5.2}$$

holds, the notion of orientation introduced by Fitzpatrick, Pejsachowicz and Rabier [11] coincide with the one given by Benevieri and Furi in [3]. The Definition 5.1 is more appropriate for establishing the generalized Schauder formula delivered in Section 6.

Since the parity of a Fredholm curve $\mathfrak{L}$ can be regarded as a generalized local detector of any change of orientation, it is natural to define an orientation ε of a subset $\mathcal{O}$ of $\Phi_0(X,Y)$ as a map satisfying (5.1). Indeed, owing to (5.1), $\sigma(\mathfrak{L},[a,b])=-1$ if $\varepsilon(\mathfrak{L}(a))$ and $\varepsilon(\mathfrak{L}(b))$ have contrary sign. Also, note that if $\mathcal{O}$ is an orientable subset of $\Phi_0(X,Y)$ with orientation ε, then ε is locally constant, i.e., ε is constant on each path connected component of $\mathcal{O}\cap GL(X,Y)$. This is a rather natural property of an orientation. As it is a difficult task to ascertain whether, or not, a subset $\mathcal{O}$ of $\Phi_0(X,Y)$ is orientable through Definition 5.1, the next result, going back to [11], characterizes this fact in a simple way, though intrinsically, by means of the admissible curves in $\mathcal{O}$. Note its similarity with the standard characterization of the gradient maps in the Euclidean space.

Proposition 5.2. *Let $\mathcal{O}$ be a a path-connected subset of $\Phi_0(X,Y)$ satisfying (5.2). Then, the next three assertions are equivalent:*

i) *$\mathcal{O}$ is orientable.*
ii) *The parity $\sigma(\mathfrak{L},[a,b])$ only depends on $\{\mathfrak{L}(a),\mathfrak{L}(b)\}$ for each $\mathfrak{L}\in\mathscr{C}([a,b],\mathcal{O})$.*
iii) *$\sigma(\mathfrak{L},[a,b])=1$ for every $\mathfrak{L}\in\mathscr{C}([a,b],\mathcal{O})$ with $\mathfrak{L}(a)=\mathfrak{L}(b)$.*

When $\mathcal{O}\subset\Phi_0(X,Y)$ is orientable, then there are, exactly, two different orientations in $\mathcal{O}$. Precisely, given $T\in\mathcal{O}\cap GL(X,Y)$, the two orientations of $\mathcal{O}$ are defined by

$$\begin{array}{rccl}\varepsilon^{\pm}: & \mathcal{O}\cap GL(X,Y) & \longrightarrow & \mathbb{Z}_2\\ & L & \mapsto & \pm\sigma(\mathfrak{L}_{LT},[a,b])\end{array} \tag{5.3}$$

where $\mathfrak{L}_{LT}\in\mathscr{C}([a,b],\mathcal{O})$ is an arbitrary Fredholm path linking L to T, and the sign $\pm$ determines the orientation of the path connected component of T, i.e., if we choose ε^+, then the orientation of the path connected component of T is 1, whereas it is -1 if ε^- is chosen. Incidentally, $\mathcal{O}$ might not have exactly two orientations when (5.2) fails, [11].

The fact that any simply connected subset $\mathcal{O}$ of $\Phi_0(X,Y)$ satisfying (5.2) is orientable (see [11]) shows that the set of orientable subsets of $\Phi_0(X,Y)$ is really large. Moreover, if $\mathcal{O}$ is path connected, satisfies (5.2) and its cohomology group $H^1(\mathcal{O},\mathbb{Z}_2)$ is trivial, then the subset $\mathcal{O}$ is orientable.

Our next result establishes formally that the parity can detect locally the change of orientability of any Fredholm curve. Note that, choosing $\mathcal{O}=\mathcal{L}_c(X)$, this result provides us with Theorem 4.2; $\mathcal{L}_c(X)$ is orientable because, being linear, it is simply connected.

