On the fermionization of the XYZ spin Heisenberg chain (algebra).

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(Dated: January 27, 2023)

We present a generalization of the Yang-Baxter relation (our first point) applicable to a one-dimensional anisotropic chain (XYZ) with creation and annihilation operators for fermions, instead of the usual relation with spins. In this way, we are able to research systems that do not satisfy the free-fermion restriction. The role of a sign associated to the *modulus* k of the Jacobi elliptic functions is crucial. We obtain a special property relating the products of local transition matrices with fermion operators and the terms of generalized Hamiltonian functions (our second point). With these two ground stages we prove the existence of a set of conserved quantities, among them our proposed Hamiltonian of an anisotropic fermion chain.

Keywords: fermion XYZ, non free fermion, Yang-Baxter, conserved quantities.

INTRODUCTION.

We are going to study a fermion chain Hamiltonian, which we denote as *fermion Heisenberg Hamiltonian*:

$$\mathcal{H}(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y}) = -\sum_{n=1}^{N}\Big\{ \mathbf{J_z}\,(\mathbf{n}_{n+1}\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n) + \frac{\mathbf{J_x}+\mathbf{J_y}}{2}\,(\mathbf{a}^\dagger_{n+1}\mathbf{a}_n - \mathbf{a}_{n+1}\mathbf{a}^\dagger_n) + \frac{\mathbf{J_x}-\mathbf{J_y}}{2}\,(\mathbf{a}_{n+1}\mathbf{a}_n - \mathbf{a}^\dagger_{n+1}\mathbf{a}^\dagger_n)\Big\}\,, \tag{1}$$

written in terms of the fermion creation and annihilation operators $(\mathbf{a}^\dagger_n, \mathbf{a}_n)$. The subindex n indicates the position in a linear chain with N sites. These operators satisfy the anticommutator relations: $\{\mathbf{a}^\dagger_{n_1}, \mathbf{a}^\dagger_{n_2}\} = \{\mathbf{a}_{n_1}, \mathbf{a}_{n_2}\} = \mathbf{0}$, $\{\mathbf{a}^\dagger_{n_1}, \mathbf{a}_{n_2}\} = \delta_{n_1 n_2}\mathbf{1}$. The $\mathbf{n}_n = \mathbf{a}^\dagger_n\mathbf{a}_n = \mathbf{n}^2_n$ represent the number operators, and we also denote the operators $\tilde{\mathbf{n}}_n = \mathbf{a}_n\mathbf{a}^\dagger_n = \mathbf{1} - \mathbf{n}_n = \tilde{\mathbf{n}}^2_n$.

This Hamiltonian is linked with the ferromagnetic Heisenberg Hamiltonian [1], which is the one for a spin quantum model in one dimension, also known as the XYZ model:

$$\begin{aligned} H(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y}) &= -\frac{1}{2}\sum_{n=1}^{N}\Big\{ \mathbf{J_z}\,\sigma^z_{n+1}\sigma^z_n + \mathbf{J_x}\,\sigma^x_{n+1}\sigma^x_n + \mathbf{J_y}\,\sigma^y_{n+1}\sigma^y_n\Big\} = \\ &= -\sum_{n=1}^{N}\Big\{ \frac{\mathbf{J_z}}{2}\,\sigma^z_{n+1}\sigma^z_n + \frac{\mathbf{J_x}+\mathbf{J_y}}{2}\,(\sigma^+_{n+1}\sigma^-_n + \sigma^-_{n+1}\sigma^+_n) + \frac{\mathbf{J_x}-\mathbf{J_y}}{2}\,(\sigma^+_{n+1}\sigma^+_n + \sigma^-_{n+1}\sigma^-_n)\Big\}\,, \end{aligned} \tag{2}$$

via a Jordan Wigner transformation [2], up to a term of the form: $\dfrac{\mathbf{J_z}}{2}\sum_{n=1}^{N}\mathbb{1}$.

The XYZ spin model was solved by R. J. Baxter [3] by relating it with a two-dimensional model in classical statistical physics, the eight-vertex model. Among the main concepts we have: the local transition matrices, the monodromy matrices and the transfer matrices. The key equations are the Yang-Baxter relations.

