

Poisson–Hopf deformations of Lie–Hamilton systems revisited: deformed superposition rules and applications to the oscillator algebra

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Abstract

The formalism for Poisson–Hopf (PH) deformations of Lie–Hamilton systems, recently proposed in [1], is refined in one of its crucial points concerning applications, namely the obtention of effective and computationally feasible PH deformed superposition rules for prolonged PH deformations of Lie–Hamilton systems. The two new notions here proposed are a generalization of the standard superposition rules and the concept of diagonal prolongations for Lie systems, which are consistently recovered under the non-deformed limit. Using a technique from superintegrability theory, we obtain a maximal number of functionally independent constants of the motion for a generic prolonged PH deformation of a Lie–Hamilton system, from which a simplified deformed superposition rule can be derived. As an application, explicit deformed superposition rules for prolonged PH deformations of Lie–Hamilton systems based on the oscillator Lie algebra \mathfrak{h}_4 are computed. Moreover, by making use that the main structural properties of the Lie subalgebra \mathfrak{b}_2 of \mathfrak{h}_4 are preserved under the PH deformation, we consider prolonged PH deformations based on \mathfrak{b}_2 as restrictions of those for \mathfrak{h}_4 -Lie–Hamilton systems, thus allowing the study of prolonged PH deformations of the complex Bernoulli equations, for which both the constants of the motion and the deformed superposition rules are explicitly presented.

MSC: 16T05, 17B66, 34A26

PACS: 02.20.Uw, 02.20.Sv, 02.60.Lj

KEYWORDS: Lie system, constant of the motion, diagonal prolongation, superposition rule, Poisson–Hopf algebra, oscillator algebra, Bernoulli differential equations.

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1 Introduction

The original approach to Poisson–Hopf (PH) deformations of Lie–Hamilton systems developed in [1] combined the classical theory of Lie systems with methods from quantum algebras and integrable systems, leading to a novel type of systems of ordinary differential equations with generalized symmetry that, despite being deprived of some of the appealing properties of Lie systems, still allowed for a systematic analysis of their constants of the motion. In essence, the method is based on the idea of deforming a Lie–Hamilton system (LH system in short) with a given Vessiot–Guldberg Lie algebra onto a Hamiltonian system depending on a quantum deformation parameter z (or $q = e^z$), the dynamics of which is described by a t -dependent vector field taking values in a linear space of vector fields spanning a smooth distribution in the sense of Stefan–Sussmann, with the particularity that the initial LH system and its Vessiot–Guldberg Lie algebra is retrieved when $z \rightarrow 0$. This allowed us, among other applications, to provide a unified geometrical description of the PH deformations of the three inequivalent LH systems on the plane based on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ (see [2, 3]). Nonetheless, the PH deformation method proposed in [1] was, to a certain extent, still incomplete, as it did not studied the existence and methods of derivation of an extension of the superposition rule concept for PH deformed LH systems, a hereafter called *PH deformed superposition rule* or, simply, a *deformed superposition rule*.

In this paper we present the way to implement such a generic procedure, by making use of a powerful tool developed in the context of superintegrable systems possessing a Hopf algebra symmetry [4]. The construction is proved to be complete and valid for the hereafter called prolonged PH deformations of LH systems, hence providing a generic prescription for the obtention of their deformed superposition principles. In a nutshell, deformed superposition rules are z -parametric families of mappings allowing

for the description of some coordinates of particular solutions of a prolonged PH deformation of a LH system in terms of the others. When $z \rightarrow 0$, deformed superposition rules and prolonged PH deformations become standard superposition rules and diagonal prolongations [5] for LH systems, respectively. It is remarkable that given a LH system on an n -dimensional manifold M , its prolonged PH deformations become Hamiltonian systems on M^{m+1} that neither need to be Lie systems nor must consist of several copies of the original Lie system for $z \neq 0$. This is a remarkable difference with respect to LH systems, whose diagonal prolongations to M^{m+1} give rise to a LH system consisting of several copies of the initial one and whose constants of the motion, for m large enough, allow for a superposition rule for the initial LH system.

The paper is structured as follows. In Section 2 the fundamental properties of LH systems and their deformations based on the notion of PH algebras, as developed in [1], are shortly reviewed. Special attention is devoted to the construction of constants of the motion of both LH systems and their prolonged PH deformations, which are hereafter called *prolonged deformations* to simplify our terminology. While in Subsection 2.3.1 we recall the method deduced from the coalgebra formalism [6], already used in [1] and [3], in Subsection 2.3.2 we present new material that completes and enlarges the previous work. Specifically, it is shown that for generic prolonged deformations of LH systems, in contrast to what happens for the undeformed systems, constructing the constants of the motion basing solely on the coalgebra formalism [6] may not supply us with a sufficient number of functionally independent constants to establish a deformed superposition rule in explicit closed form, as in this case we necessarily have to consider constants of the motion of higher ‘order’ (i.e. the dimension of the underlying tensor product space for the coproduct), which could imply that the resulting expressions are not expressible analytically in a discernible way. This rather subtle phenomenon is due to a symmetry breaking phenomena in the coproduct induced by the deformation, which also causes that the prolonged deformation of the initial LH system is not, in general, a diagonal prolongation [5]. In turn, this implies that a permutation in the tensor product space does not necessarily transform a constant of the motion for the prolonged deformation into another one. Moreover, it can even happen that some of the constants of the motion of the diagonal prolongation of an undeformed LH system do not have at all a counterpart for a prolonged deformation of the LH system. This can be seen as a gap in the formalism as given in [1], as it somewhat disturbs the correspondence between the diagonal prolongations of a LH system and its prolonged deformations through the limit with respect to the deformation parameter. Based on this observation, we reconsider the problem of determining a sufficient number of constants of the motion for both the diagonal prolongation of a LH system and its prolonged deformations, in such a manner that the correspondence through the limit $z \rightarrow 0$ is maintained, and we satisfactorily solve it in full generality, using a mechanism borrowed from the theory of superintegrable systems, namely using the superintegrability property of systems endowed with a Hopf algebra of symmetries [4]. This construction leads to the consideration of a left- and a right-set of m constants of the motion in involution (for each set) that are valid for arbitrary prolonged deformations to M^{m+1} , providing a maximal number of $(2m - 1)$ functionally independent constants of the motion for the given deformation. From them we can infer a suitable deformed superposition rule selecting those constants of the motion having minimal ‘order’, hence minimizing the analytical computation difficulties. In the classical limit $z \rightarrow 0$, the ‘additional’ right-constants of the motion are shown to be obtainable using the permutation method from the left-set, which was the one formerly considered in [6]. This new Ansatz refines in a natural way the results in [1], pointing out the relevance of working simultaneously with left- and right-coproducts in the deformation, as it constitutes a procedure that generically guarantees that the resulting functions are constants of the motion of the prolonged deformation of the initial LH system.

As an application, Section 3 analyses the LH systems based on the oscillator algebra \mathfrak{h}_4 . The (undeformed) diagonal prolongations to \mathfrak{h}_4 -LH systems, which were already studied in [7], are now considered by using a different set of constants of the motion, leading to a different albeit equivalent superposition rule. The purpose of this reformulation is to consistently deduce the superposition rule as the limit $z \rightarrow 0$ of the deformed one, whose derivation involves the use of the right-set of constants of the motion. In Section 4, the so-called nonstandard (or Jordanian) PH deformation of \mathfrak{h}_4 [8] is considered and the explicit derivation of formulae for the corresponding prolonged deformation of \mathfrak{h}_4 -LH systems is developed. The choice of the nonstandard deformation of \mathfrak{h}_4 , on the other hand, has an interesting structural consequence: the two-dimensional book Lie algebra \mathfrak{b}_2 is preserved as a Hopf subalgebra, thus allowing

us to restrict the prolonged deformation of \mathfrak{h}_4 -LH systems to this subalgebra and therefore to obtain deformed \mathfrak{b}_2 -LH systems. As a representative of such systems based on \mathfrak{b}_2 , we consider the complex Bernoulli equations, for which the constants of the motion and the superposition rules are given for both deformed and non-deformed versions. In this context, the interpretation of the deformed system as a small perturbation of the initial one, governed by the deformation parameter z , allows us to establish a connection between non-trivially coupled systems, and one of its equations corresponds to a Riccati equation. Finally, in Section 5 some conclusions are drawn, several possible future developments of the method are proposed, and its applications to the analysis of nonlinear systems of differential equations are commented.

2 The Poisson–Hopf deformation formalism revisited

This section briefly recalls the fundamental properties of Lie and LH systems. It also reviews the general deformation method of LH systems introduced in [1] (see also [3]) based on PH algebras. In contrast to [1, 3], the deformation of LH systems is developed in full generality to point out that the mechanism holds for arbitrary smooth manifolds. In particular, we focus on the problem of obtaining a sufficient number of constants of the motion for prolonged deformations of LH systems, from which formally explicit deformed superposition rules can be deduced, regardless of the initial LH system and the considered deformation. It should be emphasized that, albeit in [1] this possibility was outlined for deformed LH systems, no deformed superposition rules were explicitly given. In this work we propose an extension of the techniques in [1] that allows us to construct such deformed superposition rules for arbitrary prolonged deformations of LH systems. Unless otherwise stated, we hereafter assume all structures to be smooth and globally defined. This will simplify the presentation of our results and it will allow us to focus on its main new features.

2.1 Lie and Lie–Hamilton systems

Let $\mathbf{x} = \{x_1, \dots, x_n\}$ denote the coordinates in an n -dimensional manifold M and consider a non-autonomous system of first-order ordinary differential equations given by

$$\frac{dx_j}{dt} = f_j(t, \mathbf{x}), \quad j = 1, \dots, n, \quad (2.1)$$

for some arbitrary functions $f_j : \mathbb{R} \times M \rightarrow \mathbb{R}$. Such a system can be described equivalently via a t -dependent vector field $\mathbf{X} : \mathbb{R} \times M \rightarrow TM$ defined as:

$$\mathbf{X}(t, \mathbf{x}) = \sum_{j=1}^n f_j(t, \mathbf{x}) \frac{\partial}{\partial x_j}. \quad (2.2)$$

A system of the type (2.1) is called a *Lie system* [5, 9–17] whenever it admits a *fundamental system of solutions*, i.e., whenever its general solution, $\mathbf{x}(t)$, can be expressed in terms of m particular solutions $\{\mathbf{x}_1(t), \dots, \mathbf{x}_m(t)\}$ and n constants $\{k_1, \dots, k_n\}$ as

$$\mathbf{x}(t) = \Psi(\mathbf{x}_1(t), \dots, \mathbf{x}_m(t), k_1, \dots, k_n) \quad (2.3)$$

for a certain function $\Psi : M^m \times M \rightarrow M$. The expression (2.3) is usually referred to as a *superposition rule* of the system (2.1). Within applications to physical phenomena, the wealth of group-theoretical techniques developed from the 60's onwards revived the interest in analyzing systematically the existence of superposition rules, leading to an extensive geometrical study of Lie systems and superposition rules and their application to systems at both the classical and quantum levels (see, e.g., [5, 18–24] and references therein).

The Lie–Scheffers Theorem (see [5, 9, 10, 13]) states that a t -dependent vector field \mathbf{X} as in (2.2) determines a Lie system if and only if there exist some functions $b_1(t), \dots, b_\ell(t)$ and vector fields $\mathbf{X}_1, \dots, \mathbf{X}_\ell$

on M spanning an ℓ -dimensional real Lie algebra V such that

$$\mathbf{X}(t, \mathbf{x}) = \sum_{i=1}^{\ell} b_i(t) \mathbf{X}_i(\mathbf{x}), \quad \forall \mathbf{x} \in M. \quad (2.4)$$

It can then be proved that the system \mathbf{X} admits a superposition rule so that the constraint $\ell \leq nm$ is satisfied. In these conditions, V is called a *Vessiot–Guldberg Lie algebra* of \mathbf{X} (see also [25–27] for more recent applications of Vessiot–Guldberg Lie algebras).

A Lie system is said to be a *Lie–Hamilton system* whenever it admits a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields with respect to a Poisson structure [2, 5–7, 16, 28, 29]. Let us assume the case of a LH system on M that admits a Vessiot–Guldberg Lie algebra, V , of Hamiltonian vector fields relative to a symplectic form ω . The compatibility condition between the generators \mathbf{X}_i of V and ω is locally determined by the invariance of ω under the Lie derivative with respect to any generator \mathbf{X}_i of V , i.e.,

$$\mathcal{L}_{\mathbf{X}_i} \omega = 0, \quad i = 1, \dots, \ell. \quad (2.5)$$

Now a *Hamiltonian function* h_i is related to the vector field \mathbf{X}_i through the contraction or inner product of ω with respect to \mathbf{X}_i :

$$\iota_{\mathbf{X}_i} \omega = dh_i, \quad i = 1, \dots, \ell. \quad (2.6)$$

Recall that every symplectic form allows us to define a Poisson bracket

$$\{\cdot, \cdot\}_{\omega} : (f_1, f_2) \in C^{\infty}(M) \times C^{\infty}(M) \rightarrow \mathbf{X}_{f_2} f_1 \in C^{\infty}(M), \quad (2.7)$$

where \mathbf{X}_f is the unique Hamiltonian vector such that $\iota_{\mathbf{X}_f} \omega = df$ for an $f \in C^{\infty}(M)$. It follows that $(C^{\infty}(M), \{\cdot, \cdot\}_{\omega})$ is endowed with a Poisson algebra structure. The space $\text{Ham}(\omega)$ of Hamiltonian vector fields on M relative to ω , which is a Lie algebra with respect to the commutator of vector fields, is related to the former by means of the Lie algebra morphism [30]

$$(C^{\infty}(M), \{\cdot, \cdot\}_{\omega}) \xrightarrow{\varphi} (\text{Ham}(\omega), [\cdot, \cdot]) \quad (2.8)$$

mapping a function $f \in C^{\infty}(M)$ onto the Hamiltonian vector field $-\mathbf{X}_f$. The Hamiltonian functions h_i ($i = 1, \dots, \ell$) coming from (2.6) span, eventually together with a constant function h_0 on M , a finite-dimensional Lie algebra of functions $\varphi^{-1}(V)$ that is called a *Lie–Hamilton algebra* (LH algebra), \mathcal{H}_{ω} , of \mathbf{X} [2, 7].