Proposition 5.3. *Let $\mathcal{O}$ be an orientable subset of $\Phi_0(X,Y)$ and pick $\mathfrak{L}\in\mathscr{C}([a,b],\mathcal{O})$. Then, $\sigma(\mathfrak{L},[a,b])=-1$ if, and only if, $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ are in different path connected components of $\mathcal{O}\cap GL(X,Y)$ with opposite orientations.*

Proof. Let $\varepsilon:\mathcal{O}\cap GL(X,Y)\to\mathbb{Z}_2$ be an orientation for $\mathcal{O}$ with $\sigma(\mathfrak{L},[a,b])=-1$. Then, by the definition of orientation,

$$-1=\sigma(\mathfrak{L},[a,b])=\varepsilon(\mathfrak{L}(a))\cdot\varepsilon(\mathfrak{L}(b)).$$

Thus, $\varepsilon(\mathfrak{L}(a)) = -\varepsilon(\mathfrak{L}(b))$, which implies that $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ have opposite orientations. Since ε is constant on each path-connected component, it follows that $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ are in different path connected components of $\mathcal{O}$ with opposite orientations.

Conversely, if $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ are in different path-connected components of $\mathcal{O}$ with opposite orientations, then, by (5.1),

$$\sigma(\mathfrak{L}, [a,b]) = \varepsilon(\mathfrak{L}(a)) \cdot \varepsilon(\mathfrak{L}(b)) = -1,$$

which ends the proof. □

As a byproduct of Proposition 5.3, it becomes apparent that the algebraic multiplicity detects locally any change of orientation of a given Fredholm curve. Thus, the localized change of orientation of a Fredholm curve can be detected through χ.

Corollary 5.4. *Let $\mathcal{O}$ be an orientable subset of $\Phi_0(X,Y)$, $\mathfrak{L}_\omega \in \mathscr{C}^\omega([a,b],\mathcal{O})$, and $\lambda_0 \in \Sigma(\mathfrak{L}_\omega)$. Then, $\chi[\mathfrak{L}_\omega, \lambda_0]$ is odd if, and only if, $\mathfrak{L}_\omega(\lambda_0-\delta)$ and $\mathfrak{L}_\omega(\lambda_0+\delta)$ are in different path-connected components of $\mathcal{O} \cap GL(X,Y)$ with opposite orientations for sufficiently small $\delta > 0$.*

Proof. Since $\mathfrak{L}_\omega(a) \in GL(X,Y)$, it follows from Theorems 4.4.1 and 4.4.4 of [22], that $\Sigma(\mathfrak{L}_\omega)$ is finite and $\lambda \in \mathrm{Alg}(\mathfrak{L}_\omega)$ for all $\lambda \in \Sigma(\mathfrak{L}_\omega)$. Take $\delta > 0$ such that λ_0 is δ-isolated, thus owing to Corollary 4.6

$$\sigma(\mathfrak{L}_\omega, [\lambda_0-\delta, \lambda_0+\delta]) = (-1)^{\chi[\mathfrak{L}_\omega,\lambda_0]}.$$

Therefore, $\sigma(\mathfrak{L}_\omega, [\lambda_0-\delta, \lambda_0+\delta]) = -1$ if, and only if, $\chi[\mathfrak{L}_\omega, \lambda_0] \in 2\mathbb{N}+1$. Now, the result is a direct consequence of Proposition 5.3. □

Our next result generalizes, very substantially, Corollary 5.4 to cover the case of general admissible paths and reduces the problem of detecting any change of orientation to the problem of the computation of the local multiplicity. Thus, it establishes a sharp connection between the topological notion of orientation and the algebraic one of multiplicity, making the concept of orientation computable.

Theorem 5.5. *Let $\mathcal{O} \subset \Phi_0(X,Y)$ be an orientable subset, $\mathfrak{L} \in \mathcal{C}([a,b],\mathcal{O})$ a Fredholm curve and $\lambda_0 \in \Sigma(\mathfrak{L})$ a δ-isolated eigenvalue. Then, the next assertions are equivalent:*

(a) *$\sum_{\lambda\in\Sigma(\mathfrak{L}_\omega)} \chi[\mathfrak{L}_\omega, \lambda_0]$ is odd for any analytical path $\mathfrak{L}_\omega \in \mathscr{C}^\omega([\lambda_0-\delta, \lambda_0+\delta], \Phi_0(X,Y))$ such that $\mathfrak{L}|_{[\lambda_0-\delta,\lambda_0+\delta]}$ and $\mathfrak{L}_\omega$ are $\mathscr{C}$-homotopic.*

(b) *$\mathrm{Card}(\Sigma(\tilde{\mathfrak{L}}))$ is odd for every $\mathscr{C}$-transversal path $\tilde{\mathfrak{L}} \in \mathscr{C}([\lambda_0-\delta, \lambda_0+\delta], \Phi_0(X,Y))$ sufficiently close to $\mathfrak{L}|_{[\lambda_0-\delta,\lambda_0+\delta]}$.*

