On the Evaluation of the Asymptotic Fairness of Bonus-Malus Systems.

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Abstract

In this paper we try to evaluate the asymptotic fairness of bonus-malus systems, assuming the simplest case when there is no hunger for bonus. The asymptotic fairness has to be understood as the bonus-malus system ability in assessing the individual risks in the long run (see Lemaire [1995] p.xvi). Firstly we define the asymptotic fairness of a bonus-malus system following an expression that can be found in Lemaire [1985] p. 168. Secondly, we define a measure of the global asymptotic fairness considering the structure function of the risk group. Finally we try to calculate, for each set of transition rules and a given structure function, the scale of premiums that brings the global asymptotic fairness closest to the ideal situation where each insured pays in the long run a premium corresponding to its own claim frequency. This is possible thanks to the application of a multiobjective optimization technique named Goal Programing. We give an example illustrating the fact that the ideal case could be fairly well approached.

Keywords: bonus-malus system, asymptotic fairness, Goal Programming, simplex method.

Contents

1 Introduction

Many criteria have been defined to evaluate bonus-malus systems (BMS) in an attempt to facilitate their design. These actuarial tools have been collected in Lemaire [1985, 1995, 1998]. The birth process of these criteria seems to be almost the same, consisting in the identification of some property that a BMS should verify, followed by the definition of some measure that rates the degree of fulfillment for a given BMS. For instance, it is known that the relative stationary average level (RSAL) measures the degree of clustering of policies in the lowest classes (high discount classes) of the BMS, an important phenomenon which can result in a harmful decrease of the average premium level. The RSAL is defined in such a way that, as written in Lemaire [1995] p.64, "a low value of RSAL indicates a high clustering of policies in the high-discount BMS classes. A high RSAL suggests a better spread of policies among classes." Others examples of tools for BMS evaluation are the coefficient of variation of the insured's premiums, and the elasticity or efficiency of a BMS (Loimaranta [1972], De Pril [1978], Lemaire [1995, the BMS has not yet reached the steady-state (Lemaire [1995] p.86). Both the RSAL and the elasticity are dependent on the policy risk level through the claim frequency λ; though the last one could be generalized to a measure of global elasticity if the structure function of a given risk group was introduced in the discussion (Lemaire [1995] p.89), (De Pril [1978] p.62).

Following these guidelines, the aim of this paper is to focus on another tool for BMS evaluation consisting in a measure of the BMS fairness. As stated in Lemaire [1995] p.xvi, the most important target of a BMS is "...to better asses individual risks, so that everyone will pay, in the long run, a premium corresponding to its own claim frequency". Taking this idea in mind, we could think about two extreme situations. Firstly when there is no BMS and every policy pays the same pure premium P: we consider this case as being unfair because good risk (with low risk parameter λ) will pay the same as the bad ones (those with higher λ). Secondly, the case where fairness is asymptotically attained is when every insured will pay in the long run a premium corresponding to its own risk, the discrimination between insureds being then asymptotically perfect. For instance this would be the case if Exact Credibility (also named Bayesian bonus-malus) was applied through an infinite number of periods (Lemaire [1995] p.163). We consider this second case as being representative of perfect fairness. We will try to rate all the intermediate cases where a BMS has been designed to improve the first situation (absence of BMS, unfairness) while never reaching in practice the second one (Bayesian case, perfect fairness).

As the measure of fairness will also use the mean asymptotic premium, it will only work for a BMS in the steady-state, i.e. it will evaluate the asymptotic fairness of a BMS. At a first stage we will define it depending on the policy risk level. Afterwards, considering the structure function of some risk group will allow us to define the global asymptotic fairness, in a similar way as in the referred case of the elasticity.

optimal scale of premiums. By means of the expression commercial requirements we try to refer to some conditions that the designer should like the BMS to verify in order to get it market competitive and attractive to the insureds. For instance constraints referred to the maximum and minimum premiums, and to the size of the differences between the premiums of consecutive bonus-malus classes, could be translated into linear restrictions to be included in the definition of the feasible set.

Finally it is very important to point out that we will not consider hunger for bonus or any other phenomenon that could influence the distributions of the number of claims to be explained in the next section. Thus we will assume that these distributions will remain the same along the infinity of time periods.

2 Basic assumptions. Bonus-malus systems as Markovian chains

Given a risk group, we assume that the level of risk of each policy is represented by a risk parameter $\lambda > 0$; the expected number of claims per period. We suppose that it is not possible to determine the true value of this parameter for each policy, and we suppose that across the group there exists a random variable Λ (the structure variable) whose realizations are the values of the risk parameter for policies belonging to that group. The distribution function associated to the structure variable will be noted $U(\lambda)$ and named structure function. We suppose that Λ is independent of time. The random variables $N_t \mid \Lambda = \lambda$; number of claims of a policy in successive periods $1, \ldots, t$, conditioned to some value λ of the risk parameter, are supposed to be mutually independent and identically distributed with common distribution function $p_k(\lambda)$: This distribution function is assumed to be Poisson with parameter λ: Therefore the unconditioned random variable N_t will be Poisson mixed with the structure function. We also assume that the individcompany uses a BMS when the following conditions hold:

- There exists a finite number of classes $(C_1, \ldots, C_n)$ such that each policy stays in one class through each period (usually a year).
- The premium for each policy depends only on the class where it stays.
- The class for a given period is determined by the class in the preceding period and the number of claims reported in that period (Markovian Condition).

