Expression of the 3- and 4-dimensional vectors in total polar exponential form.

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Study I,1. We express the vectors in $\mathfrak{R}^3$ and in $\mathfrak{R}^4$ $(\mathfrak{M})$ in terms of square 2x2 matrices. The objective is to write these vectors in a similar way as the complex numbers in $\mathfrak{C}$, once considered the isomorphism of $\mathfrak{R}^2$ with $\mathfrak{C}$, with the exponential polar form, including the Euler's formula. In this way we are able to write the vectors in $\mathfrak{R}^3$ and in the Minkowski space $\mathfrak{M}$ in matrix exponential form and also in an extended polar matrix exponential form. The treatment motivates a digression on the structure of the time and space. This research continues in *Study I,2.*

I. INTRODUCTION. IN THE EULER'S PATHWAY.

THE EXPONENTIAL FUNCTION:

it deserves attention the reading of the entrance in the Prologue of the book "Real and Complex Analysis" of Walter Rudin. [1]

Real vectors in **dimension two**, complex numbers and rotations with Euler's identity:

we denote the vectors in $\mathfrak{R}^2$ in a Cartesian form with an ordered pair of real numbers. We also establish a correspondence with the points of the complex plain (one complex number), same ordered pair of real numbers:

$$\mathbf{v} = (x, y) = x\,\mathbf{e_1} + y\,\mathbf{e_2} \in \mathfrak{R}^2 \;\longleftrightarrow\; z = (x, y) = x\,(\mathbf{1}) + y\,\mathbf{i} \in \mathfrak{C}\ ,$$

$\{\mathbf{e_1}, \mathbf{e_2}\}$ a base for the real plane and $\{\mathbf{1}, \mathbf{i}\}$ a $'$base $'$ for the complex plane over the reals.

We represent either one in the polar form with: $\begin{cases} x = r\ \cos\varphi \\ y = r\ \sin\varphi \end{cases}$, $r \in (0, \infty)$, $\varphi \in [0, 2\pi)$, where we have fixed for the angle a departing positive semi-axis $\mathbf{e_1}$ in $\mathfrak{R}^2$, or with the positive part of the reals $(\mathbf{1})$ in the complex plane.

With the Euler's identity we write: $z\ = r\ \cos\varphi\,(\mathbf{1}) + r\ \sin\varphi\,\mathbf{i} = r\ \mathrm{e}^{\mathbf{i}\,\varphi} = \mathrm{e}^{\ln r\,(\mathbf{1})+\varphi\,\mathbf{i}}$.

The equations defining a rotation with angle 2α are: $\mathbf{v}' = \mathcal{R}(2\alpha)\,\mathbf{v}$, $\begin{cases} x' = x\ \cos 2\alpha - y\ \sin 2\alpha = r\ \cos(\varphi + 2\alpha) \\ y' = x\ \sin 2\alpha + y\ \cos 2\alpha = r\ \sin(\varphi + 2\alpha) \end{cases}$.

Or with a simpler notation, using the complex numbers: $z' = \mathcal{R}(2\alpha)\,[z] = \mathrm{e}^{\mathbf{i}\,2\alpha}[r\ \mathrm{e}^{\mathbf{i}\,\varphi}] = r\ \mathrm{e}^{\mathbf{i}\,(\varphi+2\alpha)}$

Under rotations it is: $r'^2 = \parallel v' \parallel^2 = |z'|^2 = x'^2 + y'^2 = x^2 + y^2 = z\,\bar{z} = |z|^2 = \parallel v \parallel^2 = r^2$.

Vectors in **dimensions three and four**. The rotations will motivate our definitions.

The main names associated to the study of the rotations are: Euler (1775), Gauss (1819), Rodrigues (1840) and Hamilton (1843). See the introduction in [2]. They found formulas for expressing the rotations in Cartesian coordinates. Rodrigues and Hamilton formulations involved the division of the angle of rotations in two equal parts, each one acting at a side of the rotated vector with opposite signs, and the introduction of some special quantities which Hamilton named "quaternions": $a + x_1\mathbb{i} + x_2\mathbb{j} + x_3\mathbb{k}$ [3], represented in this *Study* by the expression in equation (2.8) without the exponential. He also named "$\mathfrak{v}$*ector*" to the triple $x_1\mathbb{i} + x_2\mathbb{j} + x_3\mathbb{k}$, which in fact is an "axial vector".

Hamilton defined the "quaternions" in his searching for the solution of the problem of obtaining results for the sums and product of squares, generalizing an old Diophantus's result in dimension two (considering the complex numbers):

$$(*) \quad (x_1^2 + y_1^2)\,(x_2^2 + y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 \qquad (\ |z_1|^2|z_2|^2 = |z_1z_2|^2\ ).$$

In general it is: $|z_1|^2|z_2|^2 = \|\mathbf{v}_1\|^2\,\|\mathbf{v}_2\|^2 = (x_1^2 + y_1^2)\,(x_2^2 + y_2^2) \neq (x_1x_2 + y_1y_2)^2 = (\,\mathbf{v}_1.\mathbf{v}_2\,)^2\ \ (\,|z_1z_2| \neq |\,\mathbf{v}_1.\mathbf{v}_2|\,)$.

Hamilton needed to jump from pairs to quadruples for obtaining a similar result to (*). See chapter 20 in [4].

In this way an algebraic method and a geometrical construction got mixed. See S3 in Appendix B.

This construction gets a plainly algebraic meaning with the introduction of the Pauli matrices (1927). See C and P in Appendix B. With them, we define the exponential of a square matrix, as in formula (2.8). To write the three dimensional vectors in an exponential form, like in (3.1), imposes us to express them considering square matrices, and not as it is usual in a row or in a column form. We obtain (3.1), (3.2) and (3.3) as an immediate result using (2.8). To generalize the method to dimension four is not trivial at all and conducts us to look at $\mathfrak{R}^4$ in the way of a Minkowski space $\mathfrak{M}$ ($\mathfrak{R}^4$ with the Minkowski metric) instead of as an euclidean space (euclidean metric). In $\mathfrak{M}$ we distinguish three different kind of vectors: time, light ray and space types. For the time and space type vectors we deduce an exponential form, (formulas (5.1), (5.2)). This is not possible for light ray vectors which have a similar role as $\mathbf{v} = \mathbf{0}$ in the plane.

In order to write the vectors in $\mathfrak{R}^3$ in polar coordinates, a norm and two angles, we start diagonalizing the matrix form of an arbitrary unit vector (equation (4.4)). This diagonalization, with the angle parametrizations defined in (4.3), is simple. The final form is Eulerian in the sense that it appears as the product of exponential forms. General vectors in $\mathfrak{R}^3$ are in the formulas (4.6) and (4.7). Formulas (5.3) and (5.4) express similar results for vectors in $\mathfrak{M}$.

We add a minor exposition about interchangings in the characteristics of the vectors in $\mathfrak{M}$. And an inquire in the norms with pairs, triples and quadruplets. This last part is more detailed in *Study I,2*.

II. VECTORS IN $\mathfrak{R}^3$ AND IN $\mathfrak{R}^4$ ($\mathfrak{M}$ - MINKOWSKI SPACE).

Let us start with the **Canonical basis** for the linear space of $2x2$ matrices over the real or the complex field:

$$\mathbf{B}_C \equiv \left\{ \sigma^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \overset{\wedge}{\sigma} \equiv \sigma^+ \sigma^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overset{\vee}{\sigma} \equiv \sigma^- \sigma^+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{2.1}$$

The sets of four real or complex coefficients, expanding this basis as a linear space, can be taken as the four real or complex coordinates of the real or complex vectors in $\mathfrak{R}^4$ or $\mathfrak{C}^4$.

The Pauli spin matrices with the identity $\mathbb{1}$ constitute another basis for this linear space, the **Pauli basis**: [5]

$$\mathbf{B}_P \equiv \left\{ \mathbb{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y \equiv \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \right\}. \tag{2.2}$$

As matrices, they are Hermitian. These Pauli matrices with the identity are the result of doing a simple linear change in the matrices in (2.1): $\mathbb{1} = \overset{\wedge}{\sigma} + \overset{\vee}{\sigma}$, $\sigma^z = \overset{\wedge}{\sigma} - \overset{\vee}{\sigma}$, $\sigma^x = \sigma^+ + \sigma^-$ and $\sigma^y = -\mathbf{i}\,(\sigma^+ - \sigma^-)$. It has to be remarked the introduction of the imaginary unit $\mathbf{i}$, even working only with real coefficients.

We write unit space vectors $\mathbf{n} = (n_x, n_y, n_y) \in \mathfrak{R}^3$, $(-1 \le n_j \le 1, \quad j = \{x, y, z\})$. With norm: $\|\mathbf{n}\| \equiv +\sqrt{n_x^2 + n_y^2 + n_z^2} = 1$. In terms of the Pauli matrices in the form:

$$\mathbb{n} \equiv [\mathbf{n} \,.\, \boldsymbol{\sigma}] \equiv n_x \sigma^x + n_y \sigma^y + n_z \sigma^z = \begin{pmatrix} n_z & n_x - \mathbf{i} n_y \\ n_x + \mathbf{i} n_y & -n_z \end{pmatrix} \qquad \Longrightarrow \qquad \mathbb{n}^2 = \mathbb{1}. \tag{2.3}$$

Observe that $\mathbb{1}$ does not have a correspondence with any vector in $\mathfrak{R}^3$. Expressed in the Pauli basis, the vectors are represented by squared matrices. Therefore we can define their traces, and also their determinants which have opposite values to the square of the norms of the vectors:

$$-\det \mathbb{n} = \tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{n}^2) = n_x^2 + n_y^2 + n_z^2 \equiv \|\mathbb{n}\|^2 = 1 \qquad \Longrightarrow \qquad \|\mathbb{n}\| \equiv +\sqrt{\tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{n}^2)} = +\sqrt{-\det \mathbb{n}} = 1. \tag{2.4}$$

It is straightforward to write 3-dimensional (space) vectors with an arbitrary norm. We consider two different types of vectors.

1) Polar vectors $\mathbf{v} = (v_x, v_y, v_z) \in \mathfrak{R}^3$, in matrix form:

$$\mathbb{v} \equiv [\mathbf{v} \,.\, \boldsymbol{\sigma}] \equiv v_x \sigma^x + v_y \sigma^y + v_z \sigma^z = \begin{pmatrix} v_z & v_x - \mathbf{i} v_y \\ v_x + \mathbf{i} v_y & -v_z \end{pmatrix}, \qquad \mathbb{v}^2 = (v_x^2 + v_y^2 + v_z^2)\,\mathbb{1}. \tag{2.5}$$

We define a $'$scalar$\,'$ and a norm:

$$\langle \mathbb{v}_1 , \mathbb{v}_2 \rangle \equiv \frac{1}{2}\,\mathrm{Tr}\,(\mathbb{v}_1 \mathbb{v}_2)\,, \qquad \|\mathbb{v}\| \equiv +\sqrt{\tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{v}^2)} = +\sqrt{-\det \mathbb{v}} = +\sqrt{v_x^2 + v_y^2 + v_z^2} \ge 0\,, \tag{2.6}$$

with $v_j = \|\mathbb{v}\|\, n_j$, it is: $\mathbb{v} = \|\mathbb{v}\|\, \mathbb{n}$.

2) Axial vectors. We write the rotations in $\mathfrak{R}^3$ using the matrix formulation (Rodrigues, Hamilton, Pauli): [2]

$$\mathbb{v}' \equiv \mathcal{R}[\mathbb{v}] \equiv \mathcal{R}(\mathbf{2}\boldsymbol{\alpha})[\mathbb{v}] \equiv \mathbb{R}\, \mathbb{v}\, \mathbb{R}^\dagger\,, \qquad \mathcal{R}[\mathbb{1}] = \mathbb{1} \tag{2.7}$$

The rotation operator $\mathcal{R}$ is formally self-adjoint: $(\mathbb{R}\, \mathbb{v}\, \mathbb{R}^\dagger)^\dagger = \mathbb{R}\, \mathbb{v}\, \mathbb{R}^\dagger$. And $\mathbb{R}^\dagger$ is the unitary matrix:

$$\mathbb{R}^\dagger \equiv \mathbb{R}^\dagger(\boldsymbol{\alpha}) \equiv \mathrm{e}^{\mathbf{i}\,\boldsymbol{\alpha}} \equiv \mathrm{e}^{\alpha\,\mathbf{i}\,\mathbb{n}} = \cos\alpha\,\mathbb{1} + \mathbf{i}\sin\alpha\,\mathbb{n} = \begin{pmatrix} \cos\alpha + \sin\alpha\,\mathbf{i}\,n_z & \sin\alpha\,\mathbf{i}\,(n_x - \mathbf{i} n_y) \\ \sin\alpha\,\mathbf{i}\,(n_x + \mathbf{i} n_y) & \cos\alpha - \sin\alpha\,\mathbf{i}\,n_z \end{pmatrix}, \tag{2.8}$$

with the notation:

$$\mathbf{i}\boldsymbol{\alpha} \equiv \alpha\,\mathbf{i}\,\mathbb{n} = \alpha_x\,\mathbf{i}\,\sigma^x + \alpha_y\,\mathbf{i}\,\sigma^y + \alpha_z\,\mathbf{i}\,\sigma^z = (-\alpha)\,(-\mathbf{i}\,\mathbb{n})\,; \qquad \alpha_j = \alpha\, n_j, \quad j = \{x, y, z\}\,. \tag{2.9}$$

In this formulation the vectors $\mathbb{n}$ appear in an exponential accompanied by the imaginary unit $\mathbf{i}$; we name *imaginary vectors* to $\mathbf{i}\,\mathbb{n}$ or $\mathbf{i}\,\boldsymbol{\alpha}$. α is the argument of 2π periodic functions. Behind, the right hand rule, with the angle of rotation 2α and the unit axis of rotation given by the triad $(\mathbf{i})(n_x, n_y, n_y)$. Alternately, we can establish a left hand rule with a 2α angle (-2α for the right hand) and the triad $(\mathbf{i})(-n_x, -n_y, -n_y)$ [6]. Customarily we say that $\boldsymbol{\alpha}$ is an axial vector (relation with a parity transformation). These vectors do not belong to $\mathrm{lin}(\mathbf{B}_P)$ over the reals.