2.2 The Poisson–Hopf deformation approach

The remarkable point is that the space $C^{\infty}(\mathcal{H}_{\omega}^*)$ of smooth functions on the dual \mathcal{H}_{ω}^* of the LH algebra \mathcal{H}_{ω} can be endowed with a Hopf algebra structure [31–33]. For our purposes, it suffices to consider the *coalgebra structure* of the Hopf algebra determined by the coproduct map, as the remaining structural maps, namely the counit and antipode, can be deduced from the axioms defining the Hopf algebra. In particular, for an associative algebra A , the coproduct $\Delta : A \rightarrow A \otimes A$ must be an algebra homomorphism and satisfy the coassociativity condition

$$(\text{Id} \otimes \Delta)\Delta(a) = (\Delta \otimes \text{Id})\Delta(a), \quad \forall a \in A. \quad (2.9)$$

If A is a commutative Poisson algebra, the coproduct Δ satisfying (2.9) is required to be a Poisson algebra morphism, so that the Poisson bracket on $A \otimes A$ is given by

$$\{a \otimes b, c \otimes d\}_{A \otimes A} = \{a, c\} \otimes bd + ac \otimes \{b, d\}, \quad \forall a, b, c, d \in A. \quad (2.10)$$

For the case of $C^{\infty}(\mathcal{H}_{\omega}^*)$, the coalgebra structure is determined by the coproduct $\Delta(f)(x_1, x_2) := f(x_1 + x_2)$, where $x_1, x_2 \in \mathcal{H}_{\omega}$ and $f \in C^{\infty}(\mathcal{H}_{\omega}^*)$. The details concerning the complete Hopf algebra structure can be found in [1]; here we just recall that $C^{\infty}(\mathcal{H}_{\omega}^*)$ turns out to be a PH algebra through the Poisson structure defined by the Kirillov–Kostant–Souriau bracket related to a Lie algebra structure on \mathcal{H}_{ω} .

Given these algebraic preliminaries, we summarize the notion of PH deformation introduced in [1] (see also [3]) in four steps:

1. Let \mathbf{X} be a LH system of type (2.4) on an n -dimensional manifold M with symplectic form ω , so that the LH algebra \mathcal{H}_ω is spanned by a set of functions $\{h_1, \dots, h_\ell\} \subset C^\infty(\mathcal{M})$ satisfying the condition (2.6), with \mathcal{M} being a suitable submanifold of M that ensures that each h_i is well defined. Let the Poisson bracket of the functions h_i be given by:

$$\{h_i, h_j\}_\omega = \sum_{l=1}^{\ell} C_{ij}^l h_l, \quad i, j = 1, \dots, \ell, \quad (2.11)$$

for certain structure constants C_{ij}^l .

2. Consider a PH deformation of $C^\infty(\mathcal{H}_\omega^*)$, denoted by $C^\infty(\mathcal{H}_{z,\omega}^*)$, with deformation parameter $z \in \mathbb{R}$ (or $q = e^z$) as the space of smooth functions $F_{z,ij}(h_{z,1}, \dots, h_{z,\ell})$ for a family of functions $\{h_{z,1}, \dots, h_{z,\ell}\}$ on $C^\infty(\mathcal{M})$ with Poisson bracket (with respect to ω) given by

$$\{h_{z,i}, h_{z,j}\}_\omega = F_{z,ij}(h_{z,1}, \dots, h_{z,\ell}), \quad i, j = 1, \dots, \ell, \quad (2.12)$$

and satisfying the non-deformed limits

$$\lim_{z \rightarrow 0} h_{z,i} = h_i, \quad \lim_{z \rightarrow 0} F_{z,ij}(h_{z,1}, \dots, h_{z,\ell}) = \sum_{l=1}^{\ell} C_{ij}^l h_l. \quad (2.13)$$

3. Obtain the deformed vector fields $\mathbf{X}_{z,i}$ according to the relation (2.6), that is,

$$\iota_{\mathbf{X}_{z,i}} \omega = dh_{z,i}, \quad i = 1, \dots, \ell. \quad (2.14)$$

4. And, finally, define the PH deformation \mathbf{X}_z of the LH system \mathbf{X} (2.4) as

$$\mathbf{X}_z := \sum_{i=1}^{\ell} b_i(t) \mathbf{X}_{z,i}. \quad (2.15)$$

Notice that, by construction, the following non-deformed limits are consistently recovered:

$$\lim_{z \rightarrow 0} \mathbf{X}_{z,i} = \mathbf{X}_i, \quad \lim_{z \rightarrow 0} \mathbf{X}_z = \mathbf{X}. \quad (2.16)$$

The essential point to be taken into account is that the deformed vector fields $\{\mathbf{X}_{z,1}, \dots, \mathbf{X}_{z,\ell}\}$ obtained through the preceding prescription do not, in general, provide neither a finite-dimensional Lie algebra nor a quantum algebra. Actually, they span an involutive Stefan–Sussmann distribution [30, 34, 35] since

$$[\mathbf{X}_{z,i}, \mathbf{X}_{z,j}] = - \sum_{l=1}^{\ell} \frac{\partial F_{z,ij}}{\partial h_{z,l}} \mathbf{X}_{z,l}, \quad i, j = 1, \dots, \ell. \quad (2.17)$$

In other words, the functions $\{h_{z,1}, \dots, h_{z,\ell}\}$ determine a PH deformation $C^\infty(\mathcal{H}_{z,\omega}^*)$ of $C^\infty(\mathcal{H}_\omega^*)$ with deformed Poisson brackets (2.12), thus providing a (deformed) Hamiltonian function

$$h_z := \sum_{i=1}^{\ell} b_i(t) h_{z,i}. \quad (2.18)$$

However, the non-autonomous system of first-order ordinary differential equations \mathbf{X}_z (2.15) does no longer correspond, in general, to a Lie system, but to a ‘perturbation’ of the initial system (2.1) with respect to the deformation parameter z , as follows at once from the conditions (2.13) and (2.16) under the limit $z \rightarrow 0$. In this context, it is conceivable to interpret z as a small perturbation parameter. This means that, once the deformed system has been obtained through either \mathbf{X}_z or h_z , a power series expansion in z can be considered, analyzing the behaviour of the deformed Hamiltonian system up to the first, second or some higher order, enabling us a comparison with the initial undeformed system.

2.3 Constants of the motion

The coalgebra formalism considered in [36, 37] in the context of integrable systems turned out to be a highly effective tool that allows to prove in a constructive way the complete integrability of systems possessing coalgebra symmetry, including the explicit construction of the corresponding integrals of the motion. This coalgebra approach was later extended in order to characterize the property of (quasi-maximal) superintegrability [4, 38, 39]. These results covered both non-deformed integrable systems and their PH deformations. More recently, the coalgebra formalism was adapted to the framework of LH systems [6], providing a method to determine t -independent constants of the motion in a more direct way than that given by the classical methods [12, 14]. We observe that the constants of the motion of LH systems deduced by this technique are the cornerstone for the obtention of superposition rules. This procedure has been carried out systematically for LH systems on the plane \mathbb{R}^2 in [7] as well as on two-dimensional spaces of constant curvature (with different signatures of the metric tensor) in [29].

At this point, it is of capital importance to realize that the results presented in [6] (and further considered in [7, 29]) focused on non-deformed LH systems, that is, for cases with a trivial or primitive coalgebra structure. This approach turns out to be unsatisfactory, as its straightforward extension to non-primitive coproducts, which is precisely the case for prolonged deformations of LH systems, provides less constants of the motion than in the primitive case. The aim of this section is to enlarge and complete such previous work, proposing a general procedure for the explicit construction of the constants of the motion for prolonged deformations of LH systems.

2.3.1 Undeformed constants of the motion

Let us first briefly summarize the coalgebra approach for constructing t -independent constants of the motion of non-deformed LH systems [6] (see also [1, 7]). Consider the LH algebra \mathcal{H}_ω of a LH system \mathbf{X} (2.4), expressed as a Lie–Poisson algebra with generators $\{v_1, \dots, v_\ell\}$ fulfilling the Poisson brackets (see (2.11)):

$$\{v_i, v_j\} = \sum_{l=1}^{\ell} C_{ij}^l v_l, \quad i, j = 1, \dots, \ell. \quad (2.19)$$

Let $S(\mathcal{H}_\omega)$ be the symmetric algebra of \mathcal{H}_ω (i.e., the associative unital algebra of polynomials in the elements of \mathcal{H}_ω) understood as a Poisson algebra, thus with fundamental Poisson brackets (2.19). Under these conditions, $S(\mathcal{H}_\omega)$ can always be endowed with a coalgebra structure with a non-deformed (trivial or primitive) coproduct map Δ defined by

$$\Delta : S(\mathcal{H}_\omega) \rightarrow S(\mathcal{H}_\omega) \otimes S(\mathcal{H}_\omega), \quad \Delta(v_i) := v_i \otimes 1 + 1 \otimes v_i, \quad i = 1, \dots, \ell, \quad (2.20)$$

which is a Poisson algebra homomorphism of (2.19). Notice that the (trivial) counit and antipode can also be defined giving rise to the non-deformed Hopf structure corresponding to any Lie algebra [31–33].

The 2-coproduct $\Delta \equiv \Delta^{(2)}$ can be extended to a third-order coproduct through the coassociativity condition (2.9):

$$\begin{aligned} \Delta^{(3)} &:= (\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta, \\ \Delta^{(3)} &: S(\mathcal{H}_\omega) \rightarrow S(\mathcal{H}_\omega) \otimes S(\mathcal{H}_\omega) \otimes S(\mathcal{H}_\omega) \equiv S^{(3)}(\mathcal{H}_\omega), \\ \Delta^{(3)}(v_i) &= v_i \otimes 1 \otimes 1 + 1 \otimes v_i \otimes 1 + 1 \otimes 1 \otimes v_i, \quad i = 1, \dots, \ell. \end{aligned} \quad (2.21)$$

A k^{th} -order coproduct map can be defined recursively by the rule

$$\begin{aligned} \Delta^{(k)} &: S(\mathcal{H}_\omega) \rightarrow \overbrace{S(\mathcal{H}_\omega) \otimes \dots \otimes S(\mathcal{H}_\omega)}^{k \text{ times}} \equiv S^{(k)}(\mathcal{H}_\omega), \\ \Delta^{(k)} &:= \overbrace{(\text{Id} \otimes \dots \otimes \text{Id}) \otimes \Delta^{(2)}}^{(k-2) \text{ times}} \circ \Delta^{(k-1)}, \end{aligned} \quad (2.22)$$

which is also a Poisson algebra homomorphism for any $k \geq 3$.

Any element of $S(\mathcal{H}_\omega)$ can be seen as a function on \mathcal{H}_ω^* so that the coproduct (2.20) in $S(\mathcal{H}_\omega)$ can be extended to

$$\Delta : C^\infty(\mathcal{H}_\omega^*) \rightarrow C^\infty(\mathcal{H}_\omega^*) \otimes C^\infty(\mathcal{H}_\omega^*) \subset C^\infty(\mathcal{H}_\omega^* \times \mathcal{H}_\omega^*). \quad (2.23)$$

A similar extension holds for the k^{th} -order coproduct defined in (2.22). Therefore, $C^\infty(\mathcal{H}_\omega^*)$ becomes a non-deformed Poisson coalgebra, and the corresponding extension of the counit and antipode maps turns $C^\infty(\mathcal{H}_\omega^*)$ into a PH algebra [1].

Now consider the LH algebra \mathcal{H}_ω spanned by the Hamiltonian functions $\{h_1, \dots, h_\ell\}$ satisfying the Poisson brackets (2.11). In agreement with equation (2.19), we define the Lie algebra morphism

$$\phi : \mathcal{H}_\omega \rightarrow C^\infty(\mathcal{M}), \quad h_i := \phi(v_i), \quad i = 1, \dots, \ell, \quad (2.24)$$

where $\mathcal{M} \subset M$ is chosen in order to ensure that the functions h_i , and their PH deformations, to be defined shortly, are well defined. Basing on this result, we construct a family of Poisson algebra morphisms

$$D : C^\infty(\mathcal{H}_\omega^*) \rightarrow C^\infty(\mathcal{M}), \quad D^{(k)} : \overbrace{C^\infty(\mathcal{H}_\omega^*) \otimes \dots \otimes C^\infty(\mathcal{H}_\omega^*)}^{k \text{ times}} \rightarrow \overbrace{C^\infty(\mathcal{M}) \otimes \dots \otimes C^\infty(\mathcal{M})}^{k \text{ times}} \subset C^\infty(\mathcal{M}^k), \quad (2.25)$$

that are given by

$$D(v_i) = h_i(\mathbf{x}_1) := h_i^{(1)}, \quad D^{(k)}(\Delta^{(k)}(v_i)) = h_i(\mathbf{x}_1) + \dots + h_i(\mathbf{x}_k) := h_i^{(k)}, \quad i = 1, \dots, \ell, \quad (2.26)$$

where $\mathbf{x}_j = \{(x_1)_j, \dots, (x_n)_j\}$ denotes the coordinates in the j -copy submanifold $\mathcal{M} \subset M$ within \mathcal{M}^k . In the generic case with a k^{th} -order tensor product of elements $u_j(v_1, \dots, v_\ell) \in C^\infty(\mathcal{H}_\omega^*)$ the morphism $D^{(k)}$ gives rise to the following product of functions

$$D^{(k)}(u_1(v_1, \dots, v_\ell) \otimes \dots \otimes u_k(v_1, \dots, v_\ell)) = u_1(h_1(\mathbf{x}_1), \dots, h_\ell(\mathbf{x}_1)) \dots u_k(h_1(\mathbf{x}_k), \dots, h_\ell(\mathbf{x}_k)). \quad (2.27)$$

Let us finally assume that $C^\infty(\mathcal{H}_\omega^*)$ possesses a Casimir invariant

$$C = C(v_1, \dots, v_\ell), \quad (2.28)$$

that is, an element C that Poisson-commutes with all v_i with respect to the Poisson bracket given in (2.19). As proved in [6], it follows that the functions constructed through the family of coproducts (2.22) and Poisson morphisms (2.25) defined by

$$F := D(C), \quad F^{(k)}(h_1^{(k)}, \dots, h_\ell^{(k)}) := D^{(k)}[\Delta^{(k)}(C(v_1, \dots, v_\ell))], \quad k = 2, \dots, m+1, \quad (2.29)$$

are t -independent constants of the motion for the diagonal prolongation $\tilde{\mathbf{X}}^{m+1}$ of the LH system \mathbf{X} (2.4) to the product manifold \mathcal{M}^{m+1} , i.e., the t -dependent vector field on \mathcal{M}^{m+1} of the form

$$\tilde{\mathbf{X}}^{m+1}(t, \mathbf{x}_1, \dots, \mathbf{x}_{m+1}) := \sum_{k=1}^{m+1} \sum_{j=1}^n X^j(t, \mathbf{x}_k) \frac{\partial}{\partial x^j} = \sum_{i=1}^{\ell} b_i(t) \mathbf{X}_{h_i^{(m+1)}}. \quad (2.30)$$

Note that the functions (2.29) can also be considered as constants of the motion for the LH system \mathbf{X} . The right-hand side of expression (2.30) shall be called the *prolonged PH deformation* of \mathbf{X} to \mathcal{M}^{m+1} , or simply the *prolonged deformation*. As we shall see shortly, this notion can immediately be extended to non-primitive coproducts which will invalidate, in general, the equality of the right-hand side of (2.30) with the standard diagonal prolongation of \mathbf{X} .