On one side, in the process from Heisenberg to Baxter it is worth mentioning the step for the solving of the XXZ spin model, corresponding to the six-vertex model (Lieb) [4]. There is a detailed exposition of these steps in the introduction in Takhtadzhan and Faddeev [5]. On the other hand, concerning our interest for the fermionization, the work of Felderhof [6] in an anisotropic XY model (a free fermion model with N even) and also the study of the fermion XXZ and of the Hubbard models [7] [8] [9] deserve attention.

In order to obtain some similar results for the general anisotropic fermion model (XYZ) to the ones obtained by Baxter with the spin model, we depart from the parametrization obtained by Baxter. We define the values of $\{\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y}\}$ in terms of Jocobi elliptic functions with *modulus* k:

$$\begin{cases} -\,\mathbf{J_z} = -\,\mathbf{J}\,(\,cn(2\eta,\,k)\,dn(2\eta,\,k)\,) = \beta^{-1}\,\alpha_1 \\ -\dfrac{\mathbf{J_x}+\mathbf{J_y}}{2} = -\,\mathbf{J}\,(\,1\,) = \beta^{-1}\,\gamma_1 \\ -\dfrac{\mathbf{J_x}-\mathbf{J_y}}{2} = -\,\mathbf{J}\,(\,k\,sn^2(2\eta,\,k)\,) = \beta^{-1}\,\delta_1\,,, \qquad (\mathbf{J_x}-\mathbf{J_y} = -2\,k\,\beta) \end{cases} , \qquad -\mathbf{J} = \beta^{-1} = \frac{1}{sn(2\eta,\,k))}\,. \tag{3}$$

and we rewrite the fermion Heisenberg Hamiltonian (1):

$$\begin{aligned} \mathcal{H}(\alpha_1,\beta,\ \delta_1) &= \beta^{-1}\sum_{n=1}^{N}\Big\{ \alpha_1(\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n) + (\mathbf{a}^\dagger_{n+1}\,\mathbf{a}_n - \mathbf{a}_{n+1}\,\mathbf{a}^\dagger_n) + \delta_1\,(\mathbf{a}_{n+1}\,\mathbf{a}_n - \mathbf{a}^\dagger_{n+1}\,\mathbf{a}^\dagger_n)\Big\} = \\ &= \beta^{-1}\sum_{n=1}^{N}\Big\{ \alpha_1(\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n) + (\mathbf{a}^\dagger_{n+1} + \delta_1\,\mathbf{a}_{n+1})\,\mathbf{a}_n - (\mathbf{a}_{n+1} + \delta_1\,\mathbf{a}^\dagger_{n+1})\,\mathbf{a}^\dagger_n\Big\}\,. \end{aligned} \tag{4}$$

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Accordingly, we define the self-adjoint *generating functions* $\widetilde{\mathcal{H}}(\pm k)$:

$$\widetilde{\mathcal{H}}(\pm k) = \widetilde{\mathcal{H}}\,(\,\alpha,\gamma,\beta,\ \delta(\pm k)\,) = \widetilde{\mathcal{H}}_{N+1,N} + \cdots + \widetilde{\mathcal{H}}_{n+1,n} + \cdots + \widetilde{\mathcal{H}}_{2,1} = \sum_{n=1}^{N}\widetilde{\mathcal{H}}_{n+1,n}(\,\alpha,\gamma,\beta,\ \delta(\pm k)\,) = \sum_{n=1}^{N}\widetilde{\mathcal{H}}_{n+1,n} =$$

$$= \beta^{-1}\sum_{n=1}^{N}\Big\{\beta\,(\mathbf{n}_{n+1}\,\tilde{\mathbf{n}}_n + \tilde{\mathbf{n}}_{n+1}\,\mathbf{n}_n) + \alpha\,(\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\,\tilde{\mathbf{n}}_n) + \gamma\,(\mathbf{a}^{\dagger}_{n+1}\,\mathbf{a}_n - \mathbf{a}_{n+1}\,\mathbf{a}^{\dagger}_n) + \delta(\pm k)\,(\mathbf{a}_{n+1}\,\mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\,\mathbf{a}^{\dagger}_n)\Big\} =$$