Every bonus-malus system is determined by three elements:

- The initial class, where the new policies are assigned.
- The premium scale $\overline{b} = (b_1, \ldots, b_n)$, where b_i is the premium for policies in the class C_i. The highest discount class will be C_1.
- The transition rules, that is, the rules that define the conditions for a policy in class C_i to be transferred to class C_j in the next period.

It is useful to express the transition rules by means of transformations T_k such that $T_k(i) = j$ when insureds in class C_i reporting k claims are transferred to class C_j in the next period. Transformations T_k are usually described by means of matrices,

$$T_k = \left(t_{ij}^k \right)$$

where

$$\begin{aligned} t_{ij}^k &= 1 \text{ if } T_k(i) = j \\ t_{ij}^k &= 0 \text{ if } T_k(i) \neq j \end{aligned}$$

The conditional transition matrix given that $\Lambda = \lambda$, is defined as

$$M(\lambda) = (p_{ij}(\lambda))$$

Therefore, it is possible to interpret a BMS as a Markov chain. This chain is homogeneous, since we have assumed that each claim frequency λ is stationary in time. If we also assume that the chain is ergodic and not cyclic then it is a regular chain (see for instance Kemeny and Snell [1992] p.37), and there exists a stationary conditional probability distribution $\overline{\pi}(\lambda) = (\pi_1(\lambda); \ldots \pi_n(\lambda))$, where $\pi_i(\lambda)$ is defined as the limit value, when the number of periods tends to infinity, of the conditional probability for a policy to belong to class C_i, given that $\Lambda = \lambda$. It is known that the stationary distribution coincides with the L^1-normalized left eigenvector associated with the unit eigenvalue of $M(\lambda)$; that is: $\overline{\pi}(\lambda) = \overline{\pi}(\lambda) M(\lambda)$ and $\sum_{i=1}^{n} \pi_i(\lambda) = 1$. It is also possible to define the stationary unconditional probability distribution $\overline{\pi} = (\pi_1; \ldots; \pi_n)$ for an arbitrary policy belonging to the risk group, as the mean values of the stationary conditional probability distributions $(\pi_1(\lambda); \ldots \pi_n(\lambda))$:

$$\pi_i = \int_0^{+\infty} \pi_i(\lambda)\, dU(\lambda) \quad i = 1; \ldots; n: \tag{1}$$

The distributions π_i and $\pi_i(\lambda)$ can be interpreted as the probability for a policy (independently of the value of the risk parameter and conditioned to $\Lambda = \lambda$ respectively) to belong to class C_i when stationarity in the strict sense is approached.

A concept of the utmost importance is the mean asymptotic premium paid by a policy with risk parameter λ: Following Lemaire (1985, 1995), it is defined as:

$$\sum_{i=1} b_i \pi_i(\lambda); \quad \lambda > 0:$$

It represents the average premium paid by a policy in the steady state so every insured pays the same pure premium $E\{\Lambda\}$. This case is unfair because the difference $E\{\Lambda\} - \lambda$ is positive for the good risks with parameter $\lambda \leq E\{\Lambda\}$ denoting that they are paying too much, while the same difference is negative for the bad risks with parameter $\lambda \geq E\{\Lambda\}$; meaning that they are not paying enough. The second case (Bayesian case) is told perfectly fair because the difference between the premium paid and the insured's risk parameter λ is asymptotically null. We can therefore represent graphically both situations in figure 1, where the straight line $y_1(\lambda) = E\{\Lambda\} - \lambda$ (taking $E\{\Lambda\} \simeq 0.5$) stands for the first one while the second is drawn by means of the constant function $y_0(\lambda) = 0$ $(\lambda > 0)$:

FIGURE 1

If we considered a BMS in the stationary state, the situation should be an intermediate one between those two extreme cases. Recalling the commentaries made in Lemaire [1985] p. 168, the asymptotic fairness could then be represented by means of the following function:

$$y(\lambda) = \sum_{i=1} b_i \pi_i(\lambda) - \lambda \qquad (2)$$

This is the difference between the mean asymptotic premium paid by a policy with risk parameter λ and the insured's risk parameter.

Let us suppose the usual situation where the transition rules have been defined with the aim of discriminating between the good risks and the bad ones when stationarity in the strict sense is approached: the best ones, with λ neighboring zero, will nourish the high discount class with probability near to one (i.e., $\lim_{\lambda \to 0^+} \pi_1(\lambda) = 1$), while the worst ones will get into the high penalized class with probability tending to one (i.e., $\lim_{\lambda \to +\infty} \pi_n(\lambda) = 1$). It is not difficult to check that the asymptotic fairness $y(\lambda)$ will then verify the following limits:

risk parameter. We define these two mean values as

$$y^{+}(\lambda) = \max\left\{\sum_i b_i \pi_i(\lambda) - \lambda; 0\right\}; \quad (4)$$

$$y^{-}(\lambda) = -\min\left\{\sum_i b_i \pi_i(\lambda) - \lambda; 0\right\}:$$

and we will refer to them as the deviations or rating errors for each λ. These mean values could be helpful in BMS evaluation, because different BMS will furnish different $y^{\pm}(\lambda)$, opening the possibility to compare them for each λ-value. The aim of this comparison would be to clarify which system succeeds in achieving the goal of better assessing individual risks in the long run.

