

STABILITY OF LIPSCHITZ-TYPE FUNCTIONS UNDER POINTWISE PRODUCT AND RECIPROCATION

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ABSTRACT. This article provides necessary and sufficient conditions on the structure of a metric space such that for various vector lattices of real-valued Lipschitz-type functions defined on the metric space, the vector lattice is stable under pointwise product, and such that the reciprocal of each non-vanishing member of the vector lattice remains in the vector lattice. In each case the family of metric spaces for which the first property holds contains the family of metric spaces for which the second property holds. At the end we prove some extension theorems for classes of locally Lipschitz functions that complement known results for Cauchy continuous functions and for uniformly continuous functions.

1. INTRODUCTION

Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces. A function $f : X \rightarrow Y$ is called *Lipschitz* if for some $\lambda > 0$ and each $x_1, x_2 \in X$ we have $\rho(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$. For added precision, we can say that f is λ -Lipschitz. The real-valued Lipschitz functions $\text{Lip}(X)$ on X form a vector lattice that contains the constant functions but which is not in general stable under pointwise products. However, if f and g are bounded Lipschitz functions when restricted to a subset A of X , then their product so restricted is Lipschitz. In particular, this is true if A is a bounded subset of the metric space. Furthermore, given a non-vanishing member f of $\text{Lip}(X)$, it may be true that $\frac{1}{f}$ is no longer Lipschitz. For example $f(x) = \frac{1}{1+x^2}$ having bounded derivative is a Lipschitz function on \mathbb{R} , whereas its reciprocal is not Lipschitz. We will say that a class Ω of real-valued functions on $\langle X, d \rangle$ is *stable under reciprocation* if whenever $f \in \Omega$ is never zero, then $\frac{1}{f} \in \Omega$ ensues.

Of course, $f : X \rightarrow Y$ is called *locally Lipschitz* if it is Lipschitz when restricted to some neighborhood of each point of the space. We denote the real-valued locally Lipschitz functions on X by $LL(X)$. It is well known that a function is locally Lipschitz if and only if it is Lipschitz when restricted to each compact subset of X (see, e.g., [7, 14, 30]). Since compact subsets are bounded, such functions are stable under pointwise product. Furthermore, if f is Lipschitz on each compact subset and

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non-vanishing, then $|f|$ when restricted to each compact subset attains a positive minimum value, whence the reciprocal of f is Lipschitz on each compact subset. Thus the locally Lipschitz functions in general are stable under both pointwise product and reciprocation.

In this article we begin by showing that $\text{Lip}(X)$ is stable under pointwise product if and only if X is bounded, and stable under reciprocation if and only if X is compact. The first fact is a standard exercise, and we view the second as a folk-theorem for which we have no reference. We also look at the real-valued Cauchy-Lipschitz functions and uniformly locally Lipschitz functions as studied by Beer and Garrido [6, 7] and the real-valued Lipschitz in the small functions as introduced by Luukkainen [25]. The analysis leads us to consider classes of metric spaces that may not be familiar to all readers: the cofinally complete metric spaces, the UC or Atsuji metric spaces, and a class of metric spaces introduced recently by J. Cabello-Sánchez [13] characterized by the stability of the uniformly continuous real-valued functions on X under pointwise product.

For real-valued functions, each class of Lipschitz-type functions forms a vector lattice containing the constant functions (where the pointwise order is understood). In this setting, the metric spaces for which the class is stable under pointwise product must include the spaces for which the class is stable under reciprocation. We confirm this assertion, in which metric structure plays no role, presently. A vector lattice of real-valued functions containing the constants that is stable under pointwise product forms a ring with multiplicative identity.

To show that a vector space V of real-valued functions is stable under pointwise product, it suffices to show that $f^2 \in V$ whenever $f \in V$, in view of the identity

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2).$$

Theorem 1.1. *Let V be a vector lattice of real-valued functions on a set X containing the constant functions. If V is stable under reciprocation, then V is stable under pointwise product.*

Proof. We first remind the reader that a vector lattice of functions is stable under taking the absolute value of its members because $|f| = \max\{f, 0\} - \min\{f, 0\}$.

We first claim that if $g \in V$ and there exists $\alpha > 0$ such that for all $x \in X$, $|g(x)| \neq \alpha$, then $g^2 \in V$. Indeed, note that

$$\frac{1}{g^2 - \alpha^2} = \frac{1}{2\alpha(g - \alpha)} - \frac{1}{2\alpha(g + \alpha)}$$

from which $g^2 - \alpha^2$ and then g^2 belong to V .

Now let $f \in V$ be arbitrary and put $g := |f| + 2$ which fulfills the above condition with $\alpha = 1$. Then $g^2 \in V$ and $f^2 = g^2 - 4|f| - 4$. This means that $f^2 \in V$ as well. \square

Corollary 1.2. *Let V be a vector lattice of real-valued functions on a set X containing the constant functions. If V is stable under reciprocation, then whenever $f \in V$ is non-vanishing and $g \in V$, we have $\frac{g}{f} \in V$.*

In the last section of this article, we prove some extension theorems for Cauchy-Lipschitz functions and for Lipschitz in the small functions anticipated by established extension results for Cauchy continuous functions and for uniformly continuous functions. In the process, we give necessary and sufficient conditions for a

Lipschitz in the small function between metric spaces to be Lipschitz in terms of its modulus of continuity function.

2. PRELIMINARIES

All metric spaces are assumed to contain at least two points. We denote the set of limit points of the space $\langle X, d \rangle$ by X' . We write $S_d(x, \varepsilon)$ for the open ball of center x and radius $\varepsilon > 0$ in X . We write $S_d(A, \varepsilon)$ for $\cup_{a \in A} S_d(a, \varepsilon)$, and call this set the ε -enlargement of A [2]. For $x \in X$ and A a nonempty subset of $\langle X, d \rangle$, we put $d(x, A) := \inf\{d(x, a) : a \in A\}$. We define the *isolation functional* I_d for $\langle X, d \rangle$ by the formula $I_d(x) := d(x, X \setminus \{x\})$. Of course $I_d(x) = 0$ means that $x \in X'$. All distance functionals and isolation functionals are 1-Lipschitz.

We denote the usual d -diameter of a nonempty subset A of X by $\text{diam}_d(A)$. A subset A of X is called *bounded* if it is either empty or has finite diameter; equivalently, A is a subset of some open ball in X . A subset A of X is called *totally bounded* if for each $\varepsilon > 0$ there exists a finite subset F of X with $A \subseteq S_d(F, \varepsilon)$. It is called *relatively compact* if it has compact closure. A metric space is called *boundedly compact* if each bounded subset is relatively compact. Each relatively compact subset of X is totally bounded, and the converse is true if and only if the metric d is complete. If A is a nonempty subset of X and $\varepsilon > 0$, we define $S_d^n(A, \varepsilon)$ recursively by $S_d^1(A, \varepsilon) = S_d(A, \varepsilon)$ and $S_d^{n+1}(A, \varepsilon) = S_d(S_d^n(A, \varepsilon), \varepsilon)$. A subset A of X is called *Bourbaki bounded* [5, 18] if for every $\varepsilon > 0$ there exists a nonempty finite subset F of X and $n \in \mathbb{N}$ with $A \subseteq S_d^n(F, \varepsilon)$. Each totally bounded subset is Bourbaki bounded and each Bourbaki bounded subset is metrically bounded. Each relatively compact set is Bourbaki bounded as each relatively compact set is totally bounded. A nonempty subset A of a metric space $\langle X, d \rangle$ is Bourbaki bounded if and only if each uniformly continuous real-valued function on X is bounded when restricted to A [5, 19].