$$= \beta^{-1}\sum_{n=1}^{N}\Big\{(\alpha\mathbf{n}_{n+1} + \beta\,\tilde{\mathbf{n}}_{n+1})\,\mathbf{n}_n + (\alpha\tilde{\mathbf{n}}_{n+1} + \beta\,\mathbf{n}_{n+1})\,\tilde{\mathbf{n}}_n + (\gamma\,\mathbf{a}^{\dagger}_{n+1} + \delta(\pm k)\,\mathbf{a}_{n+1})\,\mathbf{a}_n - (\gamma\,\mathbf{a}_{n+1} + \delta(\pm k)\,\mathbf{a}^{\dagger}_{n+1})\,\mathbf{a}^{\dagger}_n\Big\} = \tag{5}$$

$$= \sum_{n=1}^{N}\Big\{\mathbf{1}\Big\} + \mathcal{H}\,(\alpha_1,\beta,\delta_1(\pm k))\,(\lambda-\eta) + \beta^{-1}\sum_{n=1}^{N}\Big\{\alpha_2(\,\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\,\tilde{\mathbf{n}}_n) + \delta_2(\pm k)\,(\,\mathbf{a}_{n+1}\,\mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\,\mathbf{a}^{\dagger}_n\,)\Big\}(\lambda-\eta)^2 + \dots\,.$$

We write the relative $\pm$ signs for its usefulness in the expressions that follows (up to the equations in (27)). The functions $\alpha,\ \beta,\ \gamma$ and δ parametrized in the following way with Jacobi elliptic functions and with their series developments are:

$$\begin{array}{llll}
\alpha(\lambda,\eta,\ k) = & sn(\,\lambda+\eta\,,\,k\,) & = \displaystyle\sum_{m=0}^{\infty}\alpha_m(\eta\,,\,k)\,(\lambda-\eta)^m \\
\gamma(\lambda,\eta,\ k) = & sn(\,\lambda-\eta\,,\,k\,) & = \displaystyle\sum_{m=1}^{\infty}\gamma_m(\eta\,,\,k)\,(\lambda-\eta)^m & . \\
\beta(\ \eta\,,\ k) = & sn(\ 2\eta\,,\,k\,) & \\
\delta(\lambda,\eta,\pm k) = & \pm k\ \beta(\eta,k)\ \alpha(\lambda,\eta,k)\ \gamma(\lambda,\eta,k) & = \displaystyle\sum_{m=1}^{\infty}\delta_m(\eta,\pm k)\,(\lambda-\eta)^m
\end{array} \tag{6}$$

An important remark: the functions α , β and γ are even in the *modulus* k (actually they are functions of k^2), meanwhile δ is odd: $\alpha(k) = \alpha(-k)$, $\beta(k) = \beta(-k)$ and $\gamma(k) = \gamma(-k)$, but $\delta(k) = -\delta(-k)$. This is one of the keystones of our construction.

The functions $\widetilde{\mathcal{H}}_{n+1,n}(\pm k)$ satisfy:

$$\widetilde{\mathcal{H}}_{n+1,n}(\pm k) \ = \ \widetilde{\mathcal{H}}_{n,n+1}(\mp k)\,. \tag{7}$$

THE RELATION WITH THE STATISTICAL MODEL.

Going back to the eight-vertex model, we define the *fermion local transition matrices*:

$$\mathcal{L}_n(\lambda,\pm k) = \begin{pmatrix} \mathrm{i}\,\alpha\ \mathbf{n}_n + \gamma\ \tilde{\mathbf{n}}_n & \mathrm{i}\,\delta(\pm k)\ \mathbf{a}^{\dagger}_n + \beta\ \mathbf{a}_n \\ -\mathrm{i}\beta\ \mathbf{a}^{\dagger}_n + \delta(\pm k)\ \mathbf{a}_n & -\mathrm{i}\,\gamma\ \mathbf{n}_n + \alpha\ \tilde{\mathbf{n}}_n \end{pmatrix}\ . \tag{8}$$

