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Optimal designs of constant-stress accelerated life-tests for one-shot devices with model misspecification analysis

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Abstract
The design of constant-stress accelerated life-test (CSALT) is important in reliability estimation. In reliability studies, practitioners usually rely on underlying distribution to design CSALTs. However, model misspecification analysis of optimal designs has not been examined extensively. This paper considers one-shot device testing data by assuming gamma, Weibull, lognormal and Birnbaum–Saunders (BS) lifetime distributions, which are popular lifetime distributions in reliability studies. We then investigate the effect of model misspecification between these lifetime distributions in the design of optimal CSALTs, in which the asymptotic variance of the estimate of reliability of the device at a specific mission time is minimized subject to a prefixed budget and a termination time of the life-test. The inspection frequency, number of inspections at each stress level, and allocation of the test devices are determined in optimal design for one-shot device testing. Finally, a numerical example involving a grease-based magnetorheological fluids (G-MRF) data set is used to illustrate the developed methods. Results suggest the assumption of lifetime distribution as Weibull or lognormal to be more robust to model misspecification, while the assumption of gamma lifetime distribution seems to be the most non-robust (or most sensitive) one.

KEYWORDS
accelerated life-test, asymptotic variance, best test plan, BS distribution, gamma distribution, lognormal distribution, model misspecification, one-shot device, reliability, Weibull distribution

1 | INTRODUCTION

One-shot devices, such as automobile air bags, fuel injectors, missiles,[1] and fire extinguishers,[2] are all products that will perform their intended function only once and, after use, will get destroyed immediately. As the exact failure times of such one-shot devices cannot be obtained from the test, and that only the condition at an inspection time can be observed, binary data are collected indicating whether the lifetime is less than the inspection time (failure) or more than the inspection time (success). In the last decade, considerable work has been carried out on one-shot device analysis, primarily motivated by the work of Fan et al.[3] One may refer to elsewhere,[4–7] among others. All these works proceed by assuming that the lifetimes of devices follow a specific statistical distribution, such as exponential,[6,8] Weibull,[4] or gamma.[5,7] In particular, gamma, Weibull, lognormal, and Birnbaum–Saudners (BS) distributions are commonly used for fitting lifetime data in reliability and survival studies due to the flexibility of their hazard functions, namely, having increasing, constant, and decreasing behaviors. However, lognormal and BS distributions have not been studied much in the context of one-shot devices yet. In a recent paper of Zhu et al.,[9] BS distribution was considered on the basis of cyclic loading principle for the testing of one-shot devices. Lognormal distribution for the lifetime of devices was studied for the first time in a recent unpublished work.[10]

Due to significant advances in manufacturing technology, devices with high reliability may have a very long life, without any failure at normal operating conditions. In this case, accelerated life-tests (ALTs) are commonly used as they induce rapid failures in a short period of time, by testing items under higher-than-normal stress levels; see elsewhere.[11,12] There are many types of ALTs. For example, constant-stress ALTs (CSALTs) assume that each device is subject to only one prespecified stress level, while step-stress ALTs (SSALTs) apply stress to devices in such a way that stress levels will get changed at prespecified times. Optimal design is of importance in ALTs as it would result in great savings in both time and cost. Tseng et al.,[13] for example, developed optimal test plans for step-stress accelerated degradation data under a gamma degradation process. In recent years, there has been an increasing amount of literature on SSALTs; see Han[14] and Han and Bai.[15,16] To design efficient CSALTs for one-shot devices under Weibull lifetime distribution, subject to a prespecified budget and a termination time, Balakrishnan and Ling[17] considered the minimization of the asymptotic variance of the maximum likelihood estimator (MLE) of reliability at a mission time under normal operating conditions. In a similar manner, Ling[18] and Ling and Hu[19] designed optimal SSALTs for one-shot devices under exponential and Weibull lifetime distributions, respectively. For an elaborate review of all these developments concerning the analysis of one-shot device data, one may refer to the recent book by Balakrishnan et al.[20]

As gamma, Weibull, lognormal, and BS lifetime distributions have been used quite extensively in reliability and survival literature, model misspecification between these four prominent lifetime distributions becomes of great importance in practice. Different choices of lifetime distributions for fitting lifetime data may result in substantially different inferential results on some lifetime characteristics of interest as well as in ALT planning. In this regard, attention has been paid to model misspecification analysis recently. Peng and Tseng[21] found that model misspecification is not a serious issue in linear degradation models, when the sample size is sufficiently large for estimating the mean lifetime. However, Tsai et al.[22] observed, from simulation studies, that the effects of model misspecification between gamma and Wiener processes may be serious. Liu et al.[23] subsequently studied the model misspecification problem for two-phase Wiener and gamma processes with a known change point. Khakifirooz et al.[24] considered generalized gamma, lognormal, and Weibull distributions for model misspecification under ALTs with censored data. Ling and Balakrishnan[25] also found that the effect of model misspecification between gamma and Weibull lifetime distributions can be severe and should not be ignored for one-shot device testing data, and suggested implementing a specification test to improve the accuracy of estimation. On the other hand, it will be of interest to study the model misspecification problem on the design of ALTs for one-shot devices.

In this paper, we discuss the determination of optimal CSALT in the context of one-shot device testing by assuming gamma, Weibull, lognormal, and BS lifetime distributions. In particular, we focus on determining the inspection frequency, number of inspections at each stress level, and allocation of the test devices as decision variables while minimizing the asymptotic variance of the estimate of reliability of the device at a specific mission time under normal operating conditions, under a prefixed budget and a termination time of the life-test. To minimize the asymptotic variance of the MLE of reliability, we need the first-order derivatives of the reliability function with respect to the model parameters as well as the Fisher information matrix for the model parameters. Section 2 formulates the problem of ALTs for one-shot devices with the corresponding likelihood function, as well as the asymptotic variance of the MLE of reliability in the case when the lifetimes follow the above four mentioned lifetime distributions. Section 3 formulates the problem for the determination of optimal CSALT, subject to a termination time, when a prespecified budget or a constraint on the standard error of the estimate of the reliability, is provided. A simulation study is carried out in Section 4, which illustrates the developed algorithms, and also examines the sensitivity of optimal plans to misspecification of the model parameters. In Section 5, we examine the effect of model misspecification between these lifetime distributions in the design of optimal CSALTs, while an illustrative example involving a grease-based magnetorheological fluid (G-MRF) data set is analyzed in Section 6. Finally, some concluding remarks are made in Section 7. The first-order derivatives of the reliability function with respect to the model parameters for the lognormal and BS lifetime distribution are presented in Appendix A.

2 | DESCRIPTION OF THE MODEL

Suppose the data are stratified into I testing conditions $S_1, \ldots, S_I$, and that in testing condition S_i, N_i devices are tested with J types of stress factors being maintained at certain levels, and inspected at K_i equally spaced time points. Specifically, N_{ik}

devices are drawn and inspected at a specific time T_{ik} with $\sum_{k=1}^{K_i} N_{ik} = N_i$. Then, n_{ik} failures are observed from the test at inspection time T_{ik}. Let $\boldsymbol{x}_i = (1, x_{i1}, \dots, x_{iJ})^T$ be the vector of stress factors associated to testing condition S_i $(i = 1, \dots, I)$. The log-likelihood function of the lifetime data, obtained from the above one-shot device testing in an ALT experiment, is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{I} \sum_{k=1}^{K_i} [n_{ik} \log(1 - R(T_{ik}; S_i)) \tag{1}$$

$$+ (N_{ik} - n_{ik}) \log(R(T_{ik}; S_i))] + constant,$$

where $R(T_{ik}; S_i)$ is the reliability function evaluated at inspection time T_{ik} and test condition S_i. The MLE of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}$, is then determined by maximizing Equation (1) with respect to the model parameter $\boldsymbol{\theta}$.

Let G denote the lifetime of a unit that follows a gamma distribution with probability density function and cumulative distribution function as

$$f_G(t; S_i) = \frac{t^{\alpha_i - 1}}{\lambda_i^{\alpha_i} \Gamma(\alpha_i)} \exp\left(-\frac{t}{\lambda_i}\right),\ t > 0, \tag{2}$$

and

$$F_G(t; S_i) = \int_0^t \frac{y^{\alpha_i - 1}}{\lambda_i^{\alpha_i} \Gamma(\alpha_i)} \exp\left(-\frac{y}{\lambda_i}\right) dy,\ t > 0, \tag{3}$$

respectively, where $\alpha_i > 0$ and $\lambda_i > 0$ are the shape and scale parameters, respectively, and $\Gamma(\cdot)$ is the complete gamma function. These parameters are related to the stress factors in log-linear forms as

$$\lambda_i = \exp\left\{ \sum_{j=0}^{J} a_j x_{ij} \right\} = \exp(\boldsymbol{x}_i^T \boldsymbol{a})$$

and

$$\alpha_i = \exp\left\{ \sum_{j=0}^{J} b_j x_{ij} \right\} = \exp(\boldsymbol{x}_i^T \boldsymbol{b}),$$

where $\boldsymbol{a} = (a_0, a_1, \dots a_J)^T$, $\boldsymbol{b} = (b_0, b_1, \dots b_J)^T$, and $\boldsymbol{\theta} = \boldsymbol{\theta}_G = (a_0, \dots, a_J, b_0, \dots, b_J)^T \in \Theta = \mathbb{R}^{2(J+1)}$ is the model parameter vector. The reliability function is given by

$$R_G(t; S_i) = 1 - F_G(t; S_i) \tag{4}$$

$$= 1 - \int_0^t \frac{y^{\alpha_i - 1}}{\lambda_i^{\alpha_i} \Gamma(\alpha_i)} \exp\left(-\frac{y}{\lambda_i}\right) dy,\ t > 0.$$

Let T denote the lifetime of a unit that follows a Weibull distribution with probability density function and cumulative distribution function as

$$f_T(t; S_i) = \frac{\eta_i t^{\eta_i - 1}}{\beta_i^{\eta_i}} e^{-\left(\frac{t}{\beta_i}\right)^{\eta_i}},\ t > 0,$$

and

$$F_T(t; S_i) = 1 - e^{-\left(\frac{t}{\beta_i}\right)^{\eta_i}},\ t > 0,$$

where $\beta_i > 0$ and $\eta_i > 0$ are, respectively, the scale and shape parameters at condition S_i, which we assume are again related to the stress factors in log-linear forms as

$$\eta_i = \exp\left\{\sum_{j=0}^{J} r_j x_{ij}\right\}$$

and

$$\beta_i = \exp\left\{\sum_{j=0}^{J} s_j x_{ij}\right\}.$$

In this case, however, it is often more convenient to work with the extreme value distribution for the log-lifetimes, as it belongs to the location-scale family.[26,27] Let $W = \log(T)$ denote the log-lifetime. The probability density function, cumulative distribution function, and reliability function of the corresponding the extreme value distribution are

$$f_W(\omega; S_i) = \frac{1}{\sigma_i}\xi_i e^{-\xi_i}, \quad -\infty < \omega < \infty, \tag{5}$$

$$F_W(\omega; S_i) = 1 - e^{-\xi_i}, \quad -\infty < \omega < \infty, \tag{6}$$

$$R_W(\omega; S_i) = e^{-\xi_i}, \quad -\infty < \omega < \infty, \tag{7}$$

where $\xi_i = e^{\frac{\omega-\mu_i}{\sigma_i}}$, $\mu_i = \log(\beta_i)$, and $\sigma_i = \eta_i^{-1}$. Hence, $\boldsymbol{\theta} = \boldsymbol{\theta}_W = (r_0, \ldots, r_J, s_0, \ldots, s_J)^T \in \Theta = \mathbb{R}^{2(J+1)}$ is the model parameter vector. For analyzing reliability data, the Weibull distribution is commonly used as a lifetime model in engineering and physical sciences. In particular, it has been considered for one-shot devices testing; see elsewhere.[4,28] Balakrishnan and Ling[17] presented a sequential method for planning optimal CSALTs for one-shot devices with Weibull lifetime distribution for the devices.

