Yang-Mills model for centrally extended 2D gravity

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A Yang-Mills theory linear in the scalar curvature for two-dimensional gravity with symmetry generated by the semidirect product formed with the Lie derivative of the algebra of diffeomorphisms with the two-dimensional Abelian algebra is formulated. As compared with dilaton models, the role of the dilaton is played by the dual field strength of a $U(1)$ gauge field. All vacuum solutions are found. They are either black holes or have constant scalar curvature. Those with constant scalar curvature have constant dual field strength. In particular, solutions with vanishing cosmological constant but nonzero scalar curvature exist. In the conformal-Lorenz gauge, the model has a conformal field theory interpretation whose residual symmetry combines holomorphic diffeomorphisms with a subclass of $U(1)$ gauge transformations while preserving two-dimensional de Sitter and anti-de Sitter boundary conditions. This is the same symmetry as in Jackiw-Teitelboim-Maxwell gravity considered by Hartman and Strominger. It is argued that this is the only nontrivial Yang-Mills model linear in the scalar curvature that exists for real Lie algebras of dimension four.

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I. INTRODUCTION

Two-dimensional (2D) dilaton gravity models provide effective theories to study regimes of interest in higher-dimensional gravity. Among them are Jackiw-Teitelboim (JT) gravity [1,2], with a linear coupling ϕR between the dilaton and the scalar curvature and which accounts for near-horizon theories in higher-dimensional near-extremal black holes; the Almheiri-Polchinski [3] models, with quadratic coupling $\phi^2 R$, that consistently explain the holographic flow to $\mathrm{AdS}_2 \times X$ of many theories; and the Callan-Giddings-Harvey-Strominger model [4], with exponential coupling $e^{-\phi}R$, that provides a 2D setting to analytically understand the formation and subsequent evaporation of a black hole.

Here, we propose a nondilaton model in which the role of the dilaton is played by the dual field strength $*F$ of an Abelian gauge field A_μ. The model has classical action

$$S = \frac{1}{2\kappa}\int d^2x\sqrt{|g|}\left(R * F - \frac{1}{4\ell^2}F^2 + \frac{\gamma}{\ell^2}\right), \qquad (1.1)$$

with $*F = \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The square F^2 stands for $F^{\mu\nu}F_{\mu\nu}$, ℓ is a characteristic length, κ and γ are dimensionless constants, and A_μ has dimensions of length. The term $R * F$ couples the scalar curvature to a $U(1)$ gauge field in an unusual fashion, with $*F$ a gravity source linear in the gauge field. This point of view can be turned around to regard $\epsilon^{\mu\nu}\partial_\nu R$ as a gauge current.

The idea that motivated this investigation was to formulate a 2D gravity model as a Yang-Mills theory whose classical action is linear in the Ricci scalar. In two dimensions, for a gauge symmetry generated by the 2D Poincaré algebra, the resulting Yang-Mills action is quadratic in the scalar curvature. However, as we discuss in Sec. II, for the centrally extended Poincaré algebra $\mathfrak{p}_1$, the Utiyama-Kibble-Sciama approach [5–7], modified along the lines of Refs. [8–10], leads to the action S above. The modification consists in no longer considering plane gauge transformations but a variant of them that can be understood as the semidirect product formed by the Lie derivative of diffeomorphisms with Abelian gauge transformations. This ensures that the zweibein postulate that maps the torsion and Riemann curvature to the gauge field strengths is valid for arbitrary torsion.

Coming back to the dilaton picture, one may think of S in the following terms. Consider models with Lagrangian density $\mathcal{L} = \phi R + V(\phi)$, JT gravity corresponding to $V(\phi) = \gamma\phi/\ell^2$. The action S above is obtained by setting ϕ equal to $*F$ and taking $V(\phi) = \gamma/\ell^2 + \phi^2/2\ell^2$. This changes the field content, hence the model itself, but leads to S. From this point of view, including in $V(\phi)$ a linear

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term $\phi = *F$ contributes to the action with a total derivative that we ignore.

The occurrence of the term F^2 in the action (1.1) ensures that the model has black hole solutions similar to those in 2D dilaton gravities [11–13]. This is discussed in Sec. III, in which all vacuum solutions to the model are found. Besides black holes, we find spacetimes with constant scalar curvature $R = R_0/\ell^2$ and constant dual field strength $*F = F_0$, with R_0 and F_0 satisfying $F_0^2 + 2F_0R_0 - 2\gamma = 0$. For a given cosmological constant γ, both dS_2 and AdS_2 are possible. Having one or the other depends on the value of F_0. This scenario occurs even for zero cosmological constant, $\gamma = 0$, in which case $R_0 = -F_0/2$. If the term F^2 in the action S is removed, the classical theory still makes sense, but then only vacuum solutions with constant scalar curvature exist, R_0 and F_0 being related through $R_0F_0 = \gamma$.

We wish to study the model (1.1) in relation with other 2D gravity-Maxwell models in the literature. A particularly interesting one has been considered by Hartman and Strominger [14], who have added to the JT Lagrangian a term $-F^2/4$. This results in a JT-Maxwell model that has an AdS_2 vacuum solution for constant $*F = E$. After fixing the conformal gauge for the metric, the model has a conformal field theory (CFT) interpretation, with a residual symmetry that combines conformal diffeomorphisms and gauge transformations and that is generated by a Witt algebra. If matter is included so that the AdS_2 background is preserved at the boundary and if, upon quantization, the $U(1)$ matter current becomes anomalous, the Witt algebra becomes a Virasoro algebra with nonzero central charge. The model (1.1) shares the same symmetry. Hence, we expect it to also allow for a central charge. This is shown in Sec. IV.

We close by arguing in Sec. V that the action S is unique in the sense that it is the only Yang-Mills action linear in the scalar curvature that can be written for symmetries generated by semidirect products obtained from four-dimensional (4D) real Lie algebras.

II. CLASSICAL ACTION AND ITS SYMMETRIES

A. Local symmetry

The starting point in our analysis is the central extension $\mathfrak{p}_1$ of the Poincaré algebra in two spacetime dimensions, spanned by the generators P_0 and P_1 of translations, the generator $J \coloneqq M_{01}$ of boosts, and a central element Q, with Lie bracket

$$[P_0, P_1] = Q, \qquad [P_0, J] = P_1, \qquad [P_1, J] = P_0, \qquad [Q, P_a] = [Q, J] = 0. \tag{2.1}$$

Consider a Lie algebra valued one-form,

$$B_\mu = e^a{}_\mu P_a + \omega_\mu J + A_\mu Q, \quad a = 0, 1, \tag{2.2}$$

whose components are the zweibein $e^a{}_\mu(x)$, the spin connection $\omega_\mu(x)$, and a one-form $A_\mu(x)$. If we assign dimensions of $(\text{length})^{-1}$ to P_0 and P_1, then J is dimensionless, and Q has dimensions of $(\text{length})^2$. Taking $e^a{}_\mu$ to be dimensionless, ω_μ and A_μ carry respectively dimensions of $(\text{length})^{-1}$ and length. The corresponding two-form field strength

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \\ &\eqqcolon T^a{}_{\mu\nu} P_a + \Omega_{\mu\nu} J + Z_{\mu\nu} Q \end{aligned} \tag{2.3}$$

has components

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu - \epsilon^a{}_b(\omega_\mu e^b{}_\nu - \omega_\nu e^b{}_\mu), \tag{2.4}$$

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu, \tag{2.5}$$

$$Z_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon_{ab} e^a{}_\mu e^b{}_\nu, \tag{2.6}$$

where we have used the conventions

$$\epsilon^0{}_1 = \epsilon^1{}_0 = 1, \quad \epsilon_{ab} = \eta_{ac}\epsilon^c{}_b, \quad \eta_{ab} = \mathrm{diag}(-1, +1). \tag{2.7}$$

We next follow Refs. [8,9] and, instead of conventional gauge transformations, consider local transformations of the form

$$\delta_{(\xi,\Sigma)} B_\mu = \mathcal{L}_\xi B_\mu + \partial_\mu \Sigma + [B_\mu, \Sigma]. \tag{2.8}$$

Here, $\mathcal{L}_\xi$ is the Lie derivative along an arbitrary vector field $\xi = \xi^\mu \partial_\mu$ generating the diffeomorphism $x^\mu \to x^\mu + \xi^\mu(x)$, and

$$\Sigma = \theta J + \tau Q \tag{2.9}$$

is a function that takes values in the Abelian subalgebra spanned by J and Q, with $\theta(x)$ and $\tau(x)$ arbitrary functions of dimensions 0 and $(\text{length})^2$. Altogether, there are four independent local parameters, the two components of ξ^μ and the two functions θ and τ. Under $\delta_{(\xi,\Sigma)}$, the field strength $G_{\mu\nu}$ transforms as

$$\delta_{(\xi,\Sigma)} G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} + [G_{\mu\nu}, \Sigma]. \tag{2.10}$$

The transformation $\delta_{(\xi,\Sigma)}$ is the combination

$$\delta_{(\xi,\Sigma)} = \mathcal{L}_\xi + \tilde{\delta}_\Sigma \tag{2.11}$$

of an arbitrary change of coordinates implemented by the Lie derivative $\mathcal{L}_\xi$ and a conventional gauge transformation generated by $\tilde{\delta}_\Sigma$.