(c) *$\mathfrak{L}(\lambda_0-\delta)$ and $\mathfrak{L}(\lambda_0+\delta)$ live in different path-connected components of $\mathcal{O}\cap GL(X,Y)$ with opposite orientations.*

Proof. According to Proposition 5.3, the assertion (c) is equivalent to $\sigma(\mathfrak{L}, [\lambda_0-\delta, \lambda_0+\delta]) = -1$. Thus, by Theorem 4.3,

$$-1 = \sigma(\mathfrak{L}, [\lambda_0-\delta, \lambda_0+\delta]) = (-1)^{\sum_{\lambda\in\Sigma(\mathfrak{L}_\omega)} \chi[\mathfrak{L}_\omega,\lambda_0]}$$

for any analytic path $\mathfrak{L}_\omega \in \mathscr{C}^\omega([\lambda_0-\delta, \lambda_0+\delta], \Phi_0(X,Y))$ $\mathscr{C}$-homotopic to $\mathfrak{L}|_{[\lambda_0-\delta,\lambda_0+\delta]}$. This establishes the equivalence between the assertions (a) and (c).

Now, suppose that $\tilde{\mathfrak{L}}$ is a $\mathscr{C}$-transversal path sufficiently close to $\mathfrak{L}$. Then, by the invariance by admissible homotopies of the parity applied to the homotopy of Theorem 4.8

$$\begin{array}{rccl} H: & [0,1]\times[\lambda_0-\delta, \lambda_0+\delta] & \longrightarrow & \Phi_0(X,Y) \\ & (\mu,\lambda) & \mapsto & \mu\tilde{\mathfrak{L}}(\lambda) + (1-\mu)\mathfrak{L}(\lambda) \end{array}$$

it is apparent that

$$\sigma(\mathfrak{L},[\lambda_0-\delta,\lambda_0+\delta])=\sigma(\tilde{\mathfrak{L}},[\lambda_0-\delta,\lambda_0+\delta]).$$

On the other hand, by Theorem 4.4 and since $\chi[\tilde{\mathfrak{L}},\lambda]=1$ for each transversal eigenvalue λ,

$$\sigma(\tilde{\mathfrak{L}},[\lambda_0-\delta,\lambda_0+\delta])=(-1)^{\sum_{\lambda\in\Sigma(\tilde{\mathfrak{L}})}\chi[\tilde{\mathfrak{L}},\lambda]}=(-1)^{\sum_{\lambda\in\Sigma(\tilde{\mathfrak{L}})}1}=(-1)^{\mathrm{Card}(\Sigma(\tilde{\mathfrak{L}}))}.$$

Hence,

$$-1=\sigma(\mathfrak{L},[\lambda_0-\delta,\lambda_0+\delta])=\sigma(\tilde{\mathfrak{L}},[\lambda_0-\delta,\lambda_0+\delta])=(-1)^{\mathrm{Card}(\Sigma(\tilde{\mathfrak{L}}))}.$$

Therefore, the assertions (b) and (c) also are equivalent. This ends the proof. □

Once introduced the notion of orientation of subsets of $\Phi_0(X,Y)$, we will introduce the set of operators used in this paper for computing the degree of Fitzpatrick, Pejsachowicz and Rabier [11]. Let Ω be an open and bounded subset of the Banach space X. Then, a nonlinear operator $f:\overline{\Omega}\subset X\to Y$ is said to be $\mathcal{C}^1$-*Fredholm of index zero* if

$$f\in\mathcal{C}^1(\overline{\Omega},Y)\quad\text{and}\quad Df\in\mathcal{C}(\overline{\Omega},\Phi_0(X,Y)).$$

Subsequently, the set of all these operators is denoted by $\mathscr{F}_0^1(\Omega,Y)$. A given operator $f\in\mathscr{F}_0^1(\Omega,Y)$ is said to be *orientable* when the set $Df(\Omega)$ is an orientable subset of $\Phi_0(X,Y)$.