Each of the $F^{(k)}$ (2.29) can be considered as a function of $C^\infty(\mathcal{M}^{m+1})$. If all the $F^{(k)}$ are non-constant, then they form a set of m functionally independent functions in $C^\infty(\mathcal{M}^{m+1})$ that are in involution. In addition, these functions $F^{(k)}$ can be used to generate other t -independent constants of the motion by means of the prescription [6]

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \quad 1 \leq i < j \leq k, \quad k = 2, \dots, m+1, \quad (2.31)$$

where S_{ij} denotes the permutation of the variables $\mathbf{x}_i \leftrightarrow \mathbf{x}_j$ in \mathcal{M}^{m+1} . This can be viewed as a consequence of the fact that the diagonal prolongation of \mathbf{X} is invariant under such a permutation of variables. It can also be viewed as a consequence of (2.30) and [37, Proposition 1].

Recall that for obtaining a superposition rule depending on m particular solutions, as in (2.3), one searches for a set, I_1, \dots, I_n , of t -independent constants of the motion on \mathcal{M}^{m+1} for $\tilde{\mathbf{X}}^{m+1}$ so that [5]

$$\frac{\partial(I_1, \dots, I_n)}{\partial((x_1)_{m+1}, \dots, (x_n)_{m+1})} \neq 0, \quad (2.32)$$

and the diagonal prolongations $\tilde{\mathbf{X}}_1^m, \dots, \tilde{\mathbf{X}}_\ell^m$ are linearly independent at a generic point [5]. This allows us to express the coordinates $\mathbf{x}_{m+1} = \{(x_1)_{m+1}, \dots, (x_n)_{m+1}\}$ in terms of the remaining coordinates in \mathcal{M}^{m+1} and the constants k_1, \dots, k_n defined by the conditions $I_1 = k_1, \dots, I_n = k_n$. We stress that the set of constants of the motion (2.29) and (2.31) are frequently sufficient to deduce the superposition rules for the LH system \mathbf{X} (2.4) in a direct way, as it has already been explicitly shown in [7, 29] for some specific LH systems. Moreover, the existence of a large number of constants of the motion $F_{ij}^{(k)}$ (obtained through permutations) rather simplifies the computations, as it allows one to keep the number k low. Actually, in most of the explicit superposition rules worked out in [7, 29], it was sufficient to consider the function $F^{(2)}$ and its permutations $F_{ij}^{(2)}$, a fact that helped to avoid long cumbersome computations enabling to establish in closed form a superposition principle of reasonable simplicity. However, as we shall prove in the sequel, the functions (2.29) and (2.31) will not generically provide us, in the case of prolonged deformations of a LH system, with a sufficient number of functionally independent constants of the motion from which a deformed superposition rule could easily be inferred.

2.3.2 Deformed constants of the motion

As it has been already stated, the fact that $C^\infty(\mathcal{H}_{z,\omega}^*)$ is a PH deformation of $C^\infty(\mathcal{H}_\omega^*)$ enables us to apply the coalgebra formalism proposed in [6] to construct t -independent constants of motion for the deformed LH system \mathbf{X}_z (2.15) with the deformed Hamiltonian h_z as given in (2.18), for which some examples were presented in [1]. This procedure must however be applied with some caution, as there are some subtle points that, if not taken into account, may invalidate the conclusions. The key point is to observe that, whenever we are considering a deformed PH algebra, the deformed coproduct Δ_z is no longer trivial (or primitive) as Δ (2.20) for all the generators v_i . Indeed, the deformation ‘breaks’ the symmetry within the coproduct, that is, the positions in the tensor product space within the coproduct are no longer ‘equivalent’, as happens e.g. for $\Delta^{(3)}(v_i)$ in (2.21).

By the construction in [36, 37], the deformed counterpart of the constants of the motion $F^{(k)}$ (2.29) still holds, but it is not ensured that the permutations in (2.31) give rise to t -independent constants of the motion for the prolonged deformation $\tilde{\mathbf{X}}_z^{m+1}$ to \mathcal{M}^{m+1} of \mathbf{X} . Therefore, in the deformed case one may need to consider a higher number k with respect to the non-deformed system to deduce the deformed superposition rule, which in turns makes $m+1$ to be larger. The drawback of considering an increased $m+1$ is that the complexity of the computations grows exponentially, resulting in an extremely involved derivation of the deformed superposition rule. Fortunately, this difficulty can be circumvented by considering a second set of constants of the motion, additionally to the $F^{(k)}$, which comes from the superintegrability property of integrable systems possessing Hopf algebra symmetry [4, 38, 39]. In the following we present the explicit derivation of both sets of deformed constants of the motion.

Let $\mathcal{H}_{z,\omega}$ be the deformed LH algebra of the deformed LH system \mathbf{X}_z (2.15) with Hamiltonian h_z given by (2.18). We take a basis with generators $\{v_1, \dots, v_\ell\}$ such that the Poisson brackets are given by (see (2.12)):

$$\{v_i, v_j\}_z = F_{z,ij}(v_1, \dots, v_\ell), \quad i, j = 1, \dots, \ell. \quad (2.33)$$

Proceeding as in the non-deformed case, we consider the deformed coproduct for the generators v_i :

$$\Delta_z : C^\infty(\mathcal{H}_{z,\omega}^*) \rightarrow C^\infty(\mathcal{H}_{z,\omega}^*) \otimes C^\infty(\mathcal{H}_{z,\omega}^*), \quad (2.34)$$

along with the k^{th} -order deformed coproduct map, $\Delta_z^{(k)}$, defined exactly as in (2.22), such that the limits

$$\lim_{z \rightarrow 0} \Delta_z = \Delta, \quad \lim_{z \rightarrow 0} \Delta_z^{(k)} = \Delta^{(k)}, \quad (2.35)$$

are satisfied.

Now recall that the deformed Hamiltonian functions $\{h_{z,1}, \dots, h_{z,\ell}\}$ fulfil the relations (2.14) and the Poisson brackets (2.12) with respect to the symplectic form ω . We define the map

$$\phi_z : \mathcal{H}_{z,\omega} \rightarrow C^\infty(\mathcal{M}), \quad h_{z,i} := \phi_z(v_i), \quad i = 1, \dots, \ell, \quad (2.36)$$

where again $\mathcal{M} \subset M$ is chosen to guarantee that the functions $h_{z,i}$ are properly defined. Next, as in (2.25), we introduce the Poisson algebra morphisms

$$D_z : C^\infty(\mathcal{H}_{z,\omega}^*) \rightarrow C^\infty(\mathcal{M}), \quad D_z^{(k)} : \overbrace{C^\infty(\mathcal{H}_{z,\omega}^*) \otimes \dots \otimes C^\infty(\mathcal{H}_{z,\omega}^*)}^{k \text{ times}} \rightarrow \overbrace{C^\infty(\mathcal{M}) \otimes \dots \otimes C^\infty(\mathcal{M})}^{k \text{ times}}. \quad (2.37)$$

Now let

$$C_z = C_z(v_1, \dots, v_\ell), \quad (2.38)$$

be the Casimir function of $C^\infty(\mathcal{H}_{z,\omega}^*)$ with $\lim_{z \rightarrow 0} C_z = C$. And we define the functions (see (2.26))

$$D_z(v_i) = h_{z,i} := h_{z,i}^{(1)}, \quad D_z^{(k)}(\Delta_z^{(k)}(v_i)) := h_{z,i}^{(k)}, \quad i = 1, \dots, \ell, \quad (2.39)$$

whose explicit form does depend on the initial deformed coproduct Δ_z . Anyhow, the analogous relation to (2.27) also holds:

$$D_z^{(k)}(u_1(v_1, \dots, v_\ell) \otimes \dots \otimes u_k(v_1, \dots, v_\ell)) = u_1(h_{z,1}(\mathbf{x}_1), \dots, h_{z,\ell}(\mathbf{x}_1)) \dots u_k(h_{z,1}(\mathbf{x}_k), \dots, h_{z,\ell}(\mathbf{x}_k)), \quad (2.40)$$

with $u_j \in C^\infty(\mathcal{H}_{z,\omega}^*)$, allowing one to compute (2.39) from a given Δ_z .

The first set of constants of the motion for the prolonged deformation of the LH system \mathbf{X} , which is analogous to the right-hand side of (2.30), i.e.

$$\tilde{\mathbf{X}}_z^{m+1} = \sum_{i=1}^{\ell} b_i(t) \mathbf{X}_{h_{z,i}^{(m+1)}}, \quad (2.41)$$

is defined by

$$F_z := D_z(C_z), \quad F_z^{(k)}(h_{z,1}^{(k)}, \dots, h_{z,\ell}^{(k)}) := D_z^{(k)}\left[\Delta_z^{(k)}(C_z(v_1, \dots, v_\ell))\right], \quad k = 2, \dots, m+1, \quad (2.42)$$

which is just the deformed counterpart of (2.29). It is worth noting that $\mathbf{X}_{h_{z,i}^{(m+1)}}$ in (2.41) fulfil similar commutation relations to (2.17):

$$\left[\mathbf{X}_{h_{z,i}^{(m+1)}}, \mathbf{X}_{h_{z,j}^{(m+1)}}\right] = - \sum_{l=1}^{\ell} \frac{\partial F_{z,ij}}{\partial h_{z,l}^{(m+1)}}(h_{z,1}^{(m+1)}, \dots, h_{z,\ell}^{(m+1)}) \mathbf{X}_{h_{z,l}^{(m+1)}}, \quad i, j = 1, \dots, \ell. \quad (2.43)$$

If all $F_z^{(k)}$ (2.42) are non-constant functions, they provide a set of m functionally independent functions in involution [6, 36, 37]. Even if formally these invariants are sufficient to deduce a deformed superposition rule, it is doubtful that a closed analytical expression can be obtained, as the difficulty of the formulae increases exponentially when augmenting the order of the constants of the motion. The crucial difference with the undeformed case is that the validity of the permutation process (2.31) is not guaranteed any more, as a consequence of the ‘broken-symmetry’ of the deformed coproduct Δ_z in the tensor product space. Hence, the deformed prolongation $\tilde{\mathbf{X}}_z^{m+1}$ to \mathcal{M}^{m+1} of \mathbf{X} is not, in general, invariant relative to the interchange of variables $\mathbf{x}_i \leftrightarrow \mathbf{x}_j$. In fact, only under the non-deformed limit $z \rightarrow 0$, the coproduct Δ_z becomes primitive and $\tilde{\mathbf{X}}_z^{m+1}$ (2.41) reduces to $\tilde{\mathbf{X}}^{m+1}$ (2.30), being the latter symmetric under such

permutations. Thus, in principle, no additional constants of the motion can be obtained with this Ansatz for a generically prolonged deformation of a LH system. Nevertheless, following the approach to superintegrability of integrable systems with coalgebra symmetry [4, 38, 39], we can construct a second set of constants of the motion that is valid for any deformed LH system.

The essential point is that the k^{th} -order coproduct $\Delta_z^{(k)}$ is defined within the $(m+1)^{\text{th}}$ -order tensor product space in the form

$$\overbrace{C^\infty(\mathcal{H}_{z,\omega}^*) \otimes \dots \otimes C^\infty(\mathcal{H}_{z,\omega}^*)}^{k \text{ times}} \otimes \overbrace{1 \otimes \dots \otimes 1}^{(m+1-k) \text{ times}}, \quad (2.44)$$

and as a shorthand notation for the space where this object is defined we use

$$1 \otimes 2 \otimes \dots \otimes k. \quad (2.45)$$

However, instead of using (2.22), it is possible to define another recursion relation for the k^{th} -order coproduct, as done in [4, 38]:

$$\Delta_{zR}^{(k)} := (\Delta_z^{(2)} \otimes \overbrace{\text{Id} \otimes \dots \otimes \text{Id}}^{(k-2) \text{ times}}) \circ \Delta_{zR}^{(k-1)}, \quad k \geq 3. \quad (2.46)$$

Since we are considering products in the reversal ordering, it follows that $\Delta_{zR}^{(k)}$ lives in the $(m+1)^{\text{th}}$ -order tensor product space

$$\overbrace{1 \otimes \dots \otimes 1}^{(m+1-k) \text{ times}} \otimes \overbrace{C^\infty(\mathcal{H}_{z,\omega}^*) \otimes \dots \otimes C^\infty(\mathcal{H}_{z,\omega}^*)}^{k \text{ times}}, \quad (2.47)$$

which will be shortened as

$$(m-k+2) \otimes (m-k+3) \otimes \dots \otimes (m+1). \quad (2.48)$$

The maps $\Delta_z^{(k)}$ and $\Delta_{zR}^{(k)}$ are called left- and right-coproducts, respectively. For this reason, we call (2.42) the *set of left-constants of the motion* for the prolonged deformation $\tilde{\mathbf{X}}_z^{m+1}$ of the LH system, while the corresponding *set of right-constants of the motion* is defined by

$$F_{z(k)} \left(h_{zR,1}^{(k)}, \dots, h_{zR,\ell}^{(k)} \right) := D_{zR}^{(k)} \left[\Delta_{zR}^{(k)} (C_z(v_1, \dots, v_\ell)) \right], \quad k = 2, \dots, m+1, \quad (2.49)$$

where the morphisms $D_{zR}^{(k)}$ are defined as in (2.37), but now on the right-tensor product space (2.48), in such a manner that the functions $h_{zR,i}^{(k)}$ are defined by

$$D_{zR}^{(k)} (\Delta_{zR}^{(k)}(v_i)) := h_{zR,i}^{(k)}, \quad i = 1, \dots, \ell, \quad k = 2, \dots, m+1. \quad (2.50)$$

It is straightforward to verify that, due to the coassociativity property (2.9), the identity $\Delta_{zR}^{(m+1)} \equiv \Delta_z^{(m+1)}$ holds [4, 38], which implies that $F_{z(m+1)} \equiv F_z^{(m+1)}$. Again, if all the $F_{z(k)}$ are non-constant, they constitute a set of m functionally independent functions in involution. We stress that functional independence among all the integrals follows, by construction, from the different tensor product spaces on which they are defined. Furthermore, the two sets $F_z^{(k)}$ and $F_{z(k)}$ altogether provide a maximal number of $(2m-1)$ functionally independent constants of the motion which are valid for arbitrary PH deformations. Focusing on those functions having the lowest value of k , a closed analytical expression for the deformed superposition rule can be much more easily found that merely considering the set of left-constants of the motion.