Deepening in the BMS evaluation from the point of view of its asymptotic fairness, the two extreme cases corresponding to $y_0(\lambda); y_1(\lambda)$ may give us some helpful ideas in addition to some hints on situations that should not be recommendable at all. For instance a BMS that produced an $y(\lambda)$ such that $y(\lambda) \geq y_1(\lambda)$ for the major part of good risks belonging to the group, would be such an inappropriate tool that would increase the unfairness of $y_1(\lambda)$ instead of decreasing it. On the other side, it seems that a good property for a BMS would be to have an $y(\lambda)$ more resembling to $y_0(\lambda)$ than is $y_1(\lambda)$. This idea of $y(\lambda)$ neighboring $y_0(\lambda)$ will be precisely defined when talking about the global asymptotic fairness.

Example 1 Let us consider a risk group characterized by means of the structure function with support $\{\lambda_j = j\,0.15 : j = 1; \ldots; 10\}$ and probability masses $\{u_j : j = 1; \ldots; 10\}$ resumed in table 1:

TABLE 1

Its mean value is $E\{\Lambda\} = .4999278192$. We consider four bonus-malus

The conditioned stationary distributions $\overline{\pi}(\lambda)$ are:

$$\pi_1(\lambda) = \frac{2(e^{\lambda} - 1 - \lambda)e^{-2\lambda}}{(-\lambda^2 + 4\lambda e^{\lambda} - 2e^{2\lambda})(-1 + e^{-\lambda} + \lambda e^{-\lambda})}$$

$$\pi_2(\lambda) = \frac{-2(e^{\lambda} - 1 - \lambda)e^{-\lambda}(e^{-\lambda} - 1)}{(-\lambda^2 + 4\lambda e^{\lambda} - 2e^{2\lambda})(-1 + e^{-\lambda} + \lambda e^{-\lambda})}$$

$$\pi_3(\lambda) = \frac{-2(e^{\lambda} - 1 - \lambda)}{(-\lambda^2 + 4\lambda e^{\lambda} - 2e^{2\lambda})}$$

$$\pi_4(\lambda) = \frac{(e^{\lambda} - 1 - \lambda)(\lambda^2 e^{-2\lambda} - 4\lambda e^{-\lambda} + 2\lambda e^{-2\lambda} - 2e^{-\lambda} + 2)e^{\lambda}}{(-\lambda^2 + 4\lambda e^{\lambda} - 2e^{2\lambda})(-1 + e^{-\lambda} + \lambda e^{-\lambda})}$$

These functions $\pi_i(\lambda)$ $(i = 1, 2, 3, 4)$ are plotted in figure 2

FIGURE 2

For example the conditioned stationary distributions for policies with risk parameter $\lambda = 0.15$ and $\lambda = 1.5$ over the bonus-malus classes are respectively

$$\overline{\pi}(0.15) = (.8500328302, .1375644193, .01168746555, .0007153086397)$$
$$\overline{\pi}(1.5) = (.02873363342, .1000415774, .2551924280, .6160323610)$$

The unconditioned stationary distribution is found to be:

$$\overline{\pi} = (.5191041945, .2993384494, .1247758108, .05678154546) \qquad (6)$$

The Bayes Scale, which is a very well known tool for calculating the associated scale of premiums for a given set of transition rules (see for example Pesonen [1963] and Norberg [1976]), has been applied giving the following scale of premiums $\overline{b}^B$:

$$b_1^B = .4426318548; \quad b_2^B = .5134106322;$$
$$b_3^B = .6037333145; \quad b_4^B = .7245472036$$

FIGURE 4

The intersection between $y_B(\lambda)$ and $y_1(\lambda)$ has abscissa $\lambda = .5192668518$, and the central value (De Prill (1978)) is $\lambda = .4962203680$. The rating errors $y_j^{\pm}$ for each λ_j $(j = 1, \ldots, 10)$ are summarized in table 2:

TABLE 2

The rating errors $y_j^{\pm}$ may be interpreted as the mean value that a policy with risk parameter λ_j will pay in excess or in default in the steady-state. For example, every insured with risk parameter $\lambda_1 = 0.15$ would pay in mean an excess of $.3044530309$ monetary units (see table 2). This means that the mean asymptotic premium for that class is equal to

$$0.15 + .3044530309 = .4544530309. \tag{7}$$

As $.4544530309/0.15 = 3.029686873$ this means that the premium paid by these insureds in the steady-state would be approximately a 303% of their true risk parameter λ_1. A general look to table 2 tells us that policies with $\lambda_j \leq 0.45$ will pay too much $(y_j^{+} > 0)$ while those with $\lambda_j \geq 0.60$ will not pay enough $(y_j^{-} > 0)$. We can conclude that:

- The discrimination between good and bad risks in the steady-state is more or less achieved by the transition rules (look to the plots in figure 2).
- The best risks will pay a lower excess or even in some cases a default (see figures 3 and 4):

$$y_B(\lambda) \leq y_1(\lambda), \; \forall \lambda \in [0, 0.5].$$

- The risks with a higher claim frequency will pay a bigger or lower

Thanks to the last example we must keep conscious that the comparison between the asymptotic fairness of two different BMS could be such a harmful task. This is because we should have to compare two plots point by point, ignoring the whole perspective of the risk group. Pointwise comparisons could give way to cumbersome and rather boring explanations that, at the end, could be useless because they would not have take into account the distribution of the structure variable Λ over the risk group.