Moreover, for any open and bounded subset, Ω, of a Banach space X and any operator $f:\overline{\Omega}\subset X\to Y$ satisfying

(1) $f\in\mathscr{F}_0^1(\Omega,Y)$ is *orientable* with orientation ε,
(2) f is *proper* in $\overline{\Omega}$, i.e., $f^{-1}(K)$ is compact for every compact subset $K\subset Y$,
(3) $0\notin f(\partial\Omega)$,

it is said that (f,Ω,ε) is a *Fredholm admissible triple*. By convention, the triple $(f,\emptyset,\varepsilon)$ with $\varepsilon:\emptyset\to\mathbb{Z}_2$ is considered to be an admissible triple. The set of all Fredholm admissible triples will be denoted by $\mathscr{A}_F$. If $0\in\mathscr{R}\mathscr{V}_f$, it is said that $(f,\Omega,\varepsilon)\in\mathscr{A}_F$ is a *regular triple*; the class of regular Fredholm admissible triples will be denoted by $\mathscr{R}_F$. A map $H\in\mathcal{C}^1([0,1]\times\overline{\Omega},Y)$ is said to be a $\mathcal{C}^1$-*Fredholm homotopy* if $D_xH(t,\cdot)\in\Phi_0(X,Y)$ for every $t\in[0,1]$. Moreover, it is called *orientable* if $D_xH([0,1]\times\Omega)$ is an orientable subset of $\Phi_0(X,Y)$. Hereinafter, the notation ε_t stands for the restriction of ε to $D_xH(\{t\}\times\Omega)\cap GL(X,Y)$ for each $t\in[0,1]$. The next existence result is due to Fitzpatrick, Pejsachowicz and Rabier [11].

Theorem 5.6. *There exists an integer valued map* $\deg_{FPR}:\mathscr{A}_F\to\mathbb{Z}$ *satisfying the next properties:*

(N) ***Normalization:*** $\deg_{FPR}(L,\Omega,\varepsilon)=\varepsilon(L)$ *for all* $L\in GL(X,Y)$ *if* $0\in\Omega$.
(A) ***Additivity:*** *For every* $(f,\Omega,\varepsilon)\in\mathscr{A}_F$ *and any pair of disjoint open subsets* Ω_1 *and* Ω_2 *of* Ω *with* $0\notin f(\Omega\backslash(\Omega_1\uplus\Omega_2))$ *and* $(f,\Omega_i,\varepsilon)\in\mathscr{A}_F$ *for each* $i\in\{1,2\}$,

$$\deg_{FPR}(f,\Omega,\varepsilon)=\deg_F(f,\Omega_1,\varepsilon)+\deg_{FPR}(f,\Omega_2,\varepsilon).$$

(H) ***Homotopy Invariance:*** *For each proper* $\mathcal{C}^1$*-Fredholm homotopy* $H\in\mathcal{C}^1([0,1]\times\overline{\Omega},Y)$ *with orientation* ε *such that* $(H(t,\cdot),\Omega,\varepsilon_t)\in\mathscr{A}_F$ *for each* $t\in[0,1]$

$$\deg_{FPR}(H(0,\cdot),\Omega,\varepsilon_0)=\deg_{FPR}(H(1,\cdot),\Omega,\varepsilon_1).$$

As for the Leray–Schauder Degree, from these three properties one can easily infer that, whenever $(f,\Omega,\varepsilon)\in\mathscr{A}_F$ with $\deg_{FPR}(f,\Omega,\varepsilon)\neq 0$, there exists $x\in\Omega$ such that $f(x)=0$.

We are concluding this section with a brief sketch of the construction of $\deg_{FPR}$ carried over in [11]. It is needed for the proof of Theorem 1.1 delivered in Section 6. Let

$(f,\Omega,\varepsilon)\in\mathscr{A}_F$. By definition, $f\in\mathscr{F}_0^1(\Omega,Y)$ is $\mathcal{C}^1$-Fredholm of index zero and it is ε-orientable, i.e., $Df(\Omega)$ is an orientable subset of $\Phi_0(X,Y)$ with orientation

$$\varepsilon : Df(\Omega)\cap GL(X,Y)\longrightarrow \mathbb{Z}_2.$$

Once an orientation has been defined in $Df(\Omega)$, $\deg_{FPR}$ can be defined as $\deg_{LS}$ if $(f,\Omega,\varepsilon)\in\mathscr{R}_F$. Indeed, since in such case the set $f^{-1}(0)\cap\Omega$ is finite, we can define, in complete agreement with (N), (A) and (H),

$$\deg_{FPR}(f,\Omega,\varepsilon):=\sum_{x\in f^{-1}(0)\cap\Omega}\varepsilon(Df(x)).$$

Should it be $(f,\Omega,\varepsilon)\notin\mathscr{R}_F$, then, by definition,

$$\deg_{FPR}(f,\Omega,\varepsilon):=\deg_{FPR}(f-y,\Omega,\varepsilon)$$

where $y\in\mathscr{RV}_f$ is any regular value of f sufficiently close to 0. The existence of such regular values is guaranteed by a theorem of Quinn and Sard [28], a version of the Sard–Smale Theorem, [33], not requiring the separability of the involved Banach spaces.