We can prove the following *fermion Yang-Baxter type relations*:

$$\boxed{\mathcal{R}\,(\mu-\nu,\ \pm k)\ \ [\,\mathcal{L}_n(\mu\,,\mp k) \underset{s}{\otimes} \mathcal{L}_n(\nu\,,\pm k)\,]\ \ = \ [\,\mathcal{L}_n(\nu\,,\mp k) \underset{s}{\otimes} \mathcal{L}_n(\mu\,,\pm k)\,]\ \ \mathcal{R}\,(\mu-\nu\,,\pm k)\,,} \tag{9}$$

with:

$$\mathcal{R}(\mu-\nu\,,\pm k) = \begin{pmatrix} a(\mu-\nu) & 0 & 0 & \mathrm{i}\,d(\mu-\nu\,,\pm k) \\ 0 & b & \mathrm{i}\ c(\mu-\nu) & 0 \\ 0 & -\mathrm{i}\ c(\mu-\nu) & b & 0 \\ -\mathrm{i}\,d(\mu-\nu\,,\pm k) & 0 & 0 & a(\mu-\nu) \end{pmatrix}\ , \tag{10}$$

$$\Big\{\,a(\mu-\nu) = sn(\mu-\nu+2\eta\,,k)\,, \qquad c(\mu-\nu) = sn(\mu-\nu\,,k)\,, \qquad b = \beta\,, \qquad d(\mu-\nu,\pm k) = \pm k\ b\ a(\mu-\nu)\ c(\mu-\nu)\,\Big\}\,, \tag{11}$$

and $\det\,\mathcal{R} = (a^2-d^2)\,(b^2-c^2) = a^2(1-k^2b^2c^2)\,(b^2-c^2)$. $\mathcal{R}$ has an inverse except for a discrete set of values. The $\underset{s}{\otimes}$ is the symbol for a Grassmann direct product defined by $[\mathcal{A} \underset{s}{\otimes} \mathcal{B}]_{ik,jl} = (-1)^{[P(i)+P(j)]P(k)}\ \mathcal{A}_{ij}\mathcal{B}_{kl}$ with $P(1) = 0$ and $P(2) = 1$. [10] [11]

Using recursively the product with direct products of fermion local transition matrices in the Yang-Baxter relations, we get:

$$\mathcal{R}\,(\mu-\nu,\pm k)\ \overleftarrow{\prod_{n=1}^{N}}\Big[\,\mathcal{L}_n(\mu,\mp k) \underset{s}{\otimes} \mathcal{L}_n(\nu,\pm k)\Big] \ = \ \overleftarrow{\prod_{n=1}^{N}}\Big[\,\mathcal{L}_n(\nu,\mp k) \underset{s}{\otimes} \mathcal{L}_n(\mu,\pm k)\Big]\ \mathcal{R}\,(\mu-\nu,\pm k)\,. \tag{12}$$

'Ultra locality' of the $\mathcal{L}_n$ operators. We demonstrate the property:

$$\Big[\mathcal{L}_{n+1}(\mu,\mp k)\underset{s}{\otimes}\mathcal{L}_{n+1}(\nu,\pm k)\Big]\Big[\mathcal{L}_n(\mu,\mp k)\underset{s}{\otimes}\mathcal{L}_n(\nu,\pm k)\Big] = \Big[\mathcal{L}_{n+1}(\mu,\mp k)\ \mathcal{L}_n(\mu,\mp k)\Big]\underset{s}{\otimes}\Big[\mathcal{L}_{n+1}(\nu,\pm k)\ \mathcal{L}_n(\nu,\pm k)\Big],$$

and we also obtain:

$$\overleftarrow{\prod_{n=1}^{N}}\Big[\mathcal{L}_n(\mu,\mp k)\underset{s}{\otimes}\mathcal{L}_n(\nu,\pm k)\Big] = \Big[\overleftarrow{\prod_{n=1}^{N}}\mathcal{L}_n(\mu,\mp k)\Big]\underset{s}{\otimes}\Big[\overleftarrow{\prod_{n=1}^{N}}\mathcal{L}_n(\nu,\pm k)\Big]. \tag{13}$$

The definitions of the *fermion monodromy matrices*:

$$\overleftarrow{\mathcal{T}}_N(\lambda,\pm k) = \overleftarrow{\prod_{n=1}^{N}}\mathcal{L}_n(\lambda,\pm k) = \mathcal{L}_N(\lambda,\pm k)\ \ldots\ \mathcal{L}_n(\lambda,\pm k)\ \ldots\ \mathcal{L}_1(\lambda,\pm k), \tag{14}$$

with (12) and (13) drive to:

$$\overleftarrow{\mathcal{T}}_N(\mu,\mp k)\underset{s}{\otimes}\overleftarrow{\mathcal{T}}_N(\nu,\pm k) = \mathcal{R}^{-1}(\mu-\nu,\pm k)\Big[\overleftarrow{\mathcal{T}}_N(\nu,\mp k)\underset{s}{\otimes}\overleftarrow{\mathcal{T}}_N(\mu,\pm k)\Big]\mathcal{R}(\mu-\nu,\pm k), \tag{15}$$

valid by analytic continuation for all the values of μ and ν, which include those ones for which the determinant of $\mathcal{R}$ is zero.