In order to surpass this flaw we will have to formalize the already mentioned resemblance or neighboring between the asymptotic fairness of a given BMS (i.e. the function $y(\lambda)$) and the perfect asymptotic fairness represented by the constant function $y_0(\lambda)$. For that sake we will have to introduce the structure function in the analysis.

The way we have chosen to measure to what extent is $y(\lambda)$ resembling to $y_0(\lambda)$ is by means of its $L^1(U)$ norm, this is to say the integral of the absolute value of $y(\lambda)$ with respect to the structure function. The idea lying behind is quite simple. It is clear that

$$\int_0^{+\infty} |y_0(\lambda)| \, dU(\lambda) = 0,$$

thus the lesser will be

$$\int_0^{+\infty} |y(\lambda)| \, dU(\lambda), \tag{8}$$

the closer will be $y(\lambda)$ to $y_0(\lambda)$ in $L^1(U)$ norm. If ever we obtained an asymptotic fairness for which the integral (8) was null, then we could conclude that $y(\lambda) = y_0(\lambda)$ almost surely (that is, for every $\lambda \in S \subseteq (0, +\infty)$ such that $P\{\Lambda \in S\} = \int_S dU(\lambda) = 1$), and the situation of perfect fairness would have been achieved. Naturally, this will never occur for the reasons explained along the discussion about the y-shape (see the limits (3)), though it could be approached and this will be such an interesting skill for us.

We have chosen to integrate the absolute value of the asymptotic

Example 3 (Example 1 continued) If we came back to the BMS defined in example 1, we could calculate its global asymptotic fairness. For this sake we would only have to multiply each non null y_j^* by the probability of the respective λ_j then summing up all these products. This way we get the following result:

$$Y(R_0;\bar{b}^B;U) = .1206712878 \tag{10}$$

On the other hand, multiplying $(E\{\Lambda\} - \lambda_j)$ by the probability masses u_j and summing up these products will give us the global asymptotic fairness when no BMS has been defined:

$$\sum_{j=1}^{10} |E\{\Lambda\} - \lambda_j|\, u_j = .1450322574 \tag{11}$$

Therefore, comparing (10) and (11) we can conclude that the fairness is effectively improved by the transition rules and the Bayes scale calculated in example 1. In connection with the justification of the absolute value, observe that removing it will drive us to wrong conclusions because

$$\sum_{j=1}^{10} (E\{\Lambda\} - \lambda_j)\, u_j \simeq 0$$

Thanks to the global asymptotic fairness it would be possible to compare a single BMS applied to different risk groups and also the effect of different BMS applied to the same group, although the very interesting problem would be the minimization of the global asymptotic fairness Y with respect to any of its variables. For instance, given a structure function, it would be interesting to find the pair $(R;\bar{b})$ that produces the minimum Y-value. This problem seems so complex, that it is better to reformulate it in the following terms: given a set of transition rules and a structure function U, find the optimal scale of premiums minimizing Y:

4 Optimization of the global asymptotic fairness

As Goal Programming has been rarely applied in Actuarial Science, we think it will be useful to make a short presentation of it. This last follows largely the one made in Vilar [2000]. For a broad introduction to the topic of Goal Programming we refer to Romero [1991, 1993].

4.1 General comments on Linear Goal Programming

Suppose that the modelling process of some optimization problem has given us many objective functions $\overline{f} = (f_1, \ldots, f_m)$ that should be optimized simultaneously. When applying Goal Programming we must clearly establish what would be the best or ideal values (call them $\overline{f}^* = (f_1^*, \ldots, f_m^*)$) for these objective functions, then looking to them as goals we would like to attain keeping in mind that it is probably not possible to match them simultaneously. In other words, if a decision vector $\overline{x}$ such that $\overline{f}(\overline{x}) = \overline{f}^*$ did not exist, we would look at least for another $\overline{x}$ which gave us the nearest values to the ideal $\overline{f}^*$. This is done as follows.

Supposing that the objective functions are linear, write them as constraints of a new mathematical program, each one equated to its ideal or best value f_j^*; then add to the first member of each restriction two deviation variables $y_j^-, -y_j^+$ $(j = 1, \ldots, m)$. These variables stand for negative (how much we are down to the ideal value f_j^*) and positive deviation (how much we exceed) respectively. This way we obtain the following set of equality constraints:

$$j = 1, \ldots, m : f_j(\overline{x}) + y_j^- - y_j^+ = f_j^*.$$

We define the feasible set of a new mathematical program by means of these m linear constraints plus nonnegativity constraints on the devi-

Therefore, as a result of this process we get the following linear mathematical program:

$$\min_{(\overline{x};\overline{y})} \sum_{j=1}^{m} (y_j^{-} + y_j^{+}) \tag{12}$$

$$\text{s.t.} \quad \begin{cases} f_1(\overline{x}) + y_1^{-} - y_1^{+} = f_1^{*} \\ \vdots \\ f_m(\overline{x}) + y_m^{-} - y_m^{+} = f_m^{*} \\ y_j^{-}; y_j^{+} \geq 0. \end{cases}$$

Coordinates of the feasible solutions are $\left(\overline{x}; y_1^{-}; y_1^{+}; \ldots; y_m^{-}; y_m^{+}\right)$. In an optimal feasible solution, at least one coordinate in each pair $(y_j^{-}; y_j^{+})$ must be null; when both vanish, the ideal value f_j^{*} is attained. Program (12) is a way of presenting a linear goal program. Its simplicity consists in being linear, so the simplex algorithm can be executed to solve it.