Next, let T follow a lognormal distribution with probability density function and cumulative distribution function as

$$f_L(t; S_i) = \frac{1}{\sqrt{2\pi}\sigma_i t}\exp\left\{-\frac{(log(t) - \mu_i)^2}{2\sigma_i^2}\right\} \quad t > 0,$$

$$F_L(t; S_i) = \Phi\left(\frac{log(t) - \mu_i}{\sigma_i}\right), \quad t > 0,$$

respectively, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and μ_i and σ_i are, respectively, the scale and shape parameters, which are linked to covariates as

$$\mu_i = \sum_{j=0}^{J} p_j x_{ij} \quad \text{and} \quad \sigma_i = exp\left\{\sum_{j=0}^{J} q_j x_{ij}\right\}.$$

Here, $\boldsymbol{\theta} = \boldsymbol{\theta}_L = (p_0, \ldots, p_J, q_0, \ldots, q_J)^T \in \Theta = \mathbb{R}^{2(J+1)}$ is the model parameter vector. As in the case of Weibull, it is more convenient to work with log-lifetimes $V = \log(T)$, since this belongs to a location-scale family of distributions. In fact, the log-lifetimes follow a normal distribution with corresponding density, cumulative distribution and reliability functions as

$$f_V(\omega; S_i) = \phi\left(\frac{\omega - \mu_i}{\sigma_i}\right) = \frac{1}{\sqrt{2\pi}\sigma_i} exp\left\{-\frac{(\omega - \mu_i)^2}{2\sigma_i^2}\right\}, \quad -\infty < \omega < \infty, \tag{8}$$

$$F_V(\omega; S_i) = \Phi\left(\frac{\omega - \mu_i}{\sigma_i}\right), \quad -\infty < \omega < \infty, \tag{9}$$

$$R_V(\omega; S_i) = 1 - \Phi\left(\frac{\omega - \mu_i}{\sigma_i}\right), \quad -\infty < \omega < \infty. \tag{10}$$

While gamma and Weibull distributions have an either increasing or decreasing hazard function, lognormal distribution has an increasing–decreasing behavior of hazard, which may be more suitable in some situations involving reliability data.

Finally, we consider the case in which T follows a BS distribution with probability density function and cumulative distribution function as

$$f_{BS}(t; S_i) = \frac{1}{2\sqrt{2\pi}\gamma_i\nu_i}\left[\left(\frac{t}{\nu_i}\right)^{1/2} + \left(\frac{\nu_i}{t}\right)^{3/2}\right]\exp\left(-\frac{1}{2\gamma_i^2}\left\{\frac{t}{\nu_i} + \frac{\nu_i}{t} - 2\right\}\right), \quad t > 0, \tag{11}$$

$$F_{BS}(t; S_i) = \Phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} - \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right), \quad t > 0, \tag{12}$$

$$R_{BS}(t; S_i) = 1 - \Phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} - \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right) = \Phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{\nu_i}{t}\right)^{1/2} - \left(\frac{t}{\nu_i}\right)^{1/2}\right\}\right), \quad t > 0. \tag{13}$$

Here, $\boldsymbol{\theta} = \boldsymbol{\theta}_{BS} = (c_0, \dots, c_J, r_0, \dots, r_J)^T \in \Theta = \mathbb{R}^{2(J+1)}$ is the model parameter vector, with which we relate the the stress factors to the parameters ν_i and γ_i in log-linear forms as

$$\nu_i = exp\left\{\sum_{j=0}^{J} c_j x_{ij}\right\} \quad \text{and} \quad \gamma_i = exp\left\{\sum_{j=0}^{J} r_j x_{ij}\right\}.$$

Originally proposed elsewhere,[29,30] the BS distribution has become a commonly used lifetime model in fatigue failure analysis[31] and also in the context of ALT; see elsewhere.[32,33]

2.1 | Asymptotic variance of the MLE of reliability

The algorithm for the determination of the optimal ALT plan requires minimization of the asymptotic variance of the MLE of reliability at a specific mission time under normal operating conditions. The required optimization problem then would need the observed information matrix for model parameters as well as the first-order derivatives of reliability function with respect to the model parameters.

Let us now consider the inspection plan $\boldsymbol{\zeta} = (f, K_i, N_{ik})$, consisting of inspection frequency, number of inspections at each condition, and allocation of the devices. Balakrishnan and Ling[4] observed that, when the lifetimes are all censored, the observed information matrix obtained by the use of missing information principle is equivalent to the Fisher information matrix obtained from the expectation of the second-order derivatives of the log-likelihood function in Equation (1) with respect to the model parameters. The Fisher information matrix under $\boldsymbol{\zeta}$ is given by

$$\boldsymbol{I}(\theta; \zeta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial\theta\partial\theta^T}\right].$$

It can be shown that this is equivalent to

$$\boldsymbol{I}(\theta; \zeta) = \sum_{i=1}^{I}\sum_{k=1}^{K_i} N_{ik}\left(\frac{1}{R(T_{ik}; S_i)} + \frac{1}{1 - R(T_{ik}; S_i)}\right) \times \left(\frac{\partial R(T_{ik}; S_i)}{\partial\theta}\right)\left(\frac{\partial R(T_{ik}; S_i)}{\partial\theta^T}\right),$$

where $T_{ik} = k \times f$, for $k = 1, \dots, K_i$, for equi-spaced time points.

The asymptotic covariance matrix of the MLEs of the model parameters can then be obtained by inverting the Fisher information matrix as

$$\boldsymbol{V} \equiv \boldsymbol{V}(\theta;\boldsymbol{\zeta}) = (\boldsymbol{I}(\theta;\boldsymbol{\zeta}))^{-1}.$$

Using this expression, the asymptotic variance of the MLE of reliability under normal operating conditions at a specific mission time t_0 can be computed by delta method as

$$V_R(\boldsymbol{\zeta}) \equiv AV(\widehat{R}(t_0;S_0)) = \boldsymbol{P}_R^T \boldsymbol{V} \boldsymbol{P}_R, \tag{14}$$

where $\boldsymbol{P}_R = \frac{\partial R(t_0;S_0)}{\partial \theta}|_{\widehat{\theta}}$, with $\widehat{\theta}$ being the MLE of the model parameter θ and $\boldsymbol{x}_0 = (1, x_{01}, \dots, x_{0J})^T$ is the vector of stress factors associated with the normal operating condition S_0.

The first-order derivatives of reliability function with respect to the model parameters for gamma and Weibull lifetime distributions can be found in Balakrishnan and Ling[5] and Balakrishnan and Ling,[4] respectively. On the other hand, the first-order derivatives of reliability function with respect to the model parameters for lognormal and BS lifetime distributions are presented in Appendix A.

3 | OPTIMAL DESIGN OF EXPERIMENT

Suppose the budget for conducting a CSALT for one-shot device testing, the operation cost at testing condition S_i, the cost of devices (including the purchase of and testing cost), and the termination time are specified as $C_{budget}, C_{oper,i}, C_{item}$, and T_{ter}, respectively. Then, for a given test plan, $\boldsymbol{\zeta}$, that includes the inspection frequency, f, the number of inspections at testing condition $S_i, K_i \geq 2$, and the allocation of devices, N_{ik}, for $i = 1, 2, \dots, I$, the total cost of conducting the experiment is seen to be

$$TC(\boldsymbol{\zeta}) = C_{item} \sum_{i=1}^{I} \sum_{k=1}^{K_i} N_{ik} + f\left(\sum_{i=1}^{I} K_i C_{oper,i}\right).$$

We are now interested in the development of an efficient CSALT plan for one-shot device testing by determining the inspection frequency, number of inspections at each condition, and allocation of devices by minimizing the asymptotic variance of the MLE of reliability at a specific mission time under normal operating conditions, as given in Equation (14), subject to a prespecified budget and termination time of the life-test; that is, $TC(\boldsymbol{\zeta}) \leq C_{budget}$ (which ensures that the total cost does not exceed the total budget) and $fK_i \leq T_{ter}$ (termination time constraint). If a bound on the standard error of the estimate of the MLE of reliability is specified, one may instead be interested in minimizing the cost of conducting the experiment under such a constraint. In Balakrishnan and Ling,[17] algorithms for both problems are presented for one-shot device testing assuming Weibull lifetimes. There are different optimization criteria discussed in the literature. Most of them are concerned with the information matrix. For example, A-optimality and D-optimality seek to minimize the trace of the inverse of information matrix and the determinant of the information matrix, respectively (see[34]). The criterion presented here plans to determine a design with the minimum variance of the estimate of reliability (V-optimality).

4 | EMPIRICAL STUDY

In this section, we illustrate the proposed algorithm with test plan designs of ALT with one stress factor under different choices of budget and termination time for gamma, Weibull, lognormal, and BS distribution. In addition, we discuss the problem of minimizing the cost of the experiment given a constraint on the standard error for the estimate of the reliability at mission time under normal operating conditions. Interested readers may refer to Silvey[17] and Balakrishnan and Castilla[10] for more results for Weibull and lognormal distributions, respectively.