It is worth-mentioning that $\deg_{FPR}$ extends $\deg_{LS}$ to this more general setting. Indeed, by the definition of orientation, $\mathcal{L}_c(X)$ is orientable and the maps $\varepsilon^{\pm}: GL_c(X)\to\mathbb{Z}_2$ defined by

$$\varepsilon^{\pm}(L)=\pm\deg_{LS}(L,\Omega), \tag{5.4}$$

where the right hand side of (5.4) is given by (2.7), determine the two orientations of $\mathcal{L}_c(X)$. Therefore, if for every $(f,\Omega)\in\mathscr{R}$ we choose the orientation of f as the restriction of ε^+ to $Df(\Omega)\cap GL(X)$, then $(f,\Omega,\varepsilon^+)\in\mathscr{R}_F$ and

$$\deg_{FPR}(f,\Omega,\varepsilon^+)=\deg_{LS}(f,\Omega)$$

because

$$\deg_{LS}(f,\Omega)=\sum_{x\in f^{-1}(0)\cap\Omega}\deg_{LS}(Df(x),\Omega)=\sum_{x\in f^{-1}(0)\cap\Omega}\varepsilon^+(Df(x))=\deg_{FPR}(f,\Omega,\varepsilon^+).$$

6. Proof of Theorem 1.1. Some consequences

We begin by recalling Theorem 1.1.

Theorem 6.1. *Let $(f,\Omega,\varepsilon)\in\mathscr{A}_F$ be a Fredholm admissible triple with $\Omega\neq\emptyset$. Then, for every $L\in Df(\Omega)\cap GL(X,Y)$,*

$$\deg_{FPR}(f,\Omega,\varepsilon)=\varepsilon(L)\cdot\sum_{x\in f^{-1}(y)\cap\Omega}(-1)^{\sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})}\chi[\mathfrak{L}_{\omega,x},\lambda_x]} \tag{6.1}$$

where $\mathfrak{L}_{\omega,x}\in\mathscr{C}^{\omega}([a,b],\Phi_0(X,Y))$ is an analytical curve $\mathscr{C}$-homotopic to some curve $\mathfrak{L}_x\in\mathcal{C}([a,b],Df(\Omega))$ connecting $Df(x)$ to L, and $y=0$ if $(f,\Omega,\varepsilon)\in\mathscr{R}_F$, whereas $y\in\mathscr{RV}_f$ is any regular value of f sufficiently close to 0 if $(f,\Omega,\varepsilon)\notin\mathscr{R}_F$.

The generalized Schauder formula (6.1) expresses the degree of Fitzpatrick, Pejsachowicz and Rabier [11] in terms of the algebraic multiplicity χ. Thus, it allows to calculate $\deg_{FPR}$ algorithmically, liberating it of the topological artillery used in its definition. In particular, expressing it in an extremely versatile way from the point of view of the applications. It must be observed that it is always possible to choose an operator $L\in Df(\Omega)\cap GL(X,Y)$ since, as $Df(\Omega)$ is orientable, necessarily, $Df(\Omega)\cap GL(X,Y)\neq\emptyset$. Figure 7 illustrates the formula (6.1) in the special case when $(f,\Omega,\varepsilon)\in\mathscr{R}_F$ and

$$f^{-1}(0)\cap\Omega=\{x_1,x_2,x_3,x_4\}.$$

It shows four admissible paths $\mathfrak{L}_{x_i}\in\mathcal{C}([a,b],Df(\Omega))$, $1\leq i\leq 4$, connecting $Df(x_i)$ to L.

