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Eduardo Fernández and Fabio Gironella

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Geometry, Topology / *Géométrie, Topologie*

# A remark on the contactomorphism group of overtwisted contact spheres

*Une remarque sur le groupe des contactomorphismes  
des sphères de contact vrillées*

Eduardo Fernández<sup>a</sup> and Fabio Gironella<sup>b</sup>

<sup>a</sup> Universidad Complutense de Madrid, Departamento de Álgebra, Geometría y Topología, Facultad de Matemáticas, and Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Madrid, Spain

<sup>b</sup> Alfréd Rényi Institute of Mathematics, Budapest, Hungary.  
*E-mails:* [eduarf01@ucm.es](mailto:eduarf01@ucm.es), [fabio.gironella@renyi.hu](mailto:fabio.gironella@renyi.hu),  
[fabio.gironella.math@gmail.com](mailto:fabio.gironella.math@gmail.com).

**Abstract.** We show the existence of elements of infinite order in some homotopy groups of the contactomorphism group of overtwisted spheres. It follows in particular that the contactomorphism group of some high dimensional overtwisted spheres is not homotopically equivalent to a finite dimensional Lie group.

**Résumé.** On prouve l'existence d'éléments d'ordre infini dans certains groupes d'homotopie du groupe des contactomorphismes des sphères vrillées. En particulier, il s'en suit que le groupe des contactomorphismes de certaines sphères vrillées n'est pas homotopiquement équivalent à un groupe de Lie de dimension finie.

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## 1. Introduction and statements of the results

Let  $(M, \xi)$  be a closed contact manifold. These short notes are concerned with the relationship between the topology of the connected component  $\text{Diff}_0(M)$  of the identity in the group of diffeomorphisms of  $M$  and its subgroup  $\text{Diff}_0(M, \xi)$  consisting of contactomorphisms of  $(M, \xi)$ . More precisely, throughout the notes we will always assume contact structures to be cooriented and contactomorphisms to be coorientation-preserving.

The path components of the group of contactomorphisms of particular contact manifolds have been studied by several authors in the literature; see for instance Ding–Geiges [8], Dymara [9], Gironella [16, 17], Giroux [18], Giroux–Massot [19], Lanzat–Zapolsky [23], Massot–Niederkrüger [26], and Vogel [28]. Higher–order homotopy groups have also been studied: for instance, Casals–Presas [6], Casals–Spáčil [7] and Eliashberg [11] contain results for the case of the standard tight  $(2n + 1)$ –contact sphere. In this notes, we deal with the case of overtwisted spheres (cf. Borman et al. [2]).

The interested reader can also consult Casals et al. [4], dealing with the symplectomorphism group case using techniques similar to the ones in these notes; more precisely, there the authors use the h–principle for overtwisted structures of Borman et al. [2] to construct non trivial homotopy classes of symplectomorphisms of some exotic symplectic  $(\mathbb{R}^{2n}, \omega_{ot})$ , defined as the symplectization of an overtwisted contact structure on  $\mathbb{R}^{2n-1}$ .

Let  $(\mathbb{S}^{2n+1}, \xi_{ot})$  be any overtwisted sphere, and consider the natural inclusion

$$i: \text{Diff}_0(\mathbb{S}^{2n+1}, \xi_{ot}) \hookrightarrow \text{Diff}_0(\mathbb{S}^{2n+1}).$$

For any  $k \in \mathbb{N}$ , denote  $\mathcal{K}_k^{2n+1}$  the kernel of the homomorphism

$$\pi_k(i): \pi_k(\text{Diff}_0(\mathbb{S}^{2n+1}, \xi_{ot})) \rightarrow \pi_k(\text{Diff}_0(\mathbb{S}^{2n+1})).$$

**Theorem 1.** *Let  $k \in \mathbb{N}$  be such that  $1 \leq 4k + 1 \leq 2n - 1$ . The group  $\mathcal{K}_{4k+1}^{2n+1}$  contains an infinite cyclic subgroup.*

Under some conditions on the dimension, Theorem 1 can be improved in the case of the fundamental group and the fifth homotopy group as follows:

**Theorem 2.**

- (i) *The group  $\mathcal{K}_1^3$  contains a subgroup isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_2$ .*
- (ii) *Let  $n \geq 3$ . The group  $\mathcal{K}_1^{4n+1}$  contains a subgroup isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ .*
- (iii) *Let  $n \geq 6$ . The group  $\mathcal{K}_5^{4n+1}$  contains a subgroup isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ .*

From the methods developed in the paper we are also able to show the following

**Theorem 3.**

- (i) *Let  $n \geq 4$ . The group  $\mathcal{K}_3^{4n+3}$  contains an infinite cyclic subgroup.*
- (ii) *Let  $n \geq 2$ . The group  $\mathcal{K}_4^{8n+7}$  contains an infinite cyclic subgroup.*

As the even–order higher homotopy groups of a finite dimensional Lie group are finite (see for instance Félix et al. [13, Example 2.51]), Theorem 3 immediately implies:

**Corollary 4.** *For  $n \geq 2$ ,  $\text{Diff}_0(\mathbb{S}^{8n+7}, \xi_{ot})$  is not homotopy equivalent to a finite dimensional Lie group.*

The proofs of Theorems 1, 2 and 3 use four main ingredients. The first is the notion of *overtwisted group* introduced in Casals et al. [5], which relies on the flexibility results for overtwisted contact manifolds from Borman et al. [2] and Eliashberg [10]. The second is the existence of a long exact sequence relating the homotopy groups of the space of contact structures on  $\mathbb{S}^{2n+1}$  to those of  $\text{Diff}_0(\mathbb{S}^{2n+1}, \xi_{ot})$  and of  $\text{Diff}_0(\mathbb{S}^{2n+1})$ ; see Section 2.1. The last ingredients are the description of the rational homotopy groups of  $\text{Diff}_0(\mathbb{S}^{2n+1})$  from Farrell–Hsiang [12] and the description of some homotopy groups of the homogeneous space  $\Gamma_n = \text{SO}(2n)/U(n)$  from Bott [3], Harris [20], Kachi [22], Massey [24] and Mukai [27].

We point out that these methods could also be applied to the case of any overtwisted contact manifold  $(M^{2n+1}, \xi)$  such that both the homotopy type of the space of almost contact structures on  $M$  and the diffeomorphism group of  $M$  can be (at least partially) understood.

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## 2. Preliminaries

### 2.1. A long exact sequence of homotopy groups

Let  $(M, \xi)$  be a closed contact manifold. In this section, the spaces  $\text{Diff}_0(M)$  and  $\text{Diff}_0(M, \xi)$  are considered as pointed spaces, with base point  $\text{Id}$ . Similarly,  $\text{Cont}(M, \xi)$  is considered with base point  $\xi$ .

As shown for instance in Giroux–Massot [19] (and, more in detail, in Massot [25]), the natural map

$$\begin{array}{ccc} \text{Diff}_0(M) & \longrightarrow & \text{Cont}(M, \xi) \\ \varphi & \longmapsto & \varphi_* \xi \end{array}$$

is a locally–trivial fibration with fiber  $\text{Diff}_0(M, \xi)$ ; see also Geiges–Gonzalo [15] for a proof of the fact that the map is a Serre fibration (which is enough for what follows). In particular, it induces a long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_{k+1}(\text{Cont}(M, \xi)) \rightarrow \pi_k(\text{Diff}_0(M, \xi)) \rightarrow \pi_k(\text{Diff}_0(M)) \rightarrow \pi_k(\text{Cont}(M, \xi)) \rightarrow \cdots \quad (1)$$

### 2.2. Almost contact structures on $\mathbb{S}^{2n+1}$

Recall that, given an oriented smooth manifold  $M^{2n+1}$ , an *almost contact structure* is a triple  $(\xi, J, R)$ , where  $\xi \subseteq TM$  is a cooriented hyperplane distribution,  $J : \xi \rightarrow \xi$  is a complex structure on  $\xi$ ,  $R = \langle \nu \rangle \subseteq TM$  is a trivial line sub–bundle defining the coorientation of  $\xi$  and  $\xi \oplus R \cong TM$  as oriented vector bundles. Here, we denote  $\text{AlmCont}(M)$  the space of almost contact structures on  $M$ .

Now, recall that, given an auxiliary Riemannian metric  $g$  on  $M$ , the space  $\text{AlmCont}(M)$  is homotopy equivalent to the space of reductions of the structure group  $\text{SO}(2n + 1)$  of the principal bundle  $\text{Fr}_{\text{SO}}(M)$  of orthonormal (w.r.t.  $g$ ) oriented frames of  $TM$  to its subgroup  $\text{U}(n) = \text{U}(n) \times 1 \subseteq \text{SO}(2n + 1)$ , i.e. to the space of sections  $\Gamma(M; X)$  of the quotient fiber bundle  $\pi : X = \text{Fr}_{\text{SO}}(M) / \text{U}(n) \rightarrow M$ , with typical fiber  $\text{SO}(2n + 1) / \text{U}(n)$ .

Recall also (see Geiges [14, Lemma 8.2.1]) that there is an identification

$$\Gamma_{n+1} := \text{SO}(2n + 2) / \text{U}(n + 1) \simeq \text{SO}(2n + 1) / \text{U}(n). \quad (2)$$

In particular, the fiber bundle  $\pi$  can also be seen as a fibration

$$\begin{array}{ccc} \Gamma_{n+1} & \hookrightarrow & X \\ & & \downarrow \pi \\ & & M \end{array} \quad (3)$$

Denote the trivial real line bundle over  $M$  by  $\varepsilon = \langle w \rangle$ . Then, the Riemannian metric  $g$  on  $M$  naturally extends to a metric on  $TM \oplus \varepsilon$ , still denoted  $g$ , by declaring the vector  $w$  to be orthogonal to  $TM$  and of norm 1. Let now  $\text{Complex}(TM \oplus \varepsilon)$  be the space of complex structures on the oriented bundle  $TM \oplus \varepsilon$ . Observe that this space is homotopy equivalent to the space of complex structures which are compatible with the metric  $g$  (i.e.  $g(J \cdot, J \cdot) = g(\cdot, \cdot)$ ). Notice also that the latter can be identified with the space of sections of a fiber bundle over  $M$  with fiber the space of complex structures on  $\mathbb{R}^{2n+2}$  compatible with the standard metric, i.e.  $\Gamma_{n+1}$ .

Given any almost contact structure  $(\xi, J, R)$ , one can naturally extend  $J$  to a complex structure  $\tilde{J}: TM \oplus \varepsilon \rightarrow TM \oplus \varepsilon$  on  $TM \oplus \varepsilon$ , by defining  $\tilde{J}v = -w$ . This gives an inclusion  $j: \text{AlmCont}(M) \hookrightarrow \text{Complex}(TM \oplus \varepsilon)$ .

In fact, (2) says that  $j$  is a homotopy equivalence. More precisely, denoting the projection on the first factor by  $pr: TM \oplus \varepsilon \rightarrow TM$ , the map

$$\begin{aligned} \Phi: \text{Complex}(TM \oplus \varepsilon) &\longrightarrow \text{AlmCont}(M) \\ J &\longmapsto (TM \cap J(TM), J|_{TM \cap J(TM)}, \langle pr(Jw) \rangle) \end{aligned}$$

is the homotopy inverse of  $j$ . As a consequence:

**Lemma 5.** *If the vector bundle  $TM$  is stably trivial of type 1 over  $\mathbb{R}$ , i.e.  $TM \oplus \varepsilon$  is trivializable (as real vector bundle), the fiber bundle  $\pi: X \rightarrow M$  is trivializable.*

For the rest of the section we focus on the case of almost contact structures on  $\mathbb{S}^{2n+1}$ .

According to Lemma 5, the fiber bundle  $\pi: X \rightarrow \mathbb{S}^{2n+1}$  is trivial. Once fixed any trivialization, one can then identify  $\Gamma(M; X) = \text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$ ; in particular,  $\text{AlmCont}(\mathbb{S}^{2n+1})$  is homotopy equivalent to  $\text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$ .

**Remark 6.** The homotopy groups  $\pi_k(\Gamma_{n+1})$ , in the stable range  $1 \leq k \leq 2n$ , were computed in Bott [3]: they are of period 8 and the first eight groups are, in order,  $0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2$ . Moreover, some of the first unstable groups  $\pi_{2n+1+k}(\Gamma_{n+1})$  were computed in Harris [20], Kachi [22], Massey [24] and Mukai [27]. More precisely, we will use the fact that the following unstable homotopy groups contain a cyclic subgroup:  $\pi_{4n+3}(\Gamma_{2n+1})$ ,  $\pi_{4n+7}(\Gamma_{2n+1})$ ,  $\pi_{4n+7}(\Gamma_{2n+2})$  and  $\pi_{8n+12}(\Gamma_{4n+4})$ .

**Lemma 7.** *All the path-connected components of the space  $\text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$  are homeomorphic. In particular, all the path-connected components of  $\text{AlmCont}(\mathbb{S}^{2n+1})$  are homotopy equivalent.*

**Proof.** Let  $J_0 \in \Gamma_{n+1}$  be the standard (almost) complex structure on  $\mathbb{R}^{2n+2}$ , and

$$\begin{aligned} \xi_0: \mathbb{S}^{2n+1} &\longrightarrow \Gamma_{n+1} \\ z &\longmapsto J_0 \end{aligned}$$

the corresponding constant section in  $\Gamma(M; X) = \text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$ . Consider then any other section  $\xi: \mathbb{S}^{2n+1} \rightarrow \Gamma_{n+1}$ . Because  $\Gamma_{n+1}$  is path-connected, up to homotopy, we can moreover assume that  $\xi(N) = J_0$ , where  $N$  denotes the north pole of  $\mathbb{S}^{2n+1}$ .

Denote by  $\text{Map}_{\xi_0}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$  and  $\text{Map}_{\xi}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$  the path connected components of  $\xi_0$  and  $\xi$ , respectively. Consider the  $U(n+1)$ -principal bundle  $p: \text{SO}(2n+2) \rightarrow \Gamma_{n+1}$ ,  $A \mapsto A \cdot J_0 \cdot A^{-1}$ . By Bott periodicity,  $\pi_{2n}(U(n+1)) = 0$ . In particular, the homomorphism

$$\pi_{2n+1}(p): \pi_{2n+1}(\text{SO}(2n+2)) \longrightarrow \pi_{2n+1}(\Gamma_{n+1})$$

is surjective, so that there exists a lift  $\hat{\xi}: \mathbb{S}^{2n+1} \rightarrow \text{SO}(2n+2)$  of  $\xi$  such that  $\hat{\xi}(N) = \text{Id}$ .

The desired homeomorphism is then given by

$$\begin{aligned} \Phi_{\xi}: \text{Map}_{\xi_0}(\mathbb{S}^{2n+1}, \Gamma_{n+1}) &\longrightarrow \text{Map}_{\xi}(\mathbb{S}^{2n+1}, \Gamma_{n+1}) \\ \eta &\longmapsto \hat{\xi} \cdot \eta \end{aligned}$$

where

$$\begin{aligned} \hat{\xi} \cdot \eta: \mathbb{S}^{2n+1} &\longrightarrow \Gamma_{n+1} \\ z &\longmapsto \hat{\xi}(z) \cdot \eta(z) \end{aligned}$$

is defined by using the left action of  $\text{SO}(2n+2)$  on  $\Gamma_{n+1}$ . □

**Proposition 8.** *For each  $k \in \mathbb{N}$  there is an isomorphism*

$$\pi_k(\text{AlmCont}(\mathbb{S}^{2n+1})) \cong \pi_k(\Gamma_{n+1}) \oplus \pi_{2n+k+1}(\Gamma_{n+1})$$

**Proof.** For  $k = 0$  we argue as follows. Recall that  $[\mathbb{S}^n, X] = \pi_n(X, x)/\pi_1(X, x)$ , for any pointed topological space  $(X, x)$ . Hence,

$$\begin{aligned} \pi_0(\text{AlmCont}(\mathbb{S}^{2n+1})) &= \pi_0(\text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})) = [\mathbb{S}^{2n+1}, \Gamma_{n+1}] \\ &= \pi_{2n+1}(\Gamma_{n+1})/\pi_1(\Gamma_{n+1}) = \pi_{2n+1}(\Gamma_{n+1}), \end{aligned}$$

and the statement follows from the fact that, according to Remark 6,  $\Gamma_{n+1} = \text{SO}(2n+2)/\text{U}(n+1)$  is simply connected.

We now prove the statement for  $\pi_k$  with  $k \geq 1$ . According to Lemma 7, we can consider  $\text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})$  as a space pointed at  $\xi_0 \equiv J_0 : \mathbb{S}^{2n+1} \rightarrow \Gamma_{n+1}$ . Similarly, we consider  $\Gamma_{n+1}$  as space pointed at  $J_0$ . There is then a natural Serre fibration (of pointed spaces)

$$\begin{array}{ccc} \text{ev}_N: \text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1}) & \longrightarrow & \Gamma_{n+1} \\ \xi & \longmapsto & \xi(N) \end{array}$$

The fiber over  $J_0$  is the space  $F = \text{Map}((\mathbb{S}^{2n+1}, N), (\Gamma_{n+1}, J_0))$  of maps  $\mathbb{S}^{2n+1} \rightarrow \Gamma_{n+1}$  which evaluate at  $J_0$  on the north pole  $N$ . In particular,  $\pi_k(F, \xi_0) = \pi_{2n+k+1}(\Gamma_{2n+1}, J_0)$ .

Moreover, the map

$$\begin{array}{ccc} s: \Gamma_{n+1} & \longrightarrow & \text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1}) \\ J & \longmapsto & \xi_J \end{array}$$

where  $\xi_J \equiv J$ , defines a section of the fibration. In particular, the boundary map in the long exact sequence of homotopy groups associated to the Serre fibration  $\text{ev}_N$  is trivial, and every obtained short exact sequence of groups splits. In other words,

$$\begin{aligned} \pi_k(\text{AlmCont}(\mathbb{S}^{2n+1})) &= \pi_k(\text{Map}(\mathbb{S}^{2n+1}, \Gamma_{n+1})) \cong \pi_k(\Gamma_{n+1}) \oplus \pi_k(F) \\ &= \pi_k(\Gamma_{n+1}) \oplus \pi_{2n+k+1}(\Gamma_{2n+1}). \quad \square \end{aligned}$$

### 2.3. The overtwisted group

Let  $M$  be a  $(2n + 1)$ -dimensional manifold. We denote in this section the subspaces of contact and almost contact structures on  $M$  with a fixed overtwisted disk  $\Delta_0 \subset M$  respectively by  $\text{Cont}_{\text{OT}}(M, \Delta_0) \subseteq \text{Cont}(M)$  and  $\text{AlmCont}(M, \Delta_0) \subseteq \text{AlmCont}(M)$ .

**Theorem 9** (Borman et al. [2], Eliashberg [10]). *The following forgetful map induces a weak homotopy equivalence:*

$$\text{Cont}_{\text{OT}}(M, \Delta_0) \rightarrow \text{AlmCont}(M, \Delta_0),$$

Notice that the overtwisted disk is not allowed to move in this results. However, an easy corollary is the fact that the forgetful map

$$\text{Cont}_{\text{OT}}(M) \rightarrow \text{AlmCont}(M) \tag{4}$$

induces a bijection at  $\pi_0$ -level, where  $\text{Cont}_{\text{OT}}(M)$  denotes the space of overtwisted contact structures on  $M$ . This can be seen by introducing an overtwisted disk in a neighborhood of a (properly chosen) point of  $M$ , and using Theorem 9.

To deal with the higher-order homotopy groups, one needs the existence of a continuous choice of overtwisted disks in order to run the same argument.

**Definition 10** (Casals et al. [5]). *Let  $0 \leq k \leq 2n$ . The overtwisted  $k$ -group of  $M$ , denoted  $\text{OT}_k(M)$ , is the subgroup of  $\pi_k(\text{Cont}_{\text{OT}}(M))$  made of those classes that admit a representative  $\xi: \mathbb{S}^k \rightarrow \text{Cont}_{\text{OT}}(M)$  for which there is a certificate of overtwistedness, i.e. a continuous map*

$$\Delta: \mathbb{S}^k \rightarrow \text{Emb}_{\text{PL}}(\mathbb{D}^{2n}, M) := \{\psi: \mathbb{D}^{2n} \hookrightarrow M \text{ piece-wise linear embedding}\}$$

*such that, for each  $p \in \mathbb{S}^k$ ,  $\Delta(p)$  is overtwisted for  $\xi(p)$ .*

Homotopy classes in  $OT_k(M)$  are called *overtwisted*. In these terms, (4) says that the map  $OT_0(M) \rightarrow \pi_0(\text{AlmCont}(M))$  is a bijection. For higher-order homotopy groups one then has the following:

**Proposition 11** (Casals et al. [5, Proposition 33]). *Let  $(M, \xi_{ot})$  be any closed overtwisted contact manifold. For each  $0 \leq k \leq 2n$ , the inclusion  $\text{Cont}_{OT}(M) \hookrightarrow \text{AlmCont}(M)$  induces an isomorphism*

$$OT_k(M) \xrightarrow{\sim} \pi_k(\text{AlmCont}(M)).$$

Moreover,  $OT_k(M) < \pi_k(\text{Cont}_{OT}(M), \xi_{ot}) = \pi_k(\text{Cont}(M), \xi_{ot})$  is a normal subgroup for  $k > 0$  and, thus, the set of tight classes  $\text{Tight}_k(M) = \pi_k(\text{Cont}(M), \xi_{ot}) / OT_k(M)$  has group structure. In particular, for any  $1 \leq k \leq 2n$  there is an isomorphism

$$\pi_k(\text{Cont}(M), \xi_{ot}) \cong OT_k(M) \oplus \text{Tight}_k(M).$$

To the authors' knowledge, the only known example of a non-trivial tight class is contained in Vogel [28], where the author exhibits an order 2 loop of overtwisted contact structures on  $\mathbb{S}^3$ , based at the only overtwisted structure on  $\mathbb{S}^3$  having Hopf invariant  $-1$  (w.r.t. the standard trivialization of  $T\mathbb{S}^3$  given by the quaternions), which does not admit a certificate of overtwistedness. It follows that this tight loop cannot come from a loop of diffeomorphisms in the long exact sequence in (1). In particular, its image via the boundary map is a non-trivial element (of order 2) in the contact mapping class group.

### 3. Proofs of the statements

We start by recalling some known facts in algebraic topology. Recall the following standard homotopy equivalence (see for instance Antonelli et al. [1, Lemma 1.1.5] for a proof):

$$\text{Diff}_0(\mathbb{S}^{2n+1}) \xleftarrow{\sim} \text{Diff}_0(\mathbb{D}^{2n+1}, \partial) \times \text{SO}(2n+2). \tag{5}$$

Here, the group  $\text{Diff}_0(\mathbb{D}^{2n+1}, \partial)$  of diffeomorphisms of the disk relative to its boundary which are smoothly isotopic to the identity is understood as the subgroup of  $\text{Diff}_0(\mathbb{S}^{2n+1})$  of diffeomorphisms which fixes *pointwise* (a neighborhood of) the north hemisphere, and the arrow is the natural inclusion map. Moreover, some of the rational homotopy groups of the first factor of the right-hand side of (5) are completely characterized (see also Weiss-Williams [29, Section 6]):

**Theorem 12** (Farrell-Hsiang [12]). *Let  $0 \leq k < \min\{\frac{2n-3}{3}, n-3\}$ . Then*

$$\pi_k(\text{Diff}_0(\mathbb{D}^{2n+1}, \partial)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } k \not\equiv 3 \pmod{4}, \\ \mathbb{Q} & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Let's now go back to contact topology and prove the statements announced in the introduction.

**Proof of Theorem 1.** Let  $\xi_{ot}$  be any overtwisted structure on  $\mathbb{S}^{2n+1}$ , and  $k \in \mathbb{N}$  such that  $1 \leq 4k+1 \leq 2n-1$ . In order to simplify the notation, in the rest of the proof the spaces  $\text{Cont}(\mathbb{S}^{2n+1})$ ,  $\Gamma_{n+1}$  and each diffeomorphism/contactomorphism group are intended as pointed spaces with base points, respectively,  $\xi_{ot}$ ,  $J_0$  and  $\text{Id}$ .

The relevant part of the long exact sequence in (1) is the following:

$$\pi_{4k+2}(\text{Diff}_0(\mathbb{S}^{2n+1})) \longrightarrow \pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1})) \longrightarrow \mathcal{K}_{4k+1}^{2n+1}$$

According to Propositions 8 and 11, there is an isomorphism

$$\pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1})) \cong \pi_{4k+2}(\Gamma_{n+1}) \oplus \pi_{2n+4k+3}(\Gamma_{n+1}) \oplus \text{Tight}_k(\mathbb{S}^{2n+1}).$$

Moreover, under this isomorphism, the projection on the first factor

$$\pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1})) \rightarrow \pi_{4k+2}(\Gamma_{n+1}).$$

is just the map induced by the evaluation at the north pole  $ev_N$ . As  $\text{Diff}_0(\mathbb{D}^{2n+1}, \partial) \stackrel{i}{\subset} \text{Diff}_0(\mathbb{S}^{2n+1})$  is the subgroup of diffeomorphisms fixing the north hemisphere, it follows that the following composition is trivial:

$$\begin{array}{ccc} \pi_{4k+2}(\text{Diff}_0(\mathbb{D}^{2n+1}, \partial)) & \xrightarrow{\pi_{4k+2}^{(i)}} & \pi_{4k+2}(\text{Diff}_0(\mathbb{S}^{2n+1})) \\ \nearrow & & \searrow \\ \pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1})) & \xrightarrow{\pi_{4k+2}(ev_N)} & \pi_{4k+2}(\Gamma_{n+1}) \end{array}$$

Moreover, according to Bott periodicity,  $\pi_{4k+2}(\text{SO}(2n+2)) = 0$ . In particular, the following composition is also trivial:

$$\pi_{4k+2}(\text{Diff}_0(\mathbb{S}^{2n+1})) \longrightarrow \pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1})) \xrightarrow{\pi_{4k+2}(ev_N)} \pi_{4k+2}(\Gamma_{n+1})$$

Now, according to Remark 6,  $\pi_{4k+2}(\Gamma_{n+1})$ , hence  $\pi_{4k+2}(\text{Cont}(\mathbb{S}^{2n+1}))$ , contains a subgroup  $\mathbb{Z}$ . It then follows from the exact sequence that  $\mathcal{K}_{4k+1}^{2n+1}$  must have at least one element of infinite order, as desired.  $\square$

**Proof of Theorem 2.** According to Hatcher [21], the Smale Conjecture holds for  $\mathbb{S}^3$ ; in particular,  $\pi_2(\text{Diff}_0(\mathbb{S}^3)) = 0$ . Moreover, since  $\Gamma_2 = \text{SO}(4)/\text{U}(2) = \mathbb{S}^2$  it follows from Propositions 8 and 11 that the group

$$\text{OT}_2(\mathbb{S}^3) \cong \pi_2(\mathbb{S}^2) \oplus \pi_5(\mathbb{S}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

is a subgroup of  $\pi_2(\text{Cont}(\mathbb{S}^3, \xi_{or}))$ . (i) then follows from the exact sequence in (1).

Since  $\pi_2(\text{SO}(4n+1)) = \pi_6(\text{SO}(4n+1)) = 0$ , Theorem 12 implies that  $\pi_2(\text{Diff}_0(\mathbb{S}^{4n+1})) \otimes \mathbb{Q} = 0$  for  $n \geq 3$ , and  $\pi_6(\text{Diff}_0(\mathbb{S}^{4n+1})) \otimes \mathbb{Q} = 0$  for  $n \geq 6$ . Moreover, according to Remark 6, each of the following homotopy groups contain a cyclic subgroup:  $\pi_2(\Gamma_{2n+1})$  for  $n \geq 1$ ,  $\pi_6(\Gamma_{2n+1})$  for  $n \geq 2$ ,  $\pi_{4n+3}(\Gamma_{2n+1})$  and  $\pi_{4n+7}(\Gamma_{2n+1})$ . (ii) and (iii) then follow from the exact sequence in (1) and from Propositions 8 and 11.  $\square$

**Proof of Theorem 3.** Since  $\pi_4(\text{SO}(4n+4))$  is trivial, it follows from the identification in (5) and from Theorem 12 that  $\pi_4(\text{Diff}_0(\mathbb{S}^{4n+3})) \otimes \mathbb{Q} = 0$  for  $n \geq 4$ . Moreover, according to Remark 6,  $\pi_{4n+7}(\Gamma_{2n+2})$  contains a subgroup  $\mathbb{Z}$ .

Similarly,  $\pi_5(\text{SO}(8n+8)) = 0$  thus (5) and Theorem 12 imply that  $\pi_5(\text{Diff}_0(\mathbb{S}^{8n+7})) \otimes \mathbb{Q} = 0$  for  $n \geq 2$ . According to Remark 6,  $\pi_{8n+12}(\Gamma_{4n+4}) \cong \mathbb{Z}$ .

The statement then follow from the exact sequence in (1) and from Propositions 8 and 11.  $\square$

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