For completeness in the exposition, we display the constants of the motion corresponding to both sets in Table 1 using the shorthand notations (2.45) and (2.48). Under the non-deformed limit $z \rightarrow 0$, the left-set $F_z^{(k)}$ (2.42) reduces to $F^{(k)}$ (2.29), while the right-set $F_{z(k)}$ (2.49) provides constants of the motion $F_{(k)}$ for the undeformed LH system that are expressible in terms of the set of permutations (2.31).

Table 1: Left- and right-constants of the motion for a prolonged Poisson–Hopf deformation of a Lie–Hamilton system coming from a Casimir C_z . By construction, there is a maximal number of $(2m - 1)$ functionally independent constants of the motion since $F_{z(m+1)} \equiv F_z^{(m+1)}$.

Set of m left-constants $F_z^{(k)}$ in involution	Tensor product space for the coproduct
$F_z^{(2)} := D_z^{(2)} [\Delta_z^{(2)}(C_z)]$	$1 \otimes 2$
$F_z^{(3)} := D_z^{(3)} [\Delta_z^{(3)}(C_z)]$	$1 \otimes 2 \otimes 3$
\vdots	\vdots
$F_z^{(k)} := D_z^{(k)} [\Delta_z^{(k)}(C_z)]$	$1 \otimes 2 \otimes \dots \otimes k$
\vdots	\vdots
$F_z^{(m+1)} := D_z^{(m+1)} [\Delta_z^{(m+1)}(C_z)]$	$1 \otimes 2 \otimes \dots \otimes m \otimes (m + 1)$
Set of m right-constants $F_{z(k)}$ in involution	Tensor product space for the coproduct
$F_{z(2)} := D_{zR}^{(2)} [\Delta_{zR}^{(2)}(C_z)]$	$m \otimes (m + 1)$
$F_{z(3)} := D_{zR}^{(3)} [\Delta_{zR}^{(3)}(C_z)]$	$(m - 1) \otimes m \otimes (m + 1)$
\vdots	\vdots
$F_{z(k)} := D_{zR}^{(k)} [\Delta_{zR}^{(k)}(C_z)]$	$(m - k + 2) \otimes (m - k + 3) \otimes \dots \otimes (m + 1)$
\vdots	\vdots
$F_{z(m+1)} = F_z^{(m+1)} := D_{zR}^{(m+1)} [\Delta_{zR}^{(m+1)}(C_z)]$	$1 \otimes 2 \otimes \dots \otimes m \otimes (m + 1)$

3 Oscillator Lie–Hamilton systems

LH systems on the manifold $M \equiv \mathbb{R}^2$ were fully classified in [2], basing on a previous classification of Lie algebras of vector fields in the real plane obtained in [40]. It turns out that there are 12 equivalence classes of finite-dimensional Lie algebras of Hamiltonian vector fields on \mathbb{R}^2 . For most of these planar LH systems, the constants of the motion and the superposition rules were inspected in [7]. The simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, that appears three times in the classification, has been studied in detail from both the non-deformed and deformed viewpoints (see e.g. [1–3, 7]). In this section, we focus on the physically relevant oscillator \mathfrak{h}_4 -LH systems on \mathbb{R}^2 , reviewing the main results and applications, with the aim of introducing its Hopf algebra deformation in Section 4, where both deformed constants of the motion and deformed superposition rules will be determined, as an illustration of the refinement of the deformation procedure presented above.

Let us consider the class I_8 in the classification of real Lie algebras of Hamiltonian vector fields with global coordinates $\mathbf{x} = \{x_1, x_2\} \equiv \{x, y\}$ on \mathbb{R}^2 obtained in [2]. The Vessiot–Guldberg Lie algebra V is spanned by three generators

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad (3.1)$$

satisfying the Lie brackets

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \quad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_1, \quad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_2. \quad (3.2)$$

Hence V is isomorphic to the $(1 + 1)$ -dimensional Poincaré algebra $\mathfrak{iso}(1, 1)$. The Lie system \mathbf{X} (2.4) is given by

$$\mathbf{X}(t, x, y) = b_1(t) \frac{\partial}{\partial x} + b_2(t) \frac{\partial}{\partial y} + b_3(t) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right), \quad (3.3)$$

leading to the following first-order system

$$\begin{aligned}\frac{dx}{dt} &= b_1(t) + b_3(t)x, \\ \frac{dy}{dt} &= b_2(t) - b_3(t)y.\end{aligned}\tag{3.4}$$

The generators \mathbf{X}_i defined in (3.1) become Hamiltonian vector fields h_i with respect to the standard symplectic form

$$\omega = dx \wedge dy,\tag{3.5}$$

which, after application of (2.6), are found to be

$$h_1 = y, \quad h_2 = -x, \quad h_3 = xy, \quad h_0 = 1.\tag{3.6}$$

Note that the addition of a central generator h_0 is required to ensure that the corresponding brackets close as a Lie algebra:

$$\{h_1, h_2\}_\omega = h_0, \quad \{h_1, h_3\}_\omega = -h_1, \quad \{h_2, h_3\}_\omega = h_2, \quad \{h_0, \cdot\}_\omega = 0.\tag{3.7}$$

It follows that the resulting LH algebra \mathcal{H}_ω is isomorphic to the centrally extended Poincaré algebra $\overline{\mathfrak{iso}}(1, 1)$, which is also isomorphic to the oscillator algebra \mathfrak{h}_4 . In particular, we consider the usual basis of $\mathfrak{h}_4 = \{A_-, A_+, N, I\}$ corresponding to the ladder, number and central generators, respectively. Under the identification

$$A_- = h_1, \quad A_+ = h_2, \quad N = -h_3, \quad I = h_0,\tag{3.8}$$

it is easily verified that the relations (3.7) are brought into the usual form for \mathfrak{h}_4 :

$$\{N, A_\pm\}_\omega = \pm A_\pm, \quad \{A_-, A_+\}_\omega = I, \quad \{I, \cdot\}_\omega = 0.\tag{3.9}$$

In the following we shall denote the oscillator LH algebra \mathcal{H}_ω (3.7) by $\mathfrak{h}_{4,\omega}$.

3.1 Constants of the motion and superposition rules

We now proceed to compute t -independent constants of the motion for the \mathfrak{h}_4 -LH systems and deduce the corresponding superposition rules.

The starting point is to consider the PH algebra $C^\infty(\mathcal{H}_\omega^*) \equiv C^\infty(\mathfrak{h}_{4,\omega}^*)$ in a basis $\{v_1, v_2, v_3, v_0\}$ satisfying the same Poisson brackets (3.7). Now, besides v_0 , there exists a non-trivial Casimir element given by

$$C = v_1 v_2 + v_3 v_0.\tag{3.10}$$

From C , applying the morphism $D : C^\infty(\mathfrak{h}_{4,\omega}^*) \rightarrow C^\infty(\mathbb{R}^2)$ (2.26) to the function F in (2.29), where h_i are given in (3.6), we find that the constant of the motion is trivial:

$$F = D(C) = h_1(x_1, y_1)h_2(x_1, y_1) + h_3(x_1, y_1)h_0(x_1, y_1) = -y_1 x_1 + x_1 y_1 \times 1 = 0.\tag{3.11}$$

As the index $m + 1$ in (2.29) equals 3 (see [7]), we have that $k = 2, 3$. By making use of the morphisms $D^{(k)}$ in (2.26) and the coproducts $\Delta^{(k)}$ in (2.22), we recursively construct the constants of the motion $F^{(2)}$ and $F^{(3)}$ with the aid of the functions $h_i^{(k)}$ (2.26):

$$\begin{aligned}F^{(2)} &= D^{(2)}[\Delta^{(2)}(C)] = (h_1(x_1, y_1) + h_1(x_2, y_2))(h_2(x_1, y_1) + h_2(x_2, y_2)) \\ &\quad + (h_3(x_1, y_1) + h_3(x_2, y_2))(h_0(x_1, y_1) + h_0(x_2, y_2)) \\ &= -(y_1 + y_2)(x_1 + x_2) + (x_1 y_1 + x_2 y_2)(1 + 1) \\ &= (x_1 - x_2)(y_1 - y_2).\end{aligned}\tag{3.12}$$

In the same way, $F^{(3)}$ is found to be

$$\begin{aligned} F^{(3)} &= D^{(3)}[\Delta^{(3)}(C)] = h_1^{(3)}h_2^{(3)} + h_3^{(3)}h_0^{(3)} \\ &= (x_1 - x_2)(y_1 - y_2) + (x_1 - x_3)(y_1 - y_3) + (x_2 - x_3)(y_2 - y_3) \\ &= (2x_1 - x_2 - x_3)y_1 + (2x_2 - x_1 - x_3)y_2 + (2x_3 - x_1 - x_2)y_3. \end{aligned} \quad (3.13)$$

The elements $F^{(2)}$ and $F^{(3)}$ are left-constants of the motion for the diagonal prolongation $\tilde{\mathbf{X}}^3$ to $(\mathbb{R}^2)^3$ of the LH system \mathbf{X} (3.3). It can be checked that they are in involution in $C^\infty((\mathbb{R}^2)^3)$, that is, they Poisson-commute with respect to the symplectic form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3. \quad (3.14)$$

From $F^{(2)}$ we obtain two additional constants of the motion through the permutations (2.31) (recall that $k = 2, 3$):

$$F_{13}^{(2)} = S_{13}(F^{(2)}) = (x_3 - x_2)(y_3 - y_2), \quad F_{23}^{(2)} = S_{23}(F^{(2)}) = (x_1 - x_3)(y_1 - y_3). \quad (3.15)$$

The remaining transposition is discarded, as $F_{12}^{(2)} = S_{12}(F^{(2)}) \equiv F^{(2)}$. As far as the right-constants of the motion $F_{(k)}$ are concerned, we have that $F_{(2)} \equiv F_{13}^{(2)}$ and $F_{(3)} \equiv F^{(3)}$ which are also in involution, while $F_{23}^{(2)}$ does not belong to any of the sets $\{F^{(k)}\}, \{F_{(k)}\}$.

The functions (3.12), (3.13) and (3.15) determine four t -independent constants of the motion for $\tilde{\mathbf{X}}^3$. They can also be considered as t -independent constants of the motion for \mathbf{X} (3.3). Moreover, there exist some constants k_i such that

$$F^{(2)} = k_1, \quad F_{23}^{(2)} = k_2, \quad F_{13}^{(2)} = k_3, \quad F^{(3)} = F^{(2)} + F_{23}^{(2)} + F_{13}^{(2)} = k_1 + k_2 + k_3 \equiv k. \quad (3.16)$$

With these results, we can explicitly derive a superposition rule. We recall that in [7] this was carried out by choosing $F^{(2)}$ and $F_{23}^{(2)}$, thus expressing (x_1, y_1) in terms of (x_2, y_2, x_3, y_3) and k_1, k_2 . The resulting expression was further simplified by also introducing k_3 , explicitly

$$\begin{aligned} x_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(x_2 + x_3) + \frac{k_2 - k_1 \pm B}{2(y_2 - y_3)}, \\ y_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(y_2 + y_3) + \frac{k_2 - k_1 \mp B}{2(x_2 - x_3)}, \end{aligned} \quad (3.17)$$

$$B = \sqrt{k_1^2 + k_2^2 + k_3^2 - 2(k_1k_2 + k_1k_3 + k_2k_3)},$$

where $k_3 = k_3(x_2, y_2, x_3, y_3)$ through $F_{13}^{(2)}$ in (3.15), and such that the following inequality is satisfied:

$$k_1^2 + k_2^2 + k_3^2 \geq 2(k_1k_2 + k_1k_3 + k_2k_3). \quad (3.18)$$

As we shall prove in Subsection 4.1, the constant $F_{23}^{(2)}$ will disappear under the deformation, implying that we only have to consider the three remaining (left- and right-) constants of the motion $F^{(2)}$, $F^{(3)} \equiv F_{(3)}$ and $F_{(2)} \equiv F_{13}^{(2)}$ of (3.16). Furthermore, it will turn out that the latter relation between the right-constant of the motion and the permuted one does not hold any more in this form. Therefore, in order to obtain a superposition rule that is consistent with the limit $z \rightarrow 0$ of the corresponding rule for the prolonged deformation of our LH system, we have to proceed without using $F_{23}^{(2)}$. To this extent, we start with $F^{(2)}$ and $F^{(3)}$, now writing (x_1, y_1) in terms of (x_2, y_2, x_3, y_3) and the constants k_1 and k (instead of k_2). Next we introduce the constant k_3 to simplify the superposition rule, so that we are led to the expressions

$$\begin{aligned} x_1(x_2, y_2, x_3, y_3, k_1, k, k_3) &= x_3 + \frac{k - 2k_1 \pm B}{2(y_2 - y_3)}, \\ y_1(x_2, y_2, x_3, y_3, k_1, k, k_3) &= y_3 + \frac{k - 2k_1 \mp B}{2(x_2 - x_3)}, \\ B &= \sqrt{(k - 2(k_1 + k_3))^2 - 4k_1k_3}, \end{aligned} \quad (3.19)$$

where again $k_3 = k_3(x_2, y_2, x_3, y_3)$ through $F_{(2)} \equiv F_{13}^{(2)}$, subjected to the constraint

$$(k - 2(k_1 + k_3))^2 \geq 4k_1k_3. \quad (3.20)$$

We remark that by introducing $k = k_1 + k_2 + k_3$ and $k_3 = (x_3 - x_2)(y_3 - y_2)$ in (3.19), we easily recover the formulae (3.17).