4.2 Calculating the optimal scale of premiums.

Suppose we are given a structure function U. As our method requires this distribution to be of the discrete type, if it was of the continuous or mixed types we would have to discretize it by means of some well known discretization method discussed in the actuarial literature (see for instance Klugman, Panjer, Willmot [1998] p.607, Vilar [2000]). From hereafter the structure function will be considered of the discrete type and its probability function will be noted

$$j = 1; \ldots; m : P\{\theta = \theta_j\} = u_j$$

Taking this last comment in mind, the global asymptotic fairness is now written as:

$$Y_{R;U}(\overline{b}) = E_{*}\{|y(\theta)|\} = \sum \left| \sum b_i \pi_i(\theta_j) - \theta_j \right| u_j \tag{13}$$

where the unknowns are the premiums $\bar{b} = (b_1; \ldots; b_n)$: Next we can try to calculate the scale of premiums that brings the mean asymptotic premiums closest to the ideal situation stated in (14).

For that sake, following the presentation made in section 4.1, we only have to add to the first member of (14) the deviation variables $y_j^-; \ -y_j^+$; join the nonnegativity constraints for these variables, and proceed to minimize the weighted sum of these deviation variables. The weights are the probability masses $(u_j)_{j=1}^m$ for we have to minimize the expression (13) which is the integral of the absolute value. Following these steps we obtain the linear mathematical program:

$$\min_{(\bar{b}, y)} \sum_{j=1}^{m} (y_j^- + y_j^+)\, u_j \tag{15}$$

$$\text{s.t.:} \quad \begin{cases} \sum_{i=1}^{n} b_i \pi_i(\lambda_j) + y_j^- - y_j^+ = \lambda_j; & (j = 1; \ldots; m) \\ y_j^-; y_j^+ \geq 0; & (j = 1; \ldots; m) \end{cases}$$

Now comes a crucial stage, for it is time to be aware that we have chosen to calculate the scale of premiums $\bar{b}$ by means of the resolution of an optimization problem. As a general rule, the optimums found in the resolution of a mathematical program will satisfy the properties expressed in the definition of the feasible set. An optimum is in particular a feasible solution, therefore if substituted in the constraints it will verify them all, but it does not have to verify any other property that has not been expressed as a constraint of the mathematical program. Looking to the linear program (15), it is clear that an optimal feasible solution will satisfy the property of bringing the mean asymptotic premium closest to the ideal situation of perfect fairness (in the sense already explained in (8)), but it is not forced to furnish solutions satisfying any other requirement as could be for instance the nonnegativity and monotony of the scale, or the financial equilibrium of the BMS. In other words, program (15) is only the halfway in modelling the calculation of a scale of premiums using this multiobjective technique, because include the properties of monotony and nonnegativity of the scale of premiums.

The financial equilibrium of a BMS is such a well established concept in Actuarial Science. Roughly speaking, it states that when a BMS is acting in the long run, the mean value of the total pure premium incomes must be equal to the mean value of the total claim payments made by the insurance company. This is very important indeed from both the theoretical and practical sides, because a risk group endowed with a BMS that did not verify this property could get ruined in the long run. The Bayesian bonus-malus actually verifies this property (Lemaire [1995] p.161), as does the already mentioned Bayes Scale of premiums (Norberg [1976]). For instance the transition rules and the Bayes Scale of premiums calculated in example 1 would give a BMS in financial equilibrium. To express this property we only have to write down the two mean values then equalling them. The mean value of the total pure premiums incomes in the steady-state is

$$\sum_{i=1} b_i \pi_i \tag{16}$$

where

$$\pi_i = \sum_{j=1} \pi_i(\lambda_j)\, u_j \quad (i = 1; \ldots ; n)$$

are the unconditioned stationary probabilities (see (1)). On the other side, the mean value of the total payments in the steady-state is:

$$E\{\Lambda\} = \sum_{j=1} \lambda_j u_j. \tag{17}$$

Equalling (17) and (16) gives the financial equilibrium constraint:

Thus coming back to (15), a feasible solution such that (18) is positive (respectively negative) indicates an unbalance worth to the insurance company (respectively to the insureds).

The constraints named as commercial requirements of the scale of premiums are not so obvious as the later, calling for a more delicate treatment because of the subjectivity involved in the modelling process. Firstly, we think that it should be out of discussion that the scale of premiums ought to be nonnegative and monotone. Therefore it could be necessary to include the following set of restrictions:

$$0 \le b_1 \le \cdots \le b_n$$

Secondly, in a competitive environment the insurance company may want to have a scale of premiums attractive to its insureds. To calculate such a scale, many questions could be asked and studied. The aim of the following list is not to be exhaustive, though we think that these three questions could be considered along an analysis on what scale would be more adequate for the insurance company:

- Is it possible to set a bound for the difference between the premiums corresponding to the cheapest and dearest bonus-malus classes? For instance this could be translated into a linear restriction in two different ways. The first one is straightforward: noting the bound M we could set

  $$b_n - b_1 \le M.$$

  The second one consists in forcing the cheapest and dearest premiums to be respectively lower and greater than certain percentages $C_1; C_n$ of a given premium that we call central; noting b_{i_c} $(1 < i_c < n)$ the central premium we get

  $$b_1 \ge C_1 b_{i_c}$$

- Is it advantageous to set the premium of a determinate bonus-malus class equal to a certain level of the initial pure premium? This would be equivalent to set one of the decision variables equal to a constant: $b_i = K$.