With the definition of a *supertrace*:

$$\mathrm{str}(\mathcal{L}) = \mathrm{tr}(\sigma^Z\mathcal{L}), \tag{16}$$

it is easy to prove:

$$\mathrm{str}\Big(\mathcal{L}_N(\lambda)\ldots\mathcal{L}_n(\lambda)\ldots\mathcal{L}_1(\lambda)\Big) = \mathrm{str}\Big(\mathcal{L}_{N-1}(\lambda)\ldots\mathcal{L}_n(\lambda)\ldots\mathcal{L}_1(\lambda)\mathcal{L}_N(\lambda)\Big) = \mathrm{str}\Big(\mathcal{L}_1(\lambda)\mathcal{L}_N(\lambda)\ldots\mathcal{L}_n(\lambda)\ldots\mathcal{L}_2(\lambda)\Big), \tag{17}$$

and therefore all the cyclic possibilities. We use (17) and their cyclic possibilities for obtaining the results in equations (29).

The *fermion transfer matrices*:

$$\mathfrak{T}_N(\lambda,\pm k) = \mathrm{str}\Big(\overleftarrow{\mathcal{T}}_N(\lambda,\pm k)\Big) = \mathrm{str}\Big(\overleftarrow{\prod_{n=1}^{N}}\mathcal{L}_n(\lambda,\pm k)\Big) = \mathrm{str}\Big(\overleftarrow{\prod_{n=1}^{N}}\Big[\sum_{m_n=0}^{\infty}{}_{m_n}\mathcal{L}_n(\pm k)\,(\lambda-\eta)^{m_n}\Big]\Big). \tag{18}$$

We also apply the supertrace to equations of the type $\overleftarrow{\mathcal{T}}\underset{s}{\otimes}\overleftarrow{\mathcal{T}'}$, as in (15), in the following way:

$$\mathrm{str}\Big[\overleftarrow{\mathcal{T}}\underset{s}{\otimes}\overleftarrow{\mathcal{T}'}\Big] = \mathrm{tr}\Big[(\sigma^Z\otimes\sigma^Z)\Big(\overleftarrow{\mathcal{T}}\underset{s}{\otimes}\overleftarrow{\mathcal{T}'}\Big)\Big] = \mathrm{tr}\Big[\Big(\sigma^Z\,\overleftarrow{\mathcal{T}}\Big)\otimes\Big(\sigma^Z\,\overleftarrow{\mathcal{T}'}\Big)\Big] = \Big[\mathrm{str}\Big(\overleftarrow{\mathcal{T}}\Big)\Big\}\Big\{\mathrm{str}\Big(\overleftarrow{\mathcal{T}'}\Big)\Big]. \tag{19}$$

With the supertrace in (15), using $[\,\sigma^Z\otimes\sigma^Z,\mathcal{R}\,] = [\,\sigma^Z\otimes\sigma^Z,\mathcal{R}^{-1}\,] = 0$, we obtain:

$$\mathfrak{T}_N(\mu,\mp k)\ \mathfrak{T}_N(\nu,\pm k) = \mathfrak{T}_N(\nu,\mp k)\ \mathfrak{T}_N(\mu,\pm k). \tag{20}$$

THE CONSERVED QUANTITIES. THE FERMION HEISENBERG HAMILTONIANS.