3.2 The book algebra and Lie–Hamilton systems

We observe from (3.9) that the generator N , along with either A_+ or A_- , span a two-dimensional subalgebra of \mathfrak{h}_4 isomorphic to the so-called ‘book’ algebra \mathfrak{b}_2 , where N can be seen as a dilation and A_\pm as a translation. In the basis of the LH algebra \mathfrak{h}_4 with commutators (3.2), we choose the subalgebra \mathfrak{b}_2 as the one generated by \mathbf{X}_2 and \mathbf{X}_3 :

$$\mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_2. \quad (3.21)$$

When \mathfrak{b}_2 is seen as a Vessiot–Guldberg Lie algebra, it gives rise to the particular Lie subsystem of (3.3) with $b_1(t) \equiv 0$:

$$\begin{aligned} \frac{dx}{dt} &= b_3(t)x, \\ \frac{dy}{dt} &= b_2(t) - b_3(t)y. \end{aligned} \quad (3.22)$$

The symplectic form (3.5) is kept invariant, while the Hamiltonian vector fields (3.6) for \mathfrak{b}_2 are given by

$$h_2 = -x, \quad h_3 = xy, \quad \{h_2, h_3\}_\omega = h_2. \quad (3.23)$$

We recall that \mathfrak{b}_2 arises within the classification of planar LH systems [2, 7] as the class $I_{14A}^{r=1} \simeq \mathbb{R} \times \mathbb{R} \simeq \mathfrak{b}_2$. Although \mathfrak{b}_2 does not admit non-constant Casimir invariants, its consideration as a particular case of the \mathfrak{h}_4 -LH systems allows us to apply the above results concerning constants of the motion and superposition rules, as it was pointed out in [7]. Furthermore, in spite of the apparently naive form of the differential equations (3.22), it is worthy to be remarked that \mathfrak{b}_2 -LH systems emerge in various physical and mathematical contexts such as [2, 7]:

- *Generalised Buchdahl equations*, which are second-order differential equations appearing in the study of relativistic fluids [41, 42] and have also been studied by means of a Lagrangian approach in [43].
- Some particular two-dimensional *Lotka–Volterra systems with t -dependent coefficients* [44, 45].
- *Complex Bernoulli differential equations with t -dependent real coefficients* [46], which are the particular case of the non-autonomous complex Bernoulli differential equations with complex coefficients [47, 48].

In what follows, we focus on the third type of \mathfrak{b}_2 -systems and its PH deformation will be obtained in Subsection 4.3. The two remaining types can also be developed in similar manner, although computations are rather cumbersome due to the complicated symplectic structure that arises, as well as the change of variables required to relate such systems to the expressions (3.21)–(3.23).

3.3 Complex Bernoulli differential equations

Let us consider the family of non-autonomous complex Bernoulli differential equations

$$\frac{dw}{dt} = a_1(t)w + a_2(t)w^s, \quad s \notin \{0, 1\}, \quad (3.24)$$

where w is a complex function and $a_1(t), a_2(t)$ are arbitrary *real* valued t -dependent functions. Introducing the polar reference $w = re^{i\theta}$, we obtain that the differential equation (3.24) unfolds as the real first-order system

$$\begin{aligned}\frac{dr}{dt} &= a_1(t)r + a_2(t)r^s \cos[\theta(s-1)], \\ \frac{d\theta}{dt} &= a_2(t)r^{s-1} \sin[\theta(s-1)],\end{aligned}\tag{3.25}$$

which can be expressed through the t -dependent vector field

$$\mathbf{Y}(t, r, \theta) = a_1(t)\mathbf{Y}_1 + a_2(t)\mathbf{Y}_2,\tag{3.26}$$

where

$$\mathbf{Y}_1 = r \frac{\partial}{\partial r}, \quad \mathbf{Y}_2 = r^s \cos[\theta(s-1)] \frac{\partial}{\partial r} + r^{s-1} \sin[\theta(s-1)] \frac{\partial}{\partial \theta}.\tag{3.27}$$

The corresponding Lie bracket

$$[\mathbf{Y}_1, \mathbf{Y}_2] = (s-1)\mathbf{Y}_2,\tag{3.28}$$

shows that \mathbf{Y} is a Lie system with Vessiot–Guldberg Lie algebra V isomorphic to \mathfrak{b}_2 .

The next step is to determine a symplectic form $\omega = f(r, \theta)dr \wedge d\theta$ compatible with the vector fields (3.27) by requiring the relation (2.5) to be satisfied. A routine computation shows that ω can be chosen as

$$\omega = \frac{s-1}{r \sin^2[\theta(s-1)]} dr \wedge d\theta.\tag{3.29}$$

Therefore \mathbf{Y} (3.26) is a LH system whose Hamiltonian functions \bar{h}_i , deduced by means of the relation (2.6), are given by

$$\bar{h}_1 = -\frac{1}{\tan[\theta(s-1)]}, \quad \bar{h}_2 = -\frac{r^{s-1}}{\sin[\theta(s-1)]}.\tag{3.30}$$

The Poisson bracket with respect to the symplectic form (3.29) reads

$$\{\bar{h}_1, \bar{h}_2\}_\omega = -(s-1)\bar{h}_2.\tag{3.31}$$

Now our task consists in establishing the relationship of these results with (3.21)–(3.23). This is done considering the change of variables given by

$$\begin{aligned}x &= \frac{r^{s-1}}{\sin[\theta(s-1)]}, & y &= -\frac{\cos[\theta(s-1)]}{(s-1)r^{s-1}}, \\ r^{2(s-1)} &= \frac{x^2}{1+(s-1)^2x^2y^2}, & \tan^2[\theta(s-1)] &= \frac{1}{(s-1)^2x^2y^2}.\end{aligned}\tag{3.32}$$

Under these transformations, the symplectic form (3.29) adopts the canonical form (3.5), while the relations amongst vector fields and t -dependent coefficients are given by

$$\begin{aligned}\mathbf{Y}_1 &= (s-1)\mathbf{X}_3, & \mathbf{Y}_2 &= \mathbf{X}_2, & \bar{h}_1 &= (s-1)h_3, & \bar{h}_2 &= h_2, \\ a_1(t) &= b_3(t)/(s-1), & a_2(t) &= b_2(t).\end{aligned}\tag{3.33}$$

With the relations (3.32) at hand, it is straightforward to obtain the constants of the motion for the complex Bernoulli differential equations (3.25). The three functions $F^{(2)}$, $F^{(3)}$ and $F_{(2)} \equiv F_{13}^{(2)}$ (see (3.12), (3.13) and (3.15) respectively) have the explicit form

$$\begin{aligned}F^{(2)} &= \frac{1}{1-s} \left(\frac{r_1^{s-1}}{\sin[\theta_1(s-1)]} - \frac{r_2^{s-1}}{\sin[\theta_2(s-1)]} \right) \left(\frac{\cos[\theta_1(s-1)]}{r_1^{s-1}} - \frac{\cos[\theta_2(s-1)]}{r_2^{s-1}} \right), \\ F_{(2)} &= \frac{1}{1-s} \left(\frac{r_2^{s-1}}{\sin[\theta_2(s-1)]} - \frac{r_3^{s-1}}{\sin[\theta_3(s-1)]} \right) \left(\frac{\cos[\theta_2(s-1)]}{r_2^{s-1}} - \frac{\cos[\theta_3(s-1)]}{r_3^{s-1}} \right), \\ F^{(3)} &= \frac{1}{1-s} \sum_{1 \leq i < j \leq 3} \left(\frac{r_i^{s-1}}{\sin[\theta_i(s-1)]} - \frac{r_j^{s-1}}{\sin[\theta_j(s-1)]} \right) \left(\frac{\cos[\theta_i(s-1)]}{r_i^{s-1}} - \frac{\cos[\theta_j(s-1)]}{r_j^{s-1}} \right).\end{aligned}\tag{3.34}$$

These are the (left- and right-) constants of the motion used to derive the superposition rule for \mathfrak{h}_4 -LH systems (3.19); for the Bernoulli equations this corresponds to express (r_1, θ_1) in terms of $(r_2, \theta_2, r_3, \theta_3)$ and the constants k_1, k . By introducing (3.32) in (3.19), we directly infer the superposition rule for the complex Bernoulli differential equations (3.25) in implicit form:

$$\begin{aligned} \frac{r_1^{s-1}}{\sin[\theta_1(s-1)]} &= \frac{r_3^{s-1}}{\sin[\theta_3(s-1)]} + (1-s) \frac{k - 2k_1 \pm B}{2 \left(\frac{\cos[\theta_2(s-1)]}{r_2^{s-1}} - \frac{\cos[\theta_3(s-1)]}{r_3^{s-1}} \right)}, \\ \frac{\cos[\theta_1(s-1)]}{r_1^{s-1}} &= \frac{\cos[\theta_3(s-1)]}{r_3^{s-1}} + (1-s) \frac{k - 2k_1 \mp B}{2 \left(\frac{r_2^{s-1}}{\sin[\theta_2(s-1)]} - \frac{r_3^{s-1}}{\sin[\theta_3(s-1)]} \right)}, \end{aligned} \quad (3.35)$$

where the constants k_1, k, k_3 and the function B are the same as in (3.19).

4 Prolonged deformations of oscillator Lie–Hamilton systems

Multiparametric coboundary Lie bialgebras for the oscillator Lie algebra $\mathfrak{h}_4 = \{A_-, A_+, N, I\}$ (3.9) were classified in [8] along with their quantum deformations. This exhaustive analysis shows that mathematical and physical properties of each deformation are in direct correspondence with the generators that remain undeformed, that is, with a primitive (trivial) coproduct (2.20). As the central generator I is always primitive, one should additionally require either N or a single A_\pm to be primitive as well. It turns out that all (multiparametric) deformations with N primitive lead to quantum deformations that are governed by I [8], with N behaving as a ‘secondary’ primitive generator. In the context of LH systems this implies that these quantum deformations give rise to ‘trivial’ LH systems (recall that $I = h_0 = 1$ in (3.8)). By contrast, deformations with a primitive $A_+ = -x$ (or $A_- = y$) provide non-trivial LH systems, as in these cases A_+ plays the role of the ‘main’ primitive generator, with I playing the role of a ‘secondary’ one.

The simplest (i.e. one-parameter) quantum deformation such that A_+ is primitive corresponds to consider the classical r -matrix

$$r = z A_+ \wedge N, \quad (4.1)$$

which is a solution of the classical Yang–Baxter equation, and where z is the quantum deformation parameter such that $q = e^z$. This element underlies the so-called nonstandard (or Jordanian) quantum oscillator algebra $U_z(\mathfrak{h}_4)$, whose boson representations have been studied in [49, 50].

In the LH framework, we start with the Lie algebra \mathfrak{h}_4 in the basis $\{v_1, v_2, v_3, v_0\}$ with Lie brackets (see (3.7))

$$[v_1, v_2] = v_0, \quad [v_1, v_3] = -v_1, \quad [v_2, v_3] = v_2, \quad [v_0, \cdot] = 0, \quad (4.2)$$

as well as with the classical r -matrix

$$r = z v_3 \wedge v_2. \quad (4.3)$$

The Lie bialgebra is provided by the cocommutator map δ that is obtained from the classical r -matrix as

$$\delta(v_i) = [v_i \otimes 1 + 1 \otimes v_i, r], \quad (4.4)$$

yielding

$$\delta(v_2) = \delta(v_0) = 0, \quad \delta(v_1) = z(v_2 \wedge v_1 + v_3 \wedge v_0), \quad \delta(v_3) = z v_2 \wedge v_3, \quad (4.5)$$

which is just the skew-symmetric part of the first-order term Δ_1 in z of the full coproduct Δ_z , that is,

$$\Delta_z(v_i) = \Delta_0(v_i) + \Delta_1(v_i) + o[z^2], \quad \Delta_0(v_i) = v_i \otimes 1 + 1 \otimes v_i, \quad \delta(v_i) = \Delta_1(v_i) - \sigma \circ \Delta_1(v_i), \quad (4.6)$$

where σ is the flip operator: $\sigma(v_i \otimes v_j) = v_j \otimes v_i$.

From the complete quantum algebra $U_z(\mathfrak{h}_4)$ [8, 49], the corresponding Poisson coalgebra structure can easily be deduced giving rise to the following deformed coproduct and Poisson brackets:

$$\begin{aligned}\Delta_z(v_2) &= v_2 \otimes 1 + 1 \otimes v_2, & \Delta_z(v_0) &= v_0 \otimes 1 + 1 \otimes v_0, \\ \Delta_z(v_1) &= v_1 \otimes e^{-zv_2} + 1 \otimes v_1 + zv_3 \otimes e^{-zv_2} v_0, & \Delta_z(v_3) &= v_3 \otimes e^{-zv_2} + 1 \otimes v_3,\end{aligned}\quad (4.7)$$

$$\{v_1, v_2\}_z = e^{-zv_2} v_0, \quad \{v_1, v_3\}_z = -v_1, \quad \{v_2, v_3\}_z = \frac{1 - e^{-zv_2}}{z}, \quad \{v_0, \cdot\}_z = 0, \quad (4.8)$$

such that Δ_z (4.7) satisfies the coassociativity condition (2.9) and is a Poisson algebra homomorphism of the Poisson brackets (4.8). The deformed Casimir turns out to be

$$C_z = v_1 \left(\frac{e^{zv_2} - 1}{z} \right) + v_3 v_0. \quad (4.9)$$

Now we apply the algorithmic procedure summarized in Subsection 2.2 to construct a PH deformation $C^\infty(\mathfrak{h}_{z4,\omega}^*)$ of $C^\infty(\mathfrak{h}_{4,\omega}^*)$, hence deforming the \mathfrak{h}_4 -LH systems of Section 3. To this extent, we start from the functions $\{h_1, h_2, h_3, h_0\}$ on $C^\infty(\mathbb{R}^2)$ as given in (3.6) with Poisson brackets (3.7), where ω is the canonical symplectic form (3.5). Taking into account the boson representations of $U_z(\mathfrak{h}_4)$ given in [49, 50], we introduce the Hamiltonian functions on $C^\infty(\mathbb{R}^2)$

$$h_{z,1} = e^{zx}y, \quad h_{z,2} = -x, \quad h_{z,3} = \left(\frac{e^{zx} - 1}{z} \right) y, \quad h_{z,0} = 1, \quad (4.10)$$

which satisfy the following Poisson brackets with respect to the same symplectic form (3.5)

$$\begin{aligned}\{h_{z,1}, h_{z,2}\}_\omega &= e^{-zh_{z,2}} h_{z,0}, & \{h_{z,1}, h_{z,3}\}_\omega &= -h_{z,1}, \\ \{h_{z,2}, h_{z,3}\}_\omega &= \frac{1 - e^{-zh_{z,2}}}{z}, & \{h_{z,0}, \cdot\}_\omega &= 0,\end{aligned}\quad (4.11)$$

in agreement with the relations (4.8). In the third step, the deformed vector fields $\mathbf{X}_{z,i}$ on \mathbb{R}^2 are obtained through the relation (2.14), namely

$$\mathbf{X}_{z,1} = e^{zx} \frac{\partial}{\partial x} - ze^{zx}y \frac{\partial}{\partial y}, \quad \mathbf{X}_{z,2} = \frac{\partial}{\partial y}, \quad \mathbf{X}_{z,3} = \left(\frac{e^{zx} - 1}{z} \right) \frac{\partial}{\partial x} - e^{zx}y \frac{\partial}{\partial y}. \quad (4.12)$$

Finally, the PH deformation of the \mathfrak{h}_4 -LH system (3.3) is determined by

$$\mathbf{X}_z(t, x, y) = b_1(t) \mathbf{X}_{z,1} + b_2(t) \mathbf{X}_{z,2} + b_3(t) \mathbf{X}_{z,3}, \quad (4.13)$$

leading to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= b_1(t) e^{zx} + b_3(t) \left(\frac{e^{zx} - 1}{z} \right), \\ \frac{dy}{dt} &= b_2(t) - (b_3(t) + z b_1(t)) e^{zx}y.\end{aligned}\quad (4.14)$$

It is worth remarking that since ω is the standard symplectic form (3.5), the same system of differential equations (4.14) can, alternatively, be obtained by computing the usual Hamilton equations from the deformed Hamiltonian (2.18) with the functions (4.10),

$$h_z = b_1(t) e^{zx}y - b_2(t)x + b_3(t) \left(\frac{e^{zx} - 1}{z} \right) y + b_0(t), \quad (4.15)$$

in the form

$$\frac{dx}{dt} = \frac{\partial h_z}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial h_z}{\partial x}. \quad (4.16)$$

As we have already commented, the deformed vector fields (4.12) span a Stefan–Sussman distribution [30, 34, 35] whose commutation rules (2.17) turn out to be

$$[\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = z e^{-zh_{z,2}} h_{z,0} \mathbf{X}_{z,2}, \quad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = \mathbf{X}_{z,1}, \quad [\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = -e^{-zh_{z,2}} \mathbf{X}_{z,2}. \quad (4.17)$$

By introducing the functions (4.10) we obtain that

$$[\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = z e^{zx} \mathbf{X}_{z,2}, \quad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = \mathbf{X}_{z,1}, \quad [\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = -e^{zx} \mathbf{X}_{z,2}. \quad (4.18)$$

Note that that the expressions (4.10)–(4.18) reduce to (3.1)–(3.7) in the limit $z \rightarrow 0$. It is worth stressing that, even considering the commutation relations (4.18) up to first-order in the deformation parameter z , the vector fields $\mathbf{X}_{z,1}, \mathbf{X}_{z,2}, \mathbf{X}_{z,3}$ do not close on a finite-dimensional Lie algebra.

The remarkable feature of the deformation is the presence of the ‘interacting’ term $e^{zx}y$ in (4.14), when compared with (3.4). This nonlinear interaction or coupling between the two variables can be regarded as a perturbation of the initial system. Indeed, by considering a power series expansion in z of the system (4.14) and truncating at the first-order, we obtain

$$\begin{aligned} \frac{dx}{dt} &= b_1(t) + (b_3(t) + z b_1(t))x + \frac{1}{2} z b_3(t)x^2 + o[z^2], \\ \frac{dy}{dt} &= b_2(t) - (b_3(t) + z b_1(t))y - z b_3(t)xy + o[z^2]. \end{aligned} \quad (4.19)$$

In the first equation, the deformation introduces a quadratic term x^2 , leading to a Riccati equation with t -dependent real coefficients [12], while in the second one, we obtain the nonlinear interaction term xy .

4.1 Deformed constants of the motion and deformed superposition rules

Now we proceed to apply the approach presented in Subsection 2.3.2 in order to obtain three deformed constants of the motion $F_z^{(2)}$, $F_{z(2)}$ and $F_z^{(3)} \equiv F_{z(3)}$ (see Table 1) for the prolonged deformation of \mathfrak{h}_4 -LH systems, as in this case (see Subsection 3.1) we have the indices $m+1=3$ and $k=2,3$.

Before considering the coalgebra structure, we note that the morphisms $\phi_z : \mathfrak{h}_{z4,\omega} \rightarrow C^\infty(\mathbb{R}^2)$ in (2.36) and $D_z : C^\infty(\mathfrak{h}_{z4,\omega}^*) \rightarrow C^\infty(\mathbb{R}^2)$ in (2.37) lead to

$$D_z(v_i) = h_{z,i}(x_1, y_1) := h_{z,i}^{(1)}, \quad i = 0, 1, 2, 3, \quad (4.20)$$

where $h_{z,i}$ are the Hamiltonian functions (4.10) fulfilling (4.11). By introducing this result into the Casimir (4.9), we find that, as expected, the corresponding constant F_z of (2.42) is again trivial:

$$F_z = D_z(C_z) = h_{z,1}^{(1)} \left(\frac{e^{zh_{z,2}} - 1}{z} \right) + h_{z,3}^{(1)} h_{z,0}^{(1)} = 0. \quad (4.21)$$

Now we consider the deformed coproduct $\Delta_z \equiv \Delta_z^{(2)}$ (4.7) on the tensor product space $1 \otimes 2$ and compute the elements $D_z^{(2)}(\Delta_z^{(2)}(v_i))$ coming from the morphism $D_z^{(2)}$ in (2.37) yielding the functions $h_{z,i}^{(2)}$ (2.39) by means of (2.40):

$$\begin{aligned} D_z^{(2)}(\Delta_z^{(2)}(v_2)) &= h_{z,2}(x_1, y_1) + h_{z,2}(x_2, y_2) = -x_1 - x_2 := h_{z,2}^{(2)}, \\ D_z^{(2)}(\Delta_z^{(2)}(v_0)) &= h_{z,0}(x_1, y_1) + h_{z,0}(x_2, y_2) = 1 + 1 = 2 := h_{z,0}^{(2)}, \\ D_z^{(2)}(\Delta_z^{(2)}(v_1)) &= h_{z,1}(x_1, y_1) e^{-zh_{z,2}(x_2, y_2)} + h_{z,1}(x_2, y_2) + z h_{z,3}(x_1, y_1) e^{-zh_{z,2}(x_2, y_2)} h_{z,0}(x_2, y_2) \\ &= e^{zx_1} e^{zx_2} y_1 + e^{zx_2} y_2 + (e^{zx_1} - 1) e^{zx_2} y_1 := h_{z,1}^{(2)}, \\ D_z^{(2)}(\Delta_z^{(2)}(v_3)) &= h_{z,3}(x_1, y_1) e^{-zh_{z,2}(x_2, y_2)} + h_{z,3}(x_2, y_2) \\ &= \left(\frac{e^{zx_1} - 1}{z} \right) e^{zx_2} y_1 + \left(\frac{e^{zx_2} - 1}{z} \right) y_2 := h_{z,3}^{(2)}. \end{aligned} \quad (4.22)$$

These expressions allow us to obtain the left-constant of the motion of (2.42) for $k = 2$:

$$F_z^{(2)} = D_z^{(2)} [\Delta_z^{(2)}(C_z)] = h_{z,1}^{(2)} \left(\frac{e^{zh_{z,2}^{(2)}} - 1}{z} \right) + h_{z,3}^{(2)} h_{z,0}^{(2)}, \quad (4.23)$$

namely

$$F_z^{(2)} = \left(\frac{2 - e^{-zx_1} - e^{zx_2}}{z} \right) (y_1 - y_2). \quad (4.24)$$

Similarly, the right-constant of the motion $F_{z(2)}$ defined in (2.49) is deduced, but now working with the right-coproduct $\Delta_{zR}^{(2)}$ on the tensor product space $2 \otimes 3$ such that the functions $h_{zR,i}^{(2)}$ (2.50) turn out to be

$$\begin{aligned} D_{zR}^{(2)}(\Delta_{zR}^{(2)}(v_2)) &= h_{z,2}(x_2, y_2) + h_{z,2}(x_3, y_3) = -x_2 - x_3 =: h_{zR,2}^{(2)}, \\ D_{zR}^{(2)}(\Delta_{zR}^{(2)}(v_0)) &= h_{z,0}(x_2, y_2) + h_{z,0}(x_3, y_3) = 1 + 1 = 2 =: h_{zR,0}^{(2)}, \\ D_{zR}^{(2)}(\Delta_{zR}^{(2)}(v_1)) &= h_{z,1}(x_2, y_2)e^{-zh_{z,2}(x_3, y_3)} + h_{z,1}(x_3, y_3) + z h_{z,3}(x_2, y_2)e^{-zh_{z,2}(x_3, y_3)} h_{z,0}(x_3, y_3) \\ &= e^{zx_2} e^{zx_3} y_2 + e^{zx_3} y_3 + (e^{zx_2} - 1) e^{zx_3} y_2 =: h_{zR,1}^{(2)}, \\ D_{zR}^{(2)}(\Delta_{zR}^{(2)}(v_3)) &= h_{z,3}(x_2, y_2)e^{-zh_{z,2}(x_3, y_3)} + h_{z,3}(x_3, y_3) \\ &= \left(\frac{e^{zx_2} - 1}{z} \right) e^{zx_3} y_2 + \left(\frac{e^{zx_3} - 1}{z} \right) y_3 =: h_{zR,3}^{(2)}, \end{aligned} \quad (4.25)$$

giving rise, through the analogous expression to (4.23), to

$$F_{z(2)} = \left(\frac{2 - e^{-zx_2} - e^{zx_3}}{z} \right) (y_2 - y_3). \quad (4.26)$$

The non-deformed limit $z \rightarrow 0$ of the expressions (4.24) and (4.26) yields the functions $F^{(2)}$ (3.12) and $F_{(2)} \equiv F_{13}^{(2)} = S_{13}(F^{(2)})$ (3.15). Nevertheless, we stress that in the deformed case, the constant of the motion $F_z^{(2)}$ does not remain invariant under the permutation S_{12} and, moreover, $F_{z(2)}$ is related to $F_z^{(2)}$ through the composition of two permutations which differs from the result obtained merely applying the permutation S_{13} :

$$F_z^{(2)} \neq S_{12}(F_z^{(2)}), \quad F_{z(2)} = S_{12}(S_{23}(F_z^{(2)})) \neq S_{13}(F_z^{(2)}). \quad (4.27)$$

In addition, there is no deformed constant of the motion that corresponds in the limit $z \rightarrow 0$ to $F_{23}^{(2)} = S_{23}(F^{(2)})$ in (3.15). In fact, it is straightforward to check that the functions obtained from $F_z^{(2)}$ by means of the permutations S_{12} , S_{13} and S_{23} ,

$$\begin{aligned} S_{12}(F_z^{(2)}) &= \left(\frac{2 - e^{-zx_2} - e^{zx_1}}{z} \right) (y_2 - y_1), & S_{13}(F_z^{(2)}) &= \left(\frac{2 - e^{-zx_3} - e^{zx_2}}{z} \right) (y_3 - y_2), \\ S_{23}(F_z^{(2)}) &= \left(\frac{2 - e^{-zx_1} - e^{zx_3}}{z} \right) (y_1 - y_3), \end{aligned} \quad (4.28)$$

do not provide any constant of the motion. This shows that the range of application of the permutations S_{ij} in the form (2.31) is rather limited in the deformed case, where only left- and right-constants of the motion can be ensured to be correct.

It remains to compute $F_z^{(3)} \equiv F_{z(3)}$, which requires to construct the third-order coproduct $\Delta_z^{(3)}$ of (4.7) on the tensor product space $1 \otimes 2 \otimes 3$. In this case, with $m + 1 = 3$, $\Delta_z^{(3)}$ is obtained by means of the coassociativity condition (2.9) (corresponding to (2.22) and (2.46) with $k = m + 1 = 3$)

$$\Delta_z^{(3)} = (\text{Id} \otimes \Delta_z) \circ \Delta_z = (\Delta_z \otimes \text{Id}) \circ \Delta_z = \Delta_{zR}^{(3)}, \quad (4.29)$$

leading to

$$\begin{aligned}
\Delta_z^{(3)}(v_l) &= v_l \otimes 1 \otimes 1 + 1 \otimes v_l \otimes 1 + 1 \otimes 1 \otimes v_l, \quad l = 0, 2, \\
\Delta_z^{(3)}(v_1) &= v_1 \otimes e^{-zv_2} \otimes e^{-zv_2} + 1 \otimes v_1 \otimes e^{-zv_2} + 1 \otimes 1 \otimes v_1 \\
&\quad + z \left(v_3 \otimes e^{-zv_2} v_0 \otimes e^{-zv_2} + v_3 \otimes e^{-zv_2} \otimes e^{-zv_2} v_0 + 1 \otimes v_3 \otimes e^{-zv_2} v_0 \right), \\
\Delta_z^{(3)}(v_3) &= v_3 \otimes e^{-zv_2} \otimes e^{-zv_2} + 1 \otimes v_3 \otimes e^{-zv_2} + 1 \otimes 1 \otimes v_3,
\end{aligned} \tag{4.30}$$

provided that

$$\Delta_z(1) = 1 \otimes 1, \quad \Delta_z(e^{-zv_2}) = e^{-zv_2} \otimes e^{-zv_2}. \tag{4.31}$$

Then, by using (2.40), we obtain the Hamiltonian functions on $(\mathbb{R}^2)^3$ given by

$$\begin{aligned}
h_{z,2}^{(3)} &:= D_z^{(3)}(\Delta_z^{(3)}(v_2)) = -x_1 - x_2 - x_3, \quad h_{z,0}^{(3)} := D_z^{(3)}(\Delta_z^{(3)}(v_0)) = 3, \\
h_{z,1}^{(3)} &:= D_z^{(3)}(\Delta_z^{(3)}(v_1)) = (3e^{zx_1} - 2)e^{z(x_2+x_3)}y_1 + (2e^{zx_2} - 1)e^{zx_3}y_2 + e^{zx_3}y_3, \\
h_{z,3}^{(3)} &:= D_z^{(3)}(\Delta_z^{(3)}(v_3)) = \left(\frac{e^{zx_1} - 1}{z}\right)e^{z(x_2+x_3)}y_1 + \left(\frac{e^{zx_2} - 1}{z}\right)e^{zx_3}y_2 + \left(\frac{e^{zx_3} - 1}{z}\right)y_3.
\end{aligned} \tag{4.32}$$

By introducing them into (2.42) we get the third constant of the motion:

$$\begin{aligned}
F_z^{(3)} &= \frac{1}{z}(3 - 2e^{-zx_1} - e^{zx_2}e^{zx_3})y_1 + \frac{1}{z}(2e^{-zx_1} - e^{-zx_1}e^{-zx_2} - 2e^{zx_3} + e^{zx_2}e^{zx_3})y_2 \\
&\quad - \frac{1}{z}(3 - 2e^{zx_3} - e^{-zx_1}e^{-zx_2})y_3,
\end{aligned} \tag{4.33}$$

whose limit $z \rightarrow 0$ directly gives $F^{(3)}$ in the second form written in (3.13).

Summing up, the functions (4.32) satisfy the Poisson brackets (4.11) with respect to the symplectic form (3.14) and, with this ω , all the following Poisson brackets vanish ($i = 0, 1, 2, 3$):

$$\{F_z^{(2)}, h_{z,i}^{(3)}\}_\omega = \{F_{z(2)}, h_{z,i}^{(3)}\}_\omega = \{F_z^{(3)}, h_{z,i}^{(3)}\}_\omega = 0, \quad \{F_z^{(2)}, F_z^{(3)}\}_\omega = \{F_{z(2)}, F_z^{(3)}\}_\omega = 0. \tag{4.34}$$

Consequently, $F_z^{(2)}$, $F_{z(2)}$ and $F_z^{(3)}$, as given in (4.24), (4.26) and (4.33), are three functionally independent constants of the motion of the prolonged deformation $\tilde{\mathbf{X}}_z^3$ of \mathfrak{h}_4 -LH systems to $(\mathbb{R}^2)^3$.

Let us deduce now $\tilde{\mathbf{X}}_z^3$ in an explicit manner. By taking into account that ω (3.14) is the standard symplectic form, we consider the corresponding deformed Hamiltonian on $(\mathbb{R}^2)^3$,

$$h_z^{(3)} = b_1(t)h_{z,1}^{(3)} + b_2(t)h_{z,2}^{(3)} + b_3(t)h_{z,3}^{(3)} + b_0(t)h_{z,0}^{(3)}, \tag{4.35}$$

with the Hamiltonian functions (4.32), and compute the Hamilton equations (similarly to (4.16)), thus finding that $\tilde{\mathbf{X}}_z^3$ is given by the following system of six differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= b_1(t)(3e^{zx_1} - 2)e^{z(x_2+x_3)} + b_3(t)\left(\frac{e^{zx_1} - 1}{z}\right)e^{z(x_2+x_3)}, \\
\frac{dy_1}{dt} &= b_2(t) - (b_3(t) + 3zb_1(t))e^{z(x_1+x_2+x_3)}y_1, \\
\frac{dx_2}{dt} &= b_1(t)(2e^{zx_2} - 1)e^{zx_3} + b_3(t)\left(\frac{e^{zx_2} - 1}{z}\right)e^{zx_3}, \\
\frac{dy_2}{dt} &= b_2(t) - b_3(t)e^{z(x_2+x_3)}((e^{zx_1} - 1)y_1 + y_2) - zb_1(t)e^{z(x_2+x_3)}((3e^{zx_1} - 2)y_1 + 2y_2), \\
\frac{dx_3}{dt} &= b_1(t)e^{zx_3} + b_3(t)\left(\frac{e^{zx_3} - 1}{z}\right), \\
\frac{dy_3}{dt} &= b_2(t) - b_3(t)e^{zx_3}((e^{zx_1} - 1)e^{zx_2}y_1 + (e^{zx_2} - 1)y_2 + y_3) \\
&\quad - zb_1(t)e^{zx_3}((3e^{zx_1} - 2)e^{zx_2}y_1 + (2e^{zx_2} - 1)y_2 + y_3).
\end{aligned} \tag{4.36}$$

Under the non-deformed limit $z \rightarrow 0$, the prolonged deformation $\widetilde{\mathbf{X}}_z^3$ reduces to the diagonal prolongation $\widetilde{\mathbf{X}}^3$ of the \mathfrak{h}_4 -LH system \mathbf{X} (3.4) to $(\mathbb{R}^2)^3$ which simply corresponds to three copies of \mathbf{X} . On the contrary, it is remarkable that $\widetilde{\mathbf{X}}_z^3$ (4.36) is no longer formed by three copies of the deformed \mathfrak{h}_4 -LH system \mathbf{X}_z (4.14). Therefore, we stress that the constants of the motion of $\widetilde{\mathbf{X}}_z^3$ cannot be considered as constants of the motion of \mathbf{X}_z .

Furthermore, the deformed vector fields $\mathbf{X}_{h_{z,i}^{(3)}}$, which determine $\widetilde{\mathbf{X}}_z^3$ by means of the expression (2.41), can directly be deduced from (4.36); these are

$$\begin{aligned}\mathbf{X}_{h_{z,1}^{(3)}} &= (3e^{zx_1} - 2)e^{z(x_2+x_3)} \frac{\partial}{\partial x_1} + (2e^{zx_2} - 1)e^{zx_3} \frac{\partial}{\partial x_2} + e^{zx_3} \frac{\partial}{\partial x_3} - 3ze^{z(x_1+x_2+x_3)} y_1 \frac{\partial}{\partial y_1} \\ &\quad - ze^{z(x_2+x_3)} ((3e^{zx_1} - 2)y_1 + 2y_2) \frac{\partial}{\partial y_2} - ze^{zx_3} ((3e^{zx_1} - 2)e^{zx_2} y_1 + (2e^{zx_2} - 1)y_2 + y_3) \frac{\partial}{\partial y_3}, \\ \mathbf{X}_{h_{z,2}^{(3)}} &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \\ \mathbf{X}_{h_{z,3}^{(3)}} &= \left(\frac{e^{zx_1} - 1}{z} \right) e^{z(x_2+x_3)} \frac{\partial}{\partial x_1} + \left(\frac{e^{zx_2} - 1}{z} \right) e^{zx_3} \frac{\partial}{\partial x_2} + \left(\frac{e^{zx_3} - 1}{z} \right) \frac{\partial}{\partial x_3} - e^{z(x_1+x_2+x_3)} y_1 \frac{\partial}{\partial y_1} \\ &\quad - e^{z(x_2+x_3)} ((e^{zx_1} - 1)y_1 + y_2) \frac{\partial}{\partial y_2} - e^{zx_3} ((e^{zx_1} - 1)e^{zx_2} y_1 + (e^{zx_2} - 1)y_2 + y_3) \frac{\partial}{\partial y_3}.\end{aligned}\tag{4.37}$$

It can be checked that they fulfil the relationship (2.14) with respect to the Hamiltonian functions (4.32) and symplectic form (3.14). They satisfy the commutation relations (2.43), which are just those given in (4.17) with the functions (4.32), that is,

$$\begin{aligned}[\mathbf{X}_{h_{z,1}^{(3)}}, \mathbf{X}_{h_{z,2}^{(3)}}] &= 3ze^{z(x_1+x_2+x_3)} \mathbf{X}_{h_{z,2}^{(3)}}, & [\mathbf{X}_{h_{z,1}^{(3)}}, \mathbf{X}_{h_{z,3}^{(3)}}] &= \mathbf{X}_{h_{z,1}^{(3)}}, \\ [\mathbf{X}_{h_{z,2}^{(3)}}, \mathbf{X}_{h_{z,3}^{(3)}}] &= -e^{z(x_1+x_2+x_3)} \mathbf{X}_{h_{z,2}^{(3)}},\end{aligned}\tag{4.38}$$

to be compared with (4.18).

The three deformed constants of the motion satisfying (4.34) allow us to deduce a deformed superposition rule for $\widetilde{\mathbf{X}}_z^3$. We keep the notation of Subsection 3.1 and consider the three equations coming from (4.24), (4.26) and (4.33):

$$F_z^{(2)} = k_1, \quad F_z^{(3)} = k, \quad F_{z(2)} = k_3,\tag{4.39}$$

where k_1 , k and k_3 are constants. From the first two equations we can express (x_1, y_1) in terms of (x_2, y_2, x_3, y_3) and the constants k_1 and k . The third equation enables to write the result in a simplified manner, namely

$$\begin{aligned}e^{zx_1} &= \frac{(1 + e^{-zx_2} - e^{zx_3})(y_2 - y_3) + \frac{z}{2}(2k_1 - k \pm B)}{zk(e^{zx_2} - 2) - zk_1(e^{zx_2}e^{zx_3} - 3) + (e^{zx_2} - 2)(2e^{zx_3} - 3)(y_2 - y_3)}, \\ y_1 &= e^{-zx_2} y_3 + (1 - e^{-zx_2}) y_2 + \frac{z(k - 2k_1 \mp B)}{2e^{zx_2}(2 - e^{-zx_2} - e^{zx_3})}, \\ B &= \sqrt{(k - 2(k_1 + k_3))^2 - 4k_1 k_3}.\end{aligned}\tag{4.40}$$

The factor B is formally the same given in (3.19), while the constant k_3 , that only appears within B , should be understood as a function $k_3 = k_3(x_2, y_2, x_3, y_3)$ through $F_{z(2)}$ (4.26). Therefore, the expressions (4.40) constitute a generic deformed superposition rule corresponding to the prolonged deformation $\widetilde{\mathbf{X}}_z^3$ of \mathfrak{h}_4 -LH systems to $(\mathbb{R}^2)^3$ given by the system of differential equations (4.36). In order to compute their undeformed limit one should apply in (4.40) the limits

$$\lim_{z \rightarrow 0} \left(\frac{e^{zx_1} - 1}{z} \right), \quad \lim_{z \rightarrow 0} y_1,\tag{4.41}$$

thus recovering the proper superposition rule (3.19) for the \mathfrak{h}_4 -LH systems (3.4).

4.2 Twist maps and canonical transformations

The above results illustrate how to apply the PH deformation approach from a given quantum algebra in order to obtain the corresponding deformed LH systems. The essential tool in the formalism is the coproduct map so that one can start with either a coboundary quantum algebra (with an underlying classical r -matrix r) or with a non-coboundary one (without r -matrix). There exist two types of coboundary deformations [31]: quasitriangular (or standard) deformations whose r -matrix is a solution of the modified classical Yang–Baxter equation (like Drinfel’d–Jimbo deformations) and triangular (or nonstandard) deformations, for which the r -matrix is a solution of the classical Yang–Baxter equation (like the one considered in this paper for \mathfrak{h}_4 and also in [1, 3] for $\mathfrak{sl}(2, \mathbb{R})$). The latter quantum algebras are twist deformations [31, 51–53] and only for them it is guaranteed that there exists a basis for which the Hopf algebra structure can be written in terms of non-deformed commutation relations and a deformed coproduct. Therefore, in our example based on the nonstandard quantum algebra $U_z(\mathfrak{h}_4)$, it is rather natural to wonder whether the deformed LH systems \mathbf{X}_z (4.14) and $\tilde{\mathbf{X}}_z^3$ (4.36) could be transformed, respectively, into the classical one \mathbf{X} (3.4) and three copies of it by means of some change of variables. In what follows we solve this question although we advance that the answer is negative as it could be expected by taking into account the commutation relations (4.17) corresponding to a Stefan–Sussman distribution.

Let A be the trivial Hopf algebra of a given Lie algebra $\mathfrak{g} = \text{span}\{v_1, \dots, v_\ell\}$ defined by the Lie brackets of \mathfrak{g} and primitive coproduct map $\Delta_0(v_i) = v_i \otimes 1 + 1 \otimes v_i$. If \mathfrak{g} admits a quantum algebra deformation coming from a triangular (nonstandard) classical r -matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, then there exists the so-called twist operator F_z which is constructed as formal power series in the deformation parameter z and coefficients in $A \otimes A$ and must fulfil certain conditions [31, 51–53]. Explicitly, the deformed Hopf algebra A_z from A is given in terms of the same non-deformed commutation relations of A and a non-cocommutative coproduct map Δ_z which is obtained from the cocommutative one Δ_0 through

$$\Delta_z(v_i) = F_z \cdot \Delta_0(v_i) \cdot F_z^{-1}, \quad (4.42)$$

which ensures the coassociativity condition (2.9) for Δ_z . The twist operator for the quantum algebra $U_z(\mathfrak{h}_4)$ with classical r -matrix (4.3) is well-known and it was formerly obtained in [54] (see also [55] and references therein), namely

$$F_z = \exp(-v_3 \otimes \log(1 + zv_2)). \quad (4.43)$$

It is worth stressing that F_z was constructed in [54] on the Borel subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, which is isomorphic to the book algebra \mathfrak{b}_2 , giving rise to the nonstandard or Jordanian quantum $\mathfrak{sl}(2, \mathbb{R})$ algebra used in [1, 3] to deduce deformed $\mathfrak{sl}(2, \mathbb{R})$ -LH systems.

Now we consider \mathfrak{h}_4 with Lie brackets given by (4.2) and primitive coproduct Δ_0 . We denote the generators by \tilde{v}_i to distinguish them from (4.7) and (4.8). By applying (4.42) with the operator (4.43) we obtain the quantum algebra $U_z(\mathfrak{h}_4)$ with non-deformed commutation relations formally identical to (4.2) and deformed coproduct given by [56]

$$\begin{aligned} \Delta_z(\tilde{v}_2) &= \tilde{v}_2 \otimes 1 + 1 \otimes \tilde{v}_2 + z\tilde{v}_2 \otimes \tilde{v}_2, & \Delta_z(\tilde{v}_0) &= \tilde{v}_0 \otimes 1 + 1 \otimes \tilde{v}_0, \\ \Delta_z(\tilde{v}_1) &= \tilde{v}_1 \otimes \frac{1}{1 + z\tilde{v}_2} + 1 \otimes \tilde{v}_1 + z\tilde{v}_3 \otimes \frac{\tilde{v}_0}{1 + z\tilde{v}_2}, & \Delta_z(\tilde{v}_3) &= \tilde{v}_3 \otimes \frac{1}{1 + z\tilde{v}_2} + 1 \otimes \tilde{v}_3. \end{aligned} \quad (4.44)$$

Moreover, the invertible nonlinear map that connects both basis for $U_z(\mathfrak{h}_4)$ is given by [56]

$$\tilde{v}_2 = \frac{e^{z\tilde{v}_2} - 1}{z}, \quad v_2 = \frac{1}{z} \log(1 + z\tilde{v}_2), \quad \tilde{v}_l = v_l, \quad l = 0, 1, 3, \quad (4.45)$$

which transforms the relations (4.7) and the commutator analogues of (4.8) for v_i into the coproduct (4.44) and undeformed commutators (4.2) for \tilde{v}_i (where (4.31) has to be used).