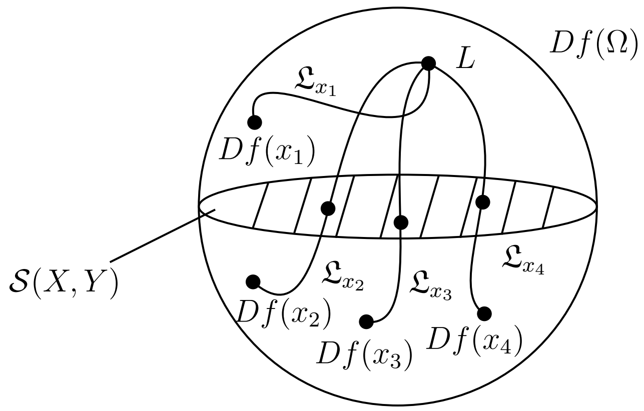

FIGURE 7. Graphic illustration of Theorem 6.1.

According to Figure 7, $Df(x_1)$ is in the same path connected component of $Df(\Omega)\cap GL(X,Y)$ as L, while $Df(x_i)$, $i\in\{2,3,4\}$, are in others. Suppose $\varepsilon(L)=1$, then since ε is constant in each path connected component

$$\varepsilon(Df(x_1))=1 \text{ and } \varepsilon(Df(x_i))=-1 \text{ for each } i\in\{2,3,4\}.$$

Thus, by definition,

$$\deg_{FPR}(f,\Omega,\varepsilon)=\sum_{i=1}^{4}\varepsilon(Df(x_i))=-3+1=-2.$$

In the practical situations, whether, or not, $Df(x_i)$ lies in the same component as L, should be determined from the algebraic multiplicity χ as described next. First, for every $i\in\{1,2,3,4\}$, one should construct an analytical curve $\mathscr{C}$-homotopic to $\mathfrak{L}_{x_i}\in\mathcal{C}([a,b],Df(\Omega))$, say $\mathfrak{L}_{\omega,x_i}\in\mathscr{C}^{\omega}([a,b],\Phi_0(X,Y))$. Then, for every $i\in\{1,2,3,4\}$, one should determine the oddities of

$$\Sigma_i\equiv\sum_{\lambda_{x_i}\in\Sigma(\mathfrak{L}_{\omega,x_i})}\chi[\mathfrak{L}_{\omega,x_i},\lambda_{x_i}].$$

In the practical example of Figure 7, one has that Σ_1 is even and Σ_2, Σ_3 and Σ_4 are odd. Therefore, by (6.1),

$$\deg_{FPR}(f,\Omega,\varepsilon)=\sum_{i=1}^{4}(-1)^{\Sigma_i}=-2.$$

When $(f,\Omega,\varepsilon)\notin\mathscr{R}_F$, one should first pick a regular value, y, of f sufficiently close to zero in order to apply the formula (6.1).

Proof of Theorem 6.1: It suffices to prove it when $(f,\Omega,\varepsilon)\in\mathscr{R}_F$. Since $Df(\Omega)$ is orientable, it is path connected and $Df(\Omega)\cap GL(X,Y)\neq\emptyset$. Therefore, there exists such $L\in Df(\Omega)\cap GL(X,Y)$. In particular, given any $x\in f^{-1}(0)\cap\Omega$, there exists a Fredholm path $\mathfrak{L}_x$ joining $Df(x)$ and L. By definition of orientation,

$$\sigma(\mathfrak{L}_x,[a,b])=\varepsilon(Df(x))\cdot\varepsilon(L). \tag{6.2}$$

Let $\mathfrak{L}_{\omega,x}\in\mathscr{C}^{\omega}([a,b],\Phi_0(X,Y))$ be an analytical Fredholm curve $\mathscr{C}$-homotopic to $\mathfrak{L}_x\in\mathcal{C}([a,b],Df(\Omega))$. The existence of $\mathfrak{L}_{\omega,x}$ has been already proven in Theorem 4.5. Thus, according to Theorem 4.5,

$$\sigma(\mathfrak{L}_x,[a,b])=\sigma(\mathfrak{L}_{\omega,x},[a,b])=(-1)^{\sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})}\chi[\mathfrak{L}_{\omega,x},\lambda_x]}.$$

Therefore, by (6.2), it follows

$$\varepsilon(Df(x)) = \varepsilon(L) \cdot (-1)^{\sum_{\lambda_x \in \Sigma(\mathfrak{L}_{\omega,x})} \chi[\mathfrak{L}_{\omega,x},\lambda_x]},$$

which concludes the proof. □

Theorem 6.1 can be simplified by introducing the following concept. A subset $\mathcal{O}$ of $\Phi_0(X,Y)$ is said to be $\mathcal{C}^\omega$-connected if for every pair of operators $L, T \in \mathcal{O}$, there exists an analytical path $\gamma \in \mathcal{C}^\omega([a,b],\mathcal{O})$ such that $\gamma(a) = L$ and $\gamma(b) = T$. Note that every open subset of $\Phi_0(X,Y)$ is $\mathcal{C}^\omega$-connected, since being locally convex, it is possible to connect every pair of points via a polygonal path and therefore one can get a $\mathcal{C}^\omega$-curve regularizing each singular point of this polygonal.