The transformations $\delta_{(\xi,\Sigma)}$ close an algebra, with closure relation

$$[\mathcal{L}_{\xi_1}+\tilde{\delta}_{\Sigma_1},\mathcal{L}_{\xi_2}+\tilde{\delta}_{\Sigma_2}]=\mathcal{L}_{[\xi_1,\xi_2]}+\tilde{\delta}_{\mathcal{L}_{\xi_2}\Sigma_1-\mathcal{L}_{\xi_1}\Sigma_2}. \tag{2.12}$$

This Lie bracket can be described in mathematical terms as follows. Consider the Lie algebra $\mathcal{X}$ of vector fields on the spacetime manifold M. and its representation provided by the Lie derivative, so that every vector field ξ is realized as a Lie derivative $\mathcal{L}_\xi$. The vector space of all pairs $(\mathcal{L}_\xi,\Sigma)\coloneqq\mathcal{L}_\xi+\tilde{\delta}_\Sigma$ equipped with the bracket (2.12) is a Lie algebra. It is, in fact, the semidirect product $\mathcal{X}\ltimes\mathfrak{a}_2$ of $\mathcal{X}$ with the Abelian algebra $\mathfrak{a}_2=\mathrm{Span}\{J,Q\}$ formed with the Lie derivative. The transformation laws of the zweibein, spin connection, and central gauge field are

$$\delta_{(\xi,\Sigma)}e^a_\mu=\mathcal{L}_\xi e^a_\mu+\epsilon^a{}_b e^b_\mu\theta, \tag{2.13}$$

$$\delta_{(\xi,\Sigma)}\omega_\mu=\mathcal{L}_\xi\omega_\mu+\partial_\mu\theta, \tag{2.14}$$

$$\delta_{(\xi,\Sigma)}A_\mu=\mathcal{L}_\xi A_\mu+\partial_\mu\tau, \tag{2.15}$$

whereas those of the field strength components take the form

$$\delta_{(\xi,\Sigma)}T^a{}_{\mu\nu}=\mathcal{L}_\xi T^a{}_{\mu\nu}+\epsilon^a{}_b T^b{}_{\mu\nu}\theta, \tag{2.16}$$

$$\delta_{(\xi,\Sigma)}\Omega_{\mu\nu}=\mathcal{L}_\xi\Omega_{\mu\nu}, \tag{2.17}$$

$$\delta_{(\xi,\Sigma)}Z_{\mu\nu}=\mathcal{L}_\xi Z_{\mu\nu}. \tag{2.18}$$

We next map the spin connection ω_μ to an affine connection $\Gamma^\alpha{}_{\mu\nu}$ through the zweibein postulate

$$\mathcal{D}_\mu e^a{}_\nu\coloneqq\partial_\mu e^a{}_\nu-\Gamma^\alpha{}_{\mu\nu}e^a{}_\alpha-\epsilon^a{}_b\omega_\mu e^b{}_\nu=0. \tag{2.19}$$

The derivative $\mathcal{D}_\mu e^a{}_\nu$ defined by the left-hand side of this equation transforms under $\delta_{(\xi,\Sigma)}$ as

$$\delta_{(\xi,\Sigma)}(\mathcal{D}_\mu e^a{}_\nu)=\mathcal{L}_\xi(\mathcal{D}_\mu e^a{}_\nu)+\epsilon^a{}_b(\mathcal{D}_\mu e^b{}_\nu)\theta, \tag{2.20}$$

so that condition $\mathcal{D}_\mu e^a{}_\nu=0$ remains invariant. It is precisely invariance under $\delta_{(\xi,\Sigma)}$ that excludes terms in Eq. (2.19) of the form $c^a{}_b A_\mu e^a{}_\nu$ with nonzero coefficients $c^a{}_b$. Using the solution to Eq. (2.19) for $\Gamma^\alpha{}_{\mu\nu}$ in terms of ω_μ, the Riemann $R^\alpha{}_{\beta\mu\nu}$ and torsion $S^\alpha{}_{\mu\nu}$ tensors[1] are mapped to $\Omega_{\mu\nu}$ and $T^a{}_{\mu\nu}$ through

$$R^\alpha{}_{\beta\mu\nu}=-E^\alpha{}_a\epsilon^a{}_b e^b{}_\beta\Omega_{\mu\nu},\qquad S^\alpha{}_{\mu\nu}=E^\alpha{}_a T^a{}_{\mu\nu}. \tag{2.21}$$

Here, $E^\mu{}_a$ is the inverse zweibein, defined by $E^\mu{}_a e^a{}_\nu=\delta^\mu{}_\nu$ and $e^a{}_\mu E^\mu{}_b=\delta^a{}_b$.

[1]We follow the convention $R^\alpha{}_{\beta\mu\nu}=\partial_\mu\Gamma^\alpha{}_{\nu\beta}+\Gamma^\alpha{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\beta}-(\mu\leftrightarrow\nu)$ and $S^\alpha{}_{\mu\nu}=2\Gamma^\alpha{}_{[\mu\nu]}$.

B. Comparison with conventional gauge transformations

Under standard $\mathfrak{p}_1$ gauge transformations, the 1-form B_μ transforms as $\tilde{\delta}_\Lambda B_\mu=\partial_\mu\Lambda+[B_\mu,\Lambda]$, with $\Lambda=\rho^a P_a+\zeta J+\sigma Q$ an arbitrary gauge parameter function. This gives for the components of B_μ the transformation laws

$$\tilde{\delta}_\Lambda e^a{}_\mu=\partial_\mu\rho^a-\epsilon^a{}_b(\omega_\mu\rho^b-e^b{}_\mu\zeta), \tag{2.22}$$

$$\tilde{\delta}_\Lambda\omega_\mu=\partial_\mu\zeta, \tag{2.23}$$

$$\tilde{\delta}_\Lambda A_\mu=\partial_\mu\sigma-\epsilon_{ab}e^a{}_\mu\rho^b. \tag{2.24}$$

It is straightforward to check that there is not any zweibein postulate linear in both ω_μ and A_μ that remains invariant under $\tilde{\delta}_\Lambda$. Furthermore, standard arguments [15] show that Eq. (2.19) remains $\tilde{\delta}_\Lambda$ invariant, modulo a change of coordinates, only if the torsion vanishes. This suggests that, to study scenarios with nonzero torsion, it is convenient to use the symmetry $\delta_{(\xi,\Sigma)}$ rather than $\tilde{\delta}_\Lambda$. Transformations of type $\delta_{(\xi,\Sigma)}$ have been used in studies of Horava-Lifshitz [8] and Carrollian [9] gravities. The two transformations are related through [10]

$$\delta_{(\xi,\Sigma)}B_\mu=\tilde{\delta}_\Lambda B_\mu+\xi^\nu G_{\nu\mu},\quad\text{with}\quad\Lambda=\xi^\mu B_\mu+\Sigma. \tag{2.25}$$

For $\delta_{(\xi,\Sigma)}$ and $\tilde{\delta}_\Lambda$ to agree, the torsion, and also $\omega_{\mu\nu}$ and $Z_{\mu\nu}$, must vanish.

C. Invariant Lagrangian

We are interested in Lagrangians that are invariant under $\delta_{(\xi,\Sigma)}$, linear in the Riemann curvature, and at most quadratic in first derivatives of the fields. Because of Eq. (2.21), linearity in the Riemann tensor is equivalent to linearity in $\Omega_{\mu\nu}$. In accordance with Eqs. (2.4)–(2.6), the most general Lagrangian of this type is

$$\mathcal{L}=\sqrt{|g|}\bigg[c_1*\Omega+\frac{c_2}{\ell^2}*Z+c_3\eta_{ab}*T^a*T^b+c_4*\Omega*Z+\frac{c_5}{\ell^2}(*Z)^2\bigg], \tag{2.26}$$

where $c_1,\ldots,c_5$ are arbitrary constants and

$$*\Phi=\frac{1}{2}\epsilon^{\mu\nu}\Phi_{\mu\nu} \tag{2.27}$$

is the dual of the two-form $\Phi_{\mu\nu}$, with $\epsilon^{\mu\nu}$ the antisymmetric pseudotensor.

In what follows, we restrict ourselves to Levi-Civita connections, for which the torsion vanishes,

$$T^a{}_{\mu\nu}=0, \tag{2.28}$$

and the metric is given in terms of the zweibein by

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu. \tag{2.29}$$

In this case, using Eq. (2.21) to write $*\Omega$ in terms of the Ricci scalar R, and introducing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.30}$$

we have

$$\begin{aligned} *\Omega &= -\frac{1}{2} R = \nabla_\mu(\epsilon^{\mu\nu}\omega_\nu), \\ *Z &= *F + 1 = \nabla_\mu(\epsilon^{\mu\nu}A_\nu) + 1. \end{aligned} \tag{2.31}$$

Substituting these equations in $\mathcal{L}$ above and discarding total derivatives, we are left with

$$\mathcal{L} = \sqrt{|g|}\left[\frac{c_2 + c_5}{\ell^2} - \frac{c_4}{2} R * F + \frac{c_5}{\ell^2}(*F)^2\right]. \tag{2.32}$$

Making the change $A_\mu \to -(c_4/4c_5)A_\mu$, and setting $\kappa = 4c_5/c_4^2$ and $\gamma/2\kappa = c_2 + c_5$, we arrive at the classical action

$$S = \frac{1}{2\kappa}\int d^2x\sqrt{|g|}\left[R * F + \frac{1}{2\ell^2}(*F)^2 + \frac{\gamma}{\ell^2}\right] + S_m. \tag{2.33}$$

This is the action in Eq. (1.1), for in two dimensions with Lorentzian signature, one has

$$\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}. \tag{2.34}$$

In Eq. (2.33), we have included a matter contribution S_m that couples $g_{\mu\nu}$ and A_μ to other fields but does not contain derivatives of $g_{\mu\nu}$ and A_μ.

III. VACUUM SOLUTIONS

Varying the action with respect to $g_{\mu\nu}$ and using that in two dimensions $2\delta_g(*F) = *F\delta g^{\mu\nu}g_{\mu\nu}$ and $2R_{\mu\nu} = g_{\mu\nu}R$, one has

$$\delta_g S = \frac{1}{2\kappa}\int d^2x\sqrt{|g|}\nabla_\mu[\nabla_\nu(v^{\mu\nu} * F) - 2v^{\mu\nu}\nabla_\nu(*F)] + \frac{1}{2}\int d^2x\sqrt{|g|}\delta g^{\mu\nu}(T^g_{\mu\nu} + T^m_{\mu\nu}). \tag{3.1}$$

The first term is a boundary term, with $v^{\mu\nu}$ given by

$$v^{\mu\nu} = -\delta g^{\mu\nu} + g^{\mu\nu}\delta g^{\alpha\beta}g_{\alpha\beta}; \tag{3.2}$$

$T^g_{\mu\nu}$ has the form

$$\kappa T^g_{\mu\nu} = \frac{1}{2} g_{\mu\nu}\left[R * F + \frac{(*F)^2}{2\ell^2} + 2\nabla^2 * F - \frac{\gamma}{\ell^2}\right] - \nabla_\mu\nabla_\nu * F; \tag{3.3}$$

and $T^m_{\mu\nu}$ is the matter energy-momentum tensor,

$$T^m_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{3.4}$$

Variation of S with respect to A_μ yields in turn

$$\delta_A S = \frac{1}{\kappa}\int d^2x\sqrt{|g|}\nabla_\mu(\epsilon^{\mu\nu}\epsilon^{\alpha\beta}[\nabla_\alpha(C_\beta\delta A_\nu) - C_\beta\nabla_\alpha\delta A_\nu]) + \int d^2x\sqrt{|g|}\delta A_\mu\left[\frac{1}{2\kappa}\epsilon^{\mu\nu}\partial_\nu\left(R + \frac{*F}{\ell^2}\right) + J^{m\mu}\right], \tag{3.5}$$

where C_μ reads

$$C_\mu = -\omega_\mu + \frac{A_\mu}{2\ell^2} \tag{3.6}$$

and $J^{m\mu}$ is the $U(1)$ matter current

$$J^{m\mu} = \frac{1}{\sqrt{|g|}}\frac{\delta S_m}{\delta A_\mu}. \tag{3.7}$$

We assume suitable boundary conditions, so that the boundary terms in Eqs. (3.1) and (3.5) vanish. The field equations are then

$$T^g_{\mu\nu} + T^m_{\mu\nu} = 0 \tag{3.8}$$

and

$$J^\mu_T \coloneqq \frac{1}{2\kappa}\epsilon^{\mu\nu}\nabla_\nu\left(R + \frac{*F}{\ell^2}\right) + J^{m\mu} = 0. \tag{3.9}$$

Acting on Eq. (3.9) with ∇_μ and using $\nabla_\mu\epsilon^{\mu\nu} = 0$, we have

$$\nabla_\mu J^{m\mu} = 0. \tag{3.10}$$

Hence, the matter contribution to the $U(1)$ gauge current must be conserved.

We wish to solve the field equations in vacuum. This is most conveniently done in the conformal gauge with light-cone coordinates

$$ds^2 = -e^{2\varphi}dx^+dx^-, \qquad x^\pm = t \pm x, \tag{3.11}$$

in which the equations take the form

$$\partial_\pm\partial_\pm * F - 2\partial_\pm\varphi\partial_\pm * F = 0, \qquad (3.12)$$

$$2\partial_+\partial_- * F - \frac{1}{2}e^{2\varphi}\left[R * F + \frac{(*F)^2}{2\ell^2} - \frac{\gamma}{\ell^2}\right] = 0, \qquad (3.13)$$

$$\partial_\pm\left(R + \frac{1}{\ell^2} * F\right) = 0, \qquad (3.14)$$

and the Ricci scalar is given by

$$R = 8e^{-2\varphi}\partial_+\partial_-\varphi. \qquad (3.15)$$

Equation (3.14) can be regarded as an integrability condition, for it is reproduced by acting with $\partial_\mp$ on Eq. (3.12) and using Eq. (3.13). It ensures that the boundary term in Eq. (3.5) vanishes, since the latter can also be written as

$$\frac{1}{\kappa}\int d^2x\sqrt{|g|}\nabla_\mu\left[\epsilon^{\mu\nu}\left(R + \frac{*F}{\ell^2}\right)\delta A_\nu\right]. \qquad (3.16)$$

To solve Eqs. (3.12)–(3.14), we distinguish between constant and nonconstant scalar curvature.

A. Solutions with constant scalar curvature

If R is constant, Eq. (3.14) implies that so is $*F$. We thus write

$$\bar{R} = \frac{R_0}{\ell^2}, \qquad (3.17)$$

$$*\bar{F} = F_0, \qquad (3.18)$$

with R_0 and F_0 dimensionless constants satisfying the constraint provided by Eq. (3.13),

$$F_0^2 + 2F_0R_0 - 2\gamma = 0. \qquad (3.19)$$

Vacuum spacetime is locally isomorphic to Minkowski, dS_2, or AdS_2. Equation (3.18) can be recast as

$$\bar{F}_{+-} = -\frac{F_0}{2}e^{2\bar{\varphi}}, \qquad (3.20)$$

with $\bar{\varphi}$ a solution to the Liouville equation (3.15) for $R = R_0/\ell^2$. An expression for the gauge potential solution $(\bar{A}_+, \bar{A}_-)$ can be found by choosing a gauge and solving Eq. (3.18). Here, we will work in the Lorenz gauge

$$\partial_+A_- + \partial_-A_+ = 0, \qquad (3.21)$$

in which $A_\pm$ and F_{+-} become [14]

$$A_\pm = \mp\,\partial_\pm a, \qquad (3.22)$$

$$F_{+-} = 2\partial_+\partial_- a, \qquad (3.23)$$

with $a = a(x^+, x^-)$ an arbitrary function of its arguments with dimensions of $(\text{length})^2$. Upon substitution in Eq. (3.18), we obtain

$$4e^{-2\bar{\varphi}}\partial_+\partial_-\bar{a} = -F_0. \qquad (3.24)$$

We note that a plays a role similar to that of φ. In fact, solving the vanishing torsion equations $T^a_{+-} = 0$ for the spin connection, we have

$$\omega_\pm = \mp\,\partial_\pm\varphi, \qquad (3.25)$$

which has the same form as Eq. (3.22).

For zero scalar curvature, $R_0 = 0$, the vacuum spacetime is locally isomorphic to Minkowski space, metric $ds^2_{R_0=0} = -dx^-dx^+$. In this case, $\bar{\varphi} = 0$, and Eq. (3.24) becomes $\partial_+\partial_-\bar{a} = -F_0/4$, with $F_0 = \pm\sqrt{2\gamma}$, which requires $\gamma > 0$. The solution for a is then

$$\bar{a}_{R_0=0} = -\frac{F_0}{8}(x^+ + x^-)^2 + f_R(x^+) + f_L(x^-), \qquad (3.26)$$

where $f_R(x^+)$ and $f_L(x^-)$ are arbitrary functions of their arguments with dimensions of length^2. The arbitrariness in f_R and f_L is reminiscent of the fact that the Lorenz condition (3.21) does not completely eliminate gauge invariance but leaves a residual gauge symmetry.

If $R_0 \neq 0$, the general solution to Eq. (3.24) is given in terms of the solution $\bar{\varphi}$ to Liouville's equation (3.15) by

$$\bar{a}_{R_0\neq 0} = -\frac{2F_0\ell^2}{R_0}\bar{\varphi} + f_R(x^+) + f_L(x^-), \qquad (3.27)$$

where F_0/R_0 on the right-hand side is the solution to Eq. (3.19),

$$\frac{F_0}{R_0} = -1 \pm \sqrt{1 + \frac{2\gamma}{R_0^2}}. \qquad (3.28)$$

These solutions are different from those of JT gravity. In our case, the Ricci scalar R_0/ℓ^2 is no longer equal to $-\gamma/\ell^2$. For a given value of γ such that $R_0^2 + 2\gamma > 0$, the scalar curvature may be positive or negative, depending on F_0. We remark that if the term $(*F)^2$ is removed from the classical action the vacuum solutions are the same, the only difference being that now $F_0R_0 = \gamma$. Furthermore, for vanishing cosmological constant, $\gamma = 0$, and provided the term $(*F)^2$ is kept, vacuum spacetime will be nonflat with constant scalar curvature $R = -F_0/2\ell^2$. In particular, a gauge field with $F_0 = \mp 4$ will generate a dS_2/AdS_2 with scalar curvature $R_0 = \pm 2$.

Coming back to the case of arbitrary γ, for $R_0 > 0$, vacuum spacetime is locally isomorphic to dS_2, whose metric in Poincaré coordinates $\{t > 0, x\}$ is

$$ds^2_{dS} = \frac{\ell^2}{t^2}(-dt^2 + dx^2) = -\frac{4\ell^2}{(x^+ + x^-)^2}dx^+dx^-. \quad (3.29)$$

In these coordinates, $R_0 = 2$, and φ becomes

$$\bar{\varphi}_{dS} = \ln\left(\frac{2\ell}{x^+ + x^-}\right). \quad (3.30)$$

The expression of $\bar{a}_{dS}$ is obtained upon substitution in Eq. (3.27). To eliminate the arbitrariness in f_R and f_L, we impose that the component C_t of C_μ in Eq. (3.6) vanishes at the boundary $t = 0$,

$$0 = C_t|_{t=0} = -(\omega_+ + \omega_-) + \frac{1}{2\ell^2}(A_+ + A_-)|_{t=0}. \quad (3.31)$$

This fixes f_R and f_L and gives

$$\bar{a}_{dS} = \frac{2F_0\ell^2}{R_0}\ln\left(\frac{x^+ + x^-}{2\ell}\right) + \ell\alpha_1(x^+ + x^-) + \ell^2\alpha_0, \quad (3.32)$$

with α_0 and α_1 arbitrary dimensionless constants. The spin connection and the gauge field are found upon substitution in Eqs. (3.25) and (3.22). Condition (3.31) and the fact that $*\bar{F}_{dS}$ is constant ensure that the boundary terms in $\delta_g S$ and $\delta_A S$ vanish on shell.

For $R_0 < 0$, vacuum spacetime is locally isomorphic to AdS_2, with metric

$$ds^2_{AdS} = \frac{\ell^2}{x^2}(-dt^2 + dx^2) = -\frac{4\ell^2}{(x^+ - x^-)^2}dx^+dx^- \quad (3.33)$$

in Poincaré coordinates $\{t, x > 0\}$. Now, $R_0 = -2$, and

$$\bar{\varphi}_{AdS} = \ln\left(\frac{2\ell}{x^+ - x^-}\right),$$
$$\bar{a}_{AdS} = -\frac{2F_0\ell^2}{R_0}\bar{\varphi}_{AdS} + \ell\alpha_1(x^+ - x^-) + \ell^2\alpha_0 \quad (3.34)$$

for a boundary condition

$$0 = C_x|_{x=0} = -(\omega_+ - \omega_-) + \frac{1}{2\ell^2}(A_+ - A_-)|_{x=0}. \quad (3.35)$$

B. Solutions with nonconstant scalar curvature: Black holes

To find the vacuum solutions with nonconstant scalar curvature, we employ similar methods to those used in the proof of Birkhoff's theorem in 2D dilaton gravity in Refs. [11,12]. Combine Eqs. (3.12) and (3.14) to write $\partial_\pm(e^{-2\varphi}\partial_\pm R) = 0$. This implies that

$$\partial_+ R = e^{2\varphi}h_L(x^-), \qquad \partial_- R = e^{2\varphi}h_R(x^+), \quad (3.36)$$

with $h_L(x^-)$ and $h_R(x^+)$ arbitrary functions of their arguments. After having fixed the conformal gauge, the model is still invariant under diffeomorphisms $x^+ \to \tilde{x}^+(x^+)$ and $x^- \to \tilde{x}^-(x^-)$, under which h_L and h_R transform as

$$\tilde{h}_L(\tilde{x}^-) = h_L(x^-)\frac{d\tilde{x}^-}{dx^-}, \qquad \tilde{h}_R(\tilde{x}^+) = h_R(x^+)\frac{d\tilde{x}^+}{dx^+}. \quad (3.37)$$

Use this residual symmetry to choose coordinates $\{\tilde{x}^+, \tilde{x}^-\}$ defined as the solutions to the equations

$$\frac{d\tilde{x}^+}{dx^+} = \frac{1}{|h_R(x^+)|}, \qquad \frac{d\tilde{x}^-}{dx^-} = \frac{1}{|h_L(x^-)|}. \quad (3.38)$$

For nonconstant curvature, h_L and h_R are different from zero, so this change is locally well defined. In the new coordinates, Eqs. (3.36) become

$$\mathrm{sign}(h_L)\widetilde{\partial_+ R} = \mathrm{sign}(h_R)\widetilde{\partial_- R} = e^{2\tilde{\varphi}}. \quad (3.39)$$

It follows that either (i) $\tilde{\varphi}(\tilde{x})$ is a function of $\tilde{x} = (\tilde{x}^+ - \tilde{x}^-)/2$ or (ii) it is a function $\tilde{\varphi}(\tilde{t})$ of $\tilde{t} = (\tilde{x}^+ + \tilde{x}^-)/2$.