TABLE 1 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets, along with the corresponding standard error and RMSE for reliability at normal operating conditions $(x_0, t_0) = (28, 65)$, when the costs of operation at the three stress levels are \$100, \$150, and \$200 per unit of time, respectively, for gamma, Weibull, lognormal, and Birnbaum–Saunders lifetime distributions

T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$RMSE$
Gamma distribution								
36	\$200,000	33;54	20;20	20;20	18	\$199,900	0.1767	0.1736
36	\$300,000	65;20;96	20;20	20;20	12	\$299,100	0.1283	0.1484
36	\$500,000	134;20;187	20;20	20;20	12	\$499,500	0.0896	0.0976
60	\$200,000	20;57	20;20	20;20	30	\$199,700	0.0670	0.0718
60	\$300,000	20;148	20;20	20;20	30	\$299,800	0.0445	0.0464
60	\$500,000	20;330	20;20	20;20	30	\$500,000	0.0310	0.0321
Weibull distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$RMSE$
36	\$200,000	20;20	20;64	23;20	18	\$199,900	0.1041	0.1152
36	\$300,000	20;20	20;132	46;20	18	\$300,000	0.0765	0.0862
36	\$500,000	20;20	20;261	87;31	18	\$499,100	0.0561	0.0621
60	\$200,000	20;57	20;20	20;20	30	\$199,700	0.0593	0.0640
60	\$300,000	20;148	20;20	20;20	30	\$299,800	0.0415	0.0414
60	\$500,000	20;20;316	20;20	20;20	20	\$499,600	0.0284	0.0276
Lognormal distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$RMSE$
36	\$200,000	20;20;20	20;20;29	20;20	12	\$199,700	0.1338	0.1355
36	\$300,000	20;20;45	20;20;71	44;20	12	\$299,800	0.0929	0.0972
36	\$500,000	20;20;98	20;20;151	93;20	12	\$500,000	0.0651	0.0707
60	\$200,000	20;20;43	20;20	20;20	20	\$199,300	0.0788	0.0775
60	\$300,000	20;20;27;111	20;20	20;20	14	\$299,200	0.0513	0.0492
60	\$500,000	20;20;20;20;20;20;240	20;20	24;20	8	\$499,600	0.0339	0.0366
BS distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$RMSE$
36	\$200,000	20;20;20	20;20;29	20;20	12	\$199,700	0.1364	0.1422
36	\$300,000	20;20;49	20;20;70	41;20	12	\$299,800	0.0943	0.0814
36	\$500,000	20;20;106	20;20;151	85;20	12	\$500,000	0.0664	0.0904
60	\$200,000	20;20;43	20;20	20;20	20	\$199,300	0.0805	0.0680
60	\$300,000	20;20;20;118	20;20	20;20	14	\$299,200	0.0519	0.0564
60	\$500,000	20;20;20;45;102;148	25;20	22;20	10	\$499,200	0.0352	0.0559

4.1 | Best CSALTs

Suppose the lifetimes of devices under stress level have a gamma, Weibull, lognormal, and BS distribution with respective parameter vectors as follows:

$$\theta_G^T = (a_0, a_1, b_0, b_1) = (5.2, -0.06, -0.36, 0.04),$$

$$\theta_W^T = (s_0, s_1, r_0, r_1) = (5.7, -0.05, -0.6, 0.03),$$

$$\theta_L^T = (p_0, p_1, q_0, q_1) = (6.9, -0.1, -0.6, 0.005),$$

$$\theta_{BS}^T = (c_0, c_1, r_0, r_1) = (7.03, -0.104, -0.75, 0.01).$$

TABLE 2 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets, along with the corresponding standard error and RMSE for reliability at normal operating conditions $(x_0, t_0) = (28, 65)$, when the costs of operation at the three stress levels are \$100, \$150, and \$500 per unit of time, respectively, for gamma, Weibull, lognormal, and Birnbaum–Saunders lifetime distributions

T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\widehat{R})$	$RMSE$
Gamma distribution								
36	\$200,000	29;48	20;20	20;20	18	\$199,700	0.1849	0.1805
36	\$300,000	63;20;92	20;20	20;20	12	\$299,700	0.1306	0.1526
36	\$500,000	136;20;201	20;20	20;20	12	\$499,900	0.0907	0.1179
60	\$200,000	20;40	20;20	20;20	30	\$199,000	0.0770	0.0810
60	\$300,000	20;131	20;20	20;20	30	\$299,100	0.0470	0.0486
60	\$500,000	20;313	20;20	39;20	30	\$499,300	0.0317	0.0345
Weibull distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\widehat{R})$	$RMSE$
36	\$200,000	20;20	20;57	20;20	18	\$199,700	0.1094	0.1202
36	\$300,000	20;20	20;125	43;20	18	\$299,800	0.0784	0.0872
36	\$500,000	20;20	20;254	86;30	18	\$500,000	0.0567	0.0644
60	\$200,000	20;42	20;20	20;20	29	\$199,700	0.0665	0.0721
60	\$300,000	20;20;123	20;20	20;20	20	\$299,300	0.0433	0.0418
60	\$500,000	20;20;305	20;20	20;20	20	\$499,500	0.0288	0.0291
Lognormal distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\widehat{R})$	$RMSE$
36	\$200,000	20;20;20	20;20;22	20;20	12	\$199,200	0.1393	0.1447
36	\$300,000	20;20;43	20;20;68	42;20	12	\$299,300	0.0948	0.1002
36	\$500,000	20;20;96	20;20;148	91;20	12	\$499,500	0.0658	0.0638
60	\$200,000	20;20;32	20;20	20;20	20	\$199,200	0.0872	0.0859
60	\$300,000	20;20;26;105	20;20	20;20	14	\$299,900	0.0525	0.0520
60	\$500,000	20;20;20;20;20;20;236	20;20	24;20	8	\$500,000	0.0341	0.0326
BS distribution								
T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\widehat{R})$	$RMSE$
36	\$200,000	20;20;20	20;20;22	20;20	12	\$199,200	0.1424	0.1589
36	\$300,000	20;20;47	20;20;67	39;20	12	\$299,300	0.0962	0.0836
36	\$500,000	20;20;104	20;20;148	83;20	12	\$499,500	0.0670	0.0711
60	\$200,000	20;20;32	20;20	20;20	20	\$199,200	0.0889	0.0755
60	\$300,000	20;20;20;111	20;20	20;20	14	\$299,900	0.0531	0.0608
60	\$500,000	20;20;20;44;101;146	24;20	22;20	10	\$499,700	0.0341	0.0368

Now, we consider the estimation of reliability of the devices at mission time $t_0 = 65$ under a normal operating stress level of $x_0 = 28$, and that the ALT has to be terminated at times 36 and 60, say. The elevated stress levels used are 30, 40, and 50. Suppose the costs of operation at these elevated stress levels are \$100, \$150, and \$200 per unit of time, respectively, and that the cost of each device is \$1100. Note that we assume here an increase in the cost of operation would result in an increase in stress level as more resources will normally be required to increase the stress level. For example, if we increase the voltage, we would also end up increasing the power so that more electricity will be consumed resulting in an increased power cost.

The best ALT plans for different budget constraints were then determined, and they are presented in Tables 1 and 2. Here, TC represents the total cost of the optimal plan, $se(\widehat{R})$ is the corresponding theoretical standard error, $\sqrt{AV(\widehat{R}(t_0; \boldsymbol{x}_0))}$, and RMSE is the root mean square error (based on 1000 simulated samples) for the estimate of reliability at time t_0. As expected, an increase in the budget and termination time results in experiments with better reliability prediction.

TABLE 3 Optimal constant-stress accelerated life-tests (CSALTs) with different standard errors (se) of the reliability at normal operating conditions $(x_0, t_0) = (28, 65)$ and termination times, minimum total cost of conducting the experiment, standard error and standadrd deviation of reliability when the costs of operation at the three stress levels are \$100, \$150, and \$200 per unit of time, respectively, for gamma, Weibull, lognormal, and Birnbaum–Saunders lifetime distributions

T_{ter}	max_{se}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$Std_{\hat{R}}$
Gamma distribution								
36	0.12	75;20;11	20;20	20;20	12	\$326, 600	0.1198	0.1265
36	0.10	111;20;163	20;20	20;20	12	\$423, 400	0.0999	0.1160
36	0.08	169;20;236	20;20	20;20	12	\$591, 900	0.0799	0.0974
60	0.12	20;20	20;20	20;20	27	\$156, 300	0.1140	0.1149
60	0.10	20;20	20;20	20;20	30	\$159, 000	0.0975	0.1017
60	0.08	20;37	20;20	20;20	30	\$177, 700	0.0792	0.0851
Weibull distribution								
T_{ter}	max_{se}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$Std_{\hat{R}}$
36	0.12	20;20	20;43	20;20	18	\$173, 500	0.1199	0.1270
36	0.10	20;20	20;71	25;20	18	\$209, 800	0.0998	0.1119
36	0.08	20;20	20;120	41;20	18	\$281, 300	0.0799	0.0805
60	0.12	20;20	20;20	20;20	21	\$150, 900	0.1197	0.1235
60	0.10	20;20	20;20	20;20	24	\$153, 600	0.0987	0.1050
60	0.08	20;21	20;20	20;20	30	\$160, 100	0.0794	0.1005
Lognormal distribution								
T_{ter}	max_{se}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$Std_{\hat{R}}$
36	0.12	20;20;25	20;20;43	20;20	12	\$222, 800	0.1200	0.1273
36	0.10	20;20;39	20;20;60	37;20	12	\$273, 400	0.0998	0.1117
36	0.08	20;20;63	20;20;98	60;20	12	\$366, 900	0.0800	0.0811
60	0.12	20;23	20;20	20;20	30	\$162, 300	0.1186	0.1205
60	0.10	20;20;21	20;20	20;20	20	\$175, 100	0.0999	0.1044
60	0.08	20;20;42	20;20	20;20	20	\$198, 200	0.0794	0.1002
BS distribution								
T_{ter}	max_{se}	N_{1k}	N_{2k}	N_{3k}	f	TC	$se(\hat{R})$	$Std_{\hat{R}}$
36	0.12	20;20;28	20;20;41	24;20	12	\$226, 100	0.1198	0.1198
36	0.10	20;20;43	20;20;62	36;20	12	\$278, 900	0.0998	0.1026
36	0.08	20;20;71	20;20;101	57;20	12	\$375, 700	0.0800	0.0756
60	0.12	20;25	20;20	20;20	30	\$164, 500	0.1193	0.1317
60	0.10	20;20;23	20;20	20;20	20	\$177, 300	0.0989	0.1031
60	0.08	20;20;44	20;20	20;20	20	\$200, 400	0.0799	0.0740

Table 2 presents the effect on the best ALT plan when the cost of operation at the highest stress level of temperature increases from \$200 to \$500. As the cost of operation is more in this case, the total number of devices inspected tends to become smaller, but there is not a big increment in the asymptotic variance of the MLE of reliability and the obtained results are quite similar to those presented in Table 1.

Table 3 presents the optimal designs and the cost of running the experiments, subject to various limits on standard error and termination times, under the four lifetime distributions. It is seen that the optimal designs under gamma, lognormal, and BS lifetime distributions generally require more test units and higher total cost, compared with those under the Weibull case.

In this study, we have chosen the minimum number of devices allocated at each condition at each inspection time to be $K_{min} = 20$, but this can be changed by the user. However, as we are dealing with one-shot devices, there is loss of information inherently present in the experiment and the data collection. For this reason, we need a large enough sample size, especially when what is required from the optimal design is stringent. Our algorithm provides a reasonable sample size under time and budget constraints to practitioners for guidance when they design a life-test for one-shot devices.