As a mater of fact, it is up to the decision maker to decide what kind of property should he ask his scale of premiums to verify and how to write it down. There are only two technical limitations: the property may be expressed by means of a linear constraint in $\bar{b} = (b_1, \ldots, b_n, y_1^-, y_1^+, \ldots, y_m^-, y_m^+)$; and the resulting linear program must be feasible, i.e. the feasible set must be different from the empty set (feasible solutions must exist).

Summing-up the discussion, we should join to the definition of the feasible set of the linear program (15), the financial equilibrium constraint and a subset of constraints aimed to model some characteristics of the scale of premiums related to the market environment. If both the constraints and the way chosen for expressing them were adequate, the resulting linear program would be feasible and we would be able to solve it by means of the simplex algorithm.

Therefore, the optimum value for the global asymptotic fairness would depend not only on the transition rules and the structure function, but also on the commercial requirements expressed as linear constraints. Thus it will be useful to express this dependence of the optimal value by means of some notation; writing C for the set of constraints that expresses the commercial requirements, from hereafter we will note $Y^*_{R,U,C}$ the optimal value for the global asymptotic fairness corresponding to it. In the following example we show how it is possible to calculate the optimal scale that satisfies all these fulfillments.

Example 4 Let us come back to the transition rules (5) and the structure function (see table 1) already used in example 1. In order to minimize the global asymptotic fairness, we consider all the scales of premiums $\bar{b} = (b_1, b_2, b_3, b_4)$ satisfying the following linear constraints and where the stationary unconditioned probabilities π_i are the same as in (6).

Commercial constraints. Following the last discussion, we arbitrarily set the following requirements for the scale of premiums:

1. Let the premium of the highest discount class (i.e. the first bonus-malus class) be at least the 30% of the premium of class 3. This is to say:

$$b_1 \geq 0.3\, b_3 \tag{19}$$

2. Let the premium of the more expensive class (i.e. class 4) be at most twice the premium of class 3. This is to say:

$$b_4 \leq 2\, b_3 \tag{20}$$

3. Let the premium in every bonus-malus class be at least 10% more expensive than the premium in the precedent class. That is:

$$b_{i+1} \geq 1.1\, b_i\,; \quad i = 1, 2, 3. \tag{21}$$

Observe that the monotony is assumed through the last restrictions (21). As we will see soon in the resolution, it is not necessary to write down the nonnegativity restrictions on the scale of premiums. We will write C_1 for the set of commercial constraints that have been just defined.

Finally, as we want our BMS to be the closer the better to perfect fairness in the steady-state, we have to minimize the same objective as in (15). Therefore, the linear goal program that we are going to solve is the following:

$$\min \sum^{10} (y_i^- + y_i^+)\, \ldots \tag{22}$$

the nonnegativity constraints on the deviation variables. Solving it by means of the simplex method, we obtain the following optimal scale of premiums $\overline{b}^*$:

$$\begin{aligned} b_1^* &= .2827527095; \quad b_2^* = .4293238448; \\ b_3^* &= .9425090315; \quad b_4^* = 1.885018063 \end{aligned}$$

We stress on the fact that scale $\overline{b}^*$ would give a BMS in financial equilibrium closest to the perfect fairness represented through $y_0(\lambda)$ (in the sense explained in (8)), given the transition rules (5) and the commercial requirements C_1. The (optimal) deviation variables associated to this scale are indicated in table 3:

TABLE 3

which clearly improve the ones obtained in example 1 (see table 2). Noting $y^*(\lambda)$ the asymptotic fairness of this new BMS, we can plot it in figure 5, where we can also find the plots of $y_1(\lambda)$ and $y_B(\lambda)$:

FIGURE 5

The optimal value for the global asymptotic fairness is

$$Y^*_{R_0;U;C_1} = .03443138919 \qquad (23)$$

Comparing (23) with (10), we conclude that this last scale gives a better global asymptotic fairness than the one calculated in example 1. We recall that it has not been necessary to assume the nonnegativity constraints on the premiums. The ratios between successive premiums are

$$\frac{b_2^*}{b_1^*} \simeq 1.52; \quad \frac{b_3^*}{b_2^*} \simeq 2.2; \quad \frac{b_4^*}{b_3^*} \simeq 2.$$

Looking to table 3 we remark that the excess paid by the best policies (those with $\lambda_1 = 0.15$) remains the greatest among the whole risk group. This can motivate the calculation of a new optimal scale of premiums

The new set of commercial constraints containing (24), (20) and (21) will be noted C_2. Therefore, the new linear program is

$$\min_{(\bar{b},\bar{y})} \sum_{j=1}^{10} (y_j^- + y_j^+)\, u_j \qquad (25)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{i=1}^{4} b_i \pi_i(\lambda_j) + y_j^- - y_j^+ = \lambda_j; & (j = 1; \ldots ; 10) \\ y_j^-; y_j^+ \geq 0; & (j = 1; \ldots ; 10) \\ \sum_{i=1}^{n} b_i \pi_i = .4999278192 \\ b_1 \geq 0.01\, b_3 \\ b_4 \leq 2\, b_3 \\ b_{i+1} \geq 1.1\, b_i; \quad i = 1; 2; 3. \end{cases}$$

Solving (25) we obtain the scale of premiums $\bar{b}^{**}$ (satisfying also the financial equilibrium constraint):

$$b_1^{**} = .01600753544; \quad b_2^{**} = .9384912706;$$
$$b_3^{**} = 1.032340398; \quad b_4^{**} = 1.442028527$$

Table 4 presents the optimal values for the deviation variables:

TABLE 4

In figure 6 we can see the plots of the functions $y_1(\lambda)$; $y^*(\lambda)$ and $y^{**}(\lambda)$. Recall that the perfect fairness would be represented graphically by means of the abscissas axe.