Consequently, if we construct deformed \mathfrak{h}_4 -LH systems from $U_z(\mathfrak{h}_4)$ where the latter is given in the basis \tilde{v}_i with undeformed commutators and deformed coproduct (4.44), it is obvious that the same non-deformed Hamiltonian vector fields h_i (3.6) hold, so that we do not actually obtain a deformed LH system but the undeformed one \mathbf{X} given by (3.4). Furthermore, since Δ_z (4.44) is a homomorphism

of the undeformed commutators (4.2) and fulfils the coassociativity condition (2.9) by construction, the deformed prolongation $\widetilde{\mathbf{X}}_z^{m+1}$ must correspond to the diagonal prolongation $\widetilde{\mathbf{X}}^{m+1}$ of \mathbf{X} (3.4) to $(\mathbb{R}^2)^{m+1}$, being both equivalent by means of some change of variables. By mimicking the methodology of the previous subsection, we can explicitly compute the Hamiltonian functions $h_{z,i}^{(2)}$ (2.39) but now with the undeformed expressions (3.6) and deformed coproduct (4.44), obtaining that

$$\begin{aligned} h_{z,2}^{(2)} &= -x_1 - x_2 + zx_1x_2, & h_{z,0}^{(2)} &= 2, \\ h_{z,1}^{(2)} &= \frac{y_1(1+zx_1)}{1-zx_2} + y_2, & h_{z,3}^{(2)} &= \frac{x_1y_1}{1-zx_2} + x_2y_2, \end{aligned} \quad (4.46)$$

which has to be compared with (4.22). These functions give rise to a ‘deformed’ Hamiltonian $h_z^{(2)}$, despite they close the undeformed Poisson brackets (3.7) with respect to the symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. This apparent contradiction is solved by introducing new variables (x'_1, y'_1, x'_2, y'_2) such that

$$\begin{aligned} x_1 &= \frac{x'_1}{1-zx'_2}, & y_1 &= y'_1(1-zx'_2), \\ x_2 &= x'_2, & y_2 &= \frac{y'_2(1-zx'_2) - zx'_1y'_1}{1-zx'_2}, \end{aligned} \quad (4.47)$$

which defines a canonical transformation preserving ω that transforms the Hamiltonian functions (4.46) into the non-deformed ones $h_i^{(2)}$. In this way we prove that the deformation parameter z is inessential since the system with $z \neq 0$ is canonically equivalent to the system with $z = 0$. In other words, if we take as starting point the quantum algebra $U_z(\mathfrak{h}_4)$ in the basis with undeformed commutation rules and deformed coproduct (4.44), no distribution (2.17) arises and we recover the undeformed LH results under a suitable canonical transformation.

However, it can be shown by direct computation that the deformed LH systems \mathbf{X}_z (4.14) that we have presented cannot be transformed into the undeformed system \mathbf{X} (3.4) through a canonical transformation. Therefore, the deformation obtained through $U_z(\mathfrak{h}_4)$ in the basis v_i with deformed commutation rules turns out to be an essential one.

In particular, if we consider the deformed Hamiltonian functions $h_{z,i}$ (4.10) in the variables (x, y) , the nonlinear twist map (4.45) that relates the two basis of the quantum algebra $U_z(\mathfrak{h}_4)$ induces a canonical transformation given by

$$\tilde{x} = \frac{1 - e^{-zx}}{z}, \quad \tilde{y} = e^{zx}y, \quad (4.48)$$

where $\omega = dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$. In the new variables the functions $h_{z,i}$ (4.10) and the system of differential equations (4.14) become

$$h_{z,1} = \tilde{y}, \quad h_{z,2} = -x = \frac{1}{z} \log(1 - z\tilde{x}), \quad h_{z,3} = \tilde{x}\tilde{y}, \quad h_{z,0} = 1, \quad (4.49)$$

$$\frac{d\tilde{x}}{dt} = b_1(t) + b_3(t)\tilde{x}, \quad \frac{d\tilde{y}}{dt} = b_2(t) \frac{1}{1 - z\tilde{x}} - b_3(t)\tilde{y}. \quad (4.50)$$

Thus, the nonlinear twist map (4.45) can be thought to be useful in the sense that it provides a ‘minimal’ LH deformation (4.50) with respect to the undeformed system (3.4), but in any case the deformation turns out to be a genuine non-trivial one, and the functions $h_{z,i}$ (4.10) are ‘truly’ deformed Hamiltonians.

4.3 Deformed book Lie–Hamilton systems and Bernoulli equations

One of the remarkable algebraic properties of the nonstandard quantum deformation of the oscillator algebra \mathfrak{h}_4 is that the book subalgebra \mathfrak{h}_2 remains as a Hopf subalgebra after the deformation, as can be inferred from the classical r -matrix in (4.3). Therefore, by construction, we obtain a Poisson subcoalgebra spanned by v_2 and v_3 within the relations (4.7) and (4.8). As a byproduct, from (4.10)–(4.18)

we directly get all the ingredients that characterize the resulting deformed \mathfrak{b}_2 -LH systems, which reduce to the expressions (3.21)–(3.23) under the limit $z \rightarrow 0$; these are

$$h_{z,2} = -x, \quad h_{z,3} = \left(\frac{e^{zx} - 1}{z} \right) y, \quad \{h_{z,2}, h_{z,3}\}_\omega = \frac{1 - e^{-zh_{z,2}}}{z}, \quad (4.51)$$

$$\mathbf{X}_{z,2} = \frac{\partial}{\partial y}, \quad \mathbf{X}_{z,3} = \left(\frac{e^{zx} - 1}{z} \right) \frac{\partial}{\partial x} - e^{zx} y \frac{\partial}{\partial y}, \quad [\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = -e^{zx} \mathbf{X}_{z,2}, \quad (4.52)$$

$$\begin{aligned} \frac{dx}{dt} &= b_3(t) \left(\frac{e^{zx} - 1}{z} \right), \\ \frac{dy}{dt} &= b_2(t) - b_3(t) e^{zx} y. \end{aligned} \quad (4.53)$$

Consequently, the prolonged deformation of \mathfrak{b}_2 -LH systems to $(\mathbb{R}^2)^3$ is straightforwardly achieved by setting $b_1(t) \equiv 0$ in $\tilde{\mathbf{X}}_z^3$ (4.36), or by only considering the deformed vector fields $\mathbf{X}_{h_{z,2}^{(3)}}$ and $\mathbf{X}_{h_{z,3}^{(3)}}$ in the expressions (4.37) and (4.38). Moreover, the corresponding deformed constants of the motion and superposition rules are exactly those given by (4.39) and (4.40) for the prolonged deformation of \mathfrak{h}_4 -LH systems. These results can further be applied to all the \mathfrak{b}_2 -LH systems mentioned in Subsection 3.2.

To illustrate the latter point, we briefly present the main results concerning the PH deformation of the complex Bernoulli differential equations studied in Subsection 3.3. We keep the symplectic form (3.29), the change of variables (3.32) and the relationships (3.33). It is easily seen that, in these conditions, the deformed Hamiltonian functions are given by

$$\begin{aligned} \bar{h}_{z,1} &= -\frac{\cos[\theta(s-1)]}{z r^{s-1}} \left(\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} - 1 \right), \\ \bar{h}_{z,2} &= -\frac{r^{s-1}}{\sin[\theta(s-1)]}, \quad \{\bar{h}_{z,1}, \bar{h}_{z,2}\}_\omega = (s-1) \frac{e^{-z\bar{h}_{z,2}} - 1}{z}, \end{aligned} \quad (4.54)$$

while the corresponding deformed vector fields $\mathbf{Y}_{z,i}$ turn out to be

$$\begin{aligned} \mathbf{Y}_{z,1} &= \left(r \cos^2[\theta(s-1)] \exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} + \frac{\sin^3[\theta(s-1)]}{z r^{s-2}} \left(\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} - 1 \right) \right) \frac{\partial}{\partial r} \\ &\quad + \sin^2[\theta(s-1)] \left(\frac{\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\}}{\tan[\theta(s-1)]} - \frac{\cos[\theta(s-1)]}{z r^{s-1}} \left(\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} - 1 \right) \right) \frac{\partial}{\partial \theta}, \\ \mathbf{Y}_{z,2} &= r^s \cos[\theta(s-1)] \frac{\partial}{\partial r} + r^{s-1} \sin[\theta(s-1)] \frac{\partial}{\partial \theta}, \\ [\mathbf{Y}_{z,1}, \mathbf{Y}_{z,2}] &= (s-1) \exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} \mathbf{Y}_{z,2}. \end{aligned} \quad (4.55)$$

The deformed Bernoulli system of differential equations adopts the form

$$\begin{aligned} \frac{dr}{dt} &= a_1(t) \left(r \cos^2[\theta(s-1)] \exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} + \frac{\sin^3[\theta(s-1)]}{z r^{s-2}} \left(\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} - 1 \right) \right) \\ &\quad + a_2(t) r^s \cos[\theta(s-1)], \\ \frac{d\theta}{dt} &= a_1(t) \sin^2[\theta(s-1)] \left(\frac{\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\}}{\tan[\theta(s-1)]} - \frac{\cos[\theta(s-1)]}{z r^{s-1}} \left(\exp \left\{ \frac{z r^{s-1}}{\sin[\theta(s-1)]} \right\} - 1 \right) \right) \\ &\quad + a_2(t) r^{s-1} \sin[\theta(s-1)], \end{aligned} \quad (4.56)$$

where the undeformed limit $z \rightarrow 0$ is given by (3.25). We also recall that it is possible to take a power series expansion in the deformation parameter z in order to interpret this result as a perturbation of the initial Bernoulli differential equations.

In spite of the apparently very cumbersome form of the resulting deformed Bernoulli system (4.56), its prolonged deformation along with the corresponding deformed constants of the motion and superposition rules can explicitly be derived from the results obtained in the Subsection 4.1. For the sake of brevity, we merely indicate that $F_z^{(2)}$ in (4.24) now becomes

$$F_z^{(2)} = \frac{1}{z(1-s)} \left(2 - \exp \left\{ -\frac{z r_1^{s-1}}{\sin[\theta_1(s-1)]} \right\} - \exp \left\{ \frac{z r_2^{s-1}}{\sin[\theta_2(s-1)]} \right\} \right) \\ \times \left(\frac{\cos[\theta_1(s-1)]}{r_1^{s-1}} - \frac{\cos[\theta_2(s-1)]}{r_2^{s-1}} \right),$$

and its undeformed limit is given in (3.34).

5 Concluding remarks

In this work, a relevant question addressed to but left open in [1], concerning the possibility of deducing a computationally feasible deformed analogue of superposition principles for PH deformations of LH systems, has been answered in the affirmative. This has been achieved by combining the formalism of PH deformations with the superintegrability property of systems having coalgebra symmetry. In this way, two separate sets of constants of the motion have been derived for the prolonged deformations, from which a sufficient number of functionally independent constants of the motion can be extracted, hence making it possible to establish a generic deformed superposition rule of the lowest possible order, regardless on the particular structure of the Hopf algebra deformation. This approach amends and generalizes the construction previously proposed in [6] based on permutations in the tensor product space, which did not take into account the symmetry breaking originated by deformed coproducts, which prevents that a constant of the motion retains its invariant character after having been transformed by a permutation of the variables. It is worth stressing that in order to develop this refinement, it has been necessary to introduce two new notions: prolonged PH deformations of LH systems and deformed superposition rules, which respectively reduce to the usual diagonal prolongations and superposition rules of the initial LH system under the non-deformed limit of the deformation parameter. Consequently, a complete correspondence between the characteristic properties of LH systems and their PH deformations has been established.

Along these lines, the LH systems based on the oscillator algebra \mathfrak{h}_4 (see [7]) and their nonstandard deformation have been studied, and an explicit deformed superposition rule for their prolonged deformation has been obtained. Besides the undeniable physical interest of the oscillator algebra \mathfrak{h}_4 , another remarkable feature has led to this choice for illustrating the generalization of the formalism. The fact that the book algebra \mathfrak{b}_2 is preserved as a Hopf subalgebra after deformation, implies that prolonged deformations of LH systems based on \mathfrak{b}_2 can easily be obtained through restriction of the prolonged deformations of \mathfrak{h}_4 -LH systems. A striking particular case is given by the prolonged deformation of complex Bernoulli equations, for which the method provides a systematic prescription for determining the constants of the motion and a deformed superposition rule. In this context, it is worth to be mentioned that PH deformations of LH systems based on \mathfrak{b}_2 , but seen as a PH subalgebra of the nonstandard deformation of $\mathfrak{sl}(2, \mathbb{R})$ considered in [1, 3], have recently been used for the description of new SIS epidemic models in [57]. This suggests in a natural way to analyze analogous models based on \mathfrak{b}_2 -LH systems obtained as restrictions of prolonged PH deformations of oscillator LH systems. Even if under the limit $z \rightarrow 0$ the algebra \mathfrak{b}_2 is obviously the same, it is expected that the properties of such deformed models should be quite distinct to those studied in [57], due to the different features of the $\mathfrak{sl}(2, \mathbb{R})$ and \mathfrak{h}_4 deformations.

The extended formalism here presented gives rise to a number of interesting questions that can be considered. A first one concerns a systematic analysis of prolonged PH deformations of LH systems in the plane, and its eventual identification with dynamical systems appearing in various applications. This

in particular applies to those systems that can be interpreted as small perturbations of LH systems, and where the deformation formalism may provide a precise insight on the exact role of the deformation parameter with respect to stability or bifurcation properties of the system, as well as concerning the geometrical and dynamical behaviour of the orbits. At a more profound level, it may be asked if PH deformations admit some kind of inverse problem. More specifically, it is conceivable that a non-autonomous parametrized nonlinear system of differential equations, without having the structure of a LH system, still allows a description in terms of a t -dependent vector field, the t -independent components of which, although not generating a finite-dimensional Lie algebra, span a distribution in the Stefan–Sussman sense. In these conditions, it would be of interest to know whether a compatible PH structure on an appropriate manifold can be found, so that one or more of the parameters in the system can be identified with deformation parameters, hence allowing the system to be associated with a PH deformation of some LH system that would be recovered by a limiting process. This problem is intimately related to the development of an unambiguous notion of equivalence classes for PH deformations, possibly focusing on certain structural properties that until now have not been inspected in full detail, so that some kind of classification parallel to that of LH systems may be established. Progress in some of the above-mentioned problems will hopefully be reported in some future work.

Acknowledgements

A.B. and F.J.H. have been partially supported by Ministerio de Ciencia e Innovación (Spain) under grant MTM2016-79639-P (AEI/FEDER, UE), by Agencia Estatal de Investigación (Spain) under grant PID2019-106802GB-I00/AEI/10.13039/501100011033, and by Junta de Castilla y León (Spain) under grants BU229P18 and BU091G19. The research of R.C.S. was financially supported by grants MTM2016-79422-P (AEI/FEDER, EU) and PID2019-106802GB-I00/AEI/10.13039/501100011033. E.F.S. acknowledges a fellowship (grant CT45/15-CT46/15) supported by the Universidad Complutense de Madrid. J. de L. acknowledges funding from the Polish National Science Centre under grant HARMONIA 2016/22/M/ST1/00542.

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