Corollary 6.2. *Let $(f,\Omega,\varepsilon) \in \mathscr{R}_F$ be a Fredholm regular pair with $\Omega \neq \emptyset$ and let $L \in Df(\Omega) \cap GL(X,Y)$. If $Df(\Omega)$ is $\mathcal{C}^\omega$-connected, then*

$$\deg_{FPR}(f,\Omega,\varepsilon) = \varepsilon(L) \cdot \sum_{x \in f^{-1}(0)\cap\Omega} (-1)^{\sum_{i=1}^{n_x} \chi[\mathfrak{L}_{\omega,x},\lambda_{x,i}]}$$

where $\mathfrak{L}_{\omega,x} \in \mathcal{C}^\omega([a,b], Df(\Omega))$ is an analytical path joining $Df(x)$ with L, and

$$\Sigma(\mathfrak{L}_{\omega,x}) = \{\lambda_{x,1}, \lambda_{x,2}, ..., \lambda_{x,n_x}\}.$$

In the special case when $Df(\Omega)$ is convex, the result is simplified in an extremely versatile form. It must be observed that since $Df(\Omega)$ is convex, it is in particular $\mathcal{C}^\omega$-connected.

Corollary 6.3. *Let $(f,\Omega,\varepsilon) \in \mathscr{R}_F$ with $\Omega \neq \emptyset$ and let $L \in Df(\Omega) \cap GL(X,Y)$. If $Df(\Omega)$ is convex, then*

$$\deg_{FPR}(f,\Omega,\varepsilon) = \varepsilon(L) \cdot \sum_{x \in f^{-1}(0)\cap\Omega} (-1)^{\sum_{i=1}^{n_x} \chi[\mathfrak{L}_{\omega,x},\lambda_{x,i}]}$$

where

$$\mathfrak{L}_{\omega,x}(\lambda) = (1-\lambda)Df(x) + \lambda L, \qquad \lambda \in [0,1],$$

and

$$\Sigma(\mathfrak{L}_{\omega,x}) = \{\lambda_{x,1}, \lambda_{x,2}, ..., \lambda_{x,n_x}\}.$$

Observe that in the particular case when the orientable subset $\mathcal{O}$ is $\mathcal{L}_c(X)$, $L = I_X$ and $\varepsilon(T) = \deg_{LS}(T,\Omega)$ for $T \in GL_c(X)$, where $\deg_{LS}(T,\Omega)$ is given by (2.7); for each $(f,\Omega) \in \mathscr{R}$, Corollary 6.3 coincides with Theorem 3.6.

Acknowledgements. We sincerely thank the three (anonymous) reviewers of this paper for their careful reading. One of them, suggested to us packaging the main theorem in a more compact way, and he/she was completely right!

References

[1] H. Amann and S. A. Weiss, On the uniqueness of the topological degree, *Math. Z.* **130** (1973), 39–54.
[2] P. Benevieri and M. Furi, A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory, *Ann. Sci. Math. Quebec* **22** (1998), 131–148.
[3] P. Benevieri and M. Furi, On the concept of orientability for Fredholm maps between real Banach manifolds, *Topol. Methods Nonlinear Anal.* **16**, Number 2 (2000), 279–306.
[4] P. Benevieri and M. Furi, On the uniqueness of the degree for nonlinear Fredholm maps of index zero between Banach manifolds, *Communications in Applied Analysis* **15**, (2011), 203–216.
[5] L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, *Math. Ann.* **71** (1911), 97–115.
[6] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* **8** (1971), 321–340.