The first terms of the fermion local transition matrices (8) with the substitutions in (6) are:

$$\mathcal{L}_n(\eta) = \beta\begin{pmatrix}\mathrm{i}\,\mathbf{n}_n & \mathbf{a}_n\\ -\mathrm{i}\,\mathbf{a}_n^\dagger & \tilde{\mathbf{n}}_n\end{pmatrix}, \qquad \text{with} \qquad \mathcal{L}_n^{-1}(\eta) = \beta^{-2}\mathcal{L}_n^\dagger(\eta) = \beta^{-1}\begin{pmatrix}-\mathrm{i}\,\mathbf{n}_n & \mathrm{i}\,\mathbf{a}_n\\ \mathbf{a}_n^\dagger & \tilde{\mathbf{n}}_n\end{pmatrix}. \tag{21}$$

We obtain:

$$\begin{cases}\Big[\mathcal{L}_{n+1}(\lambda,\pm k)\ \mathcal{L}_n(\eta)\Big] = \Big[\mathcal{L}_{n+1}(\eta)\ \mathcal{L}_n(\eta)\Big]\ \widetilde{\mathcal{H}}_{n+1,n}(\pm k)\\ \Big[\mathcal{L}_n(\eta)\ \mathcal{L}_{n-1}(\lambda,\pm k)\Big] = \widetilde{\mathcal{H}}_{n,n-1}(\mp k)\ \Big[\mathcal{L}_n(\eta)\ \mathcal{L}_{n-1}(\eta)\Big]\end{cases}. \tag{22}$$

By imposing one of the fermion local transition matrices defined in $\lambda=\eta$ (in the n^{th} position of the chain) we get:

$$\Big[\mathcal{L}_N(\lambda)\ \ldots\ \mathcal{L}_{n+1}(\lambda)\,\mathcal{L}_n(\eta)\,\mathcal{L}_{n-1}(\lambda)\ \ldots\ \mathcal{L}_1(\lambda)\Big](\pm k) = \Big\{\overleftarrow{\prod_{j=1}^{n-1}}\widetilde{\mathcal{H}}_{j+1,j}(\mp k)\Big\}\Big[\overleftarrow{\mathcal{T}}_N(\eta)\Big]\Big\{\overleftarrow{\prod_{i=n}^{N-1}}\widetilde{\mathcal{H}}_{i+1,i}(\pm k)\Big\}. \tag{23}$$

We fix for the periodic boundary case: $\mathbf{a}_{N+1}^\dagger=\mathbf{a}_1^\dagger$, $\mathbf{a}_{N+1}=\mathbf{a}_1$, $\mathcal{L}_{N+1}=\mathcal{L}_1$ and $\widetilde{\mathcal{H}}_{N+1,N}=\widetilde{\mathcal{H}}_{1,N}$. In particular:

$$\begin{cases}\Big[\Big(\mathcal{L}_N(\lambda)\ \ldots\mathcal{L}_n(\lambda)\ \ldots\mathcal{L}_2(\lambda)\Big)\mathcal{L}_1(\eta)\Big](\pm k) = \overleftarrow{\mathcal{T}}_N(\eta)\ \Big(\overleftarrow{\prod_{i=1}^{N-1}}\widetilde{\mathcal{H}}_{i+1,i}(\pm k)\Big)\\ \Big[\mathcal{L}_N(\eta)\Big(\mathcal{L}_{N-1}(\lambda)\ldots\mathcal{L}_n(\lambda)\ldots\ \mathcal{L}_1(\lambda)\Big)\Big](\mp k) = \Big(\overleftarrow{\prod_{j=1}^{N-1}}\widetilde{\mathcal{H}}_{j+1,j}(\pm k)\Big)\ \overleftarrow{\mathcal{T}}_N(\eta)\end{cases}, \quad (\,\widetilde{\mathcal{H}}_{1,N}\ \text{skipped}\,). \tag{24}$$

Using the series developments that appear in (18) and in (5) with (6):

$$\mathcal{L}_n(\lambda,\pm k) = \sum_{m_n=0}^{\infty} {}_{m_n}\mathcal{L}_n(\eta,\pm k)\,(\lambda-\eta)^{m_n} \qquad \text{and} \qquad \widetilde{\mathcal{H}}(\lambda,\pm k) = \sum_{n=1}^{N}\sum_{m_n=0}^{\infty} {}_{m_n}\widetilde{\mathcal{H}}_{n+1,n}(\eta,\pm k)\,(\lambda-\eta)^{m_n}\,, \tag{25}$$

we write the fermion transfer matrices (18) in a series development, up to the order $N-1$ (we use the symbol $\doteqdot$), by substituting (25) in (22)-(24) and their cyclic possibilities:

$$\begin{array}{l} \mathfrak{T}_N(\lambda,\pm k) \doteqdot \mathfrak{T}_N(\eta)\,\left(\sum_{l=0}^{N-1} C_l(\eta,\pm k)\,(\lambda-\eta)^l\right) \\ \mathfrak{T}_N(\lambda,\mp k) \doteqdot \left(\sum_{l=0}^{N-1} C_l(\eta,\pm k)\,(\lambda-\eta)^l\right)\,\mathfrak{T}_N(\eta) \end{array}, \qquad \mathfrak{T}_N(\eta) = \mathrm{str}\left[\overleftarrow{\mathcal{T}}_N(\eta)\right], \tag{26}$$

where the C_l are sums of products of various ${}_{m_n}\widetilde{\mathcal{H}}_{n+1,n}(\pm k)$, with the $\pm k$ in the way that they appear in (5) and (25). The ordering in the products of the terms ${}_{m_n}\widetilde{\mathcal{H}}_{n+1,n}(\pm k)$ is relevant. For the first order we have:

$$\mathfrak{T}_N(\eta)\,{}_1\widetilde{\mathcal{H}}\big(\alpha_1(\eta,k),\,\gamma=1,\,\beta(\eta,k),\,\delta_1(\eta,\pm k)\big) = {}_1\widetilde{\mathcal{H}}\big(\alpha_1(\eta,k),\,\gamma=1,\,\beta(\eta,k),\,\delta_1(\eta,\mp k)\big)\,\mathfrak{T}_N(\eta)\,. \tag{27}$$

After this point, our expressions will have the same $\pm k$ at both sides of the equal symbol, so that with the positiveness and negativeness of the value k (which affects the δ functions), we do not write the $\pm$ from now on.

The first three terms and the last one for the developments in (26) are:

$$\begin{aligned} C_0(\eta,k) &= \mathbf{1} \\ C_1(\eta,k) &= {}_1\widetilde{\mathcal{H}}\big(\alpha_1(\eta,k),\,\gamma=1,\,\beta(\eta,k),\,\delta_1(\eta,k)\big) \\ C_2(\eta,k) &= \Big\{\sum_{n=1}^{N} {}_2\widetilde{\mathcal{H}}_{n,n+1} + \Big[\sum_{m=2}^{N-1} {}_1\widetilde{\mathcal{H}}_{m+1,m}\Big(\sum_{n=1}^{m-1} {}_1\widetilde{\mathcal{H}}_{n+1,n}\Big) + {}_1\widetilde{\mathcal{H}}_{1,N}\Big(\sum_{n=2}^{N-1} {}_1\widetilde{\mathcal{H}}_{n+1,n}\Big) + {}_1\widetilde{\mathcal{H}}_{2,1}\,{}_1\widetilde{\mathcal{H}}_{1,N}\Big]\Big\}(\eta,k) \\ &\bullet\ \bullet\ \bullet \\ C_{N-1}(\eta,k) &= \Big\{\ldots+\big({}_1\widetilde{\mathcal{H}}_{N,N-1}\ldots{}_1\widetilde{\mathcal{H}}_{2,1}\big)+\ldots+\big({}_1\widetilde{\mathcal{H}}_{n-1,n-2}\ldots{}_1\widetilde{\mathcal{H}}_{1,N}\ldots{}_1\widetilde{\mathcal{H}}_{n+1,n}\big)+\ldots+\big({}_1\widetilde{\mathcal{H}}_{N-1,N-2}\ldots{}_1\widetilde{\mathcal{H}}_{1,N}\big)\Big\}(\eta,k) \\ &\quad(\ \text{without}\ {}_1\widetilde{\mathcal{H}}_{1,N} \qquad\qquad \text{without}\ {}_1\widetilde{\mathcal{H}}_{n,n-1} \qquad\qquad \text{without}\ {}_1\widetilde{\mathcal{H}}_{N,N-1}\ ) \end{aligned} \tag{28}$$

Where we have used: $\widetilde{\mathcal{H}}_{n+1,n}(\alpha_{m_n},\gamma_{m_n},\beta,\delta_{m_n}) = {}_{m_n}\widetilde{\mathcal{H}}_{n+1,n}$ and also the periodicity condition: $\widetilde{\mathcal{H}}_{N+1,N} = \widetilde{\mathcal{H}}_{1,N}$.

We introduce the equations (26) and these results in the relations (20) ($\mathfrak{T}_N(\mu,\pm k)\,\mathfrak{T}_N(\nu,\mp k) = \mathfrak{T}_N(\nu,\pm k)\,\mathfrak{T}_N(\mu,\mp k)$):

$$\begin{aligned} &\mathfrak{T}_N(\eta)\,\left(\sum_{l=0}^{N-1} C_l(\eta,k)\,(\mu-\eta)^l\right)\left(\sum_{ll=0}^{N-1} C_{ll}(\eta,k)\,(\nu-\eta)^{ll}\right)\mathfrak{T}_N(\eta) = \\ &= \mathfrak{T}_N(\eta)\,\left(\sum_{n=0}^{N-1} C_n(\eta,k)\,(\nu-\eta)^n\right)\left(\sum_{nn=0}^{N-1} C_{nn}(\eta,k)\,(\mu-\eta)^{nn}\right)\mathfrak{T}_N(\eta)\,. \end{aligned} \tag{29}$$

With the property:

$$\left\{\mathrm{str}\left[\left(\overleftarrow{\mathcal{T}}_N(\eta)\right)^{-1}\right]\right\}\left\{\mathrm{str}\left[\overleftarrow{\mathcal{T}}_N(\eta)\right]\right\} = \left\{\mathrm{str}\left[\overleftarrow{\mathcal{T}}_N(\eta)\right]\right\}\left\{\mathrm{str}\left[\left[\overleftarrow{\mathcal{T}}_N(\eta)\right]^{-1}\right]\right\} = \mathbf{1}\,,$$

we finally obtain from (29) the commutators:

$$\big[\,C_n(\eta,k)\ ,\ C_m(\eta,k)\big] = \mathbf{0}\,, \qquad \{n, m\} \in \{0,\ldots,N-1\}\,, \tag{30}$$

(and also trivially: $\big[\mathfrak{T}_N(\eta)\,,\mathfrak{T}_N(\eta)\big]=\mathbf{0}$). The results in a similar way as we have in the spin case with the existence of $\mathbf{T}_N^{-1}(\eta)$.

The $C_l(\eta,k)$ contain among them the fermion Heisenberg Hamiltonians: $C_1 = {}_1\widetilde{\mathcal{H}} = \mathcal{H}(\alpha_1(\eta,k),\,\beta(\eta,k),\,\delta_1(\eta,k))$.

DISCUSSION.

After the seminal results of Baxter, fifty years ago, there has been many researches following very different pathways. At that time one focusing point in the study of the ferromagnetism and of different models in condensed matter and in statistical mechanics.

A condition known as "free-fermion", with $\Delta = (\alpha^2+\gamma^2-\beta^2-\delta^2)(2\alpha\gamma)^{-1} = 0$ (Fan and Wu [12]) as a common departing point, then and nowadays. Also at present time in fields so apart from the initial fields of interest as there are some researches in **AdS/CFT** [13].

The results presented here have many similarities with the original ones obtained by Baxter, and they also open a possible application to a quantum inverse scattering method [5]. One main difference in the equation (27); in the spin case, the *XYZ* Hamiltonian and the transfer matrix $\mathbf{T}_N(\eta)$ commute. More possibilities in the extension to researches avoiding the free-fermion condition, and finally in the generalization of the Hubbard model, fermions with spins up and down, now with $XYZ \cup XYZ$.

Besides, with a handling in a very different setting, we observe the highly symmetrical form of our equations. In this sense, the "number" type operators $\tilde{\mathbf{n}}_n$ are specially meaningful [14]. This suggests the interest of our results in relation to the elementary particles. Work in this direction is in progress (see the program of research that follows).

PROGRAM OF THE *STUDIES* CONTAINING THIS RESEARCH.

On the fermionization of the XYZ spin Heisenberg chain (algebra).
(This study). (2022) https://eprints.ucm.es/id/eprint/72882/ *Study -2)*

The JordanWigner transformations and the fermionization of the XYZ spin Heisenberg chain.
Algebra, geometry and physics? (2022) https://eprints.ucm.es/id/eprint/74550/ *Study -1)*

A tentative model of creation and annihilation operators for neutrinos.
(2021) https://eprints.ucm.es/id/eprint/65151/ *Study 0)*

Expression of the 3- and 4-dimensional vectors in total polar