FIGURE 6

The optimal value of the global asymptotic fairness corresponding to the optimum $\bar{b}^{**}$ is

$$Y^{**}_{R_2;U;C_2} = .003095965614 \qquad (26)$$

- Restricting ourselves to the policies with risk parameter $\lambda_1 = 0.15$ the fairness has been improved, because we have gone from $y_1^+ = .1617726831$ (see table 3) to $y_1^+ = .0058068755$ (see table 4). Repeating the calculations made in example 1 (see (7)), this means that the best policies in the risk group, which were paying under the commercial requirements C_1 the 207% of their true risk parameter

$$0.15 + .1617726831 = .3117726831 \Rightarrow \frac{.3117726831}{0.15} = 2.078484554,$$

are now only paying under C_2 the 103% of λ_1.

$$0.15 + .0058068755 = .1558068755 \Rightarrow \frac{.1558068755}{0.15} = 1.038712503.$$

Recall that these calculations are valid for policies in the steady-state.

- Perfect fairness has been achieved for policies with risk parameters $\lambda_2 = 0.30$ and $\lambda_3 = 0.45$, for their optimal deviation variables are null. The frequency of these policies is equal to

$$u_2 + u_3 = .6372578$$

approximately a 63.7% of the whole risk group.

- On the other hand, it is clear from tables 3 and 4 that the scale $\overline{b}^{**}$ makes worse the fairness for the worst policies, i.e. those with risk parameters $\{\lambda_j : j = 7, 8, 9, 10\}$. But this is not relevant from the global point of view, because the frequency of these policies is equal to the individual mean claim amount $E\{X\}$. As a matter of fact, we think that a general characteristic of the scale of premiums $\overline{b}^{**}$ is to approach the perfect asymptotic fairness while preserving the financial equilibrium by means of a decreasing of the first premium at the price of increasing the second and third ones.

- Finally, a look to figure 6 and to the optimal value (26), tells us that the situation of perfect fairness symbolized by means of the constant function $y_0(\lambda) = 0$; has not been reached (as we expected) though it has been fairly well approached over the most significant values of the risk parameter.

5 Summary and conclusions

In this paper we try to show how it would be possible to analyze and evaluate the asymptotic fairness of a BMS applied to a risk group. In other words, we try to quantify to what extent does a BMS achieve the goal of better assessing individual risks in the long run. Having in mind this objective, the best case would correspond to a Bayesian bonus-malus (i.e. Exact Credibility) acting through an infinite number of periods as it is known that the difference between the premium paid by each insured and his true parameter λ would be asymptotically null. Thus we can represent this case by means of the limit null function $y_0(\lambda) = 0$; and refer to it as the case of perfect fairness. On the other hand, the worst case would be represented by the absence of any BMS, so every insured would pay the same pure premium. As we assume the simplest case where all the distributions remain the same along the infinity of periods (i.e. bonus hunger is not considered), and the mean claim amount is supposed to be $E\{X\} = 1$; we can represent this case by means of the function $y_1(\lambda) = E\{\Lambda\} - \lambda$; where Λ is the structure variable over the risk group. This case is told unfair because the good risks will pay the same pure premium as the bad ones. We can then with risk parameter λ, when approaching the steady-state. Analyzing the deviations associated to a BMS could be helpful to decide if the BMS is worth to the insureds with a certain risk parameter or to the insurance company (see example 1).

The leading idea of our analysis is to realize that the more resembling will be $y(\lambda)$ to the constant null function $y_0(\lambda)$, the higher will be the asymptotic fairness of the corresponding BMS. Nevertheless, when comparing the asymptotic fairness of two BMS we are driven to pointwise comparisons between the corresponding $y(\lambda)$ that could be harmful and useless because we would not have take into account the composition of the risk group, i.e. its structure function.

To overcome this difficulty, we define a measure $Y(R; \bar{b}; U)$ of the global asymptotic fairness (9), associated to a set consisting of some transition rules R, a scale of premiums $\bar{b}$, and a structure function U characterizing the risk group. The measure Y is defined as the integral over $[0, +\infty]$ of $|y(\lambda)|$, the absolute value of $y(\lambda)$. We try this way to quantify the degree of resemblance between an asymptotic fairness $y(\lambda)$ corresponding to a given BMS and the null function $y_0(\lambda)$. The almost sure identity between $y(\lambda)$ and $y_0(\lambda)$ would correspond to $Y = 0$, although this situation cannot actually be reached by any BMS. With this measure we are able to rate the asymptotic fairness of different BMS: the lower will be the measure, the higher will be the asymptotic fairness of the BMS introduced in a given risk group. This is what we have done in examples 3 and 4.

Although this represents an achievement, the very interesting problem would actually consist in optimizing the global asymptotic fairness $Y_{R;U}(\bar{b})$ with respect to the scale of premiums, considering given the transition rules and the structure function. If we were able to solve this problem we might compare the optimal values of the global asymptotic fairness associated to different transition rules, each one equipped with its optimal scale of premiums, applied to a given risk group characterized by means of a structure function U.