[7] J. Esquinas, Optimal multiplicity in local bifurcation theory, II: General case, *J. Diff. Equations* **75** (1988), 206–215.
[8] J. Esquinas and J. López-Gómez, Optimal multiplicity in local bifurcation theory, I: Generalized generic eigenvalues, *J. Diff. Equations* **71** (1988), 72–92.
[9] P. M. Fitzpatrick and J. Pejsachowicz, Parity and generalized multiplicity, *Trans. Amer. Math. Soc.* **326** (1991), 281–305.
[10] P. M. Fitzpatrick and J. Pejsachowicz, Orientation and the Leray–Schauder theory for fully nonlinear elliptic boundary value problems, *Mem. Amer. Math. Soc.* **483**, *Providence*, (1993).
[11] P. M. Fitzpatrick, J. Pejsachowicz and P. J. Rabier, Orientability of Fredholm families and topological degree for orientable nonlinear Fredholm mappings, *J. Functional Analysis* **124** (1994), 1–39.
[12] E. I. Fredholm, Sur une classe d'equations fonctionnelles, *Acta Math* **27**, (1903), 365390.
[13] L. Führer, Theorie des Abbildungsgrades in endlichdimensionalen Räumen, *Ph. D. Dissertation, Frei Univ. Berlin, Berlin*, (1971).
[14] I. C. Göhberg and E. I. Sigal, An Operator Generalization of the Logarithmic Residue Theorem and the Theorem of Rouché, *Math. Sbornik* **84(126)** (1971), 607–629. English Trans.: *Math. USSR Sbornik* **13** (1971), 603–625.
[15] J. Ize, Bifurcation Theory for Fredholm Operators, *Mem. Amer. Math. Soc.* **174**, *Providence*, (1976).
[16] N. Kuiper, The Homotopy Type of the Unitary Group of Hilbert Space, *Topology* **3**, (1965), 19–30.
[17] B. Laloux and J. Mawhin, Coincidence index and multiplicity, *Trans. Amer. Math. Soc.* **217** (1976), 143–162.
[18] B. Laloux and J. Mawhin, Multiplicity, Leray–Schauder formula and bifurcation, *J. Diff. Eqns.* **24** (1977), 309–322.
[19] J. Leray and J. Schauder, Topologie et équations fonctionelles, *Ann. Sci. École Norm. Sup. Sér. 3* **51** (1934), 45–78.
[20] N. G. Lloyd, Degree Theory, *Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge* (1978).
[21] J. López-Gómez, *Ecuaciones diferenciales y variable compleja, con teoría espectral y una introducción al grado topológico de Brouwer*, Prentice Hall, Madrid, 2001.
[22] J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, CRC Press, Chapman and Hall RNM vol. 426, Boca Raton, 2001.
[23] J. López-Gómez, Global bifurcation for Fredholm operators, *Rend. Istit. Mat. Univ. Trieste* **48** (2016), 539–564. DOI: 10.13137/2464-8728/13172
[24] J. López-Gómez and C. Mora-Corral, Counting zeroes of $\mathcal{C}^1$-Fredholm maps of index zero, *Bull. London Math. Soc.* **37** (2005) 778–792.
[25] J. López-Gómez and C. Mora-Corral, *Algebraic Multiplicity of Eigenvalues of Linear Operators*, Operator Theory, Advances and Applications vol. 177, Birkhäuser, Basel, 2007.
[26] R. J. Magnus, A generalization of multiplicity and the problem of bifurcation, *Proc. Lond. Math. Soc.* **32** (1976), 251–278.
[27] C. Mora-Corral, On the Uniqueness of the Algebraic Multiplicity, *J. London Math. Soc.* **69** (2004), 231–242.
[28] F. Quinn and A. Sard, Hausdorff Conullity of Critical Images of Fredholm Maps, *American Journal of Mathematics* **94**, (1972), 1101–1110.
[29] J. Pejsachowicz and P. J. Rabier, Degree theory for C^1 Fredholm mappings of index 0, *J. Anal. Math.* **76** (1998), 289.
[30] P. J. Rabier, Generalized Jordan chains and two bifurcation theorems of Krasnoselskii, *Nonl. Anal. TMA* **13** (1989), 903–934.
[31] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.
[32] A. Sard, The measure of the critical values of differentiable maps, *Bull. Amer. Math. Soc.* **48** (1942), 883–890.
[33] S. Smale, An infinite dimensional version of Sard's theorem, *Amer. J. Math.* **87** (1965), 861–866.

Institute of Interdisciplinary Mathematics, Department of Mathematical Analysis and Applied Mathematics, Complutense University of Madrid, 28040-Madrid, Spain.
E-mail address: julian@mat.ucm.es, juancsam@ucm.es