In general, the (matrix) product of two vectors belonging to $\mathrm{lin}(\mathbf{B}_P)$ over the real field also does not belong to this linear space. In order to include the *imaginary vectors* we need to consider $\mathrm{lin}(\mathbf{B}_P)$ over the complex field.

In what follows we use both types of vectors (matrices) without and with the imaginary unit.

Interpreting the $\mathbb{1}$ as a unit vector for a time coordinate, we write a vector in $\mathfrak{R}^4$ or in $\mathfrak{M}$:

$$\mathbb{w} = w_t\,\mathbb{1} + w_x\,\sigma^x + w_y\,\sigma^y + w_z\,\sigma^z = \begin{pmatrix} w_t + w_z & w_x - \mathbf{i}\,w_y \\ w_x + \mathbf{i}\,w_y & w_t - w_z \end{pmatrix} = w_t\,\mathbb{1} + |\,w_n|\,\mathbb{n}. \tag{2.10}$$

$w_n^2 \equiv w_x^2 + w_y^2 + w_z^2 \geq 0$. For a vector with $\det \mathbb{w} \neq 0$, its inverse vector is: $\mathbb{w}^{-1} = \frac{1}{-\det \mathbb{w}} \begin{pmatrix} -w_t + w_z & w_x - \mathbf{i}\,w_y \\ w_x + \mathbf{i}\,w_y & -w_t - w_z \end{pmatrix}$, $(\,\mathbb{w}\,\mathbb{w}^{-1} = \|\,\mathbb{w}\,\|_{\sim}^2\,)$. With $w_t \in \mathfrak{R}$, we write with a lower case vectors of the form: $\mathbb{w}_\mu \equiv w_t\mathbb{1} + w_x\sigma^x + w_y\sigma^y + w_z\sigma^z$, and with an upper case the related vectors of the form: $\mathbb{w}^\mu \equiv -w_t\mathbb{1} + w_x\sigma^x + w_y\sigma^y + w_z\sigma^z$, without minding about the sign of w_t. The product of these matrices is: $\mathbb{w}_\mu\,\mathbb{w}^\mu = (-\det \mathbb{w})\,\mathbb{1}$. This notation does not intent to express a contraction of the indexes, we obtain a matrix, not a scalar.

We define the pseudo-norm (later, $\|\,\mathbb{1}\,\|_{\sim}^2 = \frac{1}{2}\mathrm{Tr}\,(\,\mathbb{1}_{(\mu)}\,(-\mathbb{1})^{(\mu)}\,) = -1$, and also $\|\,\mathbb{w}_1 + \mathbb{w}_2\,\|_{\sim} \not\leq \|\,\mathbb{w}_1\,\|_{\sim} + \|\,\mathbb{w}_2\,\|_{\sim}$) and we consider the four dimensional vectors in the Minkowsky space $\mathfrak{M}$:

$$\begin{aligned} \|\,\mathbb{w}\,\|_{\sim}^2 \equiv \|\,\mathbb{w}_\mu\,\|_{\sim}^2 \equiv \|\,\mathbb{w}^\mu\,\|_{\sim}^2 \equiv -w_t^2 + w_x^2 + w_y^2 + w_z^2 = \tfrac{1}{2}\mathrm{Tr}\,(\mathbb{w}_\mu^\dagger \mathbb{w}^\mu) = \tfrac{1}{2}\mathrm{Tr}\,(\mathbb{w}^{\mu^\dagger} \mathbb{w}_\mu) = -\det \mathbb{w} \in \mathfrak{R} \implies \\ \|\,\mathbb{w}\,\|_{\sim} \equiv \|\,\mathbb{w}_\mu\,\|_{\sim} \equiv \|\,\mathbb{w}^\mu\,\|_{\sim} \equiv \sqrt{\tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{w}_\mu^\dagger \mathbb{w}^\mu)} = \sqrt{\tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{w}_\mu \mathbb{w}^\mu)} = \sqrt{-\det \mathbb{w}} \in \mathfrak{C}\,. \end{aligned} \tag{2.11}$$

We denote: $\begin{cases} \text{time type vectors, the ones with } \|\mathbb{w}\|_{\sim} \in \mathbf{i}\,\mathfrak{R} - 0 \\ \text{space type vectors, the ones with } \|\mathbb{w}\|_{\sim} \in \mathfrak{R} - 0 \\ \text{light type vectors, the ones with } \|\mathbb{w}\|_{\sim} = 0 \text{ (null vectors). There is not an inverse vector} \end{cases}$.

We define a 'scalar' and related to it the previous pseudo-norm:

$$\langle\,\mathbb{w}_1\,,\,\mathbb{w}_2\,\rangle_{\sim} \equiv \tfrac{1}{2}\mathrm{Tr}\,(\,\mathbb{w}_{1\mu}\,\mathbb{w}_2^\mu\,) = \tfrac{1}{2}\mathrm{Tr}\,(\,\mathbb{w}_1^\mu\,\mathbb{w}_{2\mu}\,)\,, \qquad \|\,\mathbb{w}\,\|_{\sim}^2 = \langle\,\mathbb{w}\,,\,\mathbb{w}\,\rangle_{\sim} = \tfrac{1}{2}\mathrm{Tr}\,(\,\mathbb{w}_\mu\,\mathbb{w}^\mu\,) = \tfrac{1}{2}\mathrm{Tr}\,(\,\mathbb{w}^\mu\,\mathbb{w}_\mu\,)\,. \tag{2.12}$$

This 'scalar' is the scalar product of the vector algebra of Gibbs and Heaviside for the vectors with $w_t = 0$. With these definitions, we have:

$$\begin{gathered} \|\,\mathbb{1}\,\|^2 = \|\,\sigma^j\,\|^2 = -\,\|\,\mathbb{1}\,\|_{\sim}^2 = \|\,\sigma^j\,\|_{\sim}^2 = 1 \\ \langle\,\mathbb{1}\,,\,\sigma^k\,\rangle = \langle\,\sigma^j\,,\,\sigma^k\,\rangle = \langle\,\mathbb{1}\,,\,\sigma^k\,\rangle_{\sim} = \langle\,\sigma^j\,,\,\sigma^k\,\rangle_{\sim} = 0\,, \quad j \neq k\,, \\ \text{and:} \quad \mathbb{0} \neq \sigma^j\,\sigma^k = \mathbf{i}\sigma^l \notin \mathrm{lin}\mathbf{B}_P \quad (j,k,l) \text{ with cyclic order} \end{gathered} \tag{2.13}$$

as the $\dot{\mathrm{T}}\mathrm{r}\ \sigma^k = 0$. These results define $\mathbf{B}_P$ as an 'orthonormal base' or as a 'pseudo-orthonormal' base.

III. VECTORS IN $\mathfrak{R}^3$ IN EXPONENTIAL FORM (CARTESIAN).

We express vectors and imaginary vectors in exponential form by using formula (2.8). We start with the triad (n_x, n_y, n_z):

$$\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,\mathbb{n}} = \pm\,\mathbf{i}\,\mathbb{n}\,, \qquad \text{and then:} \qquad \mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1} + k\,\mathbb{n}]} = k\,\mathbb{n}\,, \qquad k = \pm 1\,. \tag{3.1}$$

The $\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[\epsilon\mathbb{1} + k\,\mathbb{n}]}$, $(\epsilon \in \{+1, -1\})$ have an important role in the definition of reflections in a plane. And with $\frac{\pi}{4}$, instead of $\frac{\pi}{2}$, in relation to elementary fermions, leptons and quarks. The *imaginary vectors* $\pm\mathbf{i}\,\mathbb{v}$ are:

$$\pm\mathbf{i}\,\mathbb{v} = \pm\mathbf{i}\,\|\mathbb{v}\|\,\mathbb{n} = \mathrm{e}^{(\ln\|\mathbb{v}\|)\,\mathbb{1} \pm \mathbf{i}\,\frac{\pi}{2}\,\mathbb{n}}\,, \qquad \|\mathbb{v}\| > 0\,, \tag{3.2}$$

real time and imaginary space components in the exponent (also like a quaternion). And for (real) vectors:

$$\mathbb{v} = \|\mathbb{v}\|\,\mathbb{n} = \mathrm{e}^{(\ln\|\mathbb{v}\|)\,\mathbb{1} \,\pm\, \mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1} + \mathbb{n}]}\,, \qquad \|\mathbb{v}\| > 0\,, \tag{3.3}$$

real time and imaginary space and an added imaginary time components in the exponential. And with $\tilde{\mathbb{n}} = \mathrm{sign}(n_z)\,\mathbb{n}$ we also define $\tilde{v} \equiv \mathrm{sign}(v_z)\,\|\mathbb{v}\| = \mathrm{sign}(n_z)\,\|\mathbb{v}\| \in \{\,\|\mathbb{v}\|\,,\,-\,\|\mathbb{v}\|\,\}$ and we write:

$$\mathbb{v} = \tilde{v}\,\tilde{\mathbb{n}} = \mathrm{e}^{(\ln\tilde{v})\,\mathbb{1} \pm \mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1} + \tilde{\mathbb{n}}]}\,, \qquad \mathrm{sign}(\tilde{n}_z) = +, \qquad \tilde{v} \in \mathbb{R} - \{0\}, \tag{3.4}$$

These formulas are related to the fact that we can take two vectors, with a certain norm, in a direction in a sense and in the opposite sense, departing from a unit vector $\mathbb{n}$. This is the source of an indeterminacy in the definition of a rotation in the three dimensional space (see the comments on the axial vectors). [6]

We can not follow this simple process for the witting of the four dimensional vectors defined in $\mathfrak{R}^4$ (formula (2.10)).

IV. VECTORS IN $\mathfrak{R}^3$ IN POLAR EXPONENTIAL FORM (EULERIAN).

We use: $\frac{1}{2}(-\mathbb{1}+\sigma^z) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = -\overset{\vee}{\sigma}$ and $\frac{1}{2}(\mathbb{1}+\sigma^z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \overset{\wedge}{\sigma}$, which satisfy: $-\det(-\mathbb{1}+\sigma^z) = -\det(\mathbb{1}+\sigma^z) = 0$ and $\mathrm{Tr}\,[(-\mathbb{1}+\sigma^z)(\mathbb{1}+\sigma^z)] = \mathrm{Tr}\,(\mathbb{0}) = 0$. Like in the pseudo-norm in dimension 4. In that sense, these vectors represent null vectors. In the exponential matrix function they produce:

$$\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^z]} = \mathrm{e}^{\mp\,\mathbf{i}\,\pi\,\overset{\vee}{\sigma}} = \sigma^z\,; \qquad \mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[\mathbb{1}+\sigma^z]} = \mathrm{e}^{\pm\,\mathbf{i}\,\pi\,\overset{\wedge}{\sigma}} = -\sigma^z\,; \tag{4.1}$$

with $\det(\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^z]}) = \det(\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[\,\mathbb{1}+\sigma^z]}) = \det(\pm\sigma^z) = -1$.

We also obtain the following vectors: $\pm\mathbf{i}\,\sigma^x = \mathrm{e}^{\pm\mathbf{i}\,\frac{\pi}{2}\sigma^x}$, $\pm\mathbf{i}\,\sigma^y = \mathrm{e}^{\pm\mathbf{i}\,\frac{\pi}{2}\sigma^y}$, $\sigma^x = \mathrm{e}^{\pm\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^x]}$, $\sigma^y = \mathrm{e}^{\pm\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^y]}$. But it is not possible to write in this way, in an exponential form, vectors like: $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \mathbf{i}\,\sigma^y)$ (in a complex linear space), since the determinants of these vectors are null: $\det(A) = \det(e^X) = e^{\mathrm{tr}(X)} \neq 0$ and, if $\det(A) = 0$ then $A \neq e^X$. Similarly for light ray vectors $\pm\mathbb{1} + \sigma^z$ or $\pm\mathbb{1} + \mathbb{n}$ (in a real or complex linear space). Still remains to define some kind of norm for the vectors $\sigma^\pm$ (in a complex linear space).

Let us reinterpret (3.2) by writing:

$$\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\mathbb{1}}\,\mathbb{v} = \mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}}\,\mathbb{v} = \pm\,\mathbf{i}\,\mathbb{v}\,. \tag{4.2}$$

This last exponential represents the action of a phase, a rotation in the way we did with the polar expressions in the complex plane, the exponent in the first exponential can also be considered 'a kind of imaginary time component' that here produce jumps between real and imaginary space vectors, representing 'some kind of orthogonality ', under the $\frac{\pi}{2}$ angle in the exponent or the $\mathbf{i}$ factor at 'ground ', though as before, we need to define for the inclusion of these imaginary vectors some type of norm (see at the end of section VI) and of orthogonality. See H in Appendix B [7].

Let us go back to expression (2.3). We can express $\mathbb{n}$ by doing $n_z^2 + (n_x^2 + n_y^2) = n_z^2 + r_\tau^2 = r^2 = 1$, and introducing angle type parameters, different to the usual ones, in the following way:

$$\begin{cases} n_z = \cos 2\varphi_n \\ r_\tau = \sin 2\varphi_n \end{cases} (r=1)\,, \quad \begin{matrix} \mathbf{n} = \mathbf{n}_z + \boldsymbol{r}_\tau \\ 2\varphi_n \in [-\pi, \pi)\,, \\ r_\tau \in [-1, 1] \end{matrix} \qquad \begin{cases} n_x = r_\tau \cos\phi_\tau \\ n_y = r_\tau \sin\,\phi_\tau \end{cases}, \quad \begin{matrix} n_x + \mathbf{i}n_y = r_\tau\,\mathrm{e}^{\mathbf{i}\phi_\tau} \\ r_\tau \neq 0\,,\ \ \phi_\tau \in [\,0,\ \pi) \end{matrix}. \tag{4.3}$$

These vectors satisfy $\mathbb{n}(\phi_\tau \pm \pi, 2\varphi_n) = \mathbb{n}(\phi_\tau, -2\varphi_n)$, so that, if $\mathbb{n}(n_z, n_x, n_y) = \mathbb{n}(n_z, r_\tau, \phi_\tau) = \mathbb{n}(\phi_\tau, 2\varphi_n)$ then $\mathbb{n}(n_z, -n_x, -n_y) = \mathbb{n}(n_z, -r_\tau, \phi_\tau) = \mathbb{n}(\phi_\tau, -2\varphi_n)$. The unit vectors are:

$$\begin{aligned} \mathbb{n} &= \begin{pmatrix} n_z & n_x - \mathbf{i}n_y \\ n_x + \mathbf{i}n_y & -n_z \end{pmatrix} = \begin{pmatrix} \cos 2\varphi_n & \sin 2\varphi_n\,\mathrm{e}^{-\mathbf{i}\,\phi_\tau} \\ \sin 2\varphi_n\,\mathrm{e}^{\,\mathbf{i}\,\phi_\tau} & -\cos 2\varphi_n \end{pmatrix} = \\ &= \begin{pmatrix} \cos\varphi_n & -\sin\varphi_n\,\mathrm{e}^{-\mathbf{i}\,\phi_\tau} \\ \sin\varphi_n\,\mathrm{e}^{\,\mathbf{i}\,\phi_\tau} & \cos\varphi_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi_n & \sin\varphi_n\,\mathrm{e}^{-\mathbf{i}\,\phi_\tau} \\ -\sin\varphi_n\,\mathrm{e}^{\,\mathbf{i}\,\phi_\tau} & \cos\varphi_n \end{pmatrix} = \\ &= (\cos\varphi_n\,\mathbb{1} + \sin\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau))\ \sigma^z\ (\cos\varphi_n\,\mathbb{1} + \sin(-\varphi_n)\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)) = \\ &= \mathrm{e}^{\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,\sigma^z\,\mathrm{e}^{-\,\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)} = \mathrm{e}^{\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,\mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^z]}\,\mathrm{e}^{-\,\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,. \end{aligned} \tag{4.4}$$

with

$$\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau) \equiv \begin{pmatrix} 0 & -\mathrm{e}^{-\mathbf{i}\,\phi_\tau} \\ \mathrm{e}^{\,\mathbf{i}\,\phi_\tau} & 0 \end{pmatrix}, \qquad (\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau))^2 = -\mathbb{1}\,, \qquad \mathrm{Det}[\,\mathbf{R}_O^{\frac{\pi}{2}}\,] = 1\,. \tag{4.5}$$

And finally for the three dimensional space vectors in polar form:

$$\mathbb{v} = \mathrm{e}^{\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,\mathrm{e}^{(\ln\|\mathbb{v}\|)\,\mathbb{1}\,\pm\,\mathbf{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^z\,]}\,\mathrm{e}^{-\,\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,. \tag{4.6}$$

From here, we obtain imaginary vectors in polar form with:

$$\pm\,\mathbf{i}\,\mathbb{v} = \mathrm{e}^{\pm\,\mathbf{i}\,\frac{\pi}{2}\,\mathbb{1}}\,\mathbb{v} = \mathrm{e}^{\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,\mathrm{e}^{(\ln\|\mathbb{v}\|)\,\mathbb{1}\,\pm\,\mathbf{i}\,\frac{\pi}{2}\,\sigma^z}\,\mathrm{e}^{-\,\varphi_n\,\mathbf{R}_o^{\frac{\pi}{2}}(\phi_\tau)}\,. \tag{4.7}$$

V. VECTORS IN $\mathfrak{M}$ IN CARTESIAN AND EULERIAN EXPONENTIAL FORMS.

The proofs for the results of this section are in appendix A.

0) $\mathfrak{R}^4$ as an Euclidean space. Consider the vectors $\mathbb{w}$ defined in (2.10). Our purpose is to write them in an exponential form:

$$\mathbb{w} = \mathrm{e}^{\,\mathbb{w}'} \iff w_t\,\mathbb{1} + |w_n|\,\mathbb{n} = \mathrm{e}^{\,u\,\mathbb{1} + v\,\mathbb{n}'}, \qquad u \in \mathfrak{C},\ \ v \in \mathfrak{C}, \qquad \mathbb{n}'^2 = \mathbb{1}.$$

This is possible with: $\begin{cases} u \in \{\ \ln\sqrt{w_t^2 - |w_n^2|}\ ,\ \ln\sqrt{-(w_t^2 - |w_n^2|)} \pm \mathbf{i}\frac{\pi}{2}\ \} \\ v = \operatorname{argtanh}\dfrac{|w_n|}{w_t} = \operatorname{argtanh}\dfrac{w_t}{|w_n|} \pm \mathbf{i}\frac{\pi}{2} \end{cases}$, $\qquad (\mathbb{n}' = \mathbb{n}\ ,\ \ \mathbb{n}^2 = \mathbb{1})$.

In particular, for a simple case, with $w_t > |w_n|$, we obtain:

$$\mathbb{w} = \mathrm{e}^{\,\ln\sqrt{w_t^2 - |w_n|^2}\,\mathbb{1} + \operatorname{argtanh}\frac{|w_n|}{w_t}\,\mathbb{n}}$$

Matrices $\mathbb{w}$ satisfy: $0 < \frac{1}{2}\,\mathrm{Tr}\,(\mathbb{w}^2) = w_t^2 + w_x^2 + w_y^2 + w_z^2 \ \neq\ -w_t^2 + w_x^2 + w_y^2 + w_z^2 = -\det\mathbb{w} = \frac{1}{2}\,\mathrm{Tr}(\,\mathbb{w}_\mu \mathbb{w}^\mu)$, and in general it is: $\mathbb{w}^2 \neq k\,\mathbb{1}$. Finally, in particular for the diagonal matrices (vectors) we have:

$$\mathbb{w}_d = w_t\,\mathbb{1} + w_z\,\sigma^z = (w_t + w_z)\,\overset{\wedge}{\sigma} + (w_t - w_z)\,\overset{\vee}{\sigma}, \qquad \text{so that:} \quad \begin{cases} \mathbb{w}_{d\,\mu}\mathbb{w}_d^\mu = (-w_t^2 + w_z^2)\,\mathbb{1} \\ \mathbb{w}_d^2 = (w_t^2 + w_z^2)\,\mathbb{1} + 2w_t w_z\,\sigma^z \end{cases}, \tag{5.1}$$

and $\mathbb{w}_d^k = (w_t + w_z)^k\,\overset{\wedge}{\sigma} + (w_t - w_z)^k\,\overset{\vee}{\sigma} = \frac{1}{2}\,[(w_t + w_z)^k + (w_t - w_z)^k]\,\mathbb{1} + \frac{1}{2}\,[(w_t + w_z)^k - (w_t - w_z)^k]\,\sigma^z$.

These results suggest that it is more appropriate to treat the vectors in $\mathfrak{R}^4$ with a Minkowskian metric, therefore as belonging to a Minkowskian space rather than with an Euclidean metric (Euclidean space). Specifically:

1) $\mathfrak{R}^4$ in the way of an underlying Minkowski space $\mathcal{M}$. Consider the vectors $\mathbb{w}$ defined in (2.10) with (2.11). We classify them:

α) Time type vectors: $\qquad \mathbb{w}_{tt} \equiv w_{t_1}\,\mathbb{1} + |\,w_{n_1}|\,\mathbb{n}$, $\qquad$ with: $\quad \|\,\mathbb{w}_{tt}\,\|_{\sim}^2 \equiv -w_{t_1}^2 + w_{n_1}^2 < 0$:

$$\begin{cases} s_{tt} = +\sqrt{-\,\|\,\mathbb{w}_{tt}\,\|_{\sim}^2} \\ \rho_{tt} = \operatorname{argtanh}\dfrac{|\,w_{n_1}|}{w_{t_1}} \end{cases}, \quad \begin{cases} s_{tt} > 0 \\ |\dfrac{|\,w_{n_1}|}{w_{t_1}}| < 1 \end{cases}, \quad \begin{cases} w_{t_1} \neq 0 \\ 0 < |\,w_{n_1}| < |w_{t_1}| \end{cases}, \quad \sqrt{\frac{w_{t_1} + |\,w_{n_1}|}{w_{t_1} - |\,w_{n_1}|}} = \frac{|\,w_{t_1} + |\,w_{n_1}|\,|}{\sqrt{w_{t_1}^2 - w_{n_1}^2}} = a_1 > 0\ .$$

β) Space type vectors: $\qquad \mathbb{w}_{st} \equiv w_{t_2}\mathbb{1} + |\,w_{n_2}|\,\mathbb{n}$, $\qquad$ with: $\quad \|\,\mathbb{w}_{st}\,\|_{\sim}^2 \equiv -w_{t_2}^2 + w_{n_2}^2 > 0$:

$$\begin{cases} s_{st} = +\sqrt{\ \|\,\mathbb{w}_{st}\,\|_{\sim}^2} \\ \tilde{\rho}_{st} = \operatorname{argtanh}\dfrac{w_{t_2}}{|\,w_{n_2}|} \end{cases}, \quad \begin{cases} s_{st} > 0 \\ |\dfrac{w_{t_2}}{|\,w_{n_2}|}| < 1 \end{cases}, \quad \begin{cases} -|w_{n_2}| < w_{t_2} < |w_{n_2}| \\ |w_{n_2}| > 0 \end{cases}, \quad \sqrt{\frac{w_{t_2} + |\,w_{n_2}|}{-w_{t_2} + |\,w_{n_2}|}} = \frac{w_{t_2} + |w_{n_2}|}{\sqrt{-w_{t_2}^2 + w_{n_2}^2}} = a_2 > 0\ .$$

γ) Light ray type vectors: $\qquad \mathbb{w}_{lt} \equiv w_{t_3}\mathbb{1} + |\,w_{n_3}|\,\mathbb{n}$, $\qquad$ with: $\quad \|\,\mathbb{w}_{lt}\,\|_{\sim}^2 \equiv -w_{t_3}^2 + w_{n_3}^2 = 0$, (null vectors):

then: $\ |\,w_{n_3}| = \operatorname{sign}(w_{t_3})\,w_{t_3}$, so that we can rewrite: $\ \mathbb{w}_{lt} = w_{t_3}\,(\mathbb{1} + \operatorname{sign}(w_{t_3})\,\mathbb{n})$.

With the Minkowski metric we write $\|\,\mathbb{w}_{lt\mu}\,\|_{\sim}^2 = \|\,\mathbb{w}_{lt}^\mu\,\|_{\sim}^2 = 0$. There exist $\mathbb{w}_{lt\mu}$ and $\mathbb{w}_{lt}^\mu$, but not $\mathbb{w}_{lt\mu}^{-1}$, $\mathbb{w}_{lt}^{\mu-1}$.

It is interesting to observe that we can not express in exponential form, the light rays (they are null vectors).

In this sense null vectors are singular cases. Even more, independently of the values of $|\,w_{n_3}|$ or of w_{t_3}, the matrices $k\,(-\mathbb{1} + \mathbb{n})$ and $k'(\mathbb{1} + \mathbb{n})$ represent null vectors. Using (2.11), the definition of the pseudo norm, we multiply them to get: $(-\mathbb{1} + \mathbb{n})(\mathbb{1} + \mathbb{n}) = -\mathbb{1} + \mathbb{n}^2 = 0$. Write this last equation in the form $\mathbb{n}^2 = \mathbb{1}$, which is the condition for unit vectors or points in the sphere $n_x^2 + n_y^2 + n_z^2 = 1$. If we try to obtain the square root of $\mathbb{1}$, in order to express $\mathbb{n}$, the answer is that there are infinite values, any possible matrix $\mathbb{n}$ satisfying previous equations, i.e. all the points in the unit sphere. Same results for $\mathbb{v}^2 = w_{n_3}^2\mathbb{1} = w_{t_3}^2\mathbb{1}$.

We are in a similar situation as with the polar decomposition in dimension 2 and $r = 0$.

The exponential expressions for the vectors of time or space type are:

α) Time type vectors $\ \underline{\underline{(w_{t_1} \neq 0}},\ \ |w_{n_1}| < |w_{t_1}|)$:

$$\mathbb{w}_{tt} \equiv w_{t_1}\,\mathbb{1} + |\,w_{n_1}|\,\mathbb{n} = \operatorname{sign}(w_{t_1})\,s_{tt}\,\mathrm{e}^{\,\rho_{tt}\,\mathbb{n}} = \mathrm{e}^{\,\ln\{\operatorname{sign}(w_{t_1})\,s_{tt}\}\,\mathbb{1} + \rho_{tt}\,\mathbb{n}} \in \mathfrak{R}^4. \tag{5.2}$$

$\operatorname{sign}(w_{t_1}) = \begin{cases} +1 & \text{if } w_{t_1} > 0 \\ -1 & \text{if } w_{t_1} < 0 \end{cases}$. The vectors defining a boost are: $\ \mathbb{B} = \mathbb{B}^\dagger = \mathbb{B}(\boldsymbol{\rho}) = k\,\mathrm{e}^{\,\rho\,\mathbb{m}} = k\,\cosh\rho\,\mathbb{1} + k\sinh\rho\,\mathbb{m}$, with $k = \pm$, they satisfy: $-(k\cosh\rho)^2 + (k\sinh\rho)^2 = -1$, so that they are time type vectors. This $\operatorname{sign}(w_{t_1})$ is related to the sign in the definition of a boost, the $+$ or $-$ factor appearing with the term $\cosh\rho$ ($\cosh\rho > 0$).

β) Space type vectors $(\underline{\underline{|w_{n_2}| > 0}}, \; |w_{t_2}| < |w_{n_2}|)$:

$$a) \begin{cases} \mathsf{w}_{st} = \mathrm{e}^{\ln s_{st} \mathbb{1} + \tilde{\rho}_{st} \mathbb{m} \pm \mathbf{i}\frac{\pi}{2}[-\mathbb{1} + \mathbb{m}]} \in \mathfrak{R}^4 \\ \pm\mathbf{i}\,\mathsf{w}_{st} = \mathrm{e}^{\ln s_{st} \mathbb{1} + \tilde{\rho}_{st} \mathbb{m} \pm \mathbf{i}\frac{\pi}{2}\mathbb{m}} \in (\mathbf{i}\mathfrak{R})^4 \end{cases}, \quad \text{with } \left|\frac{w_{t_2}}{|w_{n_2}|}\right| < 1, \quad (|w_{n_2}| \neq 0) \; ; \tag{5.3}$$

$$\text{or } b) \quad \pm\mathbf{i}\,\mathsf{w}_{st} = \mathrm{e}^{\ln s_{st} \mathbb{1} + \rho_{st} \mathbb{m}} \in (\mathbf{i}\mathfrak{R})^4, \quad \text{with } \left|\frac{|w_{n_2}|}{w_{t_2}}\right| > 1, \quad \underline{\underline{w_{t_2} \neq 0}}$$

with: $\tilde{\rho}_{st} \pm \mathbf{i}\frac{\pi}{2} = \operatorname{argtanh}\frac{|w_{n_2}|}{w_{t_2}} \equiv \rho_{st} \in \mathfrak{C}$. For the meaning as a 'rotation ' of $\mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\mathbb{1}}$, and of $\mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\mathbb{m}}$ on a side, see II and I later in relation to (5.3) and (5.2).

Now, for the total polar exponential form we compulsory adopt the Minkowski space, as we need to define the norm of the vectors. In a similar way to the polar decomposition in $\mathfrak{R}^2$ with the condition $r \neq 0$ there, we have for these vectors the condition $-w_t^2 + w_n^2 \neq 0$ ('running ' inwards or outwards spheres), and we obtain:

α) Time type vectors w_{tt} ($w_{t_1} \neq 0$). We use formula (4.4) for vectors $\mathbb{m}$:

$$\begin{aligned} \mathsf{w}_{tt} &= w_{t_1}\mathbb{1} + |w_{n_1}|\,\mathbb{m} = \mathrm{e}^{\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})} (w_{t_1}\mathbb{1} + |w_{n_1}|\,\sigma^z)\, \mathrm{e}^{-\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})} = \\ &= \mathrm{e}^{\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})}\, \mathrm{e}^{\ln[\operatorname{sign}(w_{t_1})\, s_{tt}]\mathbb{1} + \rho_{tt}\,\sigma^z}\, \mathrm{e}^{-\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})}. \end{aligned} \tag{5.4}$$

β) Space type vectors $\mathsf{w}_{st}(|w_{n_2}| > 0)$. The total polar form for (5.3a) and (5.3b):

$$\{\pm\mathbf{i}\}\,\mathsf{w}_{st} = \{\mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\mathbb{1}}\}\, \mathrm{e}^{\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})}\, \mathrm{e}^{\ln s_{st}\mathbb{1} + (\tilde{\rho}_{st} \pm \mathbf{i}\frac{\pi}{2})\sigma^z \mp \mathbf{i}\frac{\pi}{2}\mathbb{1}}\, \mathrm{e}^{-\varphi_n \mathbf{R}_o^{\frac{\pi}{2}}(\phi_{\tau n})}, \tag{5.5}$$

In particular, imposing $w_{t_2} = 0$ in (5.3a) or appropriately in (5.5), we express in a complete exponential form the 3-dimensional vectors as in (3.2),(3.3) and (4.6),(4.7). We can not do it with (5.2) as this formula requires $w_{t_1} \neq 0$.

We infer two different relations for the time and space vectors in $\mathfrak{M}$:

I) *If we interchange the time and space coordinates, in the way we show, we interchange the time - space character of the vectors:*

$$\begin{aligned} \mathsf{w}_{tt}\,\{\operatorname{sign}(w_{t_1})\,\mathbb{m}\} &= (w_{t_1}\mathbb{1} + |w_{n_1}|\,\mathbb{m})\,\operatorname{sign}(w_{t_1})\,\mathbb{m} = \\ &= \operatorname{sign}(w_{t_1})\,|w_{n_1}|\,\mathbb{1} + \operatorname{sign}(w_{t_1})\,w_{t_1}\,\mathbb{m} = \tilde{w}_{t_2}\mathbb{1} + |\tilde{w}_{n_2}|\,\mathbb{m} \equiv \tilde{\mathsf{w}}_{st}, \end{aligned} \tag{5.6}$$

$$\begin{cases} \tilde{w}_{t_2} = \operatorname{sign}(w_{t_1})\,|w_{n_1}| \in \mathfrak{R} \\ |\tilde{w}_{n_2}| = \operatorname{sign}(w_{t_1})\,w_{t_1} > 0 \end{cases}, \quad \begin{cases} -|\tilde{w}_{n_2}| < \tilde{w}_{t_2} < |\tilde{w}_{n_2}| \\ |\tilde{w}_{n_2}| > 0 \end{cases}, \quad \|\tilde{\mathsf{w}}_{st}\|_{\sim}^2 \equiv -\tilde{w}_{t_2}^2 + \tilde{w}_{n_2}^2 = -\|\mathsf{w}_{tt}\|_{\sim}^2 > 0,$$

$$\begin{aligned} \mathsf{w}_{st}\,\{\operatorname{sign}(w_{t_2})\,\mathbb{m}\} &= (w_{t_2}\mathbb{1} + |w_{n_2}|\,\mathbb{m})\,\operatorname{sign}(w_{t_2})\,\mathbb{m} = \\ &= \operatorname{sign}(w_{t_2})\,|w_{n_2}|\,\mathbb{1} + \operatorname{sign}(w_{t_2})\,w_{t_2}\,\mathbb{m} = \tilde{w}_{t_1}\mathbb{1} + |\tilde{w}_{n_1}|\,\mathbb{m} \equiv \tilde{\mathsf{w}}_{tt}, \end{aligned} \tag{5.7}$$

$$\begin{cases} \tilde{w}_{t_1} = \operatorname{sign}(w_{t_2})\,|w_{n_2}| \neq 0 \\ |\tilde{w}_{n_1}| = \operatorname{sign}(w_{t_2})\,w_{t_2} \geq 0 \end{cases}, \quad \begin{cases} \tilde{w}_{t_1} \neq 0 \\ 0 \leq |\tilde{w}_{n_1}| < |\tilde{w}_{t_1}| \end{cases}, \quad \|\tilde{\mathsf{w}}_{tt}\|_{\sim}^2 \equiv -\tilde{w}_{t_1}^2 + \tilde{w}_{n_1}^2 = -\|\mathsf{w}_{st}\|_{\sim}^2 < 0.$$

Relate this with the expressions in (5.2) and (5.3) with the direct multiplication by $k\,\mathbb{m}$ (formulas in (3.1)).

II) *If we multiply a vector by* $\pm\mathbf{i}$, *or if we 'time-rotate' a vector by multiplication by* $\mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\mathbb{1}}$, *we interchange the time - space character of the metric:*

$$\mathsf{w}_{tt}\,\mathrm{e}^{\epsilon\mathbf{i}\frac{\pi}{2}\mathbb{1}} = (w_{t_1}\mathbb{1} + |w_{n_1}|\,\mathbb{m})\,\epsilon\,\mathbf{i} = \epsilon\mathbf{i}\,w_{t_1}\mathbb{1} + \epsilon\mathbf{i}\,|w_{n_1}|\,\mathbb{m} = \tilde{\tilde{w}}_{t_1}^{\epsilon}\mathbb{1} + \tilde{\tilde{w}}_{n_1}^{\epsilon}\,\mathbb{m} \equiv \tilde{\tilde{\mathsf{w}}}_{att}^{\epsilon} \in (\mathbf{i}\mathfrak{R})^4, \tag{5.8}$$

$$\begin{cases} \|\tilde{\tilde{\mathsf{w}}}_{att}^{\epsilon}\|_{\sim}^2 \equiv -\tilde{\tilde{w}}_{t_1}^{\epsilon\,2} + \tilde{\tilde{w}}_{n_1}^{\epsilon\,2} = -(\epsilon\mathbf{i}w_{t_1})^2 + (\epsilon\mathbf{i}w_{n_1})^2 = w_{t_1}^2 - w_{n_1}^2 > 0 & \textit{space type in the metric for} \\ \left|\frac{\tilde{\tilde{w}}_{n_1}^{\epsilon}}{\tilde{\tilde{w}}_{t_1}^{\epsilon}}\right| = \left|\frac{\epsilon\mathbf{i}\,|w_{n_1}|}{\epsilon\mathbf{i}\,w_{t_1}}\right| = \left|\frac{|w_{n_1}|}{w_{t_1}}\right| < 1, \quad (w_{t_1} \neq 0) & \textit{imaginary time type vectors.} \end{cases}$$

$$\mathbb{w}_{st}\,\mathrm{e}^{\epsilon\mathbf{i}\frac{\pi}{2}\mathbb{1}} = (\,w_{t_2}\,\mathbb{1} + |\,w_{n_2}|\,\mathbb{n}\,)\,\epsilon\mathbf{i} \,=\, \epsilon\mathbf{i}\,w_{t_2}\,\mathbb{1} + \epsilon\mathbf{i}\,|\,w_{n_2}|\,\mathbb{n} \,=\, \tilde{\tilde{w}}^{\epsilon}_{t_2}\,\mathbb{1} + \tilde{\tilde{w}}^{\epsilon}_{n_2}\,\mathbb{n} \,\equiv\, \tilde{\tilde{\mathbb{w}}}^{\epsilon}_{ast} \,\in (\mathbf{i}\mathfrak{R})^4, \tag{5.9}$$

$$\begin{cases} \|\,\tilde{\tilde{\mathbb{w}}}^{\epsilon}_{ast}\,\|^2_{\sim} \equiv -\tilde{\tilde{w}}^{\epsilon\,2}_{t_2} + \tilde{\tilde{w}}^{\epsilon\,2}_{n_2} = -(\epsilon\mathbf{i}w_{t_2})^2 + (\epsilon\mathbf{i}w_{n_2})^2 = w^2_{t_2} - w^2_{n_2} < 0, & \textit{time type in the metric for} \\ \left|\,\dfrac{\tilde{\tilde{w}}^{\epsilon}_{t_2}}{\tilde{\tilde{w}}^{\epsilon}_{n_2}}\,\right| = \left|\,\dfrac{\epsilon\mathbf{i}\,w_{t_2}}{\epsilon\mathbf{i}\,|\,w_{n_2}|}\,\right| = \left|\,\dfrac{w_{t_2}}{|\,w_{n_2}|}\,\right| < 1\,, \quad (|\,w_{n_2}| \neq 0) & \textit{imaginary space type vectors.} \end{cases}$$

For (5.9) look at (5.3)-b), whatsoever sign($\tilde{\tilde{w}}^{\epsilon}_{t_2}$) could be, and also sign($\tilde{\tilde{w}}^{\epsilon}_{t_1}$) for (5.8) (look at (5.2)). Pay attention that we are 'equating (substituting) formally real to imaginary'. With (5.2), time type vectors, we can not have real or imaginary 3-vectors alone ($w_t \neq 0$). We need to generalize the definition of the pseudo-norm for not hermitian vectors. See the following section and the *Studies I,2* and *I* in this set of researches. And M and G in Appendix B.

VI. A DIGRESSION ON VECTORS AND ON PAIRS, TRIPLES AND QUADRUPLES. [3] [4]

The initial motivation for the finding of the quaternions by Hamilton was to write the known property $|ab| = |a||b|$, and $|a| > 0$ with a and b one number or a pair of numbers, now with triplets or quadruples. See S4 in Appendix B.

First, the plane $\mathfrak{R}^2$ and the complex plane $-$ numbers.

We wrote in the *Introduction* $\quad \mathbf{v} = (x, y) = x\,\mathbf{e_1} + y\,\mathbf{e_2} \in \mathfrak{R}^2 \;\longleftrightarrow\; z = (x, y) = x\,(\mathbf{1}) + y\,\mathbf{i} \in \mathfrak{C}\,,$

and $\quad (x_1^2 + y_1^2)\,(x_2^2 + y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 \;\longleftrightarrow\; |z_1|^2|z_2|^2 = |z_1z_2|^2,$

but, in general: $\quad |z_1|^2|z_2|^2 = \|\,\mathbf{v_1}\,\|^2\,\|\,\mathbf{v_2}\,\|^2 = (x_1^2 + y_1^2)\,(x_2^2 + y_2^2) \neq (x_1x_2 + y_1y_2)^2 = (\mathbf{v_1}\,.\,\mathbf{v_2})^2 \neq |z_1z_2|^2.$

also: $\quad |\mathbf{v_1}.\mathbf{v_2}| = z_1z_2 = x^2 + y^2, \;$ with $\; \mathbf{v_1} = \pm\mathbf{v_2} = \mathbf{v} = (x, y), \;$ but $\; z_1 = z = x(\mathbf{1}) + y\mathbf{i} \;$ and $\; z_2 = \bar{z}.$

and under rotations: $\quad r'^2 = \|\,v'\,\|^2 = |z'|^2 = x'^2 + y'^2 = x^2 + y^2 = z\,\bar{z} = |z|^2 = \|\,v\,\|^2 = r^2.$

In the searching of integer numbers satisfying similar properties with triplets:

$|\,\mathbf{v}_1\,|\,|\,\mathbf{v}_2\,| \overset{?}{=} |\,\mathbf{v}_1\mathbf{v}_2\,| = |\,\mathbf{v}_3\,|, \quad (|a||b| = |ab|, \;$ in $\mathfrak{C}: \; |z_1|\,|z_2| = |z_1z_2|), \;$ with $\; \mathbf{v}_k = (a_k, b_k, c_k), \; k \in \{1, 2, 3\},$

in other words, is it possible to obtain: $\quad (a_1^2 + b_1^2 + c_1^2)\,(a_2^2 + b_2^2 + c_2^2) \,=\, a_3^2 + b_3^2 + c_3^2 \;?. \quad (a_k \neq 0,\, b_k \neq 0,\, c_k \neq 0),$

Hamilton did trials, for example with an extension of the complex numbers with one scalar and two different imaginary units: $a + \mathbf{i}\alpha + \mathbf{j}\beta$. He was unsuccessful. The definition of a product is problematic. Relating it with vectors, unknowns at his time, what kind of product for the vectors would be the underlying product?.

He needed to jump from pairs to quadruples and to abandon the commutation for obtaining similar results (1843):

second, the space $\mathfrak{R}^4(?) \longleftrightarrow \mathfrak{R}\mathrm{x}(\mathbf{i}\mathfrak{R})^3$: the hypercomplex numbers $-$ space. Quaternions for both:

a quaternion: $\quad \mathbf{q}_k = (a_k, \alpha_k, \beta_k, \gamma_k), \; k \in \{1, 2, 3\}, \;$ or also: $\; \mathbf{q}_k = a_k(\mathbf{1}) + \alpha_k\mathbb{i} + \beta_k\mathbb{j} + \gamma_k\mathbb{k} \equiv a_k(\mathbf{1}) + \mathbf{a_k},$

the fundamental rule: $\quad \mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = \mathbb{i}\,\mathbb{j}\,\mathbb{k} = -(\mathbf{1})\,,$ with

the anticommutative product: $\quad \mathbb{i}\,\mathbb{j} = \mathbb{k} = -\mathbb{j}\,\mathbb{i}\,, \quad \mathbb{j}\,\mathbb{k} = \mathbb{i} = -\mathbb{k}\,\mathbb{j}\,, \quad \mathbb{k}\,\mathbb{i} = \mathbb{j} = -\mathbb{i}\,\mathbb{k}\,;$ and

with the conjugate quaternion: $\quad \bar{\mathbf{q}}_k = a_k(\mathbf{1}) - \alpha_{\mathbf{k}}\mathbb{i} - \beta_{\mathbf{k}}\mathbb{j} - \gamma_{\mathbf{k}}\mathbb{k}, \;$ it is: $\; \mathbf{q}_k\,\bar{\mathbf{q}}_k = (a_k^2 + \alpha_k^2 + \beta_k^2 + \gamma_k^2)\,(\mathbf{1}) \equiv \|\,\mathbf{q}_k\,\|^2\,(\mathbf{1}),$

final result: $\quad \|\,\mathbf{q_1}\,\|\,\|\,\mathbf{q_2}\,\| = \|\,\mathbf{q_1}\,\mathbf{q_2}\,\| = \|\,\mathbf{q_3}\,\|$ $\quad$ or in components with the squares:

$$(a_1^2 + \alpha_1^2 + \beta_1^2 + \gamma_1^2)\,(a_2^2 + \alpha_2^2 + \beta_2^2 + \gamma_2^2) = (a_3^2 + \alpha_3^2 + \beta_3^2 + \gamma_3^2)\,,$$

with

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2 \quad \begin{cases} a_3 = a_1a_2 \,-\, (\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) \,\equiv\, a_1a_2 \,-\, \mathbf{a_1}\,.\,\mathbf{a_2} & (\big|_{(\mathbf{1})}) \\ \alpha_3 = a_1\alpha_2 + \alpha_1a_2 \,+\, \beta_1\gamma_2 - \gamma_1\beta_2 \,\equiv\, (a_1\,\mathbf{a_2} + \mathbf{a_1}\,a_2 + \mathbf{a_1} \times \mathbf{a_2})\big|_{\mathbb{i}} \\ \beta_3 = a_1\beta_2 + \beta_1a_2 \,-\, \alpha_1\gamma_2 + \gamma_1\alpha_2 \,\equiv\, (a_1\,\mathbf{a_2} + \mathbf{a_1}\,a_2 + \mathbf{a_1} \times \mathbf{a_2})\big|_{\mathbb{j}} \\ \gamma_3 = a_1\gamma_2 + \gamma_1a_2 \,+\, \alpha_1\beta_2 - \beta_1\alpha_2 \,\equiv\, (a_1\,\mathbf{a_2} + \mathbf{a_1}\,a_2 + \mathbf{a_1} \times \mathbf{a_2})\big|_{\mathbb{k}} \end{cases}, \tag{6.1}$$

$\mathbf{a_1}\,.\,\mathbf{a_2}$ and $\mathbf{a_1} \times \mathbf{a_2}$ similar to the usual scalar and vector products. The vectors $\mathbf{a_k}$ are axial vectors.
This norm, like the norm of the two or three dimensional vectors is a sum of squares, it is euclidean. [4]

There is a correspondence between these vectors and the vectors of the Pauli basis ($\mathbf{i}$ unit in between) [5]:

$$(\mathbf{1}) \;\longleftrightarrow\; \mathbb{1}\,, \qquad \mathbb{i} \;\longleftrightarrow\; -\mathbf{i}\sigma^x\,, \qquad \mathbb{j} \;\longleftrightarrow\; -\mathbf{i}\sigma^y\,, \qquad \mathbb{k} \;\longleftrightarrow\; -\mathbf{i}\sigma^z\,. \tag{6.2}$$

$\mathbb{R}^{\dagger}$ in formula (2.8) is an example of a quaternion. This formula shows that starting with a 3 dimensional (imaginary) vector in an exponent we obtain a 4 dimensional object (the quaternion), a real and three imaginaries.

A consequence of the results of section V, vectors in $\mathfrak{M}$, consists in the possibility of obtaining the property:

the product of the squares of the pseudo-norms as the minus square of the pseudo-norm of the product in a generalized Minkowski space $(\mathfrak{M}\cup(\mathbf{i}\mathfrak{R})^3)$: $\quad \| \mathtt{w}_1 \|^2_{\sim} \| \mathtt{w}_2 \|^2_{\sim} = - \| \mathtt{w}_1 \mathtt{w}_2 \|^2_{\sim} = - \| \mathtt{w}_3 \|^2_{\sim}$,

$\mathtt{w}_1^{\dagger} = \mathtt{w}_1$, $\quad \mathtt{w}_2^{\dagger} = \mathtt{w}_2$, in general $\mathtt{w}_3^{\dagger} \neq \mathtt{w}_3$, in the way that follows.

I) We have to clarify the meaning of $\mathtt{w}_1\mathtt{w}_2$. We use it in four ways: $\quad \mathtt{w}_1\mathtt{w}_2 \;\rightarrow \begin{cases} \mathtt{w}_{1\mu}\, \mathtt{w}_2^{\mu} \\ \mathtt{w}_{1\mu}\, \mathtt{w}_{2\mu} \end{cases}$, and $\begin{cases} \mathtt{w}_1^{\mu}\, \mathtt{w}_2^{\mu} \\ \mathtt{w}_1^{\mu}\, \mathtt{w}_{2\mu} \end{cases}$.
With $\mathtt{w}_2^{\mu} = \mathtt{w}_1^{\mu}$, the first and fourth ones already stated in the definition of the pseudo-norm .

We introduce more notation in order to write in a more compact way the $'$new covariant and contravariant $'$ vectors (with the η index in a left position):

$$^{\eta}\mathtt{w}_n \equiv \eta\; w_{nt}\mathbb{1} + \mathtt{v}_n \equiv \eta\; w_{nt}\mathbb{1} + w_{nz}\sigma^{z} + w_{nx}\sigma^{x} + w_{ny}\sigma^{y} \equiv \begin{cases} \mathtt{w}_{n\mu}\,, & \eta \equiv + \\ \mathtt{w}_n^{\mu}\,, & \eta \equiv - \end{cases} ,$$

$$^{\eta}\mathtt{w}_{3\mu} \equiv \mathtt{w}_{1\mu}\,{}^{\eta}\mathtt{w}_2 = \;(w_{1t}\mathbb{1} + \mathtt{v}_1)(\eta\; w_{2t}\mathbb{1} + \mathtt{v}_2) = \;{}^{\eta}w_{3t}\mathbb{1} + {}^{\eta}\mathtt{v}_3 + (\mathbf{i}\mathtt{v})_3 \rightarrow \begin{cases} \tilde{\mathtt{w}}_{3\mu} \equiv \mathtt{w}_{1\mu}\, \mathtt{w}_2^{\mu} \;= {}^{-}w_{3t}\mathbb{1} + {}^{-}\mathtt{v}_3 + (\mathbf{i}\mathtt{v})_3 \\ \hat{\mathtt{w}}_{3\mu} \equiv \mathtt{w}_{1\mu}\, \mathtt{w}_{2\mu} \;= {}^{+}w_{3t}\mathbb{1} + {}^{+}\mathtt{v}_3 + (\mathbf{i}\mathtt{v})_3 \end{cases} ,$$

$$^{\eta}\mathtt{w}_3^{\mu} \equiv -\mathtt{w}_1^{\mu}\,{}^{\eta}\mathtt{w}_2 = -(-w_{1t}\mathbb{1} + \mathtt{v}_1)(\eta\; w_{2t}\mathbb{1} + \mathtt{v}_2) = -{}^{\eta}w_{3t}\mathbb{1} + {}^{\eta}\mathtt{v}_3 - (\mathbf{i}\mathtt{v})_3 \rightarrow \begin{cases} \tilde{\mathtt{w}}_3^{\mu} \equiv -\mathtt{w}_1^{\mu}\mathtt{w}_{2\mu} = -{}^{-}w_{3t}\mathbb{1} + {}^{-}\mathtt{v}_3 - (\mathbf{i}\mathtt{v})_3 \\ \hat{\mathtt{w}}_3^{\mu} \equiv -\mathtt{w}_1^{\mu}\, \mathtt{w}_2^{\mu} = -{}^{+}w_{3t}\mathbb{1} + {}^{+}\mathtt{v}_3 - (\mathbf{i}\mathtt{v})_3 \end{cases} , \tag{6.3}$$

$$^{\eta}w_{3t} \equiv \eta\; w_{1t}w_{2t} + (w_{1z}w_{2z} + w_{1x}w_{2x} + w_{1y}w_{2y}) \;,$$

$$^{\eta}\mathtt{v}_3 \equiv \begin{cases} {}^{\eta}w_{3z} = w_{1t}w_{2z} + \eta\; w_{1z}w_{2t} \\ {}^{\eta}w_{3x} = w_{1t}w_{2x} + \eta\; w_{1x}w_{2t} \\ {}^{\eta}w_{3y} = w_{1t}w_{2y} + \eta\; w_{1y}w_{2t} \end{cases} , \qquad (\mathbf{i}\mathtt{v})_3 \equiv \begin{cases} (\mathbf{i}w)_{3z} = \mathbf{i}(w_{1x}w_{2y} - w_{1y}w_{2x}) \\ (\mathbf{i}w)_{3x} = \mathbf{i}(w_{1y}w_{2z} - w_{1z}w_{2y}) \\ (\mathbf{i}w)_{3y} = \mathbf{i}(w_{1z}w_{2x} - w_{1x}w_{2z}) \end{cases} .$$

We have added a 'minus factor $'$ in the definitions of $\tilde{\mathtt{w}}_3^{\mu}$ and of $\hat{\mathtt{w}}_3^{\mu}$ in order to assure their contravariant character with respect to an arbitrarily fixed covariant character of $\tilde{\mathtt{w}}_{3\mu}$ and of $\hat{\mathtt{w}}_{3\mu}$. We have to include complex 3-vectors, real plus imaginary. The new not Hermitian term $(\mathbf{i}\mathtt{v})_3$ observes a similar behavior as the time term. We should pay attention to the result: departing from (4-real) vectors after the product we obtain vectors in a 7 dimensional space over the reals (4 with the reals, 3 with the imaginaries). We can not obtain an imaginary time type vector (dimension). Although this, we have seen already imaginary time type vectors in the exponents in the definitions of various vectors.

We explore the generalization of these vectors types in another *study* in the set (*Study I,2*).

And, thanks to: Tr σ^{z} = Tr σ^{x} = Tr $\sigma^{y} = 0$, it is: $\frac{1}{2}\mathrm{Tr}(\tilde{\mathtt{w}}_{3\mu}) = {}^{-}w_{3t}$, $\frac{1}{2}\mathrm{Tr}(\hat{\mathtt{w}}_{3\mu}) = {}^{+}w_{3t}$, $\frac{1}{2}\mathrm{Tr}(\tilde{\mathtt{w}}_3^{\mu}) = -{}^{-}w_{3t}$, and $\frac{1}{2}\mathrm{Tr}(\hat{\mathtt{w}}_3^{\mu}) = -{}^{+}w_{3t}$.

II) Definition of the pseudo-norms of these not Hermitian vectors and verification of the property stated above:

$$\begin{cases} \| \tilde{\mathtt{w}}_{3\mu} \|^2_{\sim} \equiv \frac{1}{2}\mathrm{Tr}\left(\tilde{\mathtt{w}}_{3\mu}^{\dagger}\, \tilde{\mathtt{w}}_3^{\mu} \right) = \frac{1}{2}\mathrm{Tr}\left((\mathtt{w}_{1\mu}\, \mathtt{w}_2^{\mu})^{\dagger}\, (-\mathtt{w}_1^{\mu}\mathtt{w}_{2\mu}) \right) = -\frac{1}{2}\mathrm{Tr}\left(\mathtt{w}_2^{\mu}\mathtt{w}_{1\mu}\, \mathtt{w}_1^{\mu}\mathtt{w}_{2\mu} \right) = \\ \qquad = -\frac{1}{2}\, \| \mathtt{w}_{1\mu} \|^2_{\sim}\, \mathrm{Tr}\left(\mathtt{w}_2^{\mu}\, \mathbb{1}\, \mathtt{w}_{2\mu} \right) = - \| \mathtt{w}_{1\mu} \|^2_{\sim}\; \| \mathtt{w}_{2\mu} \|^2_{\sim} \\ \| \hat{\mathtt{w}}_{3\mu} \|^2_{\sim} \equiv \frac{1}{2}\,\mathrm{Tr}\left(\hat{\mathtt{w}}_{3\mu}^{\dagger}\, \hat{\mathtt{w}}_3^{\mu} \right) = \frac{1}{2}\mathrm{Tr}\left((\mathtt{w}_{1\mu}\, \mathtt{w}_{2\mu})^{\dagger}\, (-\mathtt{w}_1^{\mu}\, \mathtt{w}_2^{\mu}) \right) = -\frac{1}{2}\mathrm{Tr}\left(\mathtt{w}_{2\mu}\, \mathtt{w}_{1\mu}\, \mathtt{w}_1^{\mu}\, \mathtt{w}_2^{\mu} \right) = \\ \qquad = -\frac{1}{2}\, \| \mathtt{w}_{1\mu} \|^2_{\sim}\, \mathrm{Tr}\left(\mathtt{w}_{2\mu}\, \mathbb{1}\, \mathtt{w}_2^{\mu} \right) = - \| \mathtt{w}_{1\mu} \|^2_{\sim}\; \| \mathtt{w}_{2\mu} \|^2_{\sim} \\ \| \tilde{\mathtt{w}}_3^{\mu} \|^2_{\sim} \equiv \frac{1}{2}\,\mathrm{Tr}\left(\tilde{\mathtt{w}}_3^{\mu\,\dagger}\, \tilde{\mathtt{w}}_{3\mu} \right) = \frac{1}{2}\mathrm{Tr}\left((-\mathtt{w}_1^{\mu}\mathtt{w}_{2\mu})^{\dagger}\, (\mathtt{w}_{1\mu}\, \mathtt{w}_2^{\mu}) \right) = -\frac{1}{2}\mathrm{Tr}\left(\mathtt{w}_{2\mu}\, \mathtt{w}_1^{\mu}\, \mathtt{w}_{1\mu}\, \mathtt{w}_2^{\mu} \right) = \\ \qquad = -\frac{1}{2}\, \| \mathtt{w}_{1\mu} \|^2_{\sim}\, \mathrm{Tr}\left(\mathtt{w}_{2\mu}\, \mathbb{1}\, \mathtt{w}_2^{\mu} \right) = - \| \mathtt{w}_{1\mu} \|^2_{\sim}\; \| \mathtt{w}_{2\mu} \|^2_{\sim} \\ \| \hat{\mathtt{w}}_3^{\mu} \|^2_{\sim} \equiv \frac{1}{2}\,\mathrm{Tr}\left(\hat{\mathtt{w}}_3^{\mu\,\dagger}\, \hat{\mathtt{w}}_{3\mu} \right) = \frac{1}{2}\mathrm{Tr}\left((-\mathtt{w}_1^{\mu}\, \mathtt{w}_2^{\mu})^{\dagger}\, (\mathtt{w}_{1\mu}\, \mathtt{w}_{2\mu}) \right) = -\frac{1}{2}\mathrm{Tr}\left(\mathtt{w}_2^{\mu}\, \mathtt{w}_1^{\mu}\, \mathtt{w}_{1\mu}\, \mathtt{w}_{2\mu} \right) = \\ \qquad = -\frac{1}{2}\, \| \mathtt{w}_{1\mu} \|^2_{\sim}\, \mathrm{Tr}\left(\mathtt{w}_2^{\mu}\, \mathbb{1}\, \mathtt{w}_{2\mu} \right) = - \| \mathtt{w}_{1\mu} \|^2_{\sim}\; \| \mathtt{w}_{2\mu} \|^2_{\sim} \end{cases} \tag{6.4}$$

$\mathbf{B}_P \cup \mathbf{B}_{\{\mathbf{i}\sigma^{z},\, \mathbf{i}\sigma^{x},\, \mathbf{i}\sigma^{y}\}}$ is not an orthonormal base and neither pseudo-orthonormal: $\langle\, \sigma^{j}, \mathbf{i}\sigma^{j}\,\rangle = \langle\, \sigma^{j}, \mathbf{i}\sigma^{j}\,\rangle_{\sim} \neq 0$.

Finally, in analogy with the rotations in $\mathfrak{R}^3$ we define the boosts in $\mathfrak{M}$: $\quad \mathtt{w}' \equiv\; \mathcal{B}\,[\,\mathtt{w}\,] \equiv\; \mathcal{B}(\mathbf{2}\boldsymbol{\rho})\,[\,\mathtt{w}\,] \equiv\; \mathbb{B}\;\mathtt{w}\;\mathbb{B}$, with $\mathbb{B} = \mathbb{B}^{\dagger} \equiv \mathbb{B}(\boldsymbol{\rho}) \equiv \mathrm{e}^{\,\boldsymbol{\rho}} \equiv \mathrm{e}^{\,\rho\,\mathtt{m}} = \;\cosh\rho\;\mathbb{1} + \sinh\rho\;\mathtt{m}$, $\;\rho \in \mathfrak{R}\;$ and $\;\mathtt{m} \in \mathfrak{R}^3$.

For both, rotations and boosts we have: $\quad \mathtt{w}'^{\dagger} = (\mathbb{B}\;\mathtt{w}\;\mathbb{B})^{\dagger} = \mathbb{B}\;\mathtt{w}\;\mathbb{B} = \mathtt{w}'\;$ and $\;\mathtt{w}''^{\dagger} = (\mathbb{R}\;\mathtt{w}\;\mathbb{R}^{\dagger})^{\dagger} = \mathbb{R}\;\mathtt{w}\;\mathbb{R}^{\dagger} = \mathtt{w}''$.

Rotations and boosts act, in this matrix formalism, over (real) vectors producing (real) vectors in a hermitian form, and over imaginary vectors producing imaginary vectors in an anti-hermitian form.

An interesting property of the rotations and the boosts, implied by these products is: the initial and final vectors $\mathbb{w}$, $\mathbb{w}'$, $\mathbb{w}''$ belong to $\mathfrak{R}^4$ (real 4-vectors), although the final vectors are obtained as the product of vectors with at least one of the terms in it not belonging to $\mathfrak{R}^4$.

One more question. If the same triad of real numbers can represent a displacement (polar vectors) and also a vector defining an axis of rotation (axial vector), why do we need to duplicate the dimension of the space as we do with $\mathfrak{C}$ ($\mathfrak{R}^2$) in relation to $\mathfrak{R}$?. Think in the Hamilton's constructions. See S1 and H in Appendix B.

The answer since Gibbs and Heaviside following the line of Grassmann and overall Clifford was to avoid it by introducing new related concepts for the axial vectors, for example like "oriented areas for a product of vectors" and so on. This agrees with the space and time of our "perceptual space", of our "common sense" (1+3 real dimensions).

The development of the theoretical physics in relation to elementary particles seems to suggest a different answer as it concern the dimension of the geometry of the time and the space. Also, see Penrose [8].

VII. CONCLUSIONS.

The initial purpose of this *study* consisted in the extension of the formula of Euler with the polar expression of the points in $\mathfrak{C}$. These points (pairs of real numbers) representing the coordinates of vectors in the plane $\mathfrak{R}^2$. The purposed extension, for the three and four dimensional vectors.

The usual formalism with the coordinates of the vectors expressed in the form of rows or columns is not appropriate. The best suitable one is the one with the square Pauli matrices with the identity, which let us calculate the exponentials in a simple way.

After defining the polar coordinates (formulas in (4.3)) we get the diagonalization of any unit vector in $\mathfrak{R}^3$. The diagonal form is the matrix σ^z (formula in (4.4)). With the known result for rotations, formula (2.8), we get the writing in exponential and in polar exponential form of the vectors in $\mathfrak{R}^3$: formulas in (3.1) - (3.3) and (4.6) - (4.7). Curiously, formula (2.8) is related to the quaternion formulation and in a certain way to four dimensions.

The extension to $\mathfrak{R}^4$ is less straightforward. We obtain the announced exponential forms in equations (5.2) - (5.5). In the procedure we get unexpected results, among others, the Minkowski metric. Questions about the dimension of the space and time arise. Some results are already known, some others are new. The methods are essentially new.

Let us have a look to the processes involved. **First**, a historical perspective:

Hamilton introduced the "quaternions" (October 16, 1843) and with them, the (imaginary)-"$\mathfrak{v}$*ector*", the "scalar product" and the "$\mathfrak{v}$*ector* product". Their suitable space is $\mathfrak{R}\text{x}(\mathbf{i}\mathfrak{R})^3$. The product of two quaternions also belongs to this space and contains both, the scalar and the vector product. The metric behind, the Euclidean.

Cayley and the "matrices". A relation of quaternions with "skew convertible matrices of the order 2" (1857).

Gibbs (1881) and Heaviside (1893) with the "vector calculus" (algebra). They borrowed the "$\mathfrak{v}$*ector*" from Hamilton, but now as triples of real numbers, ('real ' , "polar") "vectors". They also borrowed the "scalar product" and the "vector product", which now are separated. A new concept appeared the "axial vectors". The vectors associated to the rotations are axial vectors which in the Hamilton quaternions were the "$\mathfrak{v}$*ector*" (the imaginary vectors). The vector product of two ("polar") vectors is an axial vector. The metric behind, the Euclidean. The appropriate space is: $(\mathfrak{R}_{\text{time}}) \cup \left\{ \mathfrak{R}^3_{\text{polar}} \cup \mathfrak{R}^3_{\text{axial}} \right\}$. See S2 in Appendix B.

Einstein (1905) - Minkowski (1908). Special relativity, vectors in $\mathfrak{R}^4 = (\mathfrak{R}^{1+3}) \equiv \mathfrak{M}$. Behind, the Minkowskian pseudo-metric and the Minkowski space.

Pauli (1927) rediscovered the matrices announced by Cayley, with an explicit formulation, the "Pauli matrices" [5],

with, **second**, some implications.

a) In dimension two over the reals.

a1) $\mathfrak{R}\text{x}\mathfrak{R} \equiv \mathfrak{R}^2$ with vector calculus. $\nexists\, \mathbf{v}, \mathbf{v}' \mid \mathbf{v}' = \mathrm{e}^{\mathbf{v}}$; what is $\mathrm{e}^{\mathbf{v}}$?. Euclidean norm: $\|\mathbf{v}\| = +\sqrt{x^2+y^2} \geq 0$, with $|\mathbf{v_1}.\mathbf{v_2}| = \|\mathbf{v_1}\|\,\|\mathbf{v_2}\|\ |\cos(\widehat{\mathbf{v_1},\mathbf{v_2}})| \neq \|\mathbf{v_1}\|\,\|\mathbf{v_2}\|$. We only define the scalar product, a real number.

a2) $\mathfrak{R}\text{x}(\mathbf{i}\mathfrak{R}) \equiv \mathfrak{C}$, with added polar coordinates: $\forall z' \neq 0$ there is $z' = \mathrm{e}^z \in \mathfrak{C}$. The modulus: $|z| = +\sqrt{x^2+y^2} \geq 0$, with $|z_1\, z_2| = |z_1|\,|z_2|$. Two space (type) dimensions, one real and one imaginary. The product, is the product of complex numbers.

a3) $\mathfrak{C} \equiv \mathfrak{R}\text{x}(\mathbf{i}\mathfrak{R})^3\big|_{1+1}$. A complex number as a restricted quaternion, which means a scalar (not a space dimension, later on a time dimension), plus an imaginary multiplying a spatial dimension. The result of the product in the same space. The module - norm is Euclidean.

a4) $\mathfrak{R}^2 \equiv \mathfrak{M}\big|_{1+1} \subset \mathfrak{M}$. A vector in the Pauli matrix form. One dimension is time type, the other is space type. And, $\forall \mathbb{w}_1 \in \mathfrak{M}\big|_{1+1} \mid -w_{1t}^2 + w_{1l}^2 \neq 0$ there is $\mathbb{w}_2 = \mathrm{e}^{\,w_1} \in \mathfrak{M}\big|_{1+1}$. There is a pseudo-metric (Minkowski). The result of the product is in the same space, satisfying: $\|\,\mathbb{w}_1\,\|^2_{\sim}\,\|\,\mathbb{w}_2\,\|^2_{\sim} = -\|\,\mathbb{w}_1\mathbb{w}_2\,\|^2_{\sim}$.

b) In dimension three over the reals.

$\mathfrak{R}^3$. This is the space framework for the vector calculus (algebra). At first these are polar vectors. Properties like in a1), with an added vector product and a resulting vector which is not polar, it is an axial vector. In the context of vector calculus, the complex numbers are a useful mathematical tool. In general, for most of the classical physics, the time is added as a parameter, without geometrical considerations. This is not the appropriate place for quaternions.

c) In dimension four over the reals.

c1) $\mathfrak{R} \cup \mathfrak{R}^3$ with vector calculus. This corresponds to the statements in b), with the time one more dimension.

c2) $\mathfrak{R}^{1+3}$, the space of special relativity. The metric is Minkowskian: $\mathfrak{M}$.

c3) $\mathfrak{R}$x$(\mathbf{i}\mathfrak{R})^3$, the space of the quaternions. We identify the $\{x, y, z\}$ axes with the axes of the $\{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$. The product operation is closed; the vector part is imaginary (axial). Still the metric is Euclidean. The norm of a product is the product of the norms. The $'$fourth dimension $'$ is a scalar; in the product it appears as the scalar product of two imaginary vectors. Hamilton developed formulas for the exponential, logarithm, and the trigonometric functions.

c4) $\mathfrak{M}$ with the Pauli basis, a square matrix formalism. We get two objectives: i) the exponential of a vector, and ii) the product of the pseudo-norms in terms of the pseudo-norm of the product. But we pay a price: the product of two elements of $\mathfrak{M}$ does not belong to $\mathfrak{M}$, it belongs to $\mathfrak{R}$x$\mathfrak{R}^3$x$(\mathbf{i}\mathfrak{R})^3 \equiv \mathfrak{M}x(\mathbf{i}\mathfrak{R})^3$, still with a pseudonorm.

In a certain sense we are writing about: quaternions, a real scalar (later a real time) with imaginary space vectors and an euclidean metric, versus the real four vectors, with the inclusion of a real time type vector with (real) space vectors and a Minkowskian metric in a square matrix formalism. In this framework we prefer to denote imaginary vector instead of axial vector for the vectors defining the rotations and also obtained in the vector product.

It is the correspondence mentioned by Cayley and by Pauli (see C and P in Appendix B) that motivates to write the Pauli basis with its hermitian matrices as the natural basis for the vectors of our space when written in matrix form. This has striking consequences:

1) the necessity of the condition (still not a pseudo-norm,) $-w_t^2 + |w_n|^2 \neq 0$ as a direct result of imposing the Pauli basis in the obtainment of the exponential forms of the 4-vectors ((5.2)-(5.3)). Afterwards this condition motivates the definitions of $\mathbb{w}_\mu$, $\mathbb{w}^\mu$ and of the pseudo-norm of a 4-vector, which we use for the writing of the Eulerian polar form (a pseudo-norm, two angles and ρ). In brief, the form of the Pauli basis is a source for the Minkowski space.

2) The $\mathbb{1}$ has a two fold aspect: a) a time type vector, and b) the square of a space three dimensional vector coincide with a pure time vector ($\mathbb{n}^2 = \mathbb{1}$), which is a more general result than $\| \mathbf{n} \| = \| \mathbb{n} \| = 1$, also related to the scalar part with quaternions. ($c^2t^2 = x^2 + y^2 + z^2$).

3) With vector calculus and quaternions there is no distinction among the three dimensions of space. With the algebraic aspect of the Pauli basis we can distinguish a $'$z - direction$'$ from the other two space dimensions, as under the matrix product with itself it maintains in the diagonal, with the time dimension, meanwhile the other two do not: they are not in the diagonal but their products are. In brief, algebraically the $'$z - direction$'$ is with the $'$t - direction$'$, metrically is with the other two space directions and in one way or another the imaginary unit in between. Does this have physical consequences?.

The geometric - algebraic formulation of D. Hestenes, with his emphasis in the Clifford Algebra, [9] [7] deserves special attention. The motivation and methods for these *studies* are completely different: we pursue an algebraic-geometrical construction of the creation and annihilation operators for the elementary fermions (leptons and quarks) with a different type of $'$timespace $'$ (eight dimensional) which is not a “coordinates-free formulation” [10].

Final comments. Equations (5.2)-(5.3) are suggesting, they belong to equations of the type: $\mathbb{z}' = e^{\mathbb{z}}$, with $\{\mathbb{z}, \mathbb{z}'\} \in \mathfrak{R}^8(\mathfrak{C}^4)$. The combination of rotations and boosts acting over real plus imaginary vectors has this form, up to real and imaginary times in the exponent. In relation to the content of Hestenes's in H (Appendix B) we distinguish the positional and the operational aspects of the vectors, and we write: $\{\mathbb{z}' \leftrightarrow \textit{ground}\} = e^{\{\mathbb{z} \leftrightarrow \textit{exponent}\}}$. Let us clarify their meanings,

$\mathbb{z}'$ at *ground*, just positional:

- $\mathbb{w} \in \mathfrak{R}^4$. The usual vectors defined in (2.5), (2.10), light rays included. More specifically in (5.2), (5.3a).
- $\mathbf{i}\mathbb{v} \in (\mathbf{i}\mathfrak{R})^3$. They are in (3.2) and in (5.3) ($w_t = 0\,!$). These are axial vectors. Periodicity do not rule them.
- $\mathbf{i}w_t\mathbb{1} \in (\mathbf{i}\mathfrak{R})$. What is this?. It deserves the reading of Minkowski and of Misner, M and G in Appendix B.

$\mathbb{z}$ in the *exponent*, with an operational aspect:

- $\mathbb{v} \in \mathfrak{R}^3$. These vectors generate the boosts transformations. They participate in the definitions (5.2), (5.3).
- $\mathbf{i}\mathbb{v} \in (\mathbf{i}\mathfrak{I}_k)^3$. These vectors generate the three dimensional rotations. They are defined in finite intervals $\mathfrak{I}_k$. Could this be a part of the denoted $'$warped$'$ dimensions in some nowadays theories in physics?.
- $w_t\mathbb{1} \in (\mathfrak{R})$. They participate generating the norm of the vectors (at *ground*).
- $\mathbf{i}w_t\mathbb{1} \in (\mathbf{i}\mathfrak{I}_t)$. With $\frac{w_t}{2} = \pm\frac{\pi}{4}$. See the comments after (4.2). They will define a generalization of the rotations. They have a crucial role in our definitions of creation and annihilation operators of fermions.

Do the real $\mathbb{v}$ or the imaginary $\mathbf{i}\mathbb{v}$ vectors (also as axial vectors in *Study I,2*) force a $w_{t_1} \neq 0$ in (5,2) or an $\mathbf{i}w_{t_2} \neq 0$ in (5.3)-b), or in other case to be exclusively of the form (5.3)-a)?. Does the Minkowski space with the previous consideration drive to a geometrical 8 dimensional space? (see *Study I,2*), or a physical space? (see *Study III*).

APPENDIX A: DEMONSTRATIONS OF THE FORMULAS IN SECTION V.

$$\text{argtanh}\, z = \frac{1}{2}\ln\frac{1+z}{1-z}, \qquad \text{argtanh}\,\frac{a}{b} = \text{argtanh}\,\frac{b}{a} \pm \mathbf{i}\,\frac{\pi}{2},$$

$$\mathrm{e}^{A\,\mathbb{n}} = \sum_{k=0}^{\infty}\left[\frac{1}{(2k)!}\,(A\,\mathbb{n})^{2k} + \frac{1}{(2k+1)!}\,(A\,\mathbb{n})^{2k+1}\right] = \cosh A\,\mathbb{1} + \sinh A\,\mathbb{n}, \qquad (\mathbb{n}^2 = \mathbb{1}).$$

0) Is there a vector that satisfies $\mathbb{w} \equiv w_t\,\mathbb{1} + |w_n|\,\mathbb{n} = \mathrm{e}^{\mathbb{w}'} = \mathrm{e}^{u\,\mathbb{1} + v\,\mathbb{n}'}$, with $u \in \mathfrak{C}$, $v \in \mathfrak{C}$ and $\mathbb{n}'^2 = \mathbb{1}$? :

$$\mathrm{e}^{u\,\mathbb{1} + v\,\mathbb{n}'} = \mathrm{e}^{u\,\mathbb{1}}\sum_{k=0}^{\infty}\left[\frac{1}{(2k)!}\,v^{2k}\,\mathbb{1} + \frac{1}{(2k+1)!}\,v^{2k+1}\,\mathbb{n}'\right] = w_t\,\mathbb{1} + |w_n|\,\mathbb{n},$$

$$\overset{\mathbb{n}'=\mathbb{n}}{\Longrightarrow} \begin{cases} w_t = \mathrm{e}^{u}\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!}\,v^{2k}\right) = \mathrm{e}^{u}\cosh v \\ |w_n| = \mathrm{e}^{u}\sum_{k=0}^{\infty}\left(\frac{1}{(2k+1)!}v^{2k+1}\right) = \mathrm{e}^{u}\sinh v \end{cases}, \quad \Longrightarrow \begin{cases} \mathrm{e}^{2u} = w_t^2 - |w_n^2| \in \mathfrak{R} \\ \tanh v = \dfrac{\sinh v}{\cos v} = \dfrac{|w_n|}{w_t} \in \mathfrak{R} \end{cases},$$

$$\Longrightarrow \begin{cases} u \in \{\ \ln\sqrt{w_t^2 - |w_n^2|}\ ,\ \ln\sqrt{-(w_t^2 - |w_n^2|)} \pm \mathbf{i}\frac{\pi}{2}\ \} \\ v = \text{argtanh}\,\dfrac{|w_n|}{w_t} = \text{argtanh}\,\dfrac{w_t}{|w_n|} \pm \mathbf{i}\frac{\pi}{2} \end{cases}, \qquad (\mathbb{n}^2 = \mathbb{1})\ .$$

Simple case. Consider vectors $\mathbb{w}$ with $w_t > |w_n|$:

$$\mathbb{w} = w_t\,\mathbb{1} + |w_n|\,\mathbb{n} = \sqrt{w_t^2 - |w_n|^2}\ \left(\frac{w_t}{\sqrt{w_t^2 - |w_n|^2|}}\,\mathbb{1} + \frac{|w_n|}{\sqrt{w_t^2 - |w_n|^2|}}\,\mathbb{n}\right) =$$

using:

$$\begin{cases} \text{argtanh}\,\dfrac{|w_n|}{w_t} = \dfrac{1}{2}\ln\dfrac{1 + \frac{|w_n|}{w_t}}{1 - \frac{|w_n|}{w_t}} = \dfrac{1}{2}\ln\dfrac{w_t + |w_n|}{w_t - |w_n|} = \ln\dfrac{w_t + |w_n|}{\sqrt{w_t^2 - |w_n|^2}} \\ -\,\text{argtanh}\,\dfrac{|w_n|}{w_t} = \ln\dfrac{\sqrt{w_t^2 - |w_n|^2}}{w_t + |w_n|} = \ln\dfrac{w_t - |w_n|}{\sqrt{w_t^2 - |w_n|^2}} \end{cases}$$

$$\begin{cases} \cosh(\text{argtanh}\,\frac{|w_n|}{w_t}) = \frac{1}{2}\left(\mathrm{e}^{\text{argtanh}\,\frac{|w_n|}{w_t}} + \mathrm{e}^{-\,\text{argtanh}\,\frac{|w_n|}{w_t}}\right) = \dfrac{w_t}{\sqrt{w_t^2 - |w_n|^2|}} \\ \sinh(\text{argtanh}\,\frac{|w_n|}{w_t}) = \frac{1}{2}\left(\mathrm{e}^{\text{argtanh}\,\frac{|w_n|}{w_t}} - \mathrm{e}^{-\,\text{argtanh}\,\frac{|w_n|}{w_t}}\right) = \dfrac{|w_n|}{\sqrt{w_t^2 - |w_n|^2|}} \end{cases}$$

$$\mathrm{e}^{\text{argtanh}\,\frac{|w_n|}{w_t}\,\mathbb{n}} = \cosh(\text{argtanh}\,\frac{|w_n|}{w_t})\,\mathbb{1} + \sinh(\text{argtanh}\,\frac{|w_n|}{w_t})\,\mathbb{n} = \frac{1}{\sqrt{w_t^2 - |w_n|^2}}\,(\,w_t\,\mathbb{1} + |w_n|\,\mathbb{n}\,)\ .$$

$$\mathbb{w} = \mathrm{e}^{\ln\sqrt{w_t^2 - |w_n|^2}\,\mathbb{1} + \text{argtanh}\,\frac{|w_n|}{w_t}\,\mathbb{n}}.$$

We also have:

$$\mathbb{w}^2 = \begin{pmatrix} w_t^2 + w_x^2 + w_y^2 + w_z^2 + 2w_tw_z & 2w_t(w_x - \mathbf{i}\,w_y) \\ 2w_t(w_x + \mathbf{i}\,w_y) & w_t^2 + w_x^2 + w_y^2 + w_z^2 - 2w_tw_z \end{pmatrix} \neq k\,\mathbb{1}\ ,$$

with:

$$\tfrac{1}{2}\,\mathrm{Tr}\,(\mathbb{w}^2) = w_t^2 + w_x^2 + w_y^2 + w_z^2 \ \neq\ -w_t^2 + w_x^2 + w_y^2 + w_z^2 = -\det\mathbb{w}\ .$$

In particular:

$$\mathbb{w}_d = w_t\,\mathbb{1} + w_z\,\sigma^z = (w_t + w_z)\,\overset{\wedge}{\sigma} + (w_t - w_z)\,\overset{\vee}{\sigma}$$

$$\mathbb{w}_d^2 = (w_t^2 + w_z^2)\,\mathbb{1} + 2w_tw_z\,\sigma^z\ , \qquad \mathbb{w}_{d,\mu}\mathbb{w}_d^{\mu} = (-w_t^2 + w_z^2)\,\mathbb{1}$$

$$\mathbb{w}_d^k = (w_t + w_z)^k\,\overset{\wedge}{\sigma} + (w_t - w_z)^k\,\overset{\vee}{\sigma} = \tfrac{1}{2}\,[(w_t + w_z)^k + (w_t - w_z)^k]\,\mathbb{1} + \tfrac{1}{2}\,[(w_t + w_z)^k - (w_t - w_z)^k]\,\sigma^z\ .$$

α) Time type vectors: $\mathbb{w}_{tt} = w_{t_1}\,\mathbb{1} + |\,w_{n_1}|\,\mathbb{n}$, $\qquad \|\,\mathbb{w}_{tt}\,\|_{\sim}^{2} \equiv -w_{t_1}^2 + w_{n_1}^2 < 0$,

$$\begin{cases} s_{tt} = +\sqrt{-\,\|\,\mathbb{w}_{tt}\,\|_{\sim}^{2}} \\ \rho_{tt} = \text{argtanh}\,\dfrac{|\,w_{n_1}|}{w_{t_1}} \end{cases}, \quad \begin{cases} s_{tt} > 0 \\ |\dfrac{|\,w_{n_1}|}{w_{t_1}}| < 1 \end{cases}, \quad \begin{cases} w_{t_1} \neq 0 \\ -|w_{t_1}| < |\,w_{n_1}| < |w_{t_1}| \end{cases}, \quad \sqrt{\frac{w_{t_1} + |\,w_{n_1}|}{w_{t_1} - |\,w_{n_1}|}} = \frac{|\,w_{t_1} + |\,w_{n_1}||}{\sqrt{w_{t_1}^2 - w_{n_1}^2}} = a_1 > 0\ .$$

$$\rho_{tt} = \ \operatorname{argtanh}\frac{|\,w_{n_1}|}{w_{t_1}} \ = \ \frac{1}{2}\ln\frac{1+\dfrac{|\,w_{n_1}|}{w_{t_1}}}{1-\dfrac{|\,w_{n_1}|}{w_{t_1}}} \ = \ \ln\sqrt{\frac{w_{t_1}+|\,w_{n_1}|}{w_{t_1}-|\,w_{n_1}|}} \ = \ \ln\frac{|\,w_{t_1}+|\,w_{n_1}|\,|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} = \ln a_1 \ ,$$

$$-\rho_{tt} = -\operatorname{argtanh}\frac{|\,w_{n_1}|}{w_{t_1}} = \operatorname{argtanh}\frac{-|\,w_{n_1}|}{w_{t_1}} = \frac{1}{2}\ln\frac{1-\dfrac{|\,w_{n_1}|}{w_{t_1}}}{1+\dfrac{|\,w_{n_1}|}{w_{t_1}}} = \ln\sqrt{\frac{w_{t_1}-|\,w_{n_1}|}{w_{t_1}+|\,w_{n_1}|}} = \ln\frac{|\,w_{t_1}-|\,w_{n_1}|\,|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} = -\ln a_1 \ ,$$

$$\begin{cases} \cosh\rho_{tt} = \frac{1}{2}\,(\,\mathrm{e}^{\rho_{tt}} + \mathrm{e}^{-\rho_{tt}}\,) \ = \ \frac{1}{2}\,\dfrac{|\,w_{t_1}+|\,w_{n_1}|\,| + |\,w_{t_1}-|\,w_{n_1}|\,|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} \ = \ \dfrac{|\,w_{t_1}|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} \\ \sinh\rho_{tt} = \frac{1}{2}\,(\,\mathrm{e}^{\rho_{tt}} - \mathrm{e}^{-\rho_{tt}}\,) \ = \ \frac{1}{2}\,\dfrac{|\,w_{t_1}+|\,w_{n_1}|\,| - |\,w_{t_1}-|\,w_{n_1}|\,|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} \ = \ \operatorname{sign}(w_{t_1})\,\dfrac{|w_{n_1}|}{\sqrt{w_{t_1}^2-w_{n_1}^2}} \end{cases}$$

$$\mathrm{e}^{\rho_{tt}\,\mathbb{n}} = \cosh\rho_{tt}\,\mathbb{1} + \sinh\rho_{tt}\,\mathbb{n} = \frac{1}{\sqrt{w_{t_1}^2-w_{n_1}^2}}\,(\,|\,w_{t_1}|\mathbb{1} + \operatorname{sign}(w_{t_1})\,|\,w_{n_1}|\,\mathbb{n}) \ ,$$

$$\operatorname{sign}(w_{t_1})\ \mathrm{e}^{\ln s_{tt}\,\mathbb{1} \ + \ \rho_{tt}\,\mathbb{n}} = \operatorname{sign}(w_{t_1})\ \mathrm{e}^{\ln\sqrt{w_{t_1}^2-w_{n_1}^2}\ \mathbb{1}}\ \mathrm{e}^{\rho_{tt}\,\mathbb{n}} =$$

$$= \operatorname{sign}(w_{t_1})\,\sqrt{w_{t_1}^2-w_{n_1}^2}\ \frac{1}{\sqrt{w_{t_1}^2-w_{n_1}^2}}\,(\,|\,w_{t_1}|\mathbb{1} + \operatorname{sign}(w_{t_1})\,|\,w_{n_1}|\,\mathbb{n}) \ = \ w_{t_1}\,\mathbb{1} + |\,w_{n_1}|\,\mathbb{n} \ .$$

$$\mathbb{w}_{tt} \ = \ w_{t_1}\,\mathbb{1} + |\,w_{n_1}|\,\mathbb{n} \ = \ \operatorname{sign}(w_{t_1})\ \mathrm{e}^{\ln s_{tt}\,\mathbb{1} \ + \ \rho_{tt}\,\mathbb{n}} \ = \ \operatorname{sign}(w_{t_1})\,s_{tt}\ \mathrm{e}^{\rho_{tt}\,\mathbb{n}} .$$

β) Space type vectors: $\quad \mathbb{w}_{st} = w_{t_2}\mathbb{1} + |\,w_{n_2}|\,\mathbb{n}\ ,\qquad \|\,\mathbb{w}_{st}\,\|_{\sim}^2 \equiv -w_{t_2}^2 + w_{n_2}^2 > 0\ ,$

$$\begin{cases} s_{st} = +\,\|\,\mathbb{w}_{st}\,\|_{\sim} \\ \tilde{\rho}_{st} = \operatorname{argtanh}\dfrac{w_{t_2}}{|\,w_{n_2}|} \end{cases}, \quad \begin{cases} s_{st} > 0 \\ |\dfrac{w_{t_2}}{|\,w_{n_2}|}| < 1 \end{cases}, \quad \begin{cases} -|w_{n_2}| < w_{t_2} < |w_{n_2}| \\ |\,w_{n_2}| \ > \ 0 \end{cases}, \quad \sqrt{\frac{w_{t_2}+|\,w_{n_2}|}{-w_{t_2}+|\,w_{n_2}|}} = \frac{w_{t_2}+|w_{n_2}|}{\sqrt{-w_{t_2}^2+w_{n_2}^2}} = a_2 > 0 \ .$$

$$\tilde{\rho}_{st} = \operatorname{argtanh}\frac{w_{t_2}}{|\,w_{n_2}|} = \frac{1}{2}\ln\frac{1+\dfrac{w_{t_2}}{|w_{n_2}|}}{1-\dfrac{w_{t_2}}{|w_{n_2}|}} = \frac{1}{2}\ln\frac{w_{t_2}+|\,w_{n_2}|}{-w_{t_2}+|w_{n_2}|} = \frac{1}{2}\ln\left\{\frac{(w_{t_2}+|w_{n_2}|)^2}{-w_{t_2}^2+w_{n_2}^2}\right\} = \ln a_2 \ ,$$

$$\begin{cases} \cosh(\ln a_2) = \frac{1}{2}\,(\,\mathrm{e}^{\ln a_2} + \mathrm{e}^{\ln a_2^{-1}}\,) = \ \frac{1}{2}\ \dfrac{w_{t_2}+|\,w_{n_2}| \ - \ w_{t_2} \ + \ |\,w_{n_2}|}{\sqrt{w_{n_2}^2-w_{t_2}^2}} \ = \ \dfrac{|\,w_{n_2}\,|}{\sqrt{w_{n_2}^2-w_{t_2}^2}} \\ \sinh(\ln a_2) \ = \ \frac{1}{2}\,(\,\mathrm{e}^{\ln a_2} - \mathrm{e}^{\ln a_2^{-1}}\,) = \frac{1}{2}\ \dfrac{w_{t_2}+|\,w_{n_2}| - (-w_{t_2}+|\,w_{n_2}|)}{\sqrt{w_{n_2}^2-w_{t_2}^2}} \ = \ \dfrac{w_{t_2}}{\sqrt{w_{n_2}^2-w_{t_2}^2}} \end{cases}$$

$\mathrm{e}^{\tilde{\rho}_{st}\,\mathbb{n}} = \mathrm{e}^{\ln a_2\,\mathbb{n}} \ = \ \cosh(\ln a_2)\,\mathbb{1} + \sinh(\ln a_2)\,\mathbb{n}\ .$ Using (3.1):

$$\mathrm{e}^{\ln s_{st}\,\mathbb{1} + \tilde{\rho}_{st}\,\mathbb{n} \,\pm\, \mathbf{i}\frac{\pi}{2}\,[-\mathbb{1}+\mathbb{n}]} = \mathrm{e}^{\ln\sqrt{-w_{t_2}^2+w_{n_2}^2}\ \mathbb{1}}\ \mathrm{e}^{\ln a_2\,\mathbb{n}}\ \mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\,[-\mathbb{1}+\mathbb{n}]} =$$

$$= \sqrt{-w_{t_2}^2+w_{n_2}^2}\,\left\{\frac{|\,w_{n_2}\,|}{\sqrt{w_{n_2}^2-w_{t_2}^2}}\,\mathbb{1} + \frac{w_{t_2}}{\sqrt{w_{n_2}^2-w_{t_2}^2}}\,\mathbb{n}\right\}\,\mathbb{n} \ = \ \mathbb{w}_{st}\ ,$$

and

$$\mathrm{e}^{\ln s_{st}\,\mathbb{1} + \rho_{st}\,\mathbb{n}} \ = \ \mathrm{e}^{\ln\sqrt{-w_{t_2}^2+w_{n_2}^2}\ \mathbb{1}}\ \mathrm{e}^{(\,\ln a_2 \pm\, \mathbf{i}\frac{\pi}{2}\,)\,\mathbb{n}} = \mathrm{e}^{\ln\sqrt{-w_{t_2}^2+w_{n_2}^2}\ \mathbb{1}}\ \mathrm{e}^{\ln a_2\,\mathbb{n}}\ \mathrm{e}^{\pm\mathbf{i}\frac{\pi}{2}\,\mathbb{n}} =$$

$$= \pm\,\mathbf{i}\,(\,w_{t_2}\mathbb{1} + |\,w_{n_2}\,|\,\mathbb{n}\,) \ = \ \pm\,\mathbf{i}\,\mathbb{w}_{st}\ ,$$

with $\qquad \rho_{st} = \operatorname{argtanh}\dfrac{|\,w_{n_2}|}{w_{t_2}} \ = \operatorname{argtanh}\dfrac{w_{t_2}}{|\,w_{n_2}|} \pm \mathbf{i}\frac{\pi}{2} \equiv \tilde{\rho}_{st} \pm \mathbf{i}\frac{\pi}{2} = \ln a_2 \pm \mathbf{i}\frac{\pi}{2},$

and as $\qquad |\dfrac{w_{t_2}}{|\,w_{n_2}|}| < 1 \qquad$ it is $\qquad |\dfrac{|\,w_{n_2}|}{w_{t_2}}| > 1\ .$

APPENDIX B: THIS AUTHOR RECOMMENDS THE READING OF THE FOLLOWING PARAGRAPHS:

C) A. Cayley. [11] (45. in page 491). Anticipation of the Pauli matrices.

P) W. Pauli. [5] (The footnote in page 6). Quaternions with his matrices.

S1) J. Stillwell. [4] (From the line 9 to the line 22 in page 425). Invention of the imaginaries in geometry.

S2) J. Stillwell. [4] (From the line 9 to the line 17 and the lines 23 and 24 in page 437).
Role of the quaternions in mathematics.

S3) J. Stillwell. [4] (From the line 24 to the line 28 in page 432). Geometry into algebra (Hilbert's view).

S4) J. Stillwell. [4] (From the line 21 to the line 24 in page 420). A motivation for Hamilton's quaternions.

H) D. Hestenes. [7] (From the line 10 to the line 26 in page 60).
Numbers in a: quantitative way (Grassmann, Clifford) in contrast to an operational way (Hamilton).

M) H. Minkowski. [12] (From the line 3 to the line 14 in page 88). The meaning of an imaginary time.

G) C. W. Misner, K. S. Thorne, J. A. Wheeler. [13] (Box 2.1, first column from the line 1 to the line 7 and in the second column from the line 19 to the line 22 in page 51). The meaning of an imaginary time.

APPENDIX C: PROGRAM OF THE *STUDIES* CONTAINING THIS RESEARCH:

Study 0) Creation and annihilation operators for neutrinos (the skeleton of a method).
I,1) Geometry: Euler. 3- and 4- dimensional vectors in polar exponential form. *(This study).*
Study I,2) Geometry: Vectors.
Study I) Geometry of the time and the space.
Study II) Spin matrices and symmetries in the time and space.
Study III) Physics: creation - annihilation operators for elementary fermions.
Study IV) Magnetic charge.
Study V) Addenda.

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[1] W. Rudin. *Real and Complex Analysis.* Third edition. McGraw-Hill (1987).
[2] S. L. Altmann. *Rotations, Quaternions, and Double Groups.* Oxford, Clarendon Press (1986) and Dover Publications, Inc. Mineola, New York. (2005).
[3] W. R. Hamilton. *The Mathematical Papers of Sir William Rowan Hamilton Vol. Ill Algebra.* Edited for the Royal Irish Academy. Cambridge at the University Press (1967).
[4] J. Stillwell. *Mathematics and Its History.* Third edition. Springer (2010). (Chapter 20).
[5] W. Pauli. *Zur Quantenmechanik des magnetischen Elektrons. - On the quantum mechanics of magnetic electrons - .* Z Phys. 43, 601-623 (1927).
[6] F. Klein and A. Sommerfeld, *The Theory of the Top. Vol I. Introduction to the Kinematics and Kinetics of the Top* (R.J. Nagem, G. Sandri, translators), Birkhäuser, Boston (2008). (Pages 11-12, 36-37). Original: Theorie des Kreisels. (From 1897 to 1910).
[7] D. Hestenes. *New Foundations for Classical Mechanics.* Second edition. Kluwer Academic Publishers (2002).
[8] R. Penrose. *On the Origins of Twistor Theory.* Gravitation and Geometry, a volume in honour of I. Robinson, Biblipolis, Naples 1987. (Section 7).
[9] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus.* D. Reidel Publishing Company (1984).
[10] D. Hestenes. *Space-Time Algebra.* Springer Int. Pub. (Birkhäuser) (1966, 2015). (Page ix).
[11] A. Cayley. *A memoir on the theory of matrices.* The Collected Mathematical Papers. Vol. II. 152. 475–496. From the Philosophical Transactions of the Royal Society of London, vol. cxlviii. for the year, 1858, pp. 17-37. Received December 10, 1857, - Read January 14, 1858.
[12] H. Minkowski. *-Space and Time-. (1907).* (Page 88). In: Einstein and others, *The Principle of Relativity.* Dover Publications, Inc (1952).
[13] C. W. Misner, K. S. Thorne, J. A. Wheeler. *Gravitation.* Freeman and Company (1973).