The clue idea for solving this optimization problem is to recall that it optimums found this way are useless, because the feasible set of this linear program does not take into account very important properties that a scale of premiums should verify, as are for instance the financial equilibrium or some conditions that we have named commercial requirements.

We have found that including these conditions as linear constraints in the definition of the feasible set can produce feasible linear programs as are for instance (22) and (25). Therefore, in this paper it is shown how it is possible to calculate, by means of Linear Goal Programming, scales of premiums that jointly match the following goals:

- Bring the asymptotic fairness closest to the perfect fairness represented by the Bayesian case.
- Verify the condition of financial equilibrium.
- Verify some conditions, to be defined by the decision maker, related to the monotony and nonnegativity of the scale of premiums, to the maximum and minimum premiums, and finally to the ratios between premiums in consecutive bonus-malus classes. These conditions are named commercial requirements.

Example 4 is intended to show that the ideal situation of perfect fairness could be fairly well approached, at least for the major part of policies belonging to a risk group (see table 4 and figure 6).

When trying to minimize the global asymptotic fairness by means of this methodology, the only limitation is to translate the conditions imposed to the scale of premiums into linear constraints that finally produce a feasible program. Nevertheless, infeasibility would be also a valuable information, because it would denounce the nonexistence of the scale of premiums satisfying the conditions written down in the definition of the feasible set.

We think that matching simultaneously those three goals represents such an interesting skill for any bonus-malus designer, because modelling a practical problem by means of an optimization method, is that the optimums have not to verify any other conditions than the ones stated in the feasible set. Thus using this argument, the necessity of including the remainder conditions could be justified as inherent to the modelling process. We could also note that thinking that the asymptotic fairness is sensible to different commercial requirements is not a strange idea. Nevertheless, we have to stress that a linear goal program that included only the constraints related to the asymptotic fairness and the financial equilibrium constraint would still result in useless optimums: for instance we could get scales with the same premium in many bonus-malus classes, or non monotonic scales, or even more, scales with negative premiums.

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Figure 1 : The two functions $y_1(\theta) = E\{\Theta\} - \theta$ (taking $E\{\Theta\} \simeq 0.5$) representative of the case where no BMS has been defined, and $y_0(\theta) = 0$ where every risk is paying in the long run a premium corresponding to its own claim frequency.

Figure 2: Plots of the stationary conditioned distributions $\pi_i(\lambda)$ $(i = 1, 2, 3, 4)$ corresponding to the transition rules (5).

Figure 3 : Plots of the functions $y_1(\lambda)$; $y_B(\lambda)$; and $y_0(\lambda)$: The second one corresponds to the asymptotic fairness of the BMS defined in example 1.

Figure 4 : A detail of figure 3 over the $\theta -$ interval $[0.48, 0.55]$, where we can appreciate the central value and the intersection between $y_1(\theta)$, and $y_B(\theta)$.

Figure 5 : Plots of the functions $y^{*}(\lambda)$ (asymptotic fairness in the optimized case); $y_1(\lambda)$ and $y_B(\lambda)$ (case of the Bayes scale of premiums, see example 1). The optimal value for the global asymptotic fairness is $Y^{*}_{R_0;U;C_1} = .03443138919.$

Figure 6 : Plots of the three functions $y_1(\lambda)$, $y^{*}(\lambda)$ and $y^{**}(\lambda)$. The optimal value for the global asymptotic fairness corresponding to the last case is $Y^{**}_{R_0,U,C_2} = .003095965614$. Remember the case of $y^{*}(\lambda)$ gave us the optimal value $Y^{*}_{R_0,U,C_1} = .03443138919$.

¸ $_j$:	0.15	0.30	0.45	0.60	0.75
u_j :	:03384627	:1923699	:4448879	:1462023	:1364640
¸ $_j$:	0.90	1.05	1.20	1.35	1.50
u_j :	:02036232	:02039220	:002342893	:002511476	:000620741

Table 1: The structure function.

[illegible]$_j$:	0.15	0.30	0.45	0.60	0.75
y_j^+ :	.3044530309	.1695400958	.0391483628	0	0
y_j^{i} :	0	0	0	.0861875930	.2074495780
[illegible]$_j$:	0.90	1.05	1.20	1.35	1.50
y_j^+ :	0	0	0	0	0
y_j^{i} :	.3271380522	.4482204418	.5728807610	.7020168509	.8355064740

Table 2: The deviation variables corresponding to the Bayes scale calculated in example 1.

¸ $_j$:	0.15	0.30	0.45	0.60	0.75
y_j^+ :	.1617726831	.0609319672	0	0	0
y_j^i :	0	0	.0098250314	.043786144	.043503
¸ $_j$:	0.90	1.05	1.20	1.35	1.50
y_j^+ :	0	0	.0080295938	0	0
y_j^i :	.022555679	0	0	.0071117622	.0071117649

Table 3: The optimal deviations associated to the optimal scale of premiums $\bar{b}^{¤}$:

¸ j :	0.15	0.30	0.45	0.60	0.75
y_j^+ :	.0058068755	0	0	.0040047857	.0056127103
y_j^i :	0	0	0	0	0
¸ j :	0.90	1.05	1.20	1.35	1.50
y_j^+ :	0	0	0	0	0
y_j^i :	.0044224675	.0339965124	.0865322326	.1609985415	.2538702075

Table 4: The optimal deviations associated to the optimal scale of premiums $\bar{b}$¤¤: