

CURVE SINGULARITIES WITH ONE PUISEUX PAIR AND VALUE SETS OF MODULES OVER THEIR LOCAL RINGS

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ABSTRACT. In this paper we characterize the value set Δ of the R -modules of the form $R + zR$ for the local ring R associated to a germ ξ of an irreducible plane curve singularity with one Puiseux pair. In the particular case of the module of Kähler differentials attached to ξ , we recover some results of Delorme. From our characterization of Δ we introduce a proper subset of semimodules over the value semigroup of the ring R . Moreover, we provide a geometric algorithm to construct all possible semimodules in this subset for a given value semigroup.

1. INTRODUCTION

Let $\xi : f(x, y) = 0$ be the germ of an irreducible plane curve singularity with one Puiseux pair, i.e. a germ of plane curve with equation

$$f(x, y) = x^p - y^q + \sum_{ip+jq > pq} a_{i,j} x^i y^j,$$

where $\gcd(p, q) = 1$. For the local ring $R = \mathbb{C}\{x, y\}/(f)$ associated to the germ ξ , one has naturally a discrete valuation $v : R \rightarrow \mathbb{N} \cup \{\infty\}$ induced by the Puiseux parameterization of the curve. The set $\Gamma := v(R)$ has a natural structure of additive sub-semigroup of the monoid $(\mathbb{N}, +)$; it is called the semigroup of values of ξ and is minimally generated by p and q , i.e. $\Gamma = \langle p, q \rangle := \{n \in \mathbb{N} \mid n = ap + bq\}$. On the other hand, it is easy to check that for any semigroup of the form $\langle p, q \rangle$ with $\gcd(p, q) = 1$ there exists a germ of plane curve singularity admitting $\langle p, q \rangle$ as a semigroup of values. However, it is well known that not every numerical semigroup minimally generated by more than two elements can be realized as the semigroup of values of a plane curve. In fact, the family of numerical semigroups arising as value semigroup of an irreducible plane curve singularity was completely characterized by Teissier in [15, Chapter I, Proposition 3.2.1].

In 1978, Delorme [4] provided a complete characterization of the value set of the module of Kähler differentials $R + (dy/dx)R$ under the hypothesis of R being the local ring of an irreducible plane curve singularity with one Puiseux pair. Following his ideas, we pursue the generalization of Delorme's results for an arbitrary R -module of the form $R + zR$ with $z \in \bar{R}$, where \bar{R} is the normalization of the ring R . In general, given an R -module M its value set $\Delta_M := v(M)$ has

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a natural structure of Γ -semimodule, i.e. Δ_M satisfies the inclusion $\Gamma + \Delta_M \subset \Delta_M$ (see Section 2); the denomination “semimodule” refers to the analogy of this structure —for semigroups— with the R -module structure. Inspired by Teissier’s characterization of the semigroups of plane branches, we define a proper non-empty subset of the set of Γ -semimodule which we call *increasing semimodules* (see Definition 2.10). We prove that a Γ -semimodule Δ is an increasing semimodule if and only if there exists a plane curve singularity with local ring R such that $\Delta = v(R + zR)$ for some $z \in \bar{R}$ (see Theorem 3.6).

For a local ring R of a plane branch with one Puiseux pair, given a Γ -semimodule Δ , Piontkowsky in [12, Lemma 11, Theorem 12] provides a characterization of the possible R -modules M with $\Delta = v(M)$. After Piontkowsky results, it is easy to see that not every Γ -semimodule can be realized as value set of a module over the local ring. However, there is no combinatorial description of such a subset of Γ -semimodules, neither an effective way to compute them directly from the semigroup Γ . Thus, our main goal has been to provide a complete combinatorial classification of Γ -semimodules arising as value set of modules over the local ring. This classification also allows us to compute all such possible Γ -semimodules without the use of the Puiseux series of the branch.

First of all, we provide —with the help of the lattice path representation of the elements in $\mathbb{N} \setminus \Gamma$ due to the third author and Uliczka [9, 10]— a *constructive* algorithm to provide all possible increasing semimodules for a fixed Γ (see Section 2.3). After that, in a purely abstract way, our main Theorem (Theorem 3.1) reads as follows: given both a numerical semigroup $\Gamma = \langle p, q \rangle$ with two minimal generators and an increasing Γ -semimodule Δ , we can construct a Puiseux parameterization of a plane curve with value semigroup Γ whose local ring has an R -module M such that $\Delta_M = \Delta$. In particular, the existence of an increasing semimodule induces naturally a deformation of the parameterization of the curve, see Remark 3.2. This contrasts with the fact that —in the one Puiseux pair case— none of the results of [12, Section 3] put specific conditions over the Γ -semimodule Δ ; therefore, they do not provide a complete classification of value set of modules. From this point of view, Theorem 3.1 and Theorem 3.6 can be thought as a sort of analogue to Teissier’s result [15, Chapter I, Prop. 3.2.1] and hence an improvement of Piontkowsky results for the case of modules over the local ring associated to a branch with one Puiseux pair. These results can be considered as the main contributions of this work.

Several theorems have been obtained in the study of the moduli spaces of torsion-free R -modules of rank 1. Greuel and Pfister [6] established the general background to work with such moduli spaces. As for plane curves, the semimodule Δ_M can be thought as “the topological type of the R -module M ”; after fixing this invariant, it makes sense to talk about a moduli space of M with fixed Δ_M . Moreover, we know that the set of all possible Δ_M parametrizes up to isomorphism all torsion free R -modules of rank 1; this holds by the fact that every Δ_M has a representative as R invariant subspace of codimension $|\mathbb{N} \setminus \Gamma|$ in some Grassmanian (see Pfister and Steenbrink [11, Definition 1] together with [6, Remark 1.4] and Cherednik and Philipp [3, page 197]; see also Section 3).

An important application of the theory of moduli spaces of torsion free R -modules of rank 1 is the relation with the topology of the compactified jacobians of singular projective curves, as shown by Piontkowsky [12]. This has been extended to higher ranks by Cherednik and Philipp [3], showing its connection with some interesting topics in number theory (see [3]). Also,

Greuel and Knörrer [5], as well as Pfister and Steenbrink [11] have shown that the classification of equivalence classes in the moduli spaces of torsion free R -modules of rank 1 is useful to detect the analytic class of simple singularities in the sense of Arnol'd [2]. Our methods also allow us to identify this fact (see Section 3.2) in the plane irreducible case.

We have organized the paper as follows. Section 2 describes the main combinatorial aspects concerning our definition of increasing Γ -semimodules; these are an extension of those value sets corresponding to Kähler differentials. In Section 3 we present the realization of increasing Γ -semimodules as value sets of non-trivial R -modules; one can then see that the increasing Γ -semimodules coincide with the admissible semimodules introduced by Cherednik [3], and they provide in particular all possible “topological types of of torsion free R -modules of rank 1”. Hence we recover some results of Piontkowsky [12] and Pfister and Steenbrink [11]. As it was the original source of inspiration of this work, we finish by turning back to investigate the value sets of Kähler differentials. In particular, we show how our methods can be used to recover some constructions which appear in the literature.

2. NUMERICAL SEMIGROUPS AND SEMIMODULES

A numerical semigroup Γ is an additive sub-semigroup of the monoid $(\mathbb{N}, +)$ such that the greatest common divisor of all its elements is equal to 1. The complement $\mathbb{N} \setminus \Gamma$ is therefore finite, and its elements are called the gaps of Γ . Thus, Γ is finitely generated. The number $c(\Gamma) = \max(\mathbb{N} \setminus \Gamma) + 1$ is called the conductor of Γ .

A Γ -semimodule is a non-empty subset Δ of \mathbb{N} such that $\Delta + \Gamma \subseteq \Delta$. A system of generators of Δ is a subset \mathcal{E} of Δ with $\Delta = \bigcup_{x \in \mathcal{E}} (x + \Gamma)$; it is called minimal if no proper subset of \mathcal{E} generates Δ . Notice that, since $\Delta \setminus \Gamma$ is finite, every Γ -semimodule is finitely generated. Since $\Delta \setminus \Gamma$ is finite, there exist a conductor of Δ which is defined as $c(\Delta) = \max(\mathbb{N} \setminus \Delta) + 1$. Moreover, every Γ -semimodule Δ has a unique minimal system of generators (see e.g. [9, Lemma 2.1]). Two Γ -semimodules Δ and Δ' are called isomorphic if there is an integer n such that $x \mapsto x + n$ as bijection from Δ to Δ' ; we write then $\Delta \cong \Delta'$.

For every Γ -semimodule Δ there is a unique semimodule $\Delta' \cong \Delta$ containing 0; this semimodule is called normalized. Accordingly, the Γ -semimodule

$$\Delta^\circ := \{x - \min \Delta : x \in \Delta\}$$

is called the normalization of Δ ; the normalization of Δ is the unique Γ -semimodule isomorphic to Δ which contains 0.

The minimal system of generators $[x_0 = 0, x_1, \dots, x_s]$ of a normalized Γ -semimodule is what the third author and Uliczka called a Γ -lean set [9], i.e. it satisfies that

$$|x_i - x_j| \notin \Gamma \text{ for any } 0 \leq i < j \leq s,$$

and conversely, every Γ -lean set of \mathbb{N} minimally generates a normalized Γ -semimodule. Hence there is a bijection between the set of isomorphism classes of Γ -semimodules and the set of Γ -lean sets of \mathbb{N} , cf. [9, Corollary 2.3].

For specific material about numerical semigroups, the reader is referred to the books of Rosales and García Sánchez [14] and Ramírez Alfonsín [13].

2.1. Lattices paths. In this paper we will restrict our attention to numerical semigroups with two generators, say $\Gamma = \langle \alpha, \beta \rangle = \mathbb{N}\alpha + \mathbb{N}\beta$ for two integer numbers α, β with $1 < \alpha < \beta$ and $\gcd(\alpha, \beta) = 1$. In this case the conductor of Γ is simply $c = c(\langle \alpha, \beta \rangle) = (\alpha - 1)(\beta - 1)$. The gaps of $\langle \alpha, \beta \rangle$ are also easy to describe: they are of the form $\alpha\beta - a\alpha - b\beta$, where $a \in]0, \beta - 1]$ and $b \in]0, \alpha - 1]$. This description yields a map from the set of gaps of $\langle \alpha, \beta \rangle$ to \mathbb{N}^2 given by $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$, which allows us to identify a gap with a lattice point; since the gaps are positive numbers, the point lies inside the triangle $\mathcal{T}_{\alpha, \beta}$ with vertices $(0, 0)$, $(0, \alpha)$, $(\beta, 0)$.

In the following we will use the notation

$$x_i = \alpha\beta - a_i\alpha - b_i\beta \quad \text{or} \quad x_i = \alpha\beta - a(x_i)\alpha - b(x_i)\beta$$

for a gap x_i of the semigroup $\langle \alpha, \beta \rangle$. We can consider a partial ordering \preceq on the set of gaps as follows:

Definition 2.1. For gaps x_1, x_2 of $\langle \alpha, \beta \rangle$, we define

$$x_1 \preceq x_2 \quad :\iff \quad a_1 \leq a_2 \quad \wedge \quad b_1 \geq b_2$$

and

$$x_1 \prec x_2 \quad :\iff \quad a_1 < a_2 \quad \wedge \quad b_1 > b_2.$$

Let $\mathcal{E} = \{0, x_1, \dots, x_s\} \subseteq \mathbb{N} \setminus \Gamma$ be a subset of gaps of $\Gamma = \langle \alpha, \beta \rangle$ such that for every $i = 1, \dots, s$ it fulfills $a_1 < a_2 < \dots < a_s$. Corollary 3.3 in [9] ensures that \mathcal{E} is $\langle \alpha, \beta \rangle$ -lean if and only if $b_1 > b_2 > \dots > b_s$. This simple fact leads to an identification between an $\langle \alpha, \beta \rangle$ -lean set and a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the line joining these two points, where the lattice points identified with the gaps in \mathcal{E} mark the turns from the x -direction to the y -direction, see [9, Lemma 3.4]; these turns will be called ES-turns for abbreviation. As an example, Figure 1 shows the lattice path corresponding to the $\langle 5, 7 \rangle$ -lean set $[0, 9, 6, 8]$.

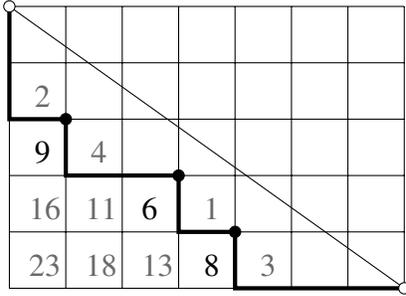


FIGURE 1. Lattice path for the $\langle 5, 7 \rangle$ -lean set $[0, 9, 6, 8]$.

Let $[g_0 = 0, g_1, \dots, g_s]$ be the minimal system of generators of a $\langle \alpha, \beta \rangle$ -semimodule Δ . In [9] the third author and Uliczka introduced the notion of syzygy of Δ as the $\langle \alpha, \beta \rangle$ -semimodule

$$\text{Syz}(\Delta) := \bigcup_{\substack{i, j \in \{0, \dots, s\} \\ i \neq j}} \left((\Gamma + g_i) \cap (\Gamma + g_j) \right).$$

According to [9, Theorem 4.2], for $\Gamma = \langle \alpha, \beta \rangle$ the syzygies of a normalized Γ -semimodule Δ can be characterized as follows:

Proposition 2.2. [9, Theorem 4.2] *Let Δ be a Γ -semimodule with minimal system of generators $[g_0 = 0, g_1, \dots, g_s]$. Assume that the minimal system of generators is ordered with the gap-order, i.e. $g_0 \prec g_1 \prec \dots \prec g_s$. Then the syzygy of Δ is the set*

$$\text{Syz}(\Delta) = \bigcup_{0 \leq k < j \leq s} \left((\Gamma + g_k) \cap (\Gamma + g_j) \right) = \bigcup_{k=0}^s (\Gamma + h_k),$$

where h_1, \dots, h_{s-1} are gaps of Γ , $h_0, h_s \leq \alpha\beta$, and

$$\begin{aligned} h_k &\equiv g_k \pmod{\alpha}, \quad h_k > g_k \text{ for } k = 0, \dots, s \\ h_k &\equiv g_{k+1} \pmod{\beta}, \quad h_k > g_{k+1} \text{ for } k = 0, \dots, s-1 \\ h_s &\equiv 0 \pmod{\beta}, \text{ and } h_s > 0. \end{aligned}$$

Remark 2.3. Observe that the modular conditions for the generators of the syzygy module $[h_0, \dots, h_s]$ give us explicit expressions for h_i in terms of the coordinates of the minimal system of generators of Δ .

Assume that we denote the minimal system of generators of Δ as $[g_0 = 0, g_1, \dots, g_s]$ and for any $i = 1, \dots, s$ we write $g_i = \alpha\beta - a_i\alpha - b_i\beta$ and $g_0 \prec g_1 \prec \dots \prec g_s$. Then,

$$h_i = \alpha\beta - a_{i-1}\alpha - b_i\beta.$$

Example 2.4. Let us consider again $\Gamma = \langle 5, 7 \rangle$ and the semimodule Δ with minimal system of generators $[0, 9, 6, 8]$ and lattice path as in Figure 1. Then the $\langle 5, 7 \rangle$ -semimodule $\text{Syz}(\Delta)$ is minimally generated by $h_0 = 15, h_1 = 13, h_2 = 16, h_3 = 14$.

The syzygies allowed the second and third authors [1] to give a formula for the conductor $c(\Delta)$ of Δ :

Theorem 2.5. [1, Theorem 1] *Let Δ be a Γ -semimodule. Let I be a minimal system of generators of Δ and J be a minimal system of generators of $\text{Syz}(\Delta)$. Let $M := \max_{\leq \mathbb{N}} \{h \in J\}$ denote the biggest, with respect to the order of the natural numbers, minimal generator of the syzygy module. Then*

$$c(\Delta) = M - \alpha - \beta + 1.$$

In particular, if we denote by (m_1, m_2) the point in the lattice \mathcal{L} representing M we have

$$c(\Delta) = c(\Gamma) - m_1\alpha - m_2\beta.$$

Remark 2.6. From now on, all minimum and maxima will be taken under the order of the natural numbers and will be denoted as $\max_{\leq \mathbb{N}}, \min_{\leq \mathbb{N}}$.

Let us order the minimal generators of Δ as $g_0 \prec g_1 \prec \dots \prec g_s$ and write

$$E_i = \bigcup_{0 \leq j \leq i} (\Gamma + g_j) \quad \text{for } 0 \leq i \leq s.$$

For any $i = 1, \dots, s$, the number $u_i := \min_{\leq \mathbb{N}} \{(\Gamma + g_i) \cap E_{i-1}\}$ will play an important role in the sequel. We may write u_i in terms of the coordinates in the lattice path \mathcal{L} :

Proposition 2.7. For any $i = 1, \dots, s$ and $g_i = \alpha\beta - a_i\alpha - b_i\beta$, let us define $a_0 = b_0 = 0$, $x_i = \alpha(\beta - a_i)$, $y_i = \beta(\alpha - b_i)$ and $m_i = \min_{\leq \mathbb{N}}(x_i, y_i)$. Let us assume that $g_0 \prec g_1 \prec \dots \prec g_s$. Then

$$u_i = \min_{\leq \mathbb{N}}\{\alpha\beta - a_{i-1}\alpha - b_i\beta, m_i\}.$$

Proof. Along the whole proof $\min := \min_{\leq \mathbb{N}}$. For every $i = 0, \dots, s$ and $j = 1, \dots, s$ set $\Omega_{i,j} := (\Gamma + g_i) \cap (\Gamma + g_j)$ and $\omega_{i,j} := \min \Omega_{i,j}$. It is easily checked that

$$\begin{aligned} \omega_{0,i} &= \min\{\beta(\alpha - b_i), \alpha(\beta - a_i)\} \\ \omega_{i,i+j} &= \alpha\beta - a_i\alpha - b_{i+j}\beta \end{aligned}$$

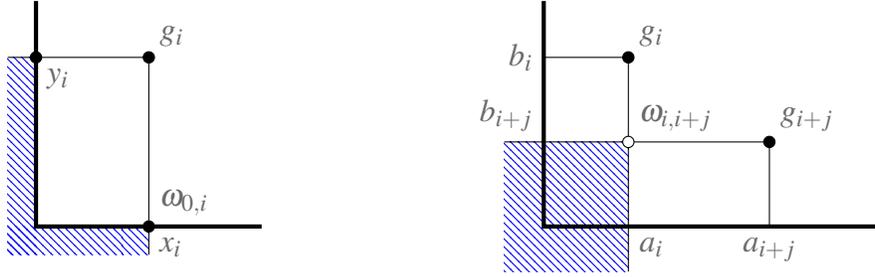


FIGURE 2. Shaded regions $(\Gamma + x_i) \cup (\Gamma + y_i)$ and $\Omega_{i,i+j}$.

Now for $i = 1, \dots, s$ we get

$$\begin{aligned} u_i &= \min\{(\Gamma + g_i) \cap E_{i-1}\} = \min\{(\Gamma + g_i) \cap \bigcup_{j=0}^{i-1} (\Gamma + g_j)\} \\ &= \min(\Omega_{0,i} \cup \Omega_{1,i} \cup \dots \cup \Omega_{i-1,i}) = \min\{\omega_{0,i}, \omega_{1,i}, \dots, \omega_{i-1,i}\}. \end{aligned}$$

Observe that $\omega_{j,i} < \omega_{j+1,i}$ for $j = 1, \dots, i-1$ since $a_k > a_{k+1}$ for all $k = 1, \dots, i-2$, therefore $u_i = \min\{\omega_{0,i}, \omega_{i-1,i}\}$, as desired. \square

Remark 2.8. Observe that if all u_i became of the form $\alpha\beta - a_{i-1}\alpha - b_i\beta$ then $u_i = h_i$, i.e. they are minimal generators of the syzygy semimodule.

Finally, we recall a few properties of the u_i 's previously defined which will be very useful in the sequel. These properties were already given by Delorme [4, Lemma 10] in a different context:

Lemma 2.9. [4, Lemma 10] Let $p, q \in \mathbb{Z}$ be such that $|p - q| \notin \Gamma$. We set

$$u := \min_{\leq \mathbb{N}}\{(\Gamma + p) \cap (\Gamma + q)\}$$

as well as $\bar{u} := u + c(\Gamma) - \alpha\beta$, $v := p + q + \alpha\beta - u$ and $\bar{v} := v + c(\Gamma) - \alpha\beta$. Then we have:

- (1) $(\Gamma + p) \cap (\Gamma + q) = (\Gamma + u) \cup (\Gamma + v)$,
- (2) $(\Gamma + p) \cup (\Gamma + q) = (\Gamma + u - \alpha\beta) \cap (\Gamma + v - \alpha\beta)$,
- (3) $\mathbb{N} + \bar{v} \subset (\Gamma + p) \cup (\Gamma + q)$.
- (4) $(\mathbb{N} + \bar{u}) \cap ((\Gamma + p) \cup (\Gamma + q)) = (\mathbb{N} + \bar{u}) \cap (\Gamma + v - \alpha\beta)$.

2.2. Increasing semimodules. From now on along the whole paper, when referring to the minimal set of generators $\{g_0, g_1, \dots, g_s\}$ of a Γ -semimodule Δ , we will assume that it is ordered by the natural order $g_0 <_{\mathbb{N}} g_1 <_{\mathbb{N}} \dots <_{\mathbb{N}} g_s$ unless we mention the contrary. For the scope of this paper we are interested in a particular subset of semimodules:

Definition 2.10. A Γ -semimodule L of a numerical semigroup S is called an *increasing semimodule* if it satisfies the following property:

(\boxtimes) If L has minimal set of generators $\{g_0 = 0, g_1, \dots, g_s\}$ and we put $g_{s+1} = \infty$, $u_0 = 0$, then for all $0 \leq i \leq s$ we have $g_{i+1} > u_i$, where $u_i = \min\{(\Gamma + g_i) \cap E_{i-1}\}$ for $1 \leq i \leq s$ and $E_i = \bigcup_{0 \leq j \leq i} (\Gamma + g_j)$ for $0 \leq i \leq s$.

Before continuing, let us mention the easy result that the class of increasing semimodules is non-empty:

Lemma 2.11. *Any normalized semimodule with two generators is increasing.*

Proof. Because of the minimal set of generators is of the form $\{0, g\}$ with $g \in \mathbb{N} \setminus \Gamma$, condition (\boxtimes) trivially holds since $g_2 = \infty$ and $u_0 = 0$. \square

Moreover, the set of increasing semimodules is a *proper* non-empty subset in the set of Γ -semimodules: consider for instance the $\langle 5, 7 \rangle$ -lean set $[0, 9, 6, 8]$ (see Figure 1); this is not an increasing semimodule since $\inf(\langle 5, 7 \rangle \cap (\langle 5, 7 \rangle + 6)) = 20 > 8$.

Lemmas 2.9 and 2.11 allow us to prove a key property which we will use in Section 3 for the characterization of increasing semimodules. This property was proven by Delorme [4, Lemma 12 (a)] for the case of an increasing semimodule with $g_1 = \beta - \alpha$. Here we present a proof for any increasing semimodule without restriction about the values of the minimal generators.

Lemma 2.12. *Let be $\Gamma = \langle \alpha, \beta \rangle$. Let L be an increasing Γ -semimodule with $[g_0 = 0, g_1, \dots, g_s]$, we set $g_{s+1} = \infty$ and $\bar{u}_i := u_i + c(\Gamma) - \alpha\beta$. Then, for any $i = 0, \dots, s-1$ there exists an element $c_i \in (-\Gamma)$, namely $c_i = c_{i-1} + g_i - u_i$, such that*

$$(\mathbb{N} + \bar{u}_i) \cap E_i = (\mathbb{N} + \bar{u}_i) \cap (\Gamma + c_i).$$

Proof. We proceed by induction on s . The case $s = 1$ is easily deduced from Lemma 2.9 with $p = g_0 = 0$, $q = g_1$; then $u_1 = \min\{(\Gamma + g_1) \cap \Gamma\} \in \Gamma$, therefore it is enough to consider

$$\begin{aligned} c_1 &= v - \alpha\beta = g_0 + g_1 + \alpha\beta - u_1 - \alpha\beta \\ &= g_1 - u_1 \in (-\Gamma). \end{aligned}$$

Since $\bar{u}_s < u_s < g_{s+1}$, we have $g_{s+1} \in \mathbb{N} + \bar{u}_s$; this together with the fact that $g_{s+1} \notin E_s$ implies that $g_{s+1} \notin \mathbb{N} + c_s$ by induction hypothesis. Therefore $g_{s+1} - c_s \notin \Gamma$ and we apply Lemma 2.9 (4) with $p = g_{s+1}$ and $q = c_s$, so that $u_{s+1} = \min\{(\Gamma + g_{s+1}) \cap E_s\}$. If we set $c_{s+1} = g_{s+1} - u_{s+1} + c_s$, then Lemma 2.9 again yields the equality

$$(\mathbb{N} + \bar{u}_{s+1}) \cap E_{s+1} = (\mathbb{N} + \bar{u}_{s+1}) \cap (\Gamma + c_{s+1}),$$

as desired. \square

Remark 2.13. From the proof of Lemma 2.12, we observe that we may relax the assumption $g_{i+1} > u_i$ and consider $g_{s+1} > \bar{u}_s$ instead.

2.3. Lattice paths of increasing semimodules. To conclude the section, we are going to show a procedure to construct any lattice path associated to an increasing semimodule. Recall that a semimodule can be represented as a lattice path in which the ES-turns correspond to the minimal set of generators of the semimodule and the SE-turns correspond to the minimal set of generators of the semimodule of syzygies.

An easy consequence of Lemma 2.11 is that any lattice path with a unique ES-turn is an increasing semimodule $\Delta^{(1)}$ generated by $g_0 = 0, g_1 \in \mathbb{N} \setminus \Gamma$. Let us write $J = [h_0, h_1]$ for the minimal set of generators of the syzygy semimodule $\text{Syz}(\Delta^{(1)})$; we will also use the writing $\text{Syz}(\Delta^{(1)}) = \langle J \rangle$. Therefore, it is a straightforward computation to check that $\min\{h_0, h_1\} = \min\{\Gamma \cap (\Gamma + g_1)\}$.

Write $u_1 = \min\{h_0, h_1\}$. There exists an increasing semimodule $\Delta^{(2)}$ with three generators containing $\Delta^{(1)}$ if and only if there is an element $g_2 \in \mathbb{N} \setminus \Delta^{(1)}$ with $g_2 > u_1$ such that $\Delta^{(2)} = \Delta^{(1)} \cup (\Gamma + g_2)$. Since u_1 is a generator of the syzygy module and g_2 must be a gap, then $g_2 > u_1$ means

$$g_2 \in \{(a, b) \in \mathbb{N}^2 : u_1 < \alpha\beta - a\alpha - b\beta\}.$$

On the other hand, condition $g_2 \in \mathbb{N} \setminus \Delta^{(1)}$ means that g_2 is a point above the lattice path associated to $\Delta^{(1)}$. Let us denote by $L(\Delta^{(1)})^+$ the region above the lattice path associated to $\Delta^{(1)}$. Hence the existence of $\Delta^{(2)}$ is equivalent to

$$L(\Delta^{(1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_1 < \alpha\beta - a\alpha - b\beta\} \neq \emptyset.$$

So, let us assume that we start with $\Delta^{(i-1)}$ minimally generated by $I = [g_0 = 0, g_1, \dots, g_{i-1}]$. Consider $u_{i-1} = \min\{(\Gamma + g_{i-1}) \cap E_{i-2}\}$. Observe that by construction

$$\text{Syz}(\Delta^{(i-1)}) = [u_1, \dots, u_{i-1}, M],$$

where $M = c(\Delta^{(i-1)}) + \alpha + \beta - 1$ by Theorem 2.5. We observe that we are ordering g_i here by the natural order in \mathbb{N} , and this ordering does not necessarily coincide with the order \prec . So, the indices in the minimal set of the syzygies may not coincide with those in Proposition 2.7.

As before, let us denote $L(\Delta^{(i-1)})^+$ the region above the lattice path associated to $\Delta^{(i-1)}$, then, there is a $g_i \in \mathbb{N} \setminus \Gamma$ with $g_i > u_{i-1}$ if and only if

$$L(\Delta^{(i-1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_{i-1} < \alpha\beta - a\alpha - b\beta\} \neq \emptyset.$$

Observe that the previous construction give us the following immediate consequence:

Proposition 2.14. *Let Δ be an increasing Γ -semimodule minimally generated by $\{g_0 = 0 <_{\mathbb{N}} \dots <_{\mathbb{N}} g_s\}$. Then, $u_1 <_{\mathbb{N}} \dots <_{\mathbb{N}} u_s <_{\mathbb{N}} M := c(\Delta) + \alpha + \beta - 1$ are the minimal set of generators of $\text{Syz}(\Delta)$.*

Remark 2.15. Propostion 2.14 shows that, in constrast to Proposition 2.7, the minimal set of generators of the syzygy semimodule of an increasing semimodule can be obtained with the natural order. Obviously the labeled may differ from the order established in the lattice path.

Example 2.16. To see how the ordering in the labeling of the minimal generators of the syzygy semimodule may differ, let us consider $\Gamma = \langle 7, 9 \rangle$ and the increasing Γ -semimodule Δ constructed in Example 2.17 with minimal set of generators $[0, 5, 20, 31]$. Observe that $\text{Syz}(\Delta)$ is minimally generated by $u_1 = 14, u_2 = 27, u_3 = 38$ and $c(\Delta) + \alpha + \beta - 1 = 40$. If we order the

minimal generators of the syzygy module with the order \prec then we have $h_0 = 14, h_1 = 40, h_2 = 38, h_3 = 27$.

The previous construction encloses a rooted tree structure over the set of increasing semimodules in terms of its first non zero minimal generator; the root corresponds to the semimodule associated to a gap of the semigroup, and represents the unique Γ -semimodule of the form $\Delta = [0, g]$. We assign this to the level 0 of the tree. The next level represent the possible increasing Γ -semimodules with three generators $[0, g_1, g_2]$ with $g_1 = g$; hence the number of leaves at this level is

$$|L(\Delta^{(1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_1 < \alpha\beta - a\alpha - b\beta\}|.$$

In general the number of nodes at a level k represent the number of increasing Γ -semimodules with g as first non zero generator and $k + 2$ minimal generators. To each node at level k we attach exactly

$$|L(\Delta^{(k+1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_{k+1} < \alpha\beta - a\alpha - b\beta\}|$$

leaves. Obviously, this tree representation is finite; observe that $k \leq \alpha$. Let us show the procedure with an example:

Example 2.17. Let us consider the semigroup $\Gamma = \langle 7, 9 \rangle$. We are going to construct all possible increasing Γ -semimodules with $g_1 = 5$. Since $u_1 = 14$, the first step of the above procedure says that there are 8 increasing Γ -semimodules with $g_1 = 5$ and 3 minimal generators, see Figure 3.

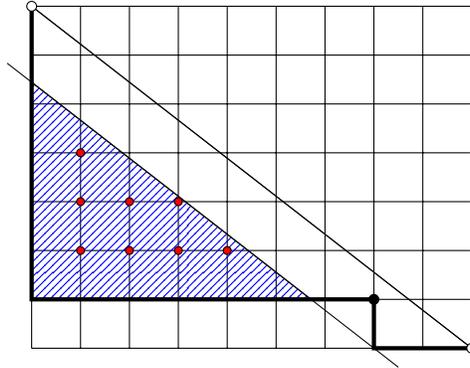


FIGURE 3. Lattice path for the $\langle 7, 9 \rangle$ -lean set $[0, 5]$ and the eight candidates for g_2 in red.

As a second step, let us choose $g_2 = 20$ as minimal generator. Then, $u_2 = 27$ and the following figure shows that there is only one increasing Γ -semimodule with $g_1 = 5, g_2 = 20$ and 4 minimal generators in total, namely $\Delta = [0, 5, 20, 31]$. Which has $u_3 = 38$.

for any $g, h \in R$. Notice that the property $g \in (f)$ reads also as the germ η contains the germ ξ , if η is the germ given by $\{g = 0\}$. Therefore, we can associate to a (germ of) plane branch the numerical semigroup defined by $\Gamma := \{v(x) \mid x \in R\}$; this is called the value semigroup of the curve. It is well known that $\delta := |\mathbb{N} \setminus \Gamma|$ coincides with $\dim_{\mathbb{C}} \bar{R}/R$ and the conductor of Γ is $c = 2\delta$ in the case of a *plane curve singularity*.

Consider a nonzero R -module $M \subset \bar{R}$, and its value set $\Delta_M := v(M)$. It is easily checked that Δ_M is a Γ -semimodule. Following [3, Sec. 2.1.2], the *degree* of M is defined as $\deg_{\bar{R}}(M) := \dim_{\mathbb{C}}(\bar{R}/M)$, which can be seen as the analogous to the delta invariant of the curve since $\deg_{\bar{R}}(R) =: \delta_R$. We also define the *deviation* of M as $\text{dev}(M) = \delta_R - \deg_{\bar{R}}(M)$. In general, Δ_M is not a normalized Γ -semimodule but this can be obtained by considering $t^{-\min(\Delta_M)}M$ instead of M . If Δ_M is a normalized Γ -semimodule we will say that M is a *standard module*. For a standard module we have $\deg_{\bar{R}}(M) = |\mathbb{N} \setminus \Delta_M|$ and $\text{dev}(M) = |\Delta_M \setminus \Gamma|$.

As pointed out by Cherednik and Philipp [3], the map $M \mapsto t^{\text{dev}(M)}M$ yields an identification between the set of standard R -modules and the set \mathcal{M} of all R -modules of degree $\delta = \delta_R$. After Pfister and Steenbrink [11, Section 2], as well as Greuel and Pfister [6, Remark 1.4], if $I(2\delta) = \{z \in \bar{R} \mid z \geq 2\delta\}$ the set \mathcal{M} can be identified with the subset $\text{Grass}(\delta, \bar{R}/I(2\delta))$ of δ -dimensional linear subspaces of the Grassmanian which are R -modules; the set \mathcal{M} parametrizes up to isomorphism all torsion-free R -modules of rank 1.

3.1. Characterization of increasing semimodules as value set of R -modules. From now on we will restrict ourselves to germs with value semigroup $\Gamma := \langle \alpha, \beta \rangle = \mathbb{N}\alpha + \mathbb{N}\beta$. The main goal of this subsection is to provide a characterization of increasing semimodules as value set of non-trivial R -modules, where R is the local ring of a branch with one Puiseux pair. Moreover, given an increasing Γ -semimodule we explicitly construct a Puiseux parameterization of a branch having an R -module with value set such a semimodule. This result can be thought as a sort of analogue to Teissier's theorem [15, Chap. I Prop. 3.2.1].

First of all, we are going to show that for fixed Γ and an increasing Γ -semimodule we can find a plane curve with value semigroup Γ , as well as non-zero functions (elements in the local ring) such that the associated R -module has as value set the given increasing semimodule.

Theorem 3.1. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with $\alpha < \beta$. Let L be an increasing Γ -semimodule, and set $b := c(\Gamma) - \beta - 1$. Then there exist a tuple $(a_1, \dots, a_b) \in \mathbb{C}^b$ and $z \in \bar{R}$ such that $L = v(R + zR)$, where R is the local ring of the germ of plane curve singularity defined by the Puiseux parameterization*

$$C: \begin{cases} x(t) := t^\alpha \\ y(t) := t^\beta + \sum_{i=1}^b a_i t^{i+\beta}. \end{cases}$$

Proof. We first introduce some notation and definitions. Let us denote by $g_0 = 0 < g_1 < \dots < g_s$ the minimal set of generators of L . Also set

$$\begin{aligned} E_i &:= \bigcup_{0 \leq j \leq i} (\Gamma + g_j), \text{ and } E_s := L \\ u_i &:= \min\{E_{i-1} \cap (\Gamma + g_i)\} \\ \sigma_{i+1} &:= \sum_{1 \leq j \leq i} (g_{j+1} - u_j) \text{ with } \sigma_0 = \sigma_1 = 0, \text{ for } i = 1, \dots, s \\ h_i &:= g_i - \sigma_i, \text{ for } i = 1, \dots, s+1 \\ I_i &:= [\sigma_{i-1}, \sigma_i] \cap \mathbb{N}, \text{ for } i = 2, \dots, s+1 \end{aligned}$$

Let us consider the polynomial ring $\mathbb{C}[X_1, \dots, X_b, T]$. We consider formal elements $Y, z \in \mathbb{C}[X_1, \dots, X_b, T]$ of the form

$$Y = T^\beta \left(1 + \sum_{i=1}^b X_i t^i\right), \quad z = T^{s_1} \left(c_0 + \sum_{i=1}^b c_i X_i t^i\right).$$

Step 1: We set $U_{\sigma_0}^0 = 1, U_{\sigma_1}^1 = z$. We will prove the existence of a family of polynomials

$$\left\{ \left\{ \{U_j^i\}_{j \in I_i} \right\}_{i=2, \dots, s+1} \right\} \subset \mathbb{C}[X_1, \dots, X_b, T]$$

such that

$$U_j^i = \left(X_j \prod_{k=2}^i A_k c_{k,j} + V_j \right) T^{j+h_i} + \sum_{r=j+1}^b \left(X_r \prod_{k=2}^i A_k c_{k,r} + V_r \right) T^{r+h_i} + \text{h.o.t.}$$

with $V_r \in \mathbb{C}[X_1, \dots, X_{r-1}]$ if $r \geq j$, $A_k \in \mathbb{C}[X_1, \dots, X_{\sigma_{i-1}}]$ and $c_{k,r} \in \mathbb{C}$.

Step 2: We consider the polynomials U_j^i as polynomials in the variable T and we observe that $A_k = \text{lc}_T(U_{\sigma_k}^k)$ the leading coefficient as polynomial in T . Set $\omega_k := U_{\sigma_k}^k$ for $k = 0, \dots, s$. By construction, we have that $\text{ord}_T(\omega_k) = g_k$ if $A_k \neq 0$ for $k = 0, \dots, s$ and for all $j \in (\sigma_{k-1}, \sigma_k)$ we have $\text{lc}_T(U_j^k) = 0$. We want to show that the system of polynomial equations defined by

$$(1) \quad \begin{cases} A_k \neq 0, & \text{for all } k \\ \text{lc}_T(U_j^k) = 0, & \text{for all } k \text{ and all } j \in (\sigma_{k-1}, \sigma_k) \\ \text{lc}_T(U_{\sigma_{s+1}}^{s+1}) = 0. \end{cases}$$

has a non-trivial compatible solution $(a_1, \dots, a_b) \in \mathbb{C}^b$.

Step 3: Finally, let us take a solution $(a_1, \dots, a_b) \in \mathbb{C}^b$ of the system (1). We can consider the ring morphism defined by

$$\begin{array}{ccc} \mathbb{C}[X_1, \dots, X_b, T] & \xrightarrow{ev} & \bar{R} = \mathbb{C}\{t\} \\ X_i & \mapsto & a_i \\ T & \mapsto & t. \end{array}$$

Therefore, we can define the germ ξ of plane curve singularity given by the following Puiseux parameterization:

$$\xi : \begin{cases} x(t) := t^\alpha \\ y(t) := ev(Y) = t^\beta + \sum_{i=1}^b a_i t^{i+\beta}. \end{cases}$$

If R stands for the local ring of the curve C , then $\Gamma = v(R)$. Moreover, it is easy to check that, by construction, the set $\{ev(\omega_k)\}$ is a minimal set of generators of the R -module $R + zR$. Therefore, we have $v(R + zR) = L$, since $\text{ord}_T(\omega_k) = \text{ord}_t(ev(\omega_k))$.

We conclude proving Steps 1 and 2.

Proof of Step 1: we apply induction. Define

$$U_{\sigma_0}^0 := 1 \quad U_{\sigma_1}^1 := T^{g_1}(c_0 + \sum_{i \geq 1} c_i X_i t^i)$$

It is easily checked that $U_{\sigma_0}^0, U_{\sigma_1}^1$ are of the required form. Let be $\varepsilon = e_1 \alpha + e_2 \beta \in \Gamma$, and write $P(\varepsilon) := T^{e_1 \alpha} Y^{e_2}$. Let us assume that for $i < k < s$ there exists this family of polynomials. We are going to construct $\{U_j^k\}$ for $j \in I_k$; first, we define $U_{\sigma_{k-1}}^k := P(u_{k-1} - g_{k-1})U_{\sigma_{k-1}}^{k-1}$. Since by induction hypothesis $U_{\sigma_{k-1}}^{k-1}$ is of the desired form, so is $U_{\sigma_{k-1}}^k$. Now, for $j \leq \sigma_k$ we define the corresponding U_j^k recursively:

- If $h_k + j \notin E_{k-1}$ we put

$$U_{j+1}^k := U_j^k - \text{lc}_T(U_j^k) T^{h_i+j}.$$

- If $h_k + j \in E_m \cap (\mathbb{N} \setminus E_m)$ for some $m < k$ we set

$$U_{j+1}^k := \text{lc}_T(U_{\sigma_m}^m) U_j^k - \text{lc}_T(U_j^k) U_{\sigma_m}^m.$$

Finally, it is a straightforward computation to check that in both cases U_{j+1}^k has the desired form.

Proof of Step 2: Observe that, by construction,

$$A_k = \prod_{j < k} A_j X_{\sigma_k} + V_{\sigma_k} \quad \text{with } V_{\sigma_k} \in \mathbb{C}[X_1, \dots, X_{\sigma_{k-1}}]$$

so that the condition $A_k \neq 0$ is equivalent to $X_{\sigma_k} \neq \frac{V_{\sigma_k}}{\prod_{j < k} A_j}$. Also observe that for $\ell \in (\sigma_{k-1}, \sigma_k)$ by definition

$$\text{lc}_T(U_\ell^k) = \prod_{j < k} c_{j,\ell} A_j X_\ell + V_\ell \quad \text{with } V_\ell \in \mathbb{C}[X_1, \dots, X_{\ell-1}]$$

so that $\text{lc}_T(U_\ell^k) = 0$ is equivalent to $X_\ell = \frac{V_\ell}{\prod_{j < k} c_{j,\ell} A_j}$.

Therefore, the system (1) can be rewritten as

$$(2) \quad \begin{cases} X_{\sigma_k} \neq \frac{V_{\sigma_k}}{\prod_{j < k} A_j} & \text{for all } k = 2, \dots, s \\ X_\ell = \frac{V_\ell}{\prod_{j < k} c_{j,\ell} A_j} & \text{for all } k = 2, \dots, s+1 \text{ and for all } \ell \in (\sigma_{k-1}, \sigma_k) \\ X_{\sigma_{s+1}} = \frac{V_{\sigma_{s+1}}}{\prod_{j < s+1} c_{j,\sigma_{s+1}} A_j}. \end{cases}$$

Finally, observe that $(\sigma_{i-1}, \sigma_i) \cap (\sigma_{j-1}, \sigma_j) = \emptyset$ if $i \neq j$. Since every isolated variable in the system is different, we can solve the system in the following recursive way. We start with $A_0, A_1 \in \mathbb{C}^*$, and for $\ell \in (\sigma_1, \sigma_2) = (0, g_2 - u_1)$ we have $X_\ell = 0$. Thus, $X_{\sigma_2} \neq 0$. Let us denote $a_{\sigma_2} = X_{\sigma_2} \in \mathbb{C}^*$. After that, since V_ℓ depends on variables of lower index than ℓ a recursive reasoning solves the system. \square

The systems (1) and (2) have more equations than needed in order to obtain the required semi-module; the reason is that, if $j \in (\sigma_{k-1}, \sigma_k)$ is such that $h_k + j \in E_{k-1}$, then U_j^k cannot be an element with $v(U_j^k) = g_k$, since $v(U_j^k) = h_k + j \in E_{k-1}$. Moreover, the last condition provides an element with $\sigma_{s+1} + h_{s+1} = c_{s+1} + \alpha\beta$. If we apply Lemma 2.12, it is trivial to see that $\mathbb{N} + c_s + \alpha\beta \subset E_s$, since $\mathbb{N} + c_s + \alpha\beta \subset \mathbb{N} + \bar{u}_s$; thus we can eliminate the last condition as well. As a result of that we may replace the system of equations given in the proof with the following:

$$(3) \quad \begin{cases} X_{\sigma_k} \neq \frac{V_{\sigma_k}}{\prod_{j < k} A_j} & \text{for all } k = 2, \dots, s \\ X_\ell = \frac{V_\ell}{\prod_{j < k} c_{j,\ell} A_j} & \text{for } k = 2, \dots, s+1 \text{ and for all } \ell \in (\sigma_{k-1}, \sigma_k) \\ & \text{such that } h_k + \ell \in \mathbb{N} \setminus E_{k-1} \end{cases}$$

Remark 3.2. A result of Zariski [16, Chap. VI Prop. 2.1] shows that any plane branch with one Puiseux pair is isomorphic to a deformation of the monomial curve given by $t \mapsto (t^\alpha, t^\beta)$. The system of equations (3) together with the proof of Theorem 3.1 show that if $(a_1, \dots, a_b) \in \mathbb{C}$ is a solution of this system, then the increasing Γ -semimodule L associated to the equation system (3) induces the deformation of the parameterization of the monomial curve defined by

$$\begin{cases} x(t) := t^\alpha \\ y(t) := t^\beta + \sum_{i=1}^b a_i t^{i+\beta}. \end{cases}$$

Remark 3.3. As pointed out by Piontkowsky [12, page 207], the free variables in the system (3) lead to different points in the Grassmanian $\text{Grass}(\delta, \bar{R}/I(2\delta))$.

Now, we prove a short of converse of Theorem 3.1. Furthermore, we show that the system (3) characterizes the increasing semimodules of semigroup Γ if the corresponding module is generated by a certain function.

Theorem 3.4. *Let $\xi : f(x, y) = 0$ denote the germ of a plane branch with value semigroup $\Gamma = \langle \alpha, \beta \rangle$, and let $R := \mathbb{C}\{x, y\}/(f)$ be its local ring. Let $h_1, h_2 \in \mathbb{C}\{x, y\}$ be such that $v(h_2) > v(h_1)$ and $v(h_2) - v(h_1) \notin \Gamma$. Then,*

- (a) $L := v(R + R\frac{h_2}{h_1})$ is an increasing semimodule.
- (b) If $\{g_0 = 0, g_1, \dots, g_s\}$ are minimal generators of L then for all $0 \leq i \leq s$ there exists a relation $\omega_{i+1} = \sum_{0 \leq j \leq i} F_{j,i} \omega_j$ where $F_{j,i} \in R$, $u_i = v(F_{i,i}) + g_i = \inf_j \{v(F_{j,i}) + g_j\}$ and $v(\omega_{i+1}) = g_{i+1}$.
- (c) If h_1, h_2 are such that

$$\frac{h_2(t)}{h_1(t)} = t^{s_1} \left(c_0 + \sum_{i \geq 1} c_i a_i t^i \right)$$

with $c_i \in \mathbb{C}^*$ for any i , and a_i are the coefficients of the Puiseux series of ξ for all i , then (a_1, \dots, a_b) satisfies the system (3).

Proof. Let us denote by ω_i the elements in $R + R\frac{h_2}{h_1}$ such that $v(\omega_i) = g_i$ for $0 \leq i \leq s$; these elements do exist because $\{g_0 = 0, g_1, \dots, g_s\}$ is the minimal set of generators of L .

First we observe that any element $z \in R + R\frac{h_2}{h_1}$ can be written as $z = z_1 + z_2\frac{h_2}{h_1}$, for $z_1, z_2 \in R$, therefore $g_0 = 0$ and $g_1 = v(h_2) - v(h_1) > u_0 = 0$. Hence the claim is true for $i = 1$ if we set $\omega_0 = 1$ and $\omega_1 = h_2/h_1$.

Before continuing, we notice that for all $i = 1, \dots, s$ there exists an element $\varpi \in R + R\frac{h_2}{h_1}$ with $v(\varpi) > u_i$ such that $\varpi = \sum_{r=0}^i \rho_r \omega_r$; this holds since, if $u_i = \min\{(\Gamma + g_i) \cap E_{i-1}\}$ then there exist $z, z' \in R$ such that $u_i = v(z) + g_i = v(z') + g_j$ for some $j < i$ and $v(z' \omega_j + z \omega_i) > u_i$. The referred element can be then constructed by taking $\varpi = z' \omega_j + z \omega_i$.

Reasoning by induction, let us assume that the result is true for $i \leq k < s$, i.e. for every $0 \leq j \leq k$ we have $g_{j+1} > u_j$ and there exists $\omega_{j+1} = \sum_{0 \leq r \leq j} F_{r,j} \omega_r$, where $F_{r,j} \in R$, $u_j = v(F_{j,j}) + g_j = \inf_r \{v(F_{r,j}) + g_r\}$ and $v(\omega_{j+1}) = g_{j+1}$; let us further suppose that the coefficients of the Puiseux series of ξ satisfies the conditions of the equation system (3) for $i \leq k < s$.

To finish the proof of (a) and (b) it remains to show that if there exist $f_0, f_1, \dots, f_r \in R$ such that $\omega_{k+1} = \sum_{r=0}^k f_r \omega_r$, then a reduction to the case $u_k \leq v(f_k) + g_k \leq v(f_j) + g_j$ with $v(f_k) + g_k \in \Gamma + u_k$ is possible, i.e. such that if any other combination of the elements ω_j has value g_{k+1} , then $f_k \neq 0$ and $v(f_k) = u_k - f_k$, hence we are in the previous situation.

In this way, let us consider $\omega_{k+1} = \sum_{r=0}^k f_r \omega_r$, if $f_k \neq 0$ there is nothing to prove and also we have

$$\min_{r \leq k} \{v(f_r \omega_r)\} = \min\{(\Gamma + g_i) \cap E_{i-1}\} = u_i.$$

So let us assume $f_k = 0$ and set $\ell(f) := \min_{r \leq k} \{v(f_r \omega_r)\}$. Let $j_1 < j_2 < k$ be the biggest indices such that $v(f_{j_1}) + g_{j_1} = v(f_{j_2}) + g_{j_2} = \ell(f)$. By induction hypothesis and Lemma 2.12 there

exists c_{j_2-1} such that $g_{j_2} - c_{j_2-1} \notin \Gamma$. In this way, we apply Lemma 2.9 with $p = g_{j_2}$, $q = c_{j_2-1}$. By part (1) of Lemma 2.9 we have $u = u_{j_2}$, $v = c_{j_2} + \alpha\beta$ and

$$(\Gamma + g_{j_2}) \cap (\Gamma + c_{j_2-1}) = (\Gamma + u_{j_2}) \cup (\Gamma + c_{j_2} + \alpha\beta).$$

Observe that $v(f_{j_2}) + g_{j_2} > \bar{u}_{j_2-1}$, hence $v(f_{j_2}) + g_{j_2} \in (\mathbb{N} + \bar{u}_{j_2-1}) \cap E_{j_2-1}$. Then Lemma 2.12 implies $v(f_{j_2}) + g_{j_2} \in \Gamma + c_{j_2-1}$. Therefore, we have two alternative cases: either $v(f_{j_2}) + g_{j_2} \in \Gamma + c_{j_2} + \alpha\beta$ or $v(f_{j_2}) + g_{j_2} \in \Gamma + u_{j_2}$. By Lemma 2.12 it holds that $v(f_{j_2}) + g_{j_2} \in \Gamma + c_{j_2}$, so the first option is not possible since $v(f_{j_2}) + g_{j_2} < g_{k+1} < \alpha\beta$. Therefore, the only case to be considered is $v(f_{j_2}) + g_{j_2} \in \Gamma + u_{j_2}$; here there exists $z \in R$ such that $v(f_{j_2} + zF_{j_2,j_2}) > v(f_{j_2})$

where $\omega_{j_2+1} = F_{j_2,j_2}\omega_{j_2} + \sum_{r=0}^{j_2-1} F_{r,j_2}\omega_r$ by induction hypothesis. From this, we can write

$$\omega_{k+1} = \sum_{r=0}^k f_r \omega_r + z\omega_{j_2+1} - z\omega_{j_2+1} = \sum_{r=0}^k f'_r \omega_r$$

with $f'_r = f_r + zF_{r,j_2}$ if $r \leq j_2$, $f'_{j_2+1} = f_{j_2+1} - z$ and $f'_r = f_r$ if $r > j_2 + 1$. In the sum on the right-hand side we have $f'_{j_2+1} \neq 0$ since $z \neq 0$, as well as $\ell(f') > \ell(f)$. Therefore, we proceed as before until the achievement of $u_k = \min_{r \leq k} \{v(f_r \omega_r)\}$.

Since $g_{i+1} > u_i$ for all $0 \leq i \leq s$, we obtain that L is an increasing semimodule.

It remains to prove (c), but this is just an application of the recursive process presented in the proof of Theorem 3.1 with $U_{\sigma_1}^1 = h_2(t)/h_1(t)$. \square

Remark 3.5. After Theorem 3.1 and Theorem 3.4, and for a local ring of a branch with one Puiseux pair, a Γ -semimodule is admissible in the sense of Cherednik [3, page 198] if and only if it is an increasing semimodule.

The previous discussion can be summarized in the following theorem.

Theorem 3.6. *Let $R \subset \mathbb{C}\{x, y\}$ be a local discrete valuation ring with $v(R) = \Gamma = \langle \alpha, \beta \rangle$. Then, Δ is an increasing Γ -semimodule if and only if there exist $z, f \in \mathbb{C}\{x, y\}$ such that R is the local ring of a germ of isolated plane curve singularity $\xi : f(x, y) = 0$ with semigroup of values Γ and $\Delta = v(R + zR)$.*

Moreover, in the particular case (c) the system of equations (3) characterizes the increasing semimodules in terms of the coefficients of the Puiseux expansion of ξ . In particular for a fixed ξ , the number of realizable increasing semimodules depends on the analytic type of ξ .

3.2. Characterization of monomial semigroups with two generators. Following Pfister and Steenbrink [11], let us define the concepts of monomial curve singularity and monomial semigroup. Along this subsection all the semigroups Γ are considered with any number of generators if the number of generators is not specified.

Definition 3.7. A *monomial curve singularity* over \mathbb{C} is an irreducible curve singularity with local ring isomorphic to $A := \mathbb{C}[[t^{a_1}, \dots, t^{a_m}]]$ for some $a_1, \dots, a_m \in \mathbb{N}$ with $\gcd(a_1, \dots, a_m) = 1$.

Definition 3.8. A numerical semigroup Γ is called *monomial* if each reduced and irreducible curve singularity with semigroup Γ is a monomial curve singularity.

Pfister and Steenbrink [11, Theorem 10] provide the following characterization of monomial semigroups:

Theorem 3.9. [11, Theorem 10] *Let Γ be a numerical semigroup. Then, Γ is a monomial semigroup if and only if the following property holds:*

(MP) *If $g \in \mathbb{N} \setminus \Gamma$ and $c(g) := \min\{n \in \mathbb{N} : [n, \infty) \subseteq \Gamma \cup (\Gamma + g)\}$, then $\Gamma \cap (\Gamma + g) \subseteq [c(g), \infty)$*

Observe, that in the case of $\Gamma = \langle \alpha, \beta \rangle$, we can translate property (MP) in terms of Γ -modules. We consider the Γ -semimodule $\Delta(g)$ minimally generated by $[0, g]$, and let $c(g) = c(\Delta(g))$ be its conductor. Recall that, by the results in Section 2.3, the set of increasing semimodules has a tree structure so that the existence of an increasing semimodule with three generators is conditioned on

$$L(\Delta^{(1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_1 < \alpha\beta - a\alpha - b\beta\} \neq \emptyset,$$

where $\Delta^{(1)} = \Delta(g)$ and $u_1 = \min\{\Gamma \cap (\Gamma + g)\}$. But the inclusion $\Gamma \cap (\Gamma + g) \subseteq [c(g), \infty)$ means that

$$L(\Delta^{(1)})^+ \cap \{(a, b) \in \mathbb{N}^2 : u_1 < \alpha\beta - a\alpha - b\beta\} = \emptyset.$$

Therefore, the property (MP) is equivalent to the fact that the only increasing semimodules are those generated by $[0, g]$. We may thus rewrite the Pfister-Steenbrink theorem [11, Theorem 10] for plane monomial curves as follows:

Theorem 3.10. *Let R be a discrete valuation ring, and write $\Gamma(R) := v(R)$ for its value set. Then R is the ring of an irreducible monomial plane curve singularity if and only if the only increasing $\Gamma(R)$ -semimodules are those of the form $[0, g]$ with $g \in \mathbb{N} \setminus \Gamma(R)$.*

Recall that in the sense of Arnol'd, a singularity is called simple if it has 0 modality. Thus, this theorem indeed provides a characterization of the unique simple irreducible plane singularities, i.e. those of the type A_s, E_6, E_8 in Arnol'd classification [2].

Moreover, Greuel and Knörrer [5] prove that those are the only singularities for which there are a finite number of isomorphism classes of torsion-free R -modules of rank 1. Observe that in this case, the equation system (3) is in fact empty.

4. APPLICATION TO KÄHLER DIFFERENTIALS

Let $\xi : f(x, y) = 0$ be an irreducible germ of plane curve singularity with one Puiseux pair, i.e. such that $\Gamma = \Gamma(\xi) = \langle \alpha, \beta \rangle$. Let us denote by R its local ring and by $R \rightarrow \bar{R}$ its normalization.

Recall that to any \mathbb{C} -algebra A we can associate the A -module $\Omega_A := I/I^2$, where I is the kernel of the diagonal surjection $A \otimes_{\mathbb{C}} A \rightarrow A$; we call Ω_A the module of Kähler differentials. The injection $R \hookrightarrow \bar{R}$ induces a morphism of R -modules $\varphi : \Omega_R \rightarrow \Omega_{\bar{R}}$. Now, the \bar{R} -module $\Omega_{\bar{R}}$ is free of rank 1 and it is generated by $dt := t \otimes 1 - 1 \otimes t$. On the other hand, the R -module Ω_R has two generators, namely

$$dx = \alpha t^{\alpha-1} \quad \text{and} \quad dy = \beta t^{\beta-1} + \sum_{j \geq \beta} j a_j t^{j-1}.$$

In this way we can see that $\varphi(\Omega_R)$ is a free sub- \bar{R} -module of rank 1 generated by $\alpha t^{\alpha-1}$.

We can define the set of values of the module of Kähler differentials of ξ as $\Delta' := v(\varphi(\Omega_R))$. Since $\min \Delta' = n - 1$ it is easy to see that

$$\Delta = (\Delta')^\circ = v\left(R + R\frac{dy}{dx}\right).$$

In the sequel, we will call $(\beta - \alpha)$ -increasing Γ -semimodule to an increasing Γ -semimodule with $g_0 = 0$, $g_1 = \beta - \alpha$. Observe that the Γ -semimodule of values of Kähler differentials, Δ , is a particular example of $(\beta - \alpha)$ -increasing Γ -semimodule. Recall that the set Δ is an analytic invariant of the curve ξ , as deduced from [4, §4].

Remark 4.1. Observe that Δ is the normalization of the value set associated to the fractional ideal $(f, \partial f/\partial x, \partial f/\partial y)$: a straightforward computation yields that

$$\frac{\partial f/\partial x(t)}{\partial f/\partial y(t)} = \frac{dy}{dx}.$$

Moreover, Hefez and Hernandes [8, Theorem 2.1] prove the following general result:

Theorem 4.2. [8, Theorem 2.1] *Let $\Gamma = \langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle$ be the semigroup of values of a plane branch ξ . Then, a Puiseux parameterization of ξ is analytically equivalent to either $(t^{\bar{\beta}_0}, t^{\bar{\beta}_1})$ or*

$$\left(t^{\bar{\beta}_0}, t^{\bar{\beta}_1} + t^\lambda + \sum_{\substack{i > \lambda \\ i \notin \Delta' - \bar{\beta}_0}} a_i t^i \right),$$

where λ is its Zariski invariant and Δ' is the non-normalized set of orders of differentials of the branch. Moreover, if φ and φ' are Puiseux parameterizations of the previous form with the same Γ and Δ' then they are analytically equivalent if and only if there is $r \in \mathbb{C}^*$ such that $r^{\lambda - \bar{\beta}_1} = 1$ and $a_i = r^{i - \bar{\beta}_1} a'_i$.

Remark 4.3. Theorem 4.2 was already stated by Delorme in [4, Proposition 6] under two restrictions (Condition (CE) and (CU) of [4, Proposition 6]) for the Λ set. Therefore, Hefez and Hernandes' theorem 4.2 could be stated as: the restrictions of Delorme for the Δ' set can be eliminated.

Observe that in the case $\Gamma = \langle \alpha, \beta \rangle$, the module $R + R(dy/dx)$ fulfils the hypothesis of our Theorem 3.6. This means that the equation system (3) fully characterizes the set of normalized values of Kähler differentials Δ . Moreover, Section 2.3 together with Theorem 3.6 shows that all the possible $\beta - \alpha$ -increasing Γ -semimodules can be realized as value sets of Kähler differentials. Hence our method provides an alternative way to compute all possible Δ sets to the one shown by Hefez and Hernandes in [7] which avoids the use of the Standard bases algorithm in the case of a local ring with one Puiseux pair (Compare [7, Algorithm 4.10] with the construction described in Section 2.3).

REFERENCES

1. P. Almirón, J.J. Moyano-Fernández, A formula for the conductor of a semimodule of a numerical semigroup with two generators, *Semigroup Forum* (2021), <https://doi.org/10.1007/s00233-021-10182-1>.
2. V.I. Arnol'd, *Critical points of smooth functions*, Proc. Int. Congress Math. Vancouver, Vol.1 (1974), 19–39.

3. I. Cherednik and I. Philipp, *Modules over plane curve singularities in any ranks and DAHA*, J. Algebra **520** (2019), 186–236.
4. C. Delorme, *Sur les modules des singularités des courbes planes*, Bull. Soc. Math. France **106** (1978), 417–446.
5. G.-M Greuel, H Knörrer *Einfache Kurvensingularitäten und torsionfreie Moduln*, Math. Ann, **270** (1985), 417-425.
6. G.-M Greuel, G Pfister *Moduli spaces for torsion free modules on curve singularities, I*, J. Algebraic Geom., **2** (1993), 81-135.
7. A. Hefez, M.E. Hernandez, *Standard bases for local rings of branches and their modules of differentials*, J. Symbolic Comput.,**42** (2007), 178-191.
8. A. Hefez, M.E. Hernandez, *The analytic classification of plane branches*, Bull. Lond. Math. Soc.,**43** (2011), 289-298.
9. J.J. Moyano-Fernández, J. Uliczka, *Lattice paths with given number of turns and numerical semigroups*, Sem. Forum **88**, no. 3, (2014), 631–646.
10. J.J. Moyano-Fernández, J. Uliczka, *Duality and syzygies for semimodules over numerical semigroups*, Sem. Forum **92**, no. 3, (2016), 675–690.
11. G. Pfister and J. H. M. Steenbrink, *Reduced Hilbert schemes for irreducible curve singularities*, J. Pure Appl. Algebra **77**, no.1 (1992), 103–116.
12. J. Piontkowski, *Topology of the compactified Jacobians of singular curves*, Math. Z. **55**, no.1 (2007), 195–226.
13. J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Lecture Series in Mathematics and Its Applications 30, Oxford U.P. (2005).
14. J. C. Rosales, P. A. García Sanchez, *Numerical Semigroups*, Springer (2009).
15. B. Teissier, *Appendix*, in [16], 1986.
16. O. Zariski, *Le probleme des modules pour les branches planes*, with an appendix by B. Teissier *Hermann*, 1986.

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