As a subset of a metric space is (a) relatively compact if and only if each sequence in it has a convergent subsequence, and (b) totally bounded if and only if each sequence in it has a Cauchy subsequence, a subset is Bourbaki bounded if and only if each sequence in it has a Bourbaki-Cauchy subsequence [18, Theorem 4], as we now define, in a way paralleling the standard definition of Cauchy sequence.

Definition 2.1. A sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$ is called *Bourbaki-Cauchy* provided for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and $m \in \mathbb{N}$ such that whenever n and j exceed k , we have $x_n \in S_d^m(\{x_j\}, \varepsilon)$.

A metric space is called *Bourbaki complete* provided each Bourbaki-Cauchy sequence in it clusters [18]. Obviously, Bourbaki completeness implies completeness of the metric as it usually understood. A metric space is Bourbaki complete if and only if each Bourbaki bounded subset is relatively compact.

The metrically bounded subsets, the relatively compact subsets, the totally bounded subsets, and the Bourbaki bounded subsets each form a *bornology* on the underlying metric space: (1) each contains the singletons; (2) each is a hereditary family of subsets; (3) each is stable under finite unions.

We call a function from $\langle X, d \rangle$ to $\langle Y, \rho \rangle$ *Cauchy continuous* [7, 11, 31] if it maps Cauchy sequences in X to Cauchy sequences in Y . Each Cauchy continuous function is continuous, and each uniformly continuous function is Cauchy continuous. It is notable that each Cauchy continuous real-valued function defined on a nonempty

subset of $\langle X, d \rangle$ can be extended to a globally defined Cauchy continuous function [31]. For this reason alone, the class of Cauchy continuous functions deserves some coverage in analysis texts.

We now formally introduce some subclasses of the locally Lipschitz functions in order of decreasing size.

- $f : X \rightarrow Y$ is called *Cauchy-Lipschitz* if its restriction to the range of each Cauchy sequence in X is Lipschitz;
- $f : X \rightarrow Y$ is called *uniformly locally Lipschitz* if for some $\varepsilon > 0$ the restriction of f to each open ball of radius ε is Lipschitz;
- $f : X \rightarrow Y$ is called *Lipschitz in the small* if there exists $\lambda > 0$ and $\delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow \rho(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$.

In the case of real-valued functions, each of these classes forms a vector lattice containing the constants. In this setting, we denote these vector lattices by $CL(X)$, $ULL(X)$ and $LS(X)$, respectively. Pairwise coincidence of our four classes of locally Lipschitz functions has been determined [6, 7]. Each locally Lipschitz function is continuous; moreover, $LL(X)$ is uniformly dense in the continuous real-valued functions [7, 14, 16, 26]; each Cauchy-Lipschitz function is Cauchy continuous and $CL(X)$ is uniformly dense in the Cauchy continuous real-valued functions [7]; each Lipschitz in the small function is uniformly continuous and $LS(X)$ is uniformly dense in the uniformly continuous real-valued functions [6, 17]. A function is Cauchy-Lipschitz if and only if its restriction to each totally bounded subset is Lipschitz [7, Proposition 3.4]. Those subsets of a metric space $\langle X, d \rangle$ on which each Lipschitz in the small function is Lipschitz when restricted to them - called the *small determined subsets* - have been internally characterized by Leung and Tang [24], who also noted that this feature of the subset is determined by real-valued Lipschitz in the small functions. Unfortunately a function can be Lipschitz on each such subset without being actually Lipschitz in the small [8]. We note that a bounded subset of X is small determined if and only if it is Bourbaki bounded [6, Theorem 4.4]. Metric spaces that are small determined were first considered by Garrido and Jaramillo [17].

We close this section with a lemma that we employ going forward.

Lemma 2.2. *Let A be a nonempty closed subset of $\langle X, d \rangle$ and let $f : A \rightarrow (0, \infty)$ be λ -Lipschitz for some positive λ . Then f has a strictly positive λ -Lipschitz extension to X .*

Proof. As is well known [20, pp. 43-44], $x \mapsto \inf_{a \in A} f(a) + \lambda d(x, a)$ is a λ -Lipschitz extension of f to the entire space, and it is easily verified that this extension majorizes all others. For $x \notin A$, we have for each $a \in A$

$$0 < \lambda d(x, A) \leq f(a) + \lambda d(x, a)$$

because A is closed. As a result, the extension has strictly positive values. \square

3. LIPSCHITZ FUNCTIONS

Necessary and sufficient conditions for stability under pointwise product for real-valued Lipschitz functions are transparent. We obtain these as a special case of a more generally applicable result.

Proposition 3.1. *Let \mathcal{A} be a family of nonempty subsets of a metric space $\langle X, d \rangle$ and let \mathfrak{F} be the family of real-valued functions that are Lipschitz when restricted to each member of \mathcal{A} . Then \mathfrak{F} is stable under pointwise product if and only if each member of \mathcal{A} is bounded.*

Proof. Sufficiency is obvious. For necessity, suppose some $A \in \mathcal{A}$ is unbounded. Fixing $a_0 \in A$, while $f(x) = d(x, a_0)$ ($x \in X$) is 1-Lipschitz when restricted to each member of \mathcal{A} , f^2 fails to be Lipschitz when restricted to A . \square

Letting $\mathcal{A} = \{X\}$ in Proposition 3.1, we immediately obtain

Theorem 3.2. *Let $\langle X, d \rangle$ be a metric space. Then $\text{Lip}(X)$ is stable under pointwise product if and only if d is a bounded metric.*

Turning to reciprocation, we will use the following lemma.