Let us consider scenario i. In this case, φ, R, and $*F$ are also functions of x, where, to ease the writing, we have removed the tildes from the notation. Upon making the change $x \to r(x)$, with $dr = e^{2\varphi(x)}dx$, the metric takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}. \quad (3.40)$$

The function $f(r)$ is given in terms of φ by $f(r) = e^{2\varphi(x(r))}$. The scalar curvature becomes $R = -f''(r)$, and the field equations (3.8)–(3.9) read

$$(*F)'' = 0, \quad (3.41)$$

$$f'(*F)' - f'' * F + \frac{(*F)^2}{2\ell^2} - \frac{\gamma}{\ell^2} = 0, \quad (3.42)$$

$$-f''' + \frac{(*F)'}{\ell^2} = 0, \quad (3.43)$$

where the prime denotes differentiation with respect to r. The solution to Eq. (3.41) is $*F = a_1(r/\ell) + a_0$, with a_1 and a_0 integration constants. We are interested in $a_1 \neq 0$, since $a_1 = 0$ corresponds to constant scalar curvature. As in 2D dilaton gravity [13], we use the invariance of the metric under $(t, r, f) \to (t/b_1, b_1 r - b_0\ell, f b_1^2)$ to set $a_0 = 0$ and $a_1 = 1$. This gives

$$*F = \frac{r}{\ell}. \quad (3.44)$$

Equations (3.42) and (3.43) then yield

$$f(r) = \frac{r^3}{6\ell^3} + c_0 \frac{r^2}{2\ell^2} + \gamma \frac{r}{\ell} + c_1, \tag{3.45}$$

with c_0 and c_1 dimensionless integration constants. Being a cubic polynomial with real coefficients, $f(r)$ has at least one real root. Call r_H to its largest real root. Since $f(r)$ is positive for $r > r_H$ and changes its sign at $r = r_H$, the solution (3.40), with f in Eq. (3.45), can be understood as a black hole with horizon at r_H. Note that ∂_t is a timelike Killing vector for $f(r) > 0$. Note also that the term $(*F)^2$ in the classical action is necessary to have solutions of this type; otherwise, the contribution $(*F)^2$ in Eq. (3.42) is absent, and Eq. (3.43) reduces to $f''' = 0$, equivalently constant scalar curvature.

The other solution to Eqs. (3.39), $\varphi(t)$ only depends on t, is analyzed similarly. After reparametrizing t, and setting $r = x$, the dual field strength is now $*F = -t/\ell$, and the metric takes the form

$$ds^2 = -\frac{dt^2}{f(t)} + f(t)dx^2, \tag{3.46}$$

with

$$f(t) = \frac{t^3}{6\ell^3} + d_0 \frac{t^2}{2\ell^2} + \gamma \frac{t}{\ell} + d_1, \tag{3.47}$$

and d_0 and d_1 dimensionless constants of integration. For $d_0 = c_0$, $d_1 = c_1$, this metric describes the interior of the black hole (3.40), since when going across the horizon r_H of Eq. (3.40) the coordinate r becomes timelike and the metric can be cast as in Eqs. (3.46) and (3.47).

IV. BOUNDARY CFT DESCRIPTION OF THE MODEL

In this section, we present a CFT interpretation of the vacuum solutions with constant scalar curvature. The classical action (2.33) in the conformal-Lorenz gauge takes the form

$$S_{\mathrm{CFT}} = \frac{1}{\kappa}\int dx^+ dx^- \left[-8e^{-2\varphi}(\partial_+\partial_- a)\partial_+\partial_-\varphi + \frac{2}{\ell^2}e^{-2\varphi}(\partial_+\partial_- a)^2 + \frac{\gamma}{4\ell^2}e^{2\varphi}\right] + S_m. \tag{4.1}$$

This action contains second derivatives with respect to time of φ and a. It is invariant under conformal diffeomorphisms

$$x^\pm \to x^\pm + \xi^\pm(x^\pm)$$

generated by arbitrary vector fields $\xi^+(x^+)\partial_+$ and $\xi^-(x^-)\partial_-$, provided e^φ transforms as a conformal field of weights $(1/2, 1/2)$ and a as a scalar. S is also invariant under residual gauge transformations $a \to a + \tau_R(x^+) + \tau_L(x^-)$, with $\tau_R(x^+)$ and $\tau_L(x^-)$ arbitrary functions of their arguments with dimensions of $(\text{length})^2$. Let us see that the combination of these two symmetries is a residual symmetry δ_r of $\delta_{(\xi,\Sigma)}$ specified by ξ^+ and ξ^-.

A. Residual symmetry

In the conformal gauge, the zweibein is given by

$$e^0{}_\pm = \frac{e^\varphi}{2}, \qquad e^1{}_\pm = \pm\frac{e^\varphi}{2}. \tag{4.2}$$

To find $\delta_r\varphi = \delta_{(\xi,\Sigma)}\varphi$ for a local parameter

$$\xi_r = \xi^+\partial_+ + \xi^-\partial_-, \tag{4.3}$$

we substitute the expressions (4.2) in Eq (2.13) and use that $\delta e^\varphi = e^\varphi\delta\varphi$. This yields a system of two equations for $\delta_r\varphi$ and θ_r, whose only solution is

$$\delta_r\varphi = (\xi^+\partial_+ + \xi^-\partial_-)\varphi + \frac{1}{2}(\partial_+\xi^+ + \partial_-\xi^-), \tag{4.4}$$

$$\theta_r = \frac{1}{2}(\partial_-\xi_r^- - \partial_+\xi_r^+). \tag{4.5}$$

In Eq. (4.4), one recognizes the variation under conformal diffeomorphisms of a field e^φ with conformal weights $(1/2, 1/2)$. Substituting the result (4.5) for θ_r in the variation (2.14) of the spin connection, we have

$$\delta_r\omega_\pm = \mathcal{L}_{\xi_r}\omega_\pm \mp \frac{1}{2}\partial_\pm^2\xi^\pm. \tag{4.6}$$

The Lie derivative

$$\mathcal{L}_{\xi_r}\omega_\pm = (\xi^+\partial_+ + \xi^-\partial_-)\omega_\pm + (\partial_\pm\xi^\pm)\omega_\pm \tag{4.7}$$

on the right-hand side accounts for the variation under conformal diffeomorphisms of the one-form (ω_+, ω_-), while $\mp\frac{1}{2}\partial_\pm^2\xi^\pm$ adds a $U(1)$ contribution generated by boosts. The transformation law (4.6) can also be obtained by using Eq. (4.4) in the solution $\omega_\pm = \mp\,\partial_\pm\varphi$ to the vanishing torsion condition.

To find $\delta_r a$, set $\xi = \xi_r$ and $A_\pm = \mp\,\partial_\pm a$ in the variations $\delta A_\pm$ in Eq. (2.15). This provides two equations for $\delta_r a$ and τ_r, whose solutions are

$$\delta_r a = (\xi^+\partial_+ + \xi^-\partial_-)a + \tau_R(x^+) + \tau_L(x^-), \tag{4.8}$$

$$\tau_r = \tau_L(x^-) - \tau_R(x^+), \tag{4.9}$$

with $\tau_R(x^+)$ and $\tau_L(x^-)$ arbitrary functions of their arguments.

To determine τ_R and τ_L, one may proceed as follows. Regard any of the vacuum solutions dS_2 or AdS_2 of Sec. II as the boundary of a model with matter. Demanding the residual symmetry to be consistent with the boundary, and recalling that at the boundary a and φ are related through Eq. (3.27), it is straightforward that

$$\tau_R(x^+) = -\frac{F_0\ell^2}{R_0}\partial_+\xi^+,$$
$$\tau_L(x^-) = -\frac{F_0\ell^2}{R_0}\partial_-\xi^-, \qquad (4.10)$$

and

$$\delta_r a = (\xi^+\partial_+ + \xi^-\partial_-)a - \frac{F_0\ell^2}{R_0}(\partial_+\xi^+ + \partial_-\xi^-). \qquad (4.11)$$

The variations $\delta_r A_\pm$ then read

$$\delta_r A_\pm = \mathcal{L}_{\xi_r} A_\pm \pm \frac{F_0\ell^2}{R_0}\partial_\pm^2\xi^\pm. \qquad (4.12)$$

Residual transformations δ_r are thus determined by the vector field $\xi_r = (\xi^+, \xi^-)$. We remark that $(R_0/2F_0\ell^2)a$ and $(R_0/2F_0\ell^2)A_\pm$ transform under δ_r as φ and $\omega_\pm$.

B. Witt algebra

Denote by δ_r^+ and δ_r^- the generators of the residual symmetries associated to $\xi^+\partial_+$ and $\xi^-\partial_-$. Assume that $\xi^+(x^+)$ and $\xi^-(x^-)$ can be expanded in power series of x^+ and x^- with coefficients $c_{n,+}$ and $c_{n,-}$, so that

$$\xi^\pm\partial_\pm = \sum_n c_{n,\pm}(x^\pm)^{n+1}\partial_\pm. \qquad (4.13)$$

In accordance with Eqs. (4.5) and (4.10), the variation δ_r can be written as

$$\delta_r = \delta_r^+ + \delta_r^- = \sum_n (c_{n,+}\delta_n^+ + c_{n,-}\delta_n^-), \qquad (4.14)$$

with $\delta_n^\pm$ the $\delta_{(\xi,\Sigma)}$ transformation with parameters

$$\xi_{n,\pm} = (x^\pm)^{n+1}\partial_\pm, \qquad \theta_{n,\pm} = \mp\frac{n+1}{2}(x^\pm)^n,$$
$$\tau_{n,\pm} = \pm\frac{F_0\ell^2}{R_0}(n+1)(x^\pm)^n. \qquad (4.15)$$

The closure relation (2.12) then implies

$$[\delta_n^\pm, \delta_m^\pm] = (n-m)\delta_{n+m}^\pm. \qquad (4.16)$$

The residual symmetry is hence generated by a Witt algebra.

The variation δ_r coincides with the combination of conformal and gauge transformations introduced in JT-Maxwell gravity [14].

C. Check of invariance of dS_2 and AdS_2 boundaries

Invariance of the dS_2 and AdS_2 boundaries under δ_r can also be checked using the same arguments as in Ref. [14]. Let us briefly see this.

Consider first the case of dS_2. The boundary is located in Poincaré coordinates at $t = 0$, equivalently $x^+ + x^- = 0$. Since the boundary must remain unchanged under conformal diffeomorphisms $x^\pm \to x^\pm + \xi^\pm(x^\pm)$, the vector fields $\xi^\pm(t,x)$ must satisfy

$$\partial_+^n\xi^+(0,x) = (-1)^{n+1}\partial_-^n\xi^-(0,x), \quad n = 0,1,2,\ldots \qquad (4.17)$$

One allows for field configurations of φ and a that behave near $t = 0$ as the dS_2 vacuum solution of Sec. III,

$$\varphi, \quad -\frac{R_0 a}{2F_0\ell^2} = \ln\left(\frac{\ell}{x^+ + x^-}\right) + O(1), \qquad (4.18)$$

which satisfy the dS_2 boundary condition (3.31),

$$0 = -(\omega_- + \omega_+) + \frac{1}{2\ell^2}(A_- + A_+)\Big|_{t=0}$$
$$= \partial_x\left(\varphi - \frac{a}{2\ell^2}\right)\Big|_{t=0}. \qquad (4.19)$$

We must check that Eq. (4.19) is invariant under δ_r. To do this, compute first $\delta_r(\omega_+ + \omega_-)|_{t=0}$. Equation (4.6) gives for $\delta_r(\omega_+ + \omega_-)$ two contributions, one from the Lie derivative $\mathcal{L}_{\xi_r}(\omega_+ + \omega_-)$ and one from the boost generated terms $-\frac{1}{2}(\partial_+^2\xi^+ - \partial_-^2\xi^-)$. Expanding in powers of t, noting that near $t = 0$

$$\partial_\pm^n\xi^\pm(t \pm x)$$
$$= \partial_\pm^n\xi^\pm(0,x) + t\partial_\pm^{n+1}\xi^\pm(0,x) + O(t^2), \qquad (4.20)$$

and recalling Eq. (4.17), it is very easy to see that the Lie derivative takes at the boundary the value

$$\mathcal{L}_{\xi_r}(\omega_+ + \omega_-)|_{t=0} = \frac{1}{2}[\partial_+^2\xi^+(0,x) - \partial_-^2\xi^-(0,x)]. \qquad (4.21)$$

This cancels the contribution from boosts and gives

$$\delta_r(\omega_+ + \omega_-)|_{t=0} = \mathcal{L}_{\xi_r}(\omega_+ + \omega_-) - \frac{1}{2}\partial_+^2\xi^+ + \frac{1}{2}\partial_-^2\xi^-\Big|_{t=0}$$
$$= 0. \qquad (4.22)$$

Analogous arguments show that $\delta_r(A_+ + A_-)|_{t=0} = 0$, thus completing the proof of invariance of condition (4.19) under δ_r.

The proof for an AdS_2 boundary goes along the same lines. The only differences are that now that the boundary is at $x = 0$, equivalently $x^+ - x^- = 0$, Eq. (4.17) is replaced with Ref. [14]

$$\partial_+^n \xi^+(t,0) = \partial_-^n \xi^-(t,0), \quad n = 0,1,2,\ldots, \tag{4.23}$$

and the boundary condition takes the form Eq. (3.35). Taking into account these changes and proceeding as for dS_2, one has $\delta_r(\omega_- - \omega_+)|_{x=0} = 0$ and $\delta_r(A_- - A_+)|_{x=0} = 0$.

D. Conserved currents, charges and Hamiltonian formalism

The field equations that result upon taking variations with respect to φ and a in the action (4.1) are

$$\partial_+\partial_- * F - \frac{1}{4} e^{2\varphi}\left[R * F + \frac{1}{2\ell^2}(*F)^2 - \frac{\gamma}{\ell^2}\right] + \kappa T^m_{+-} = 0, \tag{4.24}$$

$$\frac{1}{\kappa}\partial_+\partial_-\left(R + \frac{*F}{\ell^2}\right) + \partial_+ J^m_- - \partial_- J^m_+ = 0. \tag{4.25}$$

Equation (4.25) can be written in terms of the total $U(1)$ current

$$J^T_\pm = \mp \frac{1}{2\kappa}\partial_\pm\left(R + \frac{*F}{\ell^2}\right) + J^m_\pm \tag{4.26}$$

given by Eq. (3.9) as $\partial_- J^T_+ = \partial_+ J^T_-$. This and the conservation equation (3.10), which in the conformal gauge reads $\partial_+ J^m_- + \partial_- J^m_+ = 0$, imply

$$\partial_- J^T_+ = \partial_+ J^T_- = 0. \tag{4.27}$$

Consider the case of no additional matter. Standard methods show that the Noether currents preserved by $\delta_r^\pm$ are

$$\tilde{T}^g_{\pm\pm} = -\frac{1}{\kappa}\left[\partial_\pm\partial_\pm * F - 2\partial_\pm\varphi\partial_\pm * F - \frac{F_0\ell^2}{R_0}\partial_\pm\partial_\pm\left(R + \frac{*F}{\ell^2}\right) + \partial_\pm a\partial_\pm\left(R + \frac{*F}{\ell^2}\right)\right]. \tag{4.28}$$

In fact, using Eqs. (4.24) and (4.25), it is straightforward to see that

$$\partial_- \tilde{T}^g_{++} = \partial_+ \tilde{T}^g_{--} = 0. \tag{4.29}$$

The currents $\tilde{T}^g_{\pm\pm}$ can also be cast as

$$\tilde{T}^g_{\pm\pm} = T^g_{\pm\pm} \pm \frac{2F_0\ell^2}{R_0}\partial_\pm J^g_\pm \pm 2J^g_\pm\partial_\pm a, \tag{4.30}$$

where $T^g_{\pm\pm}$ are obtained from Eq. (3.3), and

$$J^g_\pm = \mp\frac{1}{2\kappa}\partial_\pm\left(R + \frac{*F}{\ell^2}\right) \tag{4.31}$$

are the gravity contributions to the $U(1)$ current. The corresponding conserved charges are

$$Q^\pm = \int dx^\pm T_{\pm\pm}(x^+)\xi^\pm(x^+). \tag{4.32}$$

Let us check that $Q^\pm$ generate through Poisson brackets residual transformations,

$$\delta_r^\pm \phi = \{Q^\pm, \phi\}, \qquad \phi = \varphi, a. \tag{4.33}$$

The action S_{CFT} can be regarded as describing a dynamical system with Lagrangian

$$S_{\mathrm{CFT}} = \int dt L, \qquad L = 2\int dx \mathcal{L}_{\mathrm{CFT}}, \tag{4.34}$$

where $\mathcal{L}_{\mathrm{CFT}}$ is the integrand in Eq. (4.1). Since the Lagrangian L contains second derivatives with respect to time of φ and a, the Hamiltonian formulation is a bit more involved than for dynamical systems with only first-order time derivatives; see, e.g., Refs. [16,17] for reviews. The phase space is now formed by the generalized coordinates

$$\begin{aligned} q_0^\varphi &= \varphi(t,x), & q_0^a &= a(t,x), \\ q_1^a &= \dot{a}(t,x), & q_1^\varphi &= \dot{\varphi}(t,x) \end{aligned} \tag{4.35}$$

and their conjugate momenta,

$$\begin{aligned} \pi^0_\phi(t,x) &= \frac{\partial L}{\partial \dot{q}_0^\phi(t,x)} - \partial_t \frac{\partial L}{\partial \dot{q}_1^\phi(t,x)}, \\ \pi^1_\phi(t,x) &= \frac{\partial L}{\partial \dot{q}_1^\phi(t,x)}, \end{aligned} \tag{4.36}$$

where we have introduced the index $\phi = \varphi, a$. The Poisson brackets are the usual ones,

$$\{q_i^\phi(x,t), \pi^j_{\phi'}(y,t)\} = \delta^\phi_{\phi'}\delta_i^j\delta(x-y), \tag{4.37}$$

$$\{q_i^\phi(x,t), q_j^{\phi'}(y,t)\} = \{\pi^i_\phi(x,t), \pi^j_{\phi'}(y,t)\} = 0, \tag{4.38}$$

with $\phi, \phi' = \varphi, a$ and $i, j = 0, 1$. And finally, the Hamiltonian reads

$$H = \int dx(\pi^0_\varphi \dot{q}_0^\varphi + \pi^0_a \dot{q}_0^a + \pi^1_\varphi \dot{q}_1^\varphi + \pi^1_a \dot{q}_1^a) - L, \tag{4.39}$$

and Hamilton's equations take the form

$$\dot{q}_i^\phi = \frac{\partial H}{\partial \pi_\phi^i}, \qquad \dot{\pi}_\phi^i = -\frac{\partial H}{\partial q_i^\phi}, \qquad \phi = \varphi, a, \quad i = 0, 1. \tag{4.40}$$

Some simple calculations give

$$\pi_\varphi^0 = -\frac{1}{\kappa}\partial_t * F, \qquad \pi_\varphi^1 = \frac{1}{\kappa} * F, \tag{4.41}$$

$$\pi_a^0 = \frac{1}{2\kappa}\partial_t\left(R + \frac{*F}{\ell^2}\right), \qquad \pi_a^1 = -\frac{1}{2\kappa}\left(R + \frac{*F}{\ell^2}\right) \tag{4.42}$$

for the momenta and

$$H = \int dx\bigg[\pi_\varphi^0 q_1^\varphi + \pi_a^0 q_1^a + \pi_\varphi^1 \partial_x^2 q_0^\varphi + \pi_a^1 \partial_x^2 q_0^a - \kappa e^{2\varphi}\left(\pi_a^1 + \frac{\pi_\varphi^1}{4\ell^2}\right)\pi_\varphi^1 - \frac{\gamma}{2\kappa\ell^2} e^{2\varphi}\bigg] \tag{4.43}$$

for the Hamiltonian. It is straightforward to check that the Hamilton equations reproduce the same field equations (4.24) and (4.25) as the variational approach. The Poisson brackets in turn imply that

$$\{\varphi(t,x), \partial_\pm * F(t,y)\} = -\{\partial_\pm\varphi(t,x), *F(t,y)\} = -\frac{\kappa}{2}\delta(x-y), \tag{4.44}$$

$$\left\{a(t,x), \partial_\pm\left(R + \frac{*F}{\ell^2}\right)(t,y)\right\} = \left\{\partial_\pm a(t,x), \left(R + \frac{*F}{\ell^2}\right)(t,y)\right\} = \kappa\delta(x-y). \tag{4.45}$$

Using these, one easily verifies that Eqs. (4.33) hold. Furthermore, the currents $T_{\pm\pm}$ satisfy the equal-time bracket

$$\{T_{++}(x^+), T_{++}(y^+)\} = \frac{1}{\kappa}\delta(x^+ - y^+)\partial_+ T_{++}(x^+) + \frac{2}{\kappa}T_{++}(x^+)\partial_+\delta(x^+ - y^+) \tag{4.46}$$

and a similar expression for T_{--}. This is analogous to JT-Maxwell gravity [14].

E. Matter and central charge in the quantum theory

The argument for the occurrence of a central charge in JT-Maxwell gravity [14] also holds for our model. Let us briefly go through it. If matter is included, instead of $\tilde{T}^g_{\pm\pm}$ in Eq. (4.30), one has

$$\tilde{T}_{\pm\pm} = T^g_{\pm\pm} + T^m_{\pm\pm} \pm \frac{2F_0\ell^2}{R_0}\partial_\pm J^T_\pm \pm 2J^T_\pm \partial_\pm a. \tag{4.47}$$

For reasonable choices of matter, one expects the following:

(i) $T_{\pm\pm} = T^g_{\pm\pm} + T^m_{\pm\pm}$ will be holomorphically conserved. Equation (4.27) and the constraint (3.9) then imply $\partial_\mp \tilde{T}_{\pm\pm} = 0$.

(ii) S_m will have a contribution $\sqrt{|g|}J^m A$. This produces a term $\mp 2J^m_\pm \partial_\pm a$ in $T^m_{\pm\pm}$ that cancels the contribution $\pm 2J^m_\pm \partial_\pm a$ hidden in the fourth term in Eq. (4.47).

All things together, the conserved matter current[2] $\partial_- J^m_+ = \partial_+ J^m_- = 0$ enters $\tilde{T}_{\pm\pm}$ through $\pm\partial_\pm J^m_\pm$ with coefficient $2F_0\ell^2/R_0$. Assume now, as in Ref. [14], that the current is anomalous so that in the quantum theory

$$[J^m_+(x^+), J^m_+(y^+)] = -k\partial_+\delta(x^+ - y^+). \tag{4.48}$$

The current algebra (4.46) will then have a central term

$$F_0^2\ell^4 k\partial_+^3\delta(x^+ - y^+), \tag{4.49}$$

where we have used that $R_0^2 = 4$ for our choice of Poincaré coordinates. The result is formally the same for dS_2 and AdS_2 backgrounds, but it remains to find explicit realizations.

V. FURTHER REMARKS AND CONCLUSIONS

A. Euclidean case

The same model can be formulated with Euclidean signature. The starting point for the Utiyama-Kibble-Sciama procedure is now the central extension $\mathfrak{e}_0 = \mathrm{Span}\{P_1, P_2, J, Q\}$ of the Euclidean algebra in two dimensions, or Nappi-Witten algebra [18], whose Lie bracket is

$$[P_1, P_2] = Q, \qquad [J, P_1] = P_2, \qquad [J, P_2] = -P_1, \qquad [Q, P_a] = [Q, J] = 0. \tag{5.1}$$

The classical action is the same as in Eq. (1.1), except for the sign in front of F^2, which is now positive since the right-hand side of Eq. (2.34) changes its sign for Euclidean signature. Vacuum solutions are either black hole type or have constant scalar curvature and constant $*F$ and are now locally isomorphic to 2D Euclidean space, the sphere, or the hyperbolic plane.

B. No-go results for other 2D Yang-Mills gravity models

Powers of R and/or $*F$ can be included in the action S in Eq. (2.33) without changing the symmetry of the model. The question arises as to whether there are models invariant under $\delta_{(\xi,\Sigma)} = \mathcal{L}_\xi + \tilde{\delta}_\Sigma$, with Σ taking values in the 2D non-Abelian algebra $\mathfrak{na}_2$.[3] In this case, the closure relation would no longer be (2.12) but rather

[2] As implied by Eqs. (4.27) and (3.10).

[3] Up to isomorphisms, there is only one 2D non-Abelian real Lie algebra, namely, $[X, Y] = Y$.

$$[\mathcal{L}_{\xi_1}+\tilde{\delta}_{\Sigma_1},\mathcal{L}_{\xi_2}+\tilde{\delta}_{\Sigma_2}]=\mathcal{L}_{[\xi_1,\xi_2]}+\tilde{\delta}_{[\Sigma_1,\Sigma_2]+\mathcal{L}_{\xi_2}\Sigma_1-\mathcal{L}_{\xi_1}\Sigma_2}. \quad (5.2)$$

In the sequel, we provide an answer to this question in the negative. We show in particular that there is no real 4D Lie algebra whose gauging as described in Sec. II leads to an invariant action linear in the Riemann curvature.

The proof is by inspection. We are interested in indecomposable 4D real Lie algebras that have a non-Abelian 2D algebra $\mathfrak{na}_2$ as a subalgebra. All such algebras are solvable and are listed in the literature; see, e.g., Ref [19]. Some care must be taken though, since some of them have more than one $\mathfrak{na}_2$ subalgebra and different choices for $\mathfrak{na}_2$ lead to different semidirect products $\mathcal{X}\ltimes\mathfrak{na}_2$. Let us illustrate this with an example. Consider the Lie algebra $\mathrm{Span}\{t_0,t_1,t_2,t_3\}$, with

$$\mathfrak{p}_\lambda\colon\ [t_0,t_1]=\lambda t_1,\qquad [t_0,t_2]=(1-\lambda)t_2,\qquad [t_0,t_3]=t_3,$$
$$[t_1,t_2]=t_3,\qquad \lambda\geq\frac{1}{2}. \quad (5.3)$$

Note that for $\lambda=1$ and $t_0=J,\ t_1=P_0,\ t_2=Q$ and $t_3=P_1$ the central extension of the 2D Poincaré algebra in Eqs. (2.1) is recovered. Substituting

$$B_\mu=b^0{}_\mu t_0+b^1{}_\mu t_1+b^2{}_\mu t_0+b^3{}_\mu t_3, \quad (5.4)$$

$$G_{\mu\nu}=G^0{}_{\mu\nu}t_0+G^1{}_{\mu\nu}t_1+G^2{}_{\mu\nu}t_2+G^3{}_{\mu\nu}t_3 \quad (5.5)$$

in $G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+[B_\mu,B_\nu]$, we have

$$G^0{}_{\mu\nu}=\partial_\mu b^0{}_\nu-\partial_\nu b^0{}_\mu, \quad (5.6)$$

$$G^1{}_{\mu\nu}=\partial_\mu b^1{}_\nu-\partial_\nu b^1{}_\mu+\lambda(b^0{}_\mu b^1{}_\nu-b^0{}_\nu b^1{}_\mu), \quad (5.7)$$

$$G^2{}_{\mu\nu}=\partial_\mu b^2{}_\nu-\partial_\nu b^2{}_\mu=(1-\lambda)(b^0{}_\mu b^2{}_\nu-b^0{}_\nu b^2{}_\mu), \quad (5.8)$$

$$G^3{}_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+(b^0{}_\mu b^3{}_\nu-b^0{}_\nu b^3{}_\mu)$$
$$+(b^1{}_\mu b^2{}_\nu-b^1{}_\nu b^2{}_\mu). \quad (5.9)$$

There are three possible choices for the 2D non-Abelian subalgebra $\mathfrak{na}_2$,

$$\text{(a)}\ [t_0,t_3]=t_3,\quad \text{(b)}\ [t_0,t_1]=\lambda t_1,\quad \text{(c)}\ [t_0,t_2]=(1-\lambda)t_2. \quad (5.10)$$

Making $\lambda\to 1-\lambda$, $t_1\to -t_2$ and $t_2\to t_1$, the commutator (5.10c) reduces to (5.10b) while keeping all other commutators in Eq. (5.3) unchanged. Hence, it is enough to consider cases (5.10a) and (5.10b):

(a) Case $\mathfrak{na}_2=\mathrm{Span}\{t_0,t_3\}$. Under $\delta_{(\xi,\Sigma)}$, with $\Sigma=\theta t_0+\tau t_3$, the gauge fields transform as

$$\delta b^0{}_\mu=\mathcal{L}_\xi b^0{}_\mu+\partial_\mu\theta, \quad (5.11)$$

$$\delta b^1{}_\mu=\mathcal{L}_\xi b^1{}_\mu-\lambda b^1{}_\mu\theta, \quad (5.12)$$

$$\delta b^2{}_\mu=\mathcal{L}_\xi b^2{}_\mu-(1-\lambda)b^2{}_\mu\theta, \quad (5.13)$$

$$\delta b^3{}_\mu=\mathcal{L}_\xi b^3{}_\mu+\partial_\mu\tau-b^3{}_\mu\theta+b^0{}_\mu\tau, \quad (5.14)$$

whereas the variations of the field strengths read

$$\delta G^0{}_{\mu\nu}=\mathcal{L}_\xi G^0{}_{\mu\nu}, \quad (5.15)$$

$$\delta G^1{}_{\mu\nu}=\mathcal{L}_\xi G^1{}_{\mu\nu}-\lambda G^1{}_{\mu\nu}\theta, \quad (5.16)$$

$$\delta G^2{}_{\mu\nu}=\mathcal{L}_\xi G^2{}_{\mu\nu}-(1-\lambda)G^2{}_{\mu\nu}\theta, \quad (5.17)$$

$$\delta G^3{}_{\mu\nu}=\mathcal{L}_\xi G^3{}_{\mu\nu}-G^3{}_{\mu\nu}\theta+G^0{}_{\mu\nu}\tau. \quad (5.18)$$

For $\lambda\neq 1$, the only invariants up to order 2 in the field strengths are $*G^0$ and $(*G^0)^2$. The first one is a total derivative that we ignore, while the second one gives a free theory for $b^0{}_\mu$. A zweibein postulate that linearly maps $b^0{}_\mu$ to an affine connection, $G^0{}_{\mu\nu}$, to the Riemann tensor and $(G^1{}_{\mu\nu},G^2{}_{\mu\nu})$ to the torsion does exist. However, since there is no nonfree invariant action, it will not lead to a 2D gravity model.

(b) Case $\mathfrak{na}_2=\mathrm{Span}\{t_0,t_3\}$. Taking now $\Sigma=\theta t_0+\tau t_1$, the transformation laws are

$$\delta b^0{}_\mu=\mathcal{L}_\xi b^0{}_\mu+\partial_\mu\theta, \quad (5.19)$$

$$\delta b^1{}_\mu=\mathcal{L}_\xi b^1{}_\mu+\partial_\mu\tau+\lambda(b^0{}_\mu\tau-b^1{}_\mu\tau), \quad (5.20)$$

$$\delta b^2{}_\mu=\mathcal{L}_\xi b^2{}_\mu-(1-\lambda)b^2{}_\mu\theta, \quad (5.21)$$

$$\delta b^3{}_\mu=\mathcal{L}_\xi b^3{}_\mu-b^3{}_\mu\theta-b^2{}_\mu\tau, \quad (5.22)$$

and

$$\delta G^0{}_{\mu\nu}=\mathcal{L}_\xi G^0{}_{\mu\nu}, \quad (5.23)$$

$$\delta G^1{}_{\mu\nu}=\mathcal{L}_\xi G^1{}_{\mu\nu}+\lambda(G^0{}_{\mu\nu}\tau-G^1{}_{\mu\nu}\tau), \quad (5.24)$$

$$\delta G^2{}_{\mu\nu}=\mathcal{L}_\xi G^2{}_{\mu\nu}-(1-\lambda)G^2{}_{\mu\nu}\theta, \quad (5.25)$$

$$\delta G^3{}_{\mu\nu}=\mathcal{L}_\xi G^3{}_{\mu\nu}-G^3{}_{\mu\nu}\theta-G^2{}_{\mu\nu}\tau. \quad (5.26)$$

It is clear from this last set of equation that the same conclusion as in case a holds.

Going through the list of solvable 4D real Lie algebras [19], we have found that the only invariants that occur are either a total derivative or provide a free theory for a B_μ component. All this speaks in favor of the uniqueness of the model in Sec. II within the class of Yang-Mills type models for 2D gravity.

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