TABLE 4 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $(a_0, (1+\varepsilon_2)a_1, (1+\varepsilon_3)b_0, (1+\varepsilon_4)b_1)$, with ε_i being being the departure from the true value of the parameter θ_i, $T_{ter} = 60$ and $C_{budget} = \$500,000$, for gamma lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;330	20;20	20;20	30	1	0
0	0.05	0	0	20;330	20;20	20;20	30	1.0316	0.0739
0	-0.05	0	0	20;330	20;20	20;20	30	0.9733	-0.0770
0	0	0.05	0	20;330	20;20	20;20	30	1.1509	0.3096
0	0.05	0.05	0	20;330	20;20	20;20	30	1.2271	0.3725
0	-0.05	0.05	0	20;330	20;20	20;20	30	1.0832	0.2419
0	0	-0.05	0	20;330	20;20	20;20	30	1.0128	-0.2938
0	0.05	-0.05	0	20;330	20;20	20;20	30	1.0119	-0.2180
0	-0.05	-0.05	0	20;330	20;20	20;20	30	1.0179	-0.3702
0	0	0	0.05	21;329	20;20	20;20	30	0.9556	-0.0984
0	0.05	0	0.05	20;330	20;20	20;20	30	0.9814	-0.0229
0	-0.05	0	0.05	23;327	20;20	20;20	30	0.9339	-0.1763
0	0	0.05	0.05	20;330	20;20	20;20	30	1.0481	0.2104
0	0.05	0.05	0.05	20;330	20;20	20;20	30	1.1099	0.2778
0	-0.05	0.05	0.05	20;330	20;20	20;20	30	0.9930	0.1386
0	0	-0.05	0.05	21;329	20;20	20;20	30	1.0058	-0.3800
0	0.05	-0.05	0.05	20;330	20;20	20;20	30	1.0016	-0.3054
0	-0.05	-0.05	0.05	25;325	20;20	20;20	30	1.0119	-0.4544
0	0	0	-0.05	20;330	20;20	20;20	30	1.0543	0.1002
0	0.05	0	-0.05	20;330	20;20	20;20	30	1.0947	0.1716
0	-0.05	0	-0.05	20;330	20;20	20;20	30	1.0196	0.0251
0	0	0.05	-0.05	20;330	20;20	20;20	30	1.2806	0.4059
0	0.05	0.05	-0.05	20;330	20;20	20;20	30	1.3759	0.4636
0	-0.05	0.05	-0.05	20;330	20;20	20;20	30	1.1962	0.3432
0	0	-0.05	-0.05	20;330	20;20	20;20	30	1.0234	-0.2028
0	0.05	-0.05	-0.05	20;327	20;23	20;20	30	1.0233	-0.1266
0	-0.05	-0.05	-0.05	20;330	20;20	20;20	30	1.0234	-0.2804

4.2 | Sensitivity analysis over parameter misspecification

Because the MLEs of the model parameters $(\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1)$ are likely to depart from the true model parameters (a_0, a_1, b_0, b_1), we now assume that the values used to determine optimal plans have moderate errors of the form $((1+\varepsilon_1)a_0, (1+\varepsilon_2)a_1, (1+\varepsilon_3)b_0, (1+\varepsilon_4)b_1)$, where $\varepsilon_i \in \{-0.05, 0, 0.05\}$, thus allowing for under-specification as well as over-specification ($\pm 5\%$) from the true values of the model parameters.

Consider a total budget of \$500,000 and a termination time of 60 months. Then, the optimal designs with different combinations of errors are determined under the same setup as in the previous subsection under the assumption of gamma and BS lifetime distributions, and these are presented in Tables 4–6 for gamma lifetime distribution and Tables 7–9 for BS lifetime distribution. The variance efficiency (VE) and the relative bias (RB) on the estimate of the reliability at the mission time with θ and $\hat{\theta}$, given by

$$VE = \frac{V_R(\xi_\theta)}{V_R(\xi_{\hat{\theta}})}, \tag{15}$$

$$RB = \frac{\hat{R}_\theta(t_0; \boldsymbol{x}_0) - \hat{R}_{\hat{\theta}}(t_0; \boldsymbol{x}_0)}{\hat{R}_\theta(t_0; \boldsymbol{x}_0)}, \tag{16}$$

TABLE 5 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $((1+0.05)a_0, (1+\varepsilon_2)a_1, (1+\varepsilon_3)b_0, (1+\varepsilon_4)b_1)$, with ε_i being being the departure from the true value of the parameter θ_i, $T_{ter}=60$ and $C_{budget}=\$500,000$, for gamma lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;330	20;20	20;20	30	1	0
0.05	0	0	0	20;330	20;20	20;20	30	0.9947	-0.0244
0.05	0.05	0	0	20;330	20;20	20;20	30	1.0239	0.0505
0.05	-0.05	0	0	20;330	20;20	20;20	30	0.9705	-0.1023
0.05	0	0.05	0	20;330	20;20	20;20	30	1.1317	0.2884
0.05	0.05	0.05	0	20;330	20;20	20;20	30	1.2045	0.3528
0.05	-0.05	0.05	0	20;330	20;20	20;20	30	1.0671	0.2191
0.05	0	-0.05	0	20;330	20;20	20;20	30	1.0189	-0.3183
0.05	0.05	-0.05	0	20;330	20;20	20;20	30	1.0154	-0.2423
0.05	-0.05	-0.05	0	21;329	20;20	20;20	30	1.0252	-0.3948
0.05	0	0	0.05	20;330	20;20	20;20	30	0.9549	-0.1232
0.05	0.05	0	0.05	20;330	20;20	20;20	30	0.9771	-0.0469
0.05	-0.05	0	0.05	24;326	20;20	20;20	30	0.9332	-0.2017
0.05	0	0.05	0.05	20;330	20;20	20;20	30	1.0339	0.1878
0.05	0.05	0.05	0.05	20;330	20;20	20;20	30	1.0928	0.2566
0.05	-0.05	0.05	0.05	20;330	20;20	20;20	30	0.9815	0.1146
0.05	0	-0.05	0.05	21;329	20;20	20;20	30	1.0150	-0.4039
0.05	0.05	-0.05	0.05	20;330	20;20	20;20	30	1.0080	-0.3293
0.05	-0.05	-0.05	0.05	20;323	27;20	20;20	30	1.0161	-0.4781
0.05	0	0	-0.05	20;330	20;20	20;20	30	1.0453	0.0765
0.05	0.05	0	-0.05	20;330	20;20	20;20	30	1.0830	0.1491
0.05	-0.05	0	-0.05	20;330	20;20	20;20	30	1.0133	0.0004
0.05	0	0.05	-0.05	20;330	20;20	20;20	30	1.2552	0.3863
0.05	0.05	0.05	-0.05	20;330	20;20	20;20	30	1.3462	0.4456
0.05	-0.05	0.05	-0.05	20;330	20;20	20;20	30	1.1745	0.3220
0.05	0	-0.05	-0.05	20;330	20;20	20;20	30	1.0264	-0.2276
0.05	0.05	-0.05	-0.05	20;330	20;20	20;20	30	1.0282	-0.1509
0.05	-0.05	-0.05	-0.05	20;330	20;20	20;20	30	1.0292	-0.3055

were also computed, and these are presented in the last two columns of these tables. RB is a measure of bias, which will be positive if the estimator underestimates and negative if the estimator overestimates $\widehat{R}_\theta(t_0;\boldsymbol{x}_0)$.[35] The VE is a measure of efficiency, which allows us to compare one design with another for the same situation.[36] Here, we use it to measure the loss of efficiency of the misspecified design. It is seen that, within these moderate errors of the parameters (a_0, a_1, b_0, b_1), the designs of optimal CSALTs under gamma lifetime distribution are quite robust. This also happens when considering Weibull or lognormal lifetimes.[10,17] However, it seems that the designs of optimal CSALTs under BS lifetime distribution are less robust.

5 | MODEL MISSPECIFICATION ANALYSIS

In this section, we examine the effect of model misspecification between the different lifetime distributions in the design of optimal CSALTs. Two different sets of parameters, or scenarios, are considered for each misspecified model (see Table 10). These parameters are so chosen that the reliability, mean, and variance of lifetime under normal operating conditions are similar to those under the true distribution whichever it is. As can be seen in Figure 1, statistics of lognormal and BS distributions have a very similar shape, but different than those of gamma and Weibull distributions. Note that when measuring the misspecification effect, we consider, for each of the distributions, the first scenario as the true model.

TABLE 6 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $((1-0.05)a_0, (1+\varepsilon_2)a_1, (1+\varepsilon_3)b_0, (1+\varepsilon_4)b_1)$, with ε_i being the departure from the true value of the parameter θ_i, $T_{ter} = 60$ and $C_{budget} = \$500,000$, for gamma lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;330	20;20	20;20	30	1	0
-0.05	0	0	0	20;330	20;20	20;20	30	1.0062	0.0241
-0.05	0.05	0	0	20;330	20;20	20;20	30	1.0403	0.0970
-0.05	-0.05	0	0	20;330	20;20	20;20	30	0.9770	-0.0519
-0.05	0	0.05	0	20;330	20;20	20;20	30	1.1712	0.3304
-0.05	0.05	0.05	0	20;330	20;20	20;20	30	1.2509	0.3918
-0.05	-0.05	0.05	0	20;330	20;20	20;20	30	1.1002	0.2642
-0.05	0	-0.05	0	20;330	20;20	20;20	30	1.0077	-0.2693
-0.05	0.05	-0.05	0	20;330	20;20	20;20	30	1.0093	-0.1939
-0.05	-0.05	-0.05	0	20;330	20;20	20;20	30	1.0102	-0.3456
-0.05	0	0	0.05	20;330	20;20	20;20	30	0.9598	-0.0739
-0.05	0.05	0	0.05	20;330	20;20	20;20	30	0.9866	0.0008
-0.05	-0.05	0	0.05	24;326	20;20	20;20	30	0.9332	-0.1510
-0.05	0	0.05	0.05	20;330	20;20	20;20	30	1.0632	0.2325
-0.05	0.05	0.05	0.05	20;330	20;20	20;20	30	1.1279	0.2985
-0.05	-0.05	0.05	0.05	20;330	20;20	20;20	30	1.0054	0.1621
-0.05	0	-0.05	0.05	20;330	20;20	20;20	30	0.9992	-0.3560
-0.05	0.05	-0.05	0.05	20;330	20;20	20;20	30	0.9961	-0.2815
-0.05	-0.05	-0.05	0.05	23;327	20;20	20;20	30	1.0035	-0.4305
-0.05	0	0	-0.05	20;330	20;20	20;20	30	1.0643	0.1236
-0.05	0.05	0	-0.05	20;330	20;20	20;20	30	1.1074	0.1937
-0.05	-0.05	0	-0.05	20;330	20;20	20;20	30	1.0269	0.0496
-0.05	0	0.05	-0.05	20;330	20;20	20;20	30	1.3074	0.4250
-0.05	0.05	0.05	-0.05	20;330	20;20	20;20	30	1.4069	0.4811
-0.05	-0.05	0.05	-0.05	20;330	20;20	20;20	30	1.2191	0.3639
-0.05	0	-0.05	-0.05	20;330	20;20	20;20	30	1.0213	-0.1781
-0.05	0.05	-0.05	-0.05	20;330	20;20	20;20	30	1.0284	-0.1026
-0.05	-0.05	-0.05	-0.05	20;330	20;20	20;20	30	1.0187	-0.2553

Optimal CSALTs under these misspecified scenarios are presented in Tables 11 and 12. The comparison with the optimal design under the right model is measured through the difference of total devices required, Δ_N, total cost, Δ_{TC}, and RMSE, Δ_{RMSE}. The loss in efficiency (LEff), measured by

$$\text{LEff} = 100\left(1 - \frac{\text{RMSE}_{\text{TM}}}{\text{RMSE}_{\text{WM}}}\right)\%,$$

where RMSE_{TM} is the RMSE under the true model and RMSE_{WM} is the RMSE under the wrong model, quantifies the loss in efficiency due to the misspecification.

Looking at Tables 11–14, we observe that the loss of efficiency (LEff) is generally small for all the considered lifetime distributions when the sample size is small and the experimental time is sufficiently long. In particular, the Weibull lifetime distribution is the most robust distribution if it is wrongly treated as any other lifetime distribution. On the other hand, the misspecification effect in LEff is severe when the gamma lifetime distribution is wrongly treated as any other lifetime distribution and, in particular, the LEffs are over 20% when the underlying lifetime distribution is wrongly treated as the lognormal or BS lifetime distributions. When the lifetime distribution is unknown, we may need to specify a lifetime distribution that has little impact on the design, even though the assumed lifetime distribution is misspecified. Therefore, it

TABLE 7 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $((1-0.05)c_0, (1+\varepsilon_2)c_1, (1+\varepsilon_3)r_0, (1+\varepsilon_4)r_1)$, with ε_i being the departure from the true value of the parameter θ_i, $T_{ter} = 60$ and $C_{budget} = \$500,000$, for Birnbaum–Saunders lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;20;20;45;102;148	25;20	22;20	10	1	0
0	0.05	0	0	20;20;20;29;20;251	20;20	23;20	9	0.9875	0.0011
0	-0.05	0	0	20;20;20;20;20;93;165	20;20	26;20	8	0.9736	-0.0011
0	0	0.05	0	20;20;20;20;20;260	20;20	23;20	9	1.1481	0.2493
0	0.05	0.05	0	20;20;20;20;20;260	20;20	23;20	9	1.1616	0.2557
0	-0.05	0.05	0	20;20;20;25;20;254	20;20	24;20	9	1.1218	0.2430
0	0	-0.05	0	20;20;20;36;20;20;228	20;20	20;20	8	1.1441	-0.2701
0	0.05	-0.05	0	20;20;20;42;20;20;220	20;20	22;20	8	1.1335	-0.2752
0	-0.05	-0.05	0	20;20;20;20;55;20;206	20;20	23;20	8	1.1120	-0.4651
0	0	0	0.05	20;20;22;20;129;147	24;20	20;20	10	1.0335	-0.1998
0	0.05	0	0.05	20;20;20;51;71;178	22;20	20;20	10	1.0342	-0.2015
0	-0.05	0	0.05	20;20;22;20;118;160	22;20	20;20	10	1.0385	-0.1981
0	0	0.05	0.05	20;20;20;20;102;180	20;20	20;20	10	1.1811	0.2739
0	0.05	0.05	0.05	20;20;20;20;98;184	20;20	20;20	10	1.1886	0.2786
0	-0.05	0.05	0.05	20;20;20;20;35;20;227	20;20	22;20	8	1.1406	0.2693
0	0	-0.05	0.05	20;20;20;34;20;20;225	20;20	25;20	8	1.1778	-0.4408
0	0.05	-0.05	0.05	20;20;20;23;20;20;23;22;85;112	20;20	20;20	6	1.1692	-0.4475
0	-0.05	-0.05	0.05	20;20;20;20;41;21;214	20;20	28;20	8	1.1531	-0.4342
0	0	0	-0.05	20;20;20;20;20;20;43;20;20;162	20;20	20;20	6	1.0604	0.1961
0	0.05	0	-0.05	20;20;20;20;36;59;189	20;20	20;20	8	1.0589	0.1998
0	-0.05	0	-0.05	20;20;20;20;20;20;243	20;20	21;20	8	1.0977	0.1924
0	0	0.05	-0.05	20;20;20;20;25;20;20;215	20;20	25;20	7	1.2182	0.5016
0	0.05	0.05	-0.05	20;20;20;20;20;20;244	20;20	20;20	8	1.2525	0.4088
0	-0.05	0.05	-0.05	20;20;20;33;20;247	20;20	23;20	9	1.2166	0.3144
0	0	-0.05	-0.05	20;20;20;20;20;47;20;188	20;20	30;20	7	0.9939	-0.2814
0	0.05	-0.05	-0.05	20;20;20;25;20;46;20;194	20;20	20;20	7	0.9982	-0.2842
0	-0.05	-0.05	-0.05	20;20;20;20;20;44;20;20;20;158	20;20	23;20	6	1.0088	-0.2786

would be better to design a trial experiment with a small sample under gamma lifetime distribution to collect data. We then use them to identify the lifetime distribution with detective test statistics elsewhere.[25]

6 | ILLUSTRATIVE EXAMPLE

Let us consider the G-MRF data set,[19,37] in which five samples were tested at four different inspection times and temperatures (see tab. 1 of[19]). Here, we develop optimal CSALTs under Weibull, gamma, lognormal, and BS distributions, at normal operating temperature of 293K and inspection time $t = 14,588$. We consider the lowest and highest temperatures to be 333K and 351K, respectively, and the other testing temperatures to be 339K and 345K. In a standard practice in ALTs, one usually works with standardized stress levels

$$x = \frac{s - s_0}{s_4 - s_0}, \quad s \in [s_0, s_4],$$

where s_0 is the normal operating stress level condition. In this case, we set $s_0 = 1/293$, $s_1 = 1/333$, $s_2 = 1/339$, $s_3 = 1/345$, and $s_4 = 1/351$, so that we have $x_0 = 0$ and $x_4 = 1$. Values of the model parameters used for the lifetime distributions are presented in Table 15. In Table 15, we also present the mean, reliability, and standard deviation for the settings under normal operating conditions: x_0 and $t_0 = 14,588$.

TABLE 8 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $((1-0.05)c_0, (1+\varepsilon_2)c_1, (1+\varepsilon_3)r_0, (1+\varepsilon_4)r_1)$, with ε_i being the departure from the true value of the parameter θ_i, $T_{ter} = 60$ and $C_{budget} = \$500,000$, for Birnbaum–Saunders lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;20;20;45;102;148	25;20	22;20	10	1	0
0.05	0	0	0	20;20;26;20;20;20;234	20;20	24;20	8	1.0250	-0.0028
0.05	0.05	0	0	20;20;20;24;20;20;20;216	20;20	25;20	7	1.0110	-0.0018
0.05	-0.05	0	0	20;20;20;20;20;23;20;20;28;172	20;20	22;20	6	1.0219	-0.0039
0.05	0	0.05	0	20;20;26;48;26;217	21;20	24;20	10	1.2233	0.3324
0.05	0.05	0.05	0	20;20;20;20;20;20;239	20;20	25;20	8	1.2624	0.3387
0.05	-0.05	0.05	0	20;20;20;20;20;47;215	20;20	22;20	8	1.2046	0.3262
0.05	0	-0.05	0	20;20;20;20;20;44;20;20;20;157	20;20	24;20	6	1.1170	-0.4567
0.05	0.05	-0.05	0	20;20;20;20;20;20;20;20;96;109	20;20	20;20	6	1.1485	-0.4616
0.05	-0.05	-0.05	0	20;20;23;26;130;139	22;20	22;20	10	1.1243	-0.4517
0.05	0	0	0.05	20;20;20;20;159;121	20;20	22;20	10	1.0213	-0.1953
0.05	0.05	0	0.05	20;20;20;20;33;25;217	20;20	29;20	8	1.0190	-0.1970
0.05	-0.05	0	0.05	20;20;21;20;20;20;235	20;20	28;20	8	1.0418	-0.1937
0.05	0	0.05	0.05	20;20;20;20;20;49;206	20;20	29;20	8	1.1285	0.2617
0.05	0.05	0.05	0.05	20;20;20;20;20;20;241	20;20	23;20	8	1.1712	0.2662
0.05	-0.05	0.05	0.05	20;20;20;76;31;191	20;20	24;20	10	1.1267	0.2572
0.05	0	-0.05	0.05	20;20;20;20;20;26;21;25;63;130	20;20	20;20	6	1.2395	-0.6232
0.05	0.05	-0.05	0.05	20;20;20;20;38;20;220	20;20	26;20	8	1.1668	-0.4297
0.05	-0.05	-0.05	0.05	20;20;20;20;20;20;20;219	20;20	26;20	7	1.1493	-0.3167
0.05	0	0	-0.05	20;20;20;20;20;20;20;20;20;183	20;20	22;20	6	1.0896	0.1863
0.05	0.05	0	-0.05	20;20;20;29;20;20;233	20;20	22;20	8	1.0843	0.1899
0.05	-0.05	0	-0.05	20;20;20;20;40;54;189	20;20	21;20	8	1.0442	0.1828
0.05	0	0.05	-0.05	20;20;20;20;20;262	20;20	21;20	9	1.3475	0.2823
0.05	0.05	0.05	-0.05	20;20;20;20;20;261	20;20	22;20	9	1.2011	0.1895
0.05	-0.05	0.05	-0.05	20;20;20;20;20;262	20;20	21;20	9	1.2239	0.2252
0.05	0	-0.05	-0.05	20;20;20;20;67;20;196	20;20	21;20	8	0.9839	-0.2740
0.05	0.05	-0.05	-0.05	20;20;20;20;25;20;78;158	20;20	24;20	7	0.9908	-0.2767
0.05	-0.05	-0.05	-0.05	20;20;24;20;20;72;185	20;20	23;20	8	1.0054	-0.2713

Considering a total budget of \$500, 000 and termination times of 200 and 400 h, optimal CSALTs under Weibull, gamma, lognormal, and BS distributions were obtained and these are presented in Table 16. We assume that the costs of operation at these elevated stress levels are \$100, \$150, \$200, and \$250 per unit of time, respectively, and the cost of each device is \$1100.

We can observe how lognormal produces larger standard errors and standard deviations when the termination time is not large; but, when the termination time is set to be large, then the results from all four lifetime distributions are quite close, with similar efficiency values.

7 | CONCLUDING REMARKS

In this paper, we have discussed the determination of optimal CSALT for one-shot devices for the devices by considering four prominent lifetime distributions. We have considered different scenarios, and evaluated its sensitivity against parameter misspecification. As expected, an increase in the budget and termination time results in experiments with better reliability prediction. It is also seen that, within moderate errors of the parameters, the designs of optimal CSALTs are quite robust. We have further studied the effect of model misspecification between these four lifetime distributions in the design of optimal CSALTs. Results suggest the assumption of lifetime distribution as Weibull or lognormal to be more

TABLE 9 Sensitivity analysis of optimal constant-stress accelerated life-tests (CSALTs) under various combinations of parameters $((1-0.05)c_0, (1+\varepsilon_2)c_1, (1+\varepsilon_3)r_0, (1+\varepsilon_4)r_1)$, with ε_i being the departure from the true value of the parameter θ_i, $T_{ter}=60$ and $C_{budget}=\$500,000$, for Birnbaum–Saunders lifetime distribution

ε_1	ε_2	ε_3	ε_4	N_{1k}	N_{2k}	N_{3k}	f	$VE(R)$	$RB(R)$
0	0	0	0	20;20;20;45;102;148	25;20	22;20	10	1	0
-0.05	0	0	0	20;20;20;20;32;20;230	20;20	22;20	8	1.0222	0.0030
-0.05	0.05	0	0	20;20;20;20;44;20;216	20;20	24;20	8	1.0020	0.0041
-0.05	-0.05	0	0	20;20;20;20;20;57;202	20;20	25;20	8	1.0075	0.0018
-0.05	0	0.05	0	20;20;20;20;20;260	20;20	23;20	9	1.2755	0.3666
-0.05	0.05	0.05	0	20;20;20;20;20;260	20;20	23;20	9	1.1900	0.4731
-0.05	-0.05	0.05	0	20;20;20;20;20;20;20;223	20;20	22;20	7	1.2617	0.3601
-0.05	0	-0.05	0	20;20;20;42;20;20;219	20;20	23;20	8	1.1348	-0.4840
-0.05	0.05	-0.05	0	20;20;20;23;20;20;20;222	20;20	20;20	7	1.1754	-0.4892
-0.05	-0.05	-0.05	0	20;20;20;43;20;20;217	20;20	24;20	8	1.1243	-0.4788
-0.05	0	0	0.05	20;20;20;20;20;20;237	20;20	27;20	8	1.0465	-0.2044
-0.05	0.05	0	0.05	20;20;20;20;20;20;240	20;20	24;20	8	1.0525	-0.2062
-0.05	-0.05	0	0.05	20;20;20;20;20;38;221	20;20	25;20	8	1.0309	-0.2027
-0.05	0	0.05	0.05	20;20;20;26;270	23;20	23;20	11	1.1542	0.2866
-0.05	0.05	0.05	0.05	20;20;20;20;20;259	20;20	24;20	9	1.1534	0.2914
-0.05	-0.05	0.05	0.05	20;20;20;20;20;20;20;221	20;20	24;20	7	1.1361	0.2818
-0.05	0	-0.05	0.05	20;20;20;20;50;20;209	20;20	25;20	8	1.2779	-0.3588
-0.05	0.05	-0.05	0.05	20;20;20;34;20;20;227	20;20	23;20	8	1.2269	-0.4655
-0.05	-0.05	-0.05	0.05	20;20;20;30;20;20;227	20;20	27;20	8	1.2057	-0.6520
-0.05	0	0	-0.05	20;20;20;20;47;233	21;20	22;20	9	1.0458	0.2061
-0.05	0.05	0	-0.05	20;20;23;20;20;258	21;20	21;20	9	1.0670	0.2100
-0.05	-0.05	0	-0.05	20;20;20;31;20;247	23;20	22;20	9	1.0444	0.2023
-0.05	0	0.05	-0.05	20;20;20;20;27;85;169	20;20	23;20	8	1.2803	0.3210
-0.05	0.05	0.05	-0.05	20;20;20;20;52;20;210	20;20	22;20	8	1.2689	0.3283
. -0.05	-0.05	0.05	-0.05	20;20;20;20;24;256	20;20	23;20	9	1.2674	0.3137
-0.05	0	-0.05	-0.05	20;20;20;20;43;20;20;199	20;20	23;20	7	0.9935	-0.2890
-0.05	0.05	-0.05	-0.05	20;20;20;20;20;39;20;203	20;20	23;20	7	1.0215	-0.2919
-0.05	-0.05	-0.05	-0.05	20;20;28;20;20;248	27;20	20;20	9	0.9898	-0.2861

TABLE 10 Model misspecification scenarios

	θ	$R(x_0=28, t_0=65;\theta)$	$E(x_0=28)$	$Var(x_0=28)$
Gamma. Scenario 1	(5.20,-0.06,-0.36,0.04)	0.4681	72.24	2,440.60
Gamma. Scenario 2	(5.79, -0.072,-0.95, 0.049)	0.4026	66.42	2,892.86
Weibull. Scenario 1	(5.75,-0.05,-0.36,0.025)	0.4578	70.58	2,592.11
Weibull. Scenario 2	(5, -0.021,-0.72, 0.04)	0.4958	74.47	2,582.27
Lognormal. Scenario 1	(6.9,-0.1,-0.6,0.005)	0.4531	73.64	2,655.50
Lognormal. Scenario 2	(7.8,-0.13,-1.1,0.02)	0.4901	75.93	2,331.33
BS. Scenario 1	(7.03, -0.104, -0.75, 0.01)	0.4640	73.43	2,194.31
BS. Scenario 2	(6.97, -0.102, -0.692, 0.007)	0.4604	72.54	2,032.17

robust to model misspecification among the four distributions, while the assumption of gamma lifetime distribution to be the most non-robust (or most sensitive) one. Therefore, our recommendation is to design a trial experiment with a small sample under gamma lifetime distribution to collect data, and then identify the lifetime distribution with an appropriate specification test. Finally, a numerical example involving a G-MRF data is used to illustrate the developed methods.

TABLE 11 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets when the gamma distribution (S1) is incorrectly treated as Weibull, lognormal, and Birnbaum–Saunders (BS) distributions, when the costs of operation at the three stress levels are $100, $150, and $200 per unit of time, respectively

Tter	Cbudget	N_{1k}	N_{2k}	N_{3k}	f	TC	$RMSE$	Δ_N	Δ_{TC}	Δ_{RMSE}	LEff
S1. Weibull											
36	$200, 000	20;20	20;63	24;20	18	$199, 900	0.2548	0	$0	0.0812	31.86%
36	$300, 000	20;20	20;129	49;20	18	$300, 000	0.2378	-3	$900	0.0894	37.60%
36	$500, 000	20;20	20;255	95;29	18	$499, 100	0.2365	18	-$400	0.1389	58.75%
60	$200, 000	20;57	20;20	20;20	30	$199, 700	0.0810	0	$0	0.0092	11.35%
60	$300, 000	20;148	20;20	20;20	30	$299, 800	0.0562	0	$0	0.0098	17.48%
60	$500, 000	20;20;316	20;20	20;20	20	$499, 600	0.0481	6	-$400	0.0160	33.30%
S2. Weibull											
36	$200, 000	37;50	20;20	20;20	18	$199, 900	0.2366	0	$0	0.0630	26.62%
36	$300, 000	65;20;96	20;20	20;20	12	$299, 100	0.1982	0	$0	0.0498	25.12%
36	$500, 000	130;20;169	20;20;20	20;20;20	12	$499, 100	0.1518	18	-$400	0.0543	35.74%
60	$200, 000	20;57	20;20	20;20	30	$199, 700	0.0713	0	$0	-0.0005	-0.67%
60	$300, 000	20;135	20;33	20;20	30	$299, 800	0.0539	0	$0	0.0075	13.96%
60	$500, 000	20;274	20;66	28;22	30	$500, 000	0.0401	0	$0	0.0080	19.93%
S1. Lognormal											
36	$200, 000	20;20;20	20;20;29	20;20	12	$199, 700	0.3100	2	-$200	0.1804	58.18%
36	$300, 000	20;20;45	20;20;71	44;20	12	$299, 800	0.3286	2	-$200	0.2226	67.75%
36	$500, 000	20;20;98	20;20;151	93;20	12	$500, 000	0.3363	3	$900	0.2633	78.28%
60	$200, 000	20;20;43	20;20 20;20	20	6	$199, 300	0.2514	6	-$400	0.1873	74.49%
60	$300, 000	20;20;27;111	20;20	20;20	14	$299, 200	0.3297	10	-$600	0.2867	86.95%
60	$500, 000	20;20;20;20;20;20;240	20;20	24;20	8	$499, 600	0.3951	8	$0	0.3666	92.8%
S2. Lognormal											
36	$200, 000	27;20;43	20;20	20;20	12	$199, 000	0.3361	3	-$900	0.2064	61.42%
36	$300, 000	33;20;89	20;20;36	22;20	12	$299, 800	0.3345	2	-$200	0.2285	68.32%
36	$500, 000	20;20;20;112	20;20;20;117	74;20	9	$499, 900	0.3005	4	$800	0.2274	75.69%
60	$200, 000	20;21;42	20;20	20;20	20	$199, 300	0.2856	6	-$400	0.2215	77.55%
60	$300, 000	20;20;20;20;20;82	20;20	20;20	9	$299, 900	0.3761	14	$100	0.3331	88.56%
60	$500, 000	20;20;23;30;20;234	36;20	20;20	9	$499, 000	0.4146	7	-$600	0.3861	93.14%
S1. BS											
36	$200, 000	20;20;20	20;20;29	20;20	12	$199, 700	0.3240	2	-$200	0.1944	59.98%
36	$300, 000	20;20;49	20;20;70	41;20	12	$299, 800	0.2314	2	-$200	0.1254	54.20%
36	$500, 000	20;20;106	20;20;151	85;20	12	$500, 000	0.1808	3	$900	0.1078	59.61%
60	$200, 000	20;20;43	20;20	20;20	20	$199, 300	0.084	6	-$400	0.0199	23.70%
60	$300, 000	20;20;20;118	20;20	20;20	14	$299, 200	0.0755	10	-$600	0.0325	43.00%
60	$500, 000	20;20;20;45;102;148	25;20	22;20	10	$499, 200	0.1417	6	-$400	0.1133	79.94%
S2. BS											
36	$200, 000	20;20;20	20;20;29	20;20	12	$199, 700	0.3240	2	-$200	0.1944	59.98%
36	$300, 000	20;20;47	20;20;70	43;20	12	$299, 800	0.2331	2	-$200	0.1271	54.54%
36	$500, 000	20;20;102	20;20;149	91;20	12	$500, 000	0.1861	3	$900	0.1131	60.76%
60	$200, 000	20;20;43	20;20	20;20	20	$199, 300	0.0805	6	-$400	0.0164	20.34%
60	$300, 000	20;20;20;118	20;20	20;20	14	$299, 200	0.0743	10	-$600	0.0313	42.09%
60	$500, 000	20;20;20;20;20;20;20;217	20;20	28;20	7	$500, 000	0.0544	9	$400	0.0260	47.76%

TABLE 12 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets when the Weibull distribution (S1) is incorrectly treated as gamma, lognormal, and Birnbaum–Saunders (BS) distributions, when the costs of operation at the three stress levels are \$100, \$150, and \$200 per unit of time, respectively

Tter	Cbudget	N_{1k}	N_{2k}	N_{3k}	f	TC	$RMSE$	Δ_N	Δ_{TC}	Δ_{RMSE}	LEff
S1. Gamma											
36	\$200,000	33;54	20;20	20;20	18	\$199,900	0.1105	-3	\$0	-0.0192	-17.39%
36	\$300,000	65;20;96	20;20	20;20	12	\$299,100	0.1007	0	-\$900	-0.0053	-5.27%
36	\$500,000	134;20;187	20;20	20;20;20	12	\$499,500	0.0707	-2	\$400	-0.0023	-3.25%
60	\$200,000	20;57	20;20	20;20	30	\$199,700	0.0656	-11	\$0	0.0015	2.26%
60	\$300,000	20;148	20;20	20;20	30	\$299,800	0.0453	-11	\$0	0.0023	5.08%
60	\$500,000	20;330	20;20	20;20	30	\$500,000	0.0336	-11	\$400	0.0051	15.26%
S2. Gamma											
36	\$200,000	31;56	20;20	20;20	18	\$199,900	0.1203	0	\$0	-0.0094	-7.79%
36	\$300,000	58;20;103	20;20	20;20	12	\$299,100	0.1118	3	-\$900	0.0059	5.26%
36	\$500,000	107;20;20;196	20;20	20;20;20	9	\$499,000	0.0728	4	-\$100	-0.0002	-0.29%
60	\$200,000	20;57	20;20	20;20	30	\$199,700	0.0658	0	\$0	0.0017	2.57%
60	\$300,000	20;148	20;20	20;20	30	\$299,800	0.0456	0	\$0	0.0026	5.63%
60	\$500,000	20;330	20;20	20;20	30	\$500,000	0.0339	-6	\$400	0.0055	16.24%
S1. Lognormal											
36	\$200,000	20;20;20	20;20;29	20;20	12	\$199,700	0.1488	2	-\$200	0.0191	12.85%
36	\$300,000	20;20;45	20;20;71	44;20	12	\$299,800	0.1356	2	-\$200	0.0296	21.86%
36	\$500,000	20;20;98	20;20;151	93;20	12	\$500,000	0.2125	3	\$900	0.1395	65.63%
60	\$200,000	20;20;43	20;20 20;20	20	6	\$199,300	0.0639	6	-\$400	0.0003	-0.41%
60	\$300,000	20;20;27;111	20;20	20;20	14	\$299,200	0.0749	10	-\$600	0.0319	42.56%
60	\$500,000	20;20;20;20;20;20;240	20;20	24;20	8	\$499,600	0.199	8	\$0	0.1705	85.71%
S2. Lognormal											
36	\$200,000	27;20;43	20;20	20;20	12	\$199,000	0.2529	3	-\$900	0.1233	48.74%
36	\$300,000	33;20;89	20;20;36	22;20	12	\$299,800	0.1637	2	-\$200	0.0577	35.26%
36	\$500,000	20;20;20;112	20;20;20;117	74;20	9	\$499,900	0.1459	4	\$800	0.0729	49.95%
60	\$200,000	20;21;42	20;20	20;20	20	\$199,300	0.0683	6	-\$400	0.0042	6.17%
60	\$300,000	20;20;20;20;20;82	20;20	20;20	9	\$299,900	0.2722	14	\$100	0.2292	84.19%
60	\$500,000	20;20;23;30;20;234	36;20	20;20	9	\$499,000	0.2758	7	-\$600	0.2474	89.69%
S1. BS											
36	\$200,000	20;20;20	20;20;29	20;20	12	\$199,700	0.1378	2	-\$200	0.0081	5.88%
36	\$300,000	20;20;49	20;20;70	41;20	12	\$299,800	0.1054	2	-\$200	-0.0006	-0.58%
36	\$500,000	20;20;106	20;20;151	85;20	12	\$500,000	0.0887	3	\$900	0.0156	17.64%
60	\$200,000	20;20;43	20;20	20;20	20	\$199,300	0.0634	6	-\$400	0.0007	-1.08%
60	\$300,000	20;20;20;118	20;20	20;20	14	\$299,200	0.0552	10	-\$600	0.0122	22.02%
60	\$500,000	20;20;20;45;102;148	25;20	22;20	10	\$499,200	0.0617	6	-\$400	0.0333	53.93%
S2. BS											
36	\$200,000	20;20;20	20;20;29	20;20	12	\$199,700	0.1340	2	-\$200	0.0043	3.23%
36	\$300,000	20;20;47	20;20;70	43;20	12	\$299,800	0.1028	2	-\$200	-0.0032	-3.08%
36	\$500,000	20;20;102	20;20;149	91;20	12	\$500,000	0.0823	3	\$900	0.0092	11.20%
60	\$200,000	20;20;43	20;20	20;20	20	\$199,300	0.0616	6	-\$400	-0.0025	-4.02%
60	\$300,000	20;20;20;118	20;20	20;20	14	\$299,200	0.0550	10	-\$600	0.0120	21.76%
60	\$500,000	20;20;20;20;20;20;20;217	20;20	28;20	7	\$500,000	0.0464	9	\$400	0.0180	38.72%

TABLE 13 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets when the lognormal distribution (S1) is incorrectly treated as Weibull, gamma, and Birnbaum–Saunders (BS) distributions, when the costs of operation at the three stress levels are \$100, \$150, and \$200 per unit of time, respectively

Tter	Cbudget	N_{1k}	N_{2k}	N_{3k}	f	TC	$RMSE$	Δ_N	Δ_{TC}	Δ_{RMSE}	LEff
S1. Gamma											
36	\$200, 000	33;54	20;20	20;20	18	\$199, 900	0.2054	-2	\$200	0.0699	34.03%
36	\$300, 000	65;20;96	20;20	20;20	12	\$299, 100	0.2373	1	-\$700	0.1401	59.04%
36	\$500, 000	134;20;187	20;20	20;20;20	12	\$499, 500	0.2352	-1	-\$500	0.1645	69.94%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0812	-6	\$400	0.0037	4.59%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0558	-10	\$600	0.0066	11.9%
60	\$500, 000	20;330	20;20	20;20	30	\$500, 000	0.0595	-14	\$400	0.0229	38.47%
S2. Gamma											
36	\$200, 000	31;56	20;20	20;20	18	\$199, 900	0.1814	-2	\$200	0.0459	25.32%
36	\$300, 000	58;20;103	20;20	20;20	12	\$299, 100	0.1913	1	-\$700	0.0941	49.18%
36	\$500, 000	107;20;20;196	20;20	20;20;20	9	\$499, 000	0.1724	1	-\$1000	0.1017	58.99%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0827	-6	\$400	0.0052	6.24%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0655	-10	\$600	0.0163	24.83%
60	\$500, 000	20;330	20;20	20;20	30	\$500, 000	0.0518	-14	\$400	0.0152	29.29%
S1. Weibull											
36	\$200, 000	20;20	20;63	24;20	18	\$199, 900	0.1912	-2	\$200	0.0557	29.14%
36	\$300, 000	20;20	20;129	49;20	18	\$300, 000	0.1581	-2	\$200	0.0609	38.53%
36	\$500, 000	20;20	20;255	95;29	18	\$499, 100	0.1686	-3	-\$900	0.0979	58.07%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0845	-6	\$400	0.0070	8.25%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0744	-10	\$600	0.0252	33.85%
60	\$500, 000	20;20;316	20;20	20;20	20	\$499, 600	0.0548	-8	\$0	0.0182	33.23%
S2. Weibull											
36	\$200, 000	37;50	20;20	20;20	18	\$199, 900	0.1924	-2	\$200	0.0569	29.59%
36	\$300, 000	65;20;96	20;20	20;20	12	\$299, 100	0.2327	1	-\$700	0.1355	58.23%
36	\$500, 000	130;20;169	20;20;20	20;20;20	12	\$499, 100	0.2001	-3	-\$900	0.1294	64.67%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0777	-6	\$400	0.0004	0.25%
60	\$300, 000	20;135	20;33	20;20	30	\$299, 800	0.0598	-10	\$600	0.0106	17.68%
60	\$500, 000	20;274	20;66	28;22	30	\$500, 000	0.0538	-14	\$400	0.0172	31.95%
S1. BS											
36	\$200, 000	20;20;20	20;20;29	20;20	12	\$199, 700	0.1181	-2	\$0	-0.0116	-9.79%
36	\$300, 000	20;20;49	20;20;70	41;20	12	\$299, 800	0.0835	-2	\$0	-0.0225	-26.89%
36	\$500, 000	20;20;106	20;20;151	85;20	12	\$500, 000	0.0564	-3	\$0	-0.0166	-29.40%
60	\$200, 000	20;20;43	20;20	20;20	20	\$199, 300	0.0727	-6	\$0	0.0086	11.79%
60	\$300, 000	20;20;20;118	20;20	20;20	14	\$299, 200	0.0864	-10	\$0	0.0433	50.18%
60	\$500, 000	20;20;20;45;102;148	25;20	22;20	10	\$499, 200	0.0469	-8	-\$400	0.0185	39.38%
S2. BS											
36	\$200, 000	20;20;20	20;20;29	20;20	12	\$199, 700	0.1230	2	\$0	-0.0066	-5.40%
36	\$300, 000	20;20;47	20;20;70	43;20	12	\$299, 800	0.0808	2	\$0	-0.0252	-31.21%
36	\$500, 000	20;20;102	20;20;149	91;20	12	\$500, 000	0.0587	3	\$0	-0.0144	-24.47%
60	\$200, 000	20;20;43	20;20	20;20	20	\$199, 300	0.0726	6	\$0	0.0085	11.70%
60	\$300, 000	20;20;20;118	20;20	20;20	14	\$299, 200	0.0865	10	\$0	0.0434	50.23%
60	\$500, 000	20;20;20;20;20;20;20;217	20;20	28;20	7	\$500, 000	0.0469	9	\$400	0.0185	39.38%

TABLE 14 Optimal constant-stress accelerated life-tests (CSALTs) with different termination times and under different budgets when the Birnbaum–Saunders (BS) distribution (S1) is incorrectly treated as Weibull, gamma, and lognormal distributions, when the costs of operation at the three stress levels are \$100, \$150, and \$200 per unit of time, respectively

Tter	Cbudget	N_{1k}	N_{2k}	N_{3k}	f	TC	$RMSE$	Δ_N	Δ_{TC}	Δ_{RMSE}	LEff
S1. Gamma											
36	\$200, 000	33;54	20;20	20;20	18	\$199, 900	0.2492	-2	\$200	0.1128	45.26%
36	\$300, 000	65;20;96	20;20	20;20	12	\$299, 100	0.2783	1	-\$700	0.184	66.11%
36	\$500, 000	134;20;187	20;20	20;20;20	12	\$499, 500	0.2733	-1	-\$500	0.2069	75.71%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.1096	-6	\$400	0.0291	26.53%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0945	-10	\$600	0.0426	45.09%
60	\$500, 000	20;330	20;20	20;20	30	\$500, 000	0.0789	-14	\$800	0.0437	55.39%
S2. Gamma											
36	\$200, 000	31;56	20;20	20;20	18	\$199, 900	0.2047	-2	\$200	0.0683	33.36%
36	\$300, 000	58;20;103	20;20	20;20	12	\$299, 100	0.252	1	-\$700	0.1577	62.58%
36	\$500, 000	107;20;20;196	20;20	20;20;20	9	\$499, 000	0.2299	1	-\$1000	0.1635	71.12%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0943	-6	\$400	0.0138	14.59%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0774	-10	\$600	0.0255	32.98%
60	\$500, 000	20;330	20;20	20;20	30	\$500, 000	0.0734	-14	\$800	0.0382	52.03%
S1. Weibull											
36	\$200, 000	20;20	20;63	24;20	18	\$199, 900	0.1764	-2	\$200	0.04	22.67%
36	\$300, 000	20;20	20;129	49;20	18	\$300, 000	0.1579	-2	\$200	0.0636	40.29%
36	\$500, 000	20;20	20;255	95;29	18	\$499, 100	0.1482	-3	-\$900	0.0818	55.21%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0914	-6	\$400	0.0109	11.91%
60	\$300, 000	20;148	20;20	20;20	30	\$299, 800	0.0653	-10	\$600	0.0134	20.52%
60	\$500, 000	20;20;316	20;20	20;20	20	\$499, 600	0.0535	-8	\$400	0.0183	34.20%
S2. Weibull											
36	\$200, 000	37;50	20;20	20;20	18	\$199, 900	0.1974	-2	\$200	0.061	30.91%
36	\$300, 000	65;20;96	20;20	20;20	12	\$299, 100	0.294	1	-\$700	0.1997	67.93%
36	\$500, 000	130;20;169	20;20;20	20;20;20	12	\$499, 100	0.228	-3	-\$900	0.1616	70.87%
60	\$200, 000	20;57	20;20	20;20	30	\$199, 700	0.0812	-6	\$400	0.0007	0.84%
60	\$300, 000	20;135	20;33	20;20	30	\$299, 800	0.073	-10	\$600	0.0211	28.88%
60	\$500, 000	20;274	20;66	28;22	30	\$500, 000	0.0813	-14	\$800	0.0461	56.68%
S1. Lognormal											
36	\$200, 000	20;20;20	20;20;29	20;20	12	\$199, 700	0.1423	2	\$0	0.0059	4.18%
36	\$300, 000	20;20;45	20;20;71	44;20	12	\$299, 800	0.1008	2	\$0	0.0065	6.48%
36	\$500, 000	20;20;98	20;20;151	93;20	12	\$500, 000	0.0857	3	\$0	0.0193	22.54%
60	\$200, 000	20;20;43	20;20	20;20	20	\$199, 300	0.0854	6	\$0	0.0049	5.70%
60	\$300, 000	20;20;27;111	20;20	20;20	14	\$299, 200	0.0679	10	\$0	0.016	23.52%
60	\$500, 000	20;20;20;20;20;20;240	20;20	24;20	8	\$499, 600	0.0549	8	\$400	0.0197	35.92%
S2. Lognormal											
36	\$200, 000	27;20;43	20;20	20;20	12	\$199, 000	0.1693	3	-\$700	0.0329	19.42%
36	\$300, 000	33;20;89	20;20;36	22;20	12	\$299, 800	0.117	2	\$0	0.0227	19.38%
36	\$500, 000	20;20;20;112	20;20;20;117	74;20	9	\$499, 900	0.0798	4	-\$100	0.0134	16.75%
60	\$200, 000	20;21;42	20;20	20;20	20	\$199, 300	0.0839	6	\$0	0.0034	4.01%
60	\$300, 000	20;20;20;20;20;82	20;20	20;20	9	\$299, 900	0.0567	14	\$700	0.0048	8.39%
60	\$500, 000	20;20;23;30;20;234	36;20	20;20	9	\$499, 000	0.0383	7	-\$200	0.0031	8.14%

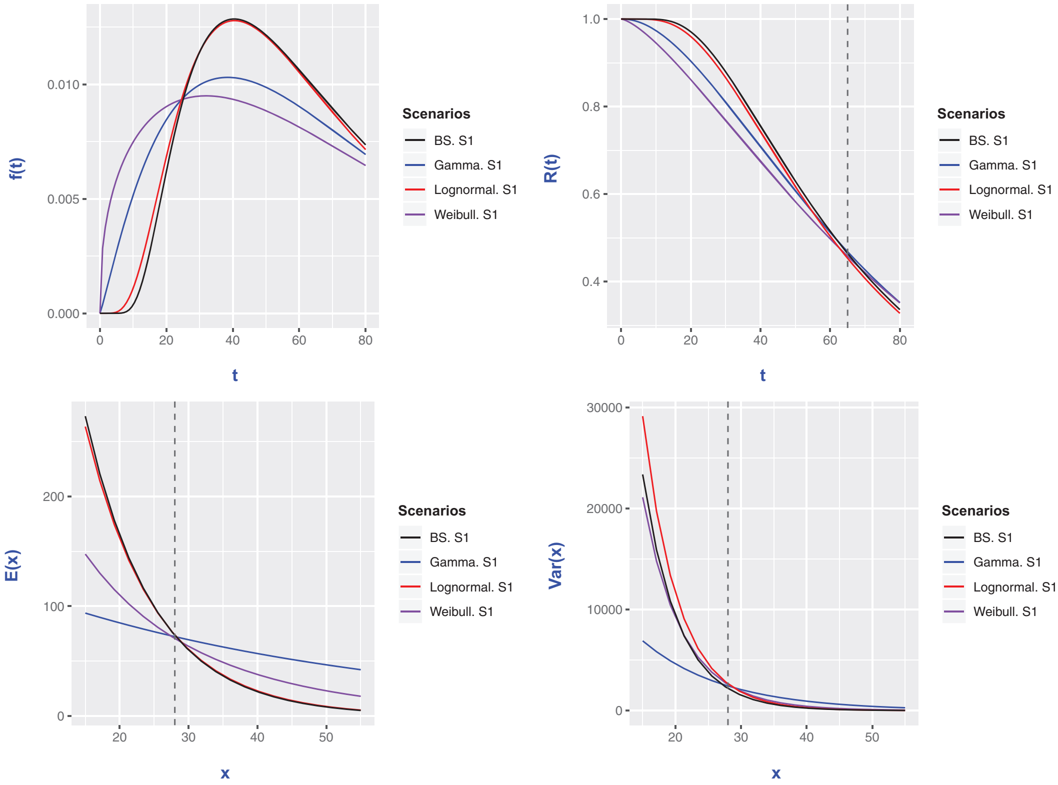

FIGURE 1 Comparison of the fitted gamma, Weibull, lognormal, and Birnbaum–Saunders (BS) models

TABLE 15 Illustrative example. Values of the model parameters used for gamma, Weibull, lognormal, and Birnbaum–Saunders (BS) lifetime distributions and mean, reliability and standard deviation for normal operating conditions ($t_0 = 14,588$, $x_0 = 0$)

	θ	$R(x_0, t_0; \theta)$	$E(x_0)$	$sd(x_0)$
Weibull distribution	(9.83,-5.7, 0.833,0)	0.5638	16,462.93	7,590.17
Gamma distribution	(8.8,-5.8, 1,0)	0.5520	18,033.74	10,938.02
Lognormal distribution	(9.6,-5.5,-0.55,0.005)	0.5083	17,438.50	10,959.47
BS distribution	(9.63, -6, -0.52, 0.005)	0.5282	17,903.24	10,861.20

TABLE 16 Optimal constant-stress-accelerated life-tests (CSALTs) for the illustrative example

Model	T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	N_{4k}	T_{ik}	TC	se	Std
Weibull	200	\$500, 000	20;20;20;76	20;20;38	20;20	75;20	49	\$499, 000	0.252	0.253
Gamma	200	\$500, 000	20;20;20;20;133	20;20	20;20	79;20	40	\$499, 200	0.215	0.222
Lognormal	200	\$500, 000	20;20;45	20;20;81	20;20	89;20	66	\$499, 400	0.291	0.304
BS	200	\$500, 000	20; 20; 20; 20; 20; 120	20;20	20;20	82;20	32	\$499, 800	0.200	0.212
Weibull	400	\$500, 000	20;20;22;21;20;105	20;20	20;20	77;20	42	\$499, 100	0.206	0.240
Gamma	400	\$500, 000	20;20;20;20;20;78;20	20;20	20;20	84;20	42	\$500, 000	0.194	0.237
Lognormal	400	\$500, 000	20;20;20;20;110	20;20	20;20	78;20	56	\$500, 000	0.207	0.251
BS	200	\$500, 000	20; 20; 20; 20; 50; 43; 31	20;20	20;20	90;20	35	\$499, 900	0.195	0.206

Misspecification effect might be theoretically studied using White's results on Quasi-MLE (see White[38]). In this pioneering work, Kullback–Leibler measure is used to measure the asymptotic bias and MSE under misspecification when the probability distribution is not correctly specified. This idea has been used effectively by Tsai et al.[22] to analyze the misspecification effect between gamma and Wiener degradation processes. We plan to carry out a similar theoretical study evaluating the robustness and misspecification effects in our future work. Of course, this will involve many scenarios to be considered, as we need to evaluate the asymptotic bias and MSE using Kullback–Leibler measure, and to do it in two ways for each pair of models considered. On the other hand, rather than fixing a budget and then looking at the LEff due to model misspecification (as done in Section 5), we may consider fixing a maximum value for the LEff and then measure the impact on the budget necessary to carry out the experiment. This loss would be evaluated in terms of cost of conducting the experiment. This is an interesting idea that would be worth studying in a future work.

Finally, it will naturally be of interest to investigate SSALTs, which include CSALTs as a special case, for one-shot device testing, and evaluate them in terms of efficiency, cost, and time. In addition, stress levels are assumed to be fixed in this study. Indeed, it will also be of interest if stress levels are optimized simultaneously as well. We are currently looking into these problems and hope to report the findings in future.

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APPENDIX A

Let us assume that the lifetimes follow a lognormal distribution. From Equation (10), we obtain the first-order derivatives of the reliability function, with respect to parameters p_j and q_j, as follows:

$$\begin{aligned}\frac{\partial R_V(\omega;S_i)}{\partial p_j} &= \frac{\partial}{\partial p_j}\left[1-\Phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)\right] = -\phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)\frac{\partial}{\partial p_j}\left[\frac{\omega-\mu_i}{\sigma_i}\right]\\ &= \frac{1}{\sigma_i}\phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)x_{ij},\\ \frac{\partial R_V(\omega;S_i)}{\partial q_j} &= \frac{\partial}{\partial q_j}\left[1-\Phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)\right] = -\phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)\frac{\partial}{\partial q_j}\left[\frac{\omega-\mu_i}{\sigma_i}\right]\\ &= \frac{\omega-\mu_i}{\sigma_i}\phi\left(\frac{\omega-\mu_i}{\sigma_i}\right)x_{ij},\end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the density and distribution functions of standard normal distribution, respectively.

Next, let us assume that the lifetimes follow a BS distribution. From Equation (13),

$$R_{BS}(t;S_i) = 1-\Phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2}-\left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right).$$

For convenience, we will denote

$$\delta_i = \left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2}-\left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right).$$

Then, applying the chain rule, we obtain

$$\frac{\partial R_{BS}(t;S_i)}{\partial c_j} = -\frac{\partial \delta_i}{\partial c_j}\phi(\delta_i) \quad \text{and} \quad \frac{\partial R_{BS}(t;S_i)}{\partial r_j} = -\frac{\partial \delta_i}{\partial r_j}\phi(\delta_i).$$

Now,

$$\begin{aligned}\frac{\partial \delta_i}{\partial c_j} &= \frac{1}{\gamma_i}\left\{-\frac{1}{2}\left(\frac{t^{1/2}}{\nu_i^{3/2}}\frac{\partial \nu_i}{\partial c_j}\right)-\frac{1}{2}\left(\frac{t^{-1/2}}{\nu_i^{1/2}}\frac{\partial \nu_i}{\partial c_j}\right)\right\}\\ &= \frac{1}{\gamma_i}\left\{-\frac{1}{2}\left(\frac{t}{\nu_i}\right)^{1/2}x_{ij}-\frac{1}{2}\left(\frac{\nu_i}{t}\right)^{1/2}x_{ij}\right\} = \frac{-1}{2\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2}+\left(\frac{\nu_i}{t}\right)^{1/2}\right\}x_{ij},\\ \frac{\partial \delta_i}{\partial r_j} &= \frac{-1}{\gamma_i^2}\left\{\left(\frac{t}{\nu_i}\right)^{1/2}-\left(\frac{\nu_i}{t}\right)^{1/2}\right\}\frac{\partial \gamma_i}{\partial r_j} = \frac{-1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2}-\left(\frac{\nu_i}{t}.\right)^{1/2}\right\}x_{ij}.\end{aligned}$$

Thus, we obtain the desired first-order derivatives as

$$\frac{\partial R_{BS}(t;S_i)}{\partial c_j} = \frac{1}{2\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} + \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} - \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right)x_{ij},$$

$$\frac{\partial R_{BS}(t;S_i)}{\partial r_j} = \frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} - \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\phi\left(\frac{1}{\gamma_i}\left\{\left(\frac{t}{\nu_i}\right)^{1/2} - \left(\frac{\nu_i}{t}\right)^{1/2}\right\}\right)x_{ij}.$$