Lemma 3.3. *Let $\langle X, d \rangle$ be a bounded metric space. Assume that $\frac{1}{f} \in \text{Lip}(X)$ whenever f is a non-vanishing Lipschitz function on X . Then every Lipschitz function on X attains its infimum.*

Proof. Let $f \in \text{Lip}(X)$; since X is bounded, $\alpha := \inf_{x \in X} f(x)$ is finite. Suppose f does not attain its infimum. By replacing f by $f - \alpha$, we may assume $\alpha = 0$. Then $\frac{1}{f}$ would be an unbounded Lipschitz function defined on a bounded metric space which is impossible. \square

Theorem 3.4. *Let $\langle X, d \rangle$ be a metric space. Then $\text{Lip}(X)$ is stable under reciprocation if and only if X is compact.*

Proof. Sufficiency follows easily from the fact that for each non-vanishing $f \in \text{Lip}(X)$, $|f|$ has an absolute minimum value. We turn to necessity. If $\text{Lip}(X)$ is stable under reciprocation, by Theorem 1.1 and Theorem 3.2, X must be bounded. It suffices to show for compactness that X is countably compact [32, p. 125]. To this end, let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of X and suppose that it does not admit a finite subcover. We may assume that $U_n \subseteq U_{n+1}$ for each n . Consider the 1-Lipschitz function

$$f := \sum_{n=1}^{\infty} \frac{1}{2^n} d(\cdot, X \setminus U_n).$$

Putting $D := \text{diam}_d(X)$, we see that for all $x \in X$, $0 < f(x) \leq D$ because $\{U_n : n \in \mathbb{N}\}$ covers X . Moreover, as there is no finite subcover, for each $n \in \mathbb{N}$ there is $x_n \in X \setminus U_n$. We compute

$$f(x_n) = \sum_{i=n+1}^{\infty} \frac{1}{2^i} d(x_n, X \setminus U_i) \leq \frac{D}{2^n}.$$

Thus, the infimum of f is zero, and it is not attained, in violation of Lemma 3.3. \square

Proposition 3.5. *Let \mathcal{A} be a family of nonempty subsets of a metric space $\langle X, d \rangle$ and let \mathfrak{F} be the family of continuous real-valued functions that are Lipschitz when restricted to each member of \mathcal{A} .*

- (a) \mathfrak{F} is stable under reciprocation if and only if every member of \mathcal{A} is relatively compact.

(b) *In the case that \mathcal{A} contains each nonempty relatively compact subset, then \mathfrak{F} is stable under reciprocation if and only if $\mathfrak{F} = LL(X)$.*

Proof. We first look at statement (a). For sufficiency, let $f \in \mathfrak{F}$ be nonvanishing and let $A \in \mathcal{A}$. By assumption there exists $\lambda > 0$ such that $f|_A$ is λ -Lipschitz. By continuity, $f|_{\text{cl}(A)}$ is also λ -Lipschitz, so by Theorem 3.4, $\frac{1}{f}$ is Lipschitz on $\text{cl}(A)$ and thus on A . For necessity, suppose $A \in \mathcal{A}$ is not relatively compact. The proof of Theorem 3.4 shows that there is a positive Lipschitz function f on $\text{cl}(A)$ whose reciprocal is not Lipschitz on $\text{cl}(A)$ and therefore is not Lipschitz on A by continuity. By Lemma 2.2, f has a positive Lipschitz-constant preserving extension g to X , and of course, $g \in \mathfrak{F}$. But $\frac{1}{g}$ cannot be in \mathfrak{F} as its restriction to A agrees with $\frac{1}{f}$.

For statement (b), if $\mathfrak{F} = LL(X)$, then clearly \mathfrak{F} is stable under reciprocation. Conversely, if each nonempty relatively compact subset belongs to \mathcal{A} and \mathfrak{F} is stable under reciprocation, then by statement (a) \mathcal{A} coincides with the nonempty relatively compact subsets, and as is well-known [30], this gives $\mathfrak{F} = LL(X)$. \square

In statement (b), the condition that \mathcal{A} contains the relatively compact sets is equivalent to requiring that each member of \mathfrak{F} is locally Lipschitz. We next give four applications of Proposition 3.1 and Proposition 3.5.

Theorem 3.6. *The family of Cauchy-Lipschitz functions on a metric space $\langle X, d \rangle$ is always stable under pointwise product, and is stable under reciprocation if and only if d is a complete metric.*

Proof. Recall that $f : X \rightarrow \mathbb{R}$ is Cauchy-Lipschitz if and only if it is Lipschitz when restricted to each nonempty totally bounded subset [7, Proposition 3.4]. Since totally bounded sets are metrically bounded, by Proposition 3.1, the pointwise product of two Cauchy-Lipschitz functions is again Cauchy-Lipschitz with no restrictions on the metric space.

We turn to reciprocation. Since each relatively compact subset is totally bounded, by Proposition 3.5 (b), stability under reciprocation occurs provided the Cauchy-Lipschitz real-valued functions coincide with the locally Lipschitz real-valued functions. This happens exactly when d is a complete metric [7, Theorem 3.5]. \square

Theorem 3.7. *Let \mathfrak{F} be the family of real-valued functions on a metric space $\langle X, d \rangle$ that are Lipschitz when restricted to each Bourbaki bounded subset of X . Then \mathfrak{F} is always stable under pointwise product, and is stable under reciprocation if and only if $\langle X, d \rangle$ is Bourbaki complete.*

Proof. Members of \mathfrak{F} are Lipschitz restricted to each nonempty relatively compact subset and thus are continuous on X . Stability under pointwise product with no restriction again follows from Proposition 3.1. From [18, Theorem 9], each Bourbaki bounded subset is relatively compact if and only if the space is Bourbaki complete. Proposition 3.5(a) yields the stated criterion for stability under reciprocation. \square

In [8], it is shown that a function is Lipschitz when restricted to each Bourbaki bounded subset of $\langle X, d \rangle$ if and only if it is Lipschitz when restricted to the range of each Bourbaki-Cauchy sequence. It is also shown in [8] that the functions that are Lipschitz when restricted to each nonempty Bourbaki bounded subset coincide with $LL(X)$ if and only if $\langle X, d \rangle$ is Bourbaki complete, allowing us to alternatively use Proposition 3.5(b) in the proof of our last result.

Theorem 3.8. *Let \mathfrak{F} be the family of real-valued functions on a metric space $\langle X, d \rangle$ that are Lipschitz when restricted to each small determined subset of X .*

- (a) *\mathfrak{F} is stable under pointwise product if and only if each small determined subset is Bourbaki bounded;*
- (b) *\mathfrak{F} is stable under reciprocation if and only if each small determined subset is Bourbaki bounded and $\langle X, d \rangle$ is Bourbaki complete.*

Proof. The first statement follows from Proposition 3.1 and [6, Theorem 4.4]. The second statement follows from the first statement, Theorem 1.1, Proposition 3.5(a) and [18, Theorem 9]. \square

We leave the simple proof of this final consequence to the reader.

Theorem 3.9. *Let \mathfrak{F} be the family of real-valued functions on a metric space that are Lipschitz when restricted to each bounded subset of $\langle X, d \rangle$. Then \mathfrak{F} is always stable under pointwise product, and is stable under reciprocation if and only if $\langle X, d \rangle$ is boundedly compact.*

4. UNIFORMLY LOCALLY LIPSCHITZ FUNCTIONS

We cannot use Proposition 3.1 and Proposition 3.5 to determine stability under pointwise product and reciprocation for $ULL(X)$ because in an arbitrary metric space, $ULL(X)$ cannot be described as the family of real-valued functions that are Lipschitz when restricted to a prescribed family of nonempty subsets \mathcal{A} of the metric space.

Example 4.1. Consider the metric space $\langle X, d \rangle$ where $X = \mathbb{N}^2$ and d is given by $d((n, k), (m, l)) = 1$ if $k \neq l$ and $d((n, k), (m, k)) = \frac{1}{k}$ if $n \neq m$. Assume \mathcal{A} were a family of subsets of \mathbb{N}^2 such that $ULL(X)$ coincides with the set of real-valued functions that are Lipschitz when restricted to each member of \mathcal{A} . We intend to show that \mathcal{A} is made up of finite sets. Let $A \in \mathcal{A}$, and consider

$$I := \{k \in \mathbb{N} : (n, k) \in A \text{ for some } n \in \mathbb{N}\}.$$

Define $f : X \rightarrow \mathbb{R}$ by $f(n, k) = k$. As f is constant on each ball of radius $\frac{1}{2}$, we have $f \in ULL(X)$. Now if I were infinite, there would be a strictly increasing sequence $\langle k_i \rangle$ in I and for each i a positive integer n_i such that $(n_i, k_i) \in A$. We have for $i > 1$

$$f(n_i, k_i) - f(n_1, k_1) = k_i - k_1 = (k_i - k_1)d((n_i, k_i), (n_1, k_1)),$$

which shows that $f|_A$ is not Lipschitz, a contradiction. We conclude that I is finite.

Now fix k in our finite set I . Define the function $g_k : X \rightarrow \mathbb{R}$ by $g_k(n, k) = n$ and $g_k(n, l) = 0$ if $l \neq k$. Then g_k is constant on each ball of radius $\frac{1}{2k}$ and so $g_k \in ULL(X)$. This implies that $g_k|_A$ is Lipschitz, and it follows that the set $A_k := \{n \in \mathbb{N} : (n, k) \in A\}$ is finite because its points are equidistant. This shows that $A = \cup_{k \in I} A_k$ is also finite.

Finally, $h : X \rightarrow \mathbb{R}$ defined by $h(n, k) = n$ evidently is not uniformly locally Lipschitz. However h is Lipschitz when restricted to each finite subset as is any real-valued function defined on X . That is a contradiction.

We note that the functions f and g_k defined above are actually Lipschitz in the small and so our construction also serves to show that $LS(X)$ cannot in general

be described as a class of functions that are Lipschitz when restricted to certain subsets, complementing the very different counterexample provided in [8].

Theorem 4.2. *Let $\langle X, d \rangle$ be a metric space. Then $ULL(X)$ is always stable under pointwise product.*

Proof. Let f and g be uniformly locally Lipschitz real-valued functions on X . Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that f restricted to each open ball of radius δ_1 is Lipschitz and g restricted to each open ball of radius δ_2 is Lipschitz. Put $\delta = \min\{\delta_1, \delta_2\}$; then f and g are both Lipschitz and bounded on each open ball of radius δ so that fg is Lipschitz on each such ball. \square

We now turn to stability of $ULL(X)$ under reciprocation. A sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$ is called *cofinally Cauchy* if for each $\varepsilon > 0$ there exists an infinite subset \mathbb{N}_ε of \mathbb{N} such that whenever $\{j, k\} \subseteq \mathbb{N}_\varepsilon$, we have $d(x_j, x_k) < \varepsilon$. The metric space is called *cofinally complete* provided each cofinally Cauchy sequence in X clusters [3]. Net cofinal completeness in the context of uniform spaces is defined in the expected way [22], and the two notions for a metric space coincide.

Some important characterizations of cofinally completeness are the following:

- the space is *uniformly paracompact* [21, 28]: for each open cover \mathcal{V} of X , there exists $\delta > 0$ and an open refinement \mathcal{U} of \mathcal{V} such that for each $x \in X$, $S_d(x, \delta)$ hits at most finitely many members of \mathcal{U} ;
- $K := \{x \in X : x \text{ has no compact neighborhood}\}$ is compact, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x \notin S_d(K, \varepsilon) \Rightarrow \{w \in X : d(w, x) \leq \delta\}$ is compact [3];
- whenever $f : X \rightarrow \mathbb{R}$ is continuous, there exists $\delta > 0$ such that f restricted to each open ball of radius δ is bounded [3].

We note that the metric space of the initial example of this section was presented by Rice [28] to show that a locally compact metric space need not be cofinally complete.

Theorem 4.3. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:*

- (1) *the metric space is cofinally complete;*
- (2) *$ULL(X)$ is stable under reciprocation;*
- (3) *whenever $g \in Lip(X)$ is non-vanishing, we have $\frac{1}{g} \in ULL(X)$.*

Proof. (1) \Rightarrow (2). With cofinal completeness, $ULL(X) = LL(X)$ [6, Theorem 3.1], and the latter vector lattice is stable under reciprocation in general.

(2) \Rightarrow (3). Since $Lip(X) \subseteq ULL(X)$, this is trivial.

(3) \Rightarrow (1). Suppose condition (1) fails. Then we can find a cofinally Cauchy sequence $\langle x_n \rangle$ in X that does not cluster. In particular, the sequence has no constant subsequence. By passing to a subsequence, we may assume that $\langle x_n \rangle$ has distinct terms yet remains cofinally Cauchy. Without loss of generality, we may write \mathbb{N} as a union of infinite subsets $\{\mathbb{M}_j : j \in \mathbb{N}\}$ such that whenever $\{i, k\} \subseteq \mathbb{M}_j$, we have $d(x_i, x_k) < \frac{1}{j}$. For each positive integer n , put $\delta_n := d(x_n, \{x_k : k \neq n\}) \in (0, 1)$.

Define $f : \{x_n : n \in \mathbb{N}\} \rightarrow (0, \infty)$ by $f(x_n) = \delta_n/n$. Whenever $n \neq k$, we get

$$|f(x_n) - f(x_k)| \leq \frac{\delta_n}{n} + \frac{\delta_k}{k} \leq \delta_n + \delta_k \leq 2d(x_n, x_k).$$

Thus, f is 2-Lipschitz. Since $\{x_n : n \in \mathbb{N}\}$ is closed, by Lemma 2.2, f has a 2-Lipschitz positive extension g to X . However, since each \mathbb{M}_j is infinite, $\frac{1}{g}$ is unbounded on $\{x_n : n \in \mathbb{M}_j\}$ for each $j \in \mathbb{N}$. Since $\text{diam}_d(\{x_n : n \in \mathbb{M}_j\}) \leq \frac{1}{j}$, the function $\frac{1}{g}$ while locally Lipschitz fails to be uniformly locally Lipschitz. \square

While we can't employ Proposition 3.5(b) here, we note that $ULL(X)$ is nevertheless stable under reciprocation precisely when $ULL(X) = LL(X)$ [6, Theorem 3.1]. With cofinal completeness of $\langle X, d \rangle$, we of course then have the uniform density of $ULL(X)$ in the real-valued continuous functions. Conversely, without cofinal completeness, we can find a continuous real-valued function f on X for which the third bullet point above fails [3, Theorem 3.4], and such a function cannot be uniformly approximated by uniformly locally Lipschitz functions.

5. LIPSCHITZ IN THE SMALL FUNCTIONS

In this case, necessary and sufficient internal conditions on the metric space $\langle X, d \rangle$ for stability under pointwise product are more subtle than they are for stability under reciprocation. In both cases, they agree with the answers recently obtained for the class of uniformly continuous real-valued functions.

J. Cabello-Sánchez made the initial advance [13]; he showed that the uniformly continuous real-valued functions are stable under pointwise product if and only if each nonempty subset A of $\langle X, d \rangle$ is either Bourbaki bounded or contains an infinite uniformly isolated subset A_0 , that is, $\inf \{I_d(x) : x \in A_0\} > 0$. A year after that, Beer, Garrido and Meroño [9] showed that the Cabello-Sánchez conditions amounted to agreement of the Bourbaki bounded subsets with a different generally larger bornology. A year later, Bouziad and Sukhacheva [12] supplied another interesting internal condition: there exists a Bourbaki bounded subset B such that for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $X \setminus S_d^n(B, \varepsilon)$ is uniformly isolated. They also came up with two attractive function space characterizations (see also [4]).

As to stability of uniform continuity under reciprocation, this was also settled in [9]: the continuous real-valued functions on X must agree with the uniformly continuous real-valued functions on X . The well-studied class of spaces for which this agreement holds are now called either the *UC-spaces* or the *Atsugi spaces* (see, e.g. [1, 2, 23, 27]). This class of metric spaces includes both the compact metric spaces and the uniformly isolated metric spaces.

Some attractive internal characterizations of UC-spaces are now given. Recall that $\alpha > 0$ is called a *Lebesgue number* for an open cover \mathcal{V} of a metric space $\langle X, d \rangle$ if whenever $A \subseteq X$ satisfies $\text{diam}_d(A) < \alpha$, there exists $V \in \mathcal{V}$ with $A \subseteq V$.

- each open cover of X has a Lebesgue number;
- whenever A and B are nonempty disjoint closed subsets of X , then $\inf\{d(a, b) : a \in A, b \in B\} > 0$;
- whenever $\langle x_n \rangle$ is a sequence in X for which $\lim_{n \rightarrow \infty} I_d(x_n) = 0$, then $\langle x_n \rangle$ clusters;

- X' is compact and $\forall \varepsilon > 0, \exists \delta > 0$ such that $x \notin S_d(X', \varepsilon) \Rightarrow I_d(x) > \delta$.

It is *not true* in a Cabello-Sánchez space that we can always find a Bourbaki bounded subset B such that $\forall \varepsilon > 0, \exists \delta > 0$ such that $x \notin S_d(B, \varepsilon) \Rightarrow I_d(x) > \delta$ [9, 12]. An example of a Cabello-Sánchez space that is neither Bourbaki bounded nor a UC-space is $(0, 1) \cup \mathbb{N}$ equipped with the usual metric of the line. Each UC-space is cofinally complete [3]; \mathbb{R} equipped with the usual metric is cofinally complete but is not a UC-space.

Our characterization of metric spaces for which $LS(X)$ is stable under pointwise product deliberately tries to parallel the statement of [4, Theorem 4].

Theorem 5.1. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:*

- (1) *each subset of X is either Bourbaki bounded or contains an infinite uniformly isolated subset;*
- (2) *$\forall f \in LS(X), \exists k \in \mathbb{N}$ such that $\{x \in X : |f(x)| > k\}$ is uniformly isolated;*
- (3) *$\forall f \in LS(X), \forall g \in LL(\mathbb{R})$ we have $g \circ f \in LS(X)$;*
- (4) *$LS(X)$ is stable under pointwise product;*
- (5) *whenever $f \in LS(X)$ and $h \in Lip(X)$ we have $fh \in LS(X)$.*

Proof. (1) \Rightarrow (2). If (1) holds, then (2) holds for the larger class of uniformly continuous real-valued functions [4, 12].

(2) \Rightarrow (3). Suppose $f \in LS(X)$ and $g \in LL(\mathbb{R})$ are arbitrary. Pick k as guaranteed by condition (2) and then $\delta_1 > 0$ such that whenever $|f(x)| > k$, we have $S_d(x, \delta_1) = \{x\}$. By the compactness of $[-k, k]$, g restricted to $[-k, k]$ is Lipschitz and so $g \circ f$ is Lipschitz in the small on $f^{-1}([-k, k])$. This means that there exists $\lambda > 0$ and $\delta \in (0, \delta_1)$ such that whenever $d(x_1, x_2) < \delta$ and $|f(x_i)| \leq k$ for $i = 1, 2$ we have

$$|(g \circ f)(x_1) - (g \circ f)(x_2)| \leq \lambda d(x_1, x_2).$$

Since $\delta < \delta_1$, the same estimate holds whenever $\{x_1, x_2\} \subseteq X$.

(3) \Rightarrow (4). Simply use $g(t) = t^2$ from which $LS(X)$ is stable under squaring.

(4) \Rightarrow (5). This trivial.

(5) \Rightarrow (1). We modify a construction used in the proof of Theorem 4 of [4]. Suppose the Cabello-Sánchez criterion (1) fails. We can find a nonempty subset A that is not Bourbaki bounded and each infinite subset E of A satisfies $\inf_{e \in E} I_d(e) = 0$. From [5, Theorem 3.3], we can find $f \in LS(X)$ that is unbounded on A , and by replacing f by $|f|$ we may assume that f has nonnegative values only. By the uniform continuity of f , pick $\delta > 0$ such $d(x, w) < \delta \Rightarrow |f(x) - f(w)| < \frac{1}{3}$. Since $\{a \in A : f(a) > 1\}$ is infinite and not uniformly isolated, we can find $a_1 \in A$ and $x_1 \in X$ such that $f(a_1) > 1$ and $0 < d(a_1, x_1) < \delta$. Similarly, $\{a \in A : f(a) > f(a_1) + 1\}$ is infinite and not uniformly isolated, and we can find $a_2 \in A$ and $x_2 \in X$ with $f(a_2) > f(a_1) + 1$ and $0 < d(a_2, x_2) < \frac{\delta}{2}$. Continuing, for each $k \in \mathbb{N}$ choose $a_k \in A$ and $x_k \in X$ with $f(a_{k+1}) > f(a_k) + 1$ and $0 < d(a_k, x_k) < \frac{\delta}{k}$. Obviously $f(a_k) > k$ for each integer k , and by the choice of δ , whenever $k \neq n$ we have

$$\min\{d(a_k, a_n), d(x_k, x_n), d(a_k, x_n)\} \geq \delta.$$

By construction, the sequence $a_1, x_1, a_2, x_2, a_3, x_3 \dots$ has distinct terms. Write E for the range of the sequence and define $h : E \rightarrow \mathbb{R}$ by $h(a_n) = d(a_n, x_n)$ and $h(x_n) = 0$. We intend to show that h is 1-Lipschitz on E . First, for each $n \in \mathbb{N}$,

$$\frac{|h(a_n) - h(x_n)|}{d(a_n, x_n)} = \frac{d(a_n, x_n)}{d(a_n, x_n)} = 1.$$

and if $n \neq k$ we have $\frac{|h(x_n) - h(x_k)|}{d(x_n, x_k)} = 0 < 1$. Also, if $n \neq k$ we have

$$\frac{|h(a_n) - h(x_k)|}{d(a_n, x_k)} = \frac{d(a_n, x_n)}{d(a_n, x_k)} < \frac{\delta/n}{\delta} \leq 1,$$

Finally, if $k < n$, we obtain

$$\frac{|h(a_n) - h(a_k)|}{d(a_n, a_k)} < \frac{\delta/k}{\delta} \leq 1.$$

These estimates taken together verify that h is 1-Lipschitz on E . We can extend h to a nonnegative 1-Lipschitz function on X [20, pp. 43-44], and to avoid extra notation, we use h to denote this extension as well. Since $f(a_n) > n$ for each $n \in \mathbb{N}$, we obtain for each $n \in \mathbb{N}$

$$|f(a_n)h(a_n) - f(x_n)h(x_n)| = f(a_n)h(a_n) > nd(a_n, x_n).$$

Since a_n and x_n can be made arbitrarily close together, the pointwise product fh cannot belong to $LS(X)$ and so condition (5) fails. \square

Moving to reciprocation for $LS(X)$, we need a fact about this vector lattice observed in [16]: a bounded Lipschitz in the small real-valued function defined on a nonempty subset A of X is already Lipschitz on the subset A . Letting δ be the distance control parameter and λ the local Lipschitz constant and choosing α with $\sup\{|f(a_1) - f(a_2)| : a_1, a_2 \in A\} < \alpha$, it is each to check that $\max\{\lambda, \frac{\alpha}{\delta}\}$ is a Lipschitz constant for f on A .

Theorem 5.2. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:*

- (1) *the metric space is a UC-space;*
- (2) *$LS(X)$ is stable under reciprocation;*
- (3) *whenever $g \in Lip(X)$ is non-vanishing, then $\frac{1}{g} \in LS(X)$.*

Proof. (1) \Rightarrow (2). If $\langle X, d \rangle$ is a UC-space, then $LS(X) = LL(X)$ [6, Theorem 3.3].

(2) \Rightarrow (3). This is trivial.

(3) \Rightarrow (1). If condition (3) holds and X is not complete, take p in $\hat{X} \setminus X$ where $\langle \hat{X}, \hat{d} \rangle$ is the completion of X . While $x \mapsto \hat{d}(x, p)$ is a Lipschitz function on X , its reciprocal is not Lipschitz in the small or even uniformly locally Lipschitz because it fails to be uniformly locally bounded.

Now consider the possibility that $\langle X, d \rangle$ while complete is not a UC-space. We can find sequences $\langle x_n \rangle$ and $\langle w_n \rangle$ such that for each n , $0 < d(x_n, w_n) < \frac{1}{n}$ but such that $\langle x_n \rangle$ fails to cluster. By completeness, $\langle x_n \rangle$ has no Cauchy subsequence

and so $\{x_n : n \in \mathbb{N}\}$ is not totally bounded. By passing to a subsequence, we may assume that for some $\delta \in (0, 1)$

- (a) whenever $n \neq j$, we have $d(x_n, x_j) > \delta$, and
- (b) for each $n \in \mathbb{N}$, $d(x_n, w_n) < \frac{\delta}{3}$.

Next put $A := \{x_n : n \in \mathbb{N}\} \cup \{w_n : n \in \mathbb{N}\}$. By conditions (a) and (b), if $\{a_1, a_2\} \subseteq A$ with $0 < d(a_1, a_2) < \frac{\delta}{3}$, then for some $n \in \mathbb{N}$, $\{a_1, a_2\} = \{x_n, w_n\}$.

Now define $f : A \rightarrow (0, \infty)$ by $f(x_n) = d(x_n, w_n)^2$ and $f(w_n) = d(x_n, w_n)$. Then if $d(a_1, a_2) < \frac{\delta}{3}$, we have $|f(a_1) - f(a_2)| \leq d(a_1, a_2)$ because for each $n \in \mathbb{N}$

$$|f(x_n) - f(w_n)| = |d(x_n, w_n) - 1|d(x_n, w_n) < d(x_n, w_n).$$

As f is Lipschitz in the small on A and f is bounded on A , f is actually Lipschitz on A . By Lemma 2.2, f has a positive Lipschitz extension g to X . But $\frac{1}{g} \notin LS(X)$ because

$$\frac{|\frac{1}{g}(x_n) - \frac{1}{g}(w_n)|}{d(x_n, w_n)} = \frac{1 - d(x_n, w_n)}{d(x_n, w_n)^3} > \frac{2/3}{d(x_n, w_n)^3}$$

while $\lim_{n \rightarrow \infty} d(x_n, w_n) = 0$. □

As might now be anticipated, our conditions on $\langle X, d \rangle$ that characterize when $LS(X)$ is stable under reciprocation are exactly those that are necessary and sufficient for $LS(X) = LL(X)$ [6, Theorem 3.3].

6. EXTENSIONS OF CAUCHY-LIPSCHITZ FUNCTIONS AND LIPSCHITZ IN THE SMALL FUNCTIONS

In his seminal article on Cauchy continuous functions [31], Snipes proved that (1) a Cauchy continuous function defined on a nonempty subset A of a metric space $\langle X, d \rangle$ into a complete metric space has a Cauchy continuous extension to $\text{cl}(A)$, and (2) if $\langle X, d \rangle$ is a complete metric space, then the Cauchy continuous functions on $\langle X, d \rangle$ agree with the continuous functions on $\langle X, d \rangle$ whatever the metric target space may be. From this it follows that each real-valued Cauchy continuous function f on a nonempty subset A of an arbitrary metric space $\langle X, d \rangle$ has a Cauchy continuous extension to X . We sketch the argument: (a) extend f to a continuous function on the closure of A relative to the completion $\langle \hat{X}, \hat{d} \rangle$ of $\langle X, d \rangle$; (b) extend this initial extension to a continuous function on the completion using the Tietze extension theorem; this second extension must be Cauchy continuous by completeness of \hat{d} ; (c) restrict this second extension back down to X .

We begin this section by showing that analogous results hold for Cauchy-Lipschitz functions.

Theorem 6.1. *Let A be a nonempty subset of $\langle X, d \rangle$ and let $\langle Y, \rho \rangle$ be a complete metric space. Suppose $f : A \rightarrow Y$ is a Cauchy-Lipschitz function. Then f has a Cauchy-Lipschitz extension to $\text{cl}(A)$.*

Proof. Since f is Cauchy continuous, by [31, Theorem 7] let \tilde{f} be a Cauchy continuous extension of f to $\text{cl}(A)$. We intend to show that \tilde{f} is Lipschitz when restricted

to the range of each Cauchy sequence $\langle x_n \rangle$ in $\text{cl}(A)$. For each $n \in \mathbb{N}$ let $\langle x_n^k \rangle_{k=1}^\infty$ be a sequence in A such that for all n and k , $d(x_n^k, x_n) < \frac{1}{nk}$. We claim that the set $B := \{x_n^k : n, k \in \mathbb{N}\}$ is totally bounded.

To see this, given $\varepsilon > 0$, there is an integer $m > \frac{2}{\varepsilon}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ whenever $n \geq m$. We also have

$$d(x_n, x_n^k) < \frac{1}{nk} \leq \frac{1}{m}$$

whenever $n \geq m$. It follows from the triangle inequality that $\{x_n^k : n \geq m \text{ and } k \in \mathbb{N}\} \subseteq S_d(x_m, \varepsilon)$. In addition, we have $d(x_n^k, x_n) < \varepsilon$ whatever n may be if $k > \frac{1}{\varepsilon}$. From these estimates,

$$\{x_n^k : n < m \text{ and } k > \frac{1}{\varepsilon}\} \subseteq \cup_{n < m} S_d(x_n, \varepsilon).$$

As only finitely many x_n^k are unaccounted for, B can be covered with a finite number of open balls of radius ε , and the claim holds.

Since $B \subseteq A$ and f is Cauchy-Lipschitz on A , $\tilde{f}|_B = f|_B$ is Lipschitz [7, Proposition 3.4]. Let $\lambda > 0$ be a Lipschitz constant for $\tilde{f}|_B$. Whenever n, m, k are positive integers, we have

$$\begin{aligned} \rho(\tilde{f}(x_n), \tilde{f}(x_m)) &\leq \rho(\tilde{f}(x_n), \tilde{f}(x_n^k)) + \rho(\tilde{f}(x_n^k), \tilde{f}(x_m^k)) + \rho(\tilde{f}(x_m^k), \tilde{f}(x_m)) \\ &\leq \rho(\tilde{f}(x_n), \tilde{f}(x_n^k)) + \lambda d(x_n^k, x_m^k) + \rho(\tilde{f}(x_m^k), \tilde{f}(x_m)). \end{aligned}$$

Letting $k \rightarrow \infty$, the continuity of \tilde{f} yields $\rho(\tilde{f}(x_n), \tilde{f}(x_m)) \leq \lambda d(x_n, x_m)$ as required. \square

For our general extension theorem for real-valued Cauchy-Lipschitz functions, in lieu of Tietze's theorem, we use this result of Czipser and Gehér [14, 15]: each real-valued locally Lipschitz function defined on a closed subset of a metric space has a locally Lipschitz extension to the entire space.

Theorem 6.2. *Let A be a nonempty subset of a metric space $\langle X, d \rangle$ and let $f \in CL(A)$. Then there exists $g \in CL(X)$ that extends f .*

Proof. Let \tilde{f} be a Cauchy-Lipschitz extension of f to $\text{cl}_{\hat{X}}(A)$ where $\langle \hat{X}, \hat{d} \rangle$ is the completion of the initial metric space. By the Czipser-Gehér result and [7, Proposition 3.2], \tilde{f} can be extended to a locally Lipschitz function \hat{g} on $\langle \hat{X}, \hat{d} \rangle$. By the completeness of \hat{d} , the second extension is Cauchy-Lipschitz [7, Theorem 3.5]. Finally, put $g := \hat{g}|_X$. \square

Since extension theorems for locally Lipschitz functions seem to parallel those for continuous functions, and extension theorems for Cauchy-Lipschitz functions seem to parallel those for Cauchy continuous functions, one would expect that extension theorems for Lipschitz in the small functions would parallel those for uniformly continuous functions. We focus on two results. First, there is a result that can be found in introductory analysis texts: a uniformly continuous function on a nonempty subset A of $\langle X, d \rangle$ to a complete metric space $\langle Y, \rho \rangle$ has a uniformly continuous extension to $\text{cl}(A)$. Second, we have a less elementary result that falls out of [17, Theorem 7]: a metric space $\langle X, d \rangle$ is small determined if and only if each

uniformly continuous real-valued function on X can be extended to a uniformly continuous function on each metric space in which $\langle X, d \rangle$ is isometrically embedded.

Theorem 6.3. *Let A be a nonempty subset of $\langle X, d \rangle$ and let $\langle Y, \rho \rangle$ be a complete metric space. Suppose $f : A \rightarrow Y$ is Lipschitz in the small. Then f has a Lipschitz in the small extension to $\text{cl}(A)$ that has the same distance control and the same uniform local Lipschitz constant.*

Proof. Suppose whenever $d(a_1, a_2) < \delta$ in A , we have $\rho(f(a_1), f(a_2)) \leq \lambda d(a_1, a_2)$. Extend f to a uniformly continuous function \tilde{f} on $\text{cl}(A)$, and let x and w be distinct points of the closure with $d(x, w) < \delta$. We can find sequences $\langle a_x^k \rangle$ and $\langle a_w^k \rangle$ in A convergent to x and w , respectively such that for each $k \in \mathbb{N}$, we have $d(a_x^k, a_w^k) < \delta$. In view of the inequality

$$\rho(\tilde{f}(x), \tilde{f}(w)) \leq \rho(\tilde{f}(x), f(a_x^k)) + \lambda d(a_x^k, a_w^k) + \rho(f(a_w^k), \tilde{f}(w))$$

the assertion of our theorem follows from the continuity of \tilde{f} . \square

We now introduce the *modulus of continuity function* $\omega_f : [0, \infty) \mapsto [0, \infty]$ [10] associated with a function f between metric spaces $\langle X, d \rangle$ and $\langle Y, \rho \rangle$:

$$\omega_f(t) := \sup\{\rho(f(x), f(w)) : x, w \in X \text{ and } d(x, w) \leq t\} \quad (t \geq 0).$$

Some obvious properties of ω_f are now listed as a courtesy to the reader:

- $\omega_f(0) = 0$ and ω_f is nondecreasing on $[0, \infty)$;
- f is uniformly continuous if and only if ω_f is continuous at $t = 0$;
- f is λ -Lipschitz if and only if $t \mapsto \lambda t$ majorizes ω_f ;
- f is Lipschitz in the small with distance control δ and uniform local Lipschitz constant λ if and only if $t \mapsto \lambda t$ majorizes ω_f on $[0, \delta)$.

We next give necessary and sufficient conditions for a given Lipschitz in the small function to be Lipschitz in terms of its modulus of continuity. Recall that a real-valued function defined on a convex subset C of a normed linear space is called *concave* if whenever $\{c_1, c_2\} \subseteq C$ and $\alpha \in [0, 1]$, we have $\alpha f(c_1) + (1 - \alpha)f(c_2) \leq f(\alpha c_1 + (1 - \alpha)c_2)$.

Proposition 6.4. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and suppose $f : X \rightarrow Y$ is Lipschitz in the small. The following conditions are equivalent:*

- (1) ω_f has a concave majorant;
- (2) ω_f has an affine majorant, that is, one of the form $t \mapsto \alpha t + \beta$;
- (3) there exists scalars α and β and $n \in \mathbb{N}$ such that $\forall t \geq n$, $\omega_f(t) \leq \alpha t + \beta$;
- (4) f is Lipschitz.

Proof. Note that (2) \Rightarrow (3) and (4) \Rightarrow (1) are trivial, since $t \mapsto \lambda t$ is a concave function.

(1) \Rightarrow (2). Let g be a concave majorant of ω_f . Now a concave function on $[0, \infty)$ is differentiable on $(0, \infty)$ except at most countably many points, and at points of differentiability, the associated tangent line to the graph supports the hypograph of the function (see, e.g., [29, pp. 5-13]); taking $t_0 \in (0, \infty)$ where $g'(t_0)$ exists, this means

$$\omega_f(t) \leq g(t) \leq g(t_0) + g'(t_0)(t - t_0) \quad (t \geq 0).$$

(3) \Rightarrow (4). Note that $\alpha \geq 0$ since $\alpha t + \beta \geq 0$ for all $t \geq n$. One can assume that $\beta \geq \omega_f(n)$ so that $t \mapsto \alpha t + \beta$ majorizes the modulus of continuity throughout $[0, \infty)$ because ω_f is nondecreasing. Let $\delta > 0$ and $\lambda > 0$ be such that $\omega_f(t) \leq \lambda t$ if $t \in [0, \delta)$. By replacing β by a larger value if needed, we may assume that $\lambda \leq \frac{\alpha\delta + \beta}{\delta} := \mu$. We claim that $\forall t \geq 0$, $\omega_f(t) \leq \mu t$. This is clear if $t \in [0, \delta)$ because $\lambda \leq \mu$. On the other hand, if $t \geq \delta$, then

$$\omega_f(t) \leq \alpha t + \beta \leq \alpha t + \beta \frac{t}{\delta} = \mu t.$$

This shows that f is μ -Lipschitz. □

Theorem 6.5. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:*

- (1) $\langle X, d \rangle$ is small determined;
- (2) for each $f \in LS(X)$, its modulus of continuity has an affine majorant;
- (3) each member of $LS(X)$ has a Lipschitz in the small extension to any metric space in which X is isometrically embedded;
- (4) there exists a normed linear space $\langle W, \|\cdot\| \rangle$ isometrically containing $\langle X, d \rangle$ for which each member of $LS(X)$ can be extended to a member of $LS(W)$.

Proof. For (1) \Rightarrow (2), if $\langle X, d \rangle$ is small determined, then by Proposition 6.4 in particular, for each real-valued Lipschitz in the small function f on X , ω_f has an affine majorant. For (2) \Rightarrow (3), if $f \in LS(X)$, then by condition (2) and Proposition 6.4, we have $f \in Lip(X)$, and so f actually has a Lipschitz extension to any metric space in which X is isometrically embedded [20, p. 43]. The implication (3) \Rightarrow (4) follows from the fact that each metric space can be isometrically embedded in some normed linear space, e.g., the bounded continuous real-valued functions on the space equipped with the supremum norm. For (4) \Rightarrow (1), it suffices to show that each real-valued Lipschitz in the small function on X is Lipschitz [17, 24]. Let $f \in LS(X)$; by condition (4), $\exists g \in LS(W)$ extending f . But by its convexity, $\langle W, \|\cdot\| \rangle$ is a small determined space, whence g is Lipschitz, and so f is Lipschitz as well. □

We now use Proposition 6.4 to derive a folk-theorem alluded to in the Wikipedia entry on the modulus of continuity. We know of no explicit reference for this result in the actual literature, but it certainly is not new.

Theorem 6.6. *Suppose f is uniformly continuous and real-valued on the metric space $\langle X, d \rangle$. The following conditions are equivalent:*

- (1) f can be uniformly approximated by elements of $Lip(X)$;
- (2) there exists $g \in Lip(X)$ at a finite uniform distance from f ;
- (3) ω_f has an affine majorant.

Proof. (1) \Rightarrow (2) is trivial. For (2) \Rightarrow (3), choose $\nu > 0$ such that

$$\sup_{x \in X} |g(x) - f(x)| < \nu.$$

It is clear that for each $t \geq 0$, $|\omega_g(t) - \omega_f(t)| \leq 2\nu$. Now if λ is a Lipschitz constant for g , then $t \mapsto \lambda t$ majorizes ω_g , whence $t \mapsto \lambda t + 2\nu$ majorizes ω_f . For (3) \Rightarrow (1), let $\varepsilon > 0$ be arbitrary. By the uniform continuity of f we can find $h \in LS(X)$ with $\sup_{x \in X} |h(x) - f(x)| < \varepsilon$ [6, 17]. If $t \mapsto \alpha t + \beta$ majorizes ω_f , then $t \mapsto \alpha t + \beta + 2\varepsilon$

majorizes ω_h . By Proposition 6.4, h is Lipschitz and uniformly approximates f within ε . \square

We remark that with some work, (3) \Rightarrow (1) of our last theorem can be obtained from [10, Proposition 2.1], where the desired subadditive function is the concave envelope of ω_f . We leave this to the imagination of the interested reader.

Our final result of this article pulls together some of our machinery in an intriguing way. We would not be surprised if the equivalence of conditions (1) and (2) has already been observed.

Theorem 6.7. *Let $\langle X, d \rangle$ be a metric space. Then following conditions are equivalent:*

- (1) $\langle X, d \rangle$ is compact;
- (2) each real-valued continuous function on X can be uniformly approximated by members of $Lip(X)$;
- (3) $\langle X, d \rangle$ is both a UC-space and small determined.

Proof. If X is compact, then $Lip(X) = LL(X)$ [30], and $LL(X)$ is uniformly dense in the real-valued continuous functions with no assumptions on X [14, 16, 26]. If $Lip(X)$ is uniformly dense in the real-valued continuous functions, then clearly each continuous real-valued function is uniformly continuous. On the other hand, by Theorem 6.6, for each $f \in LS(X)$, ω_f has an affine majorant, so that by Theorem 6.4, $LS(X) \subseteq Lip(X)$, and $\langle X, d \rangle$ is small determined.

Finally suppose $\langle X, d \rangle$ is both a UC-space and small determined. By UC-ness and Theorem 5.2, $LS(X)$ is stable under reciprocation, and so $Lip(X)$ is stable under reciprocation. By Theorem 3.4, we conclude that the metric space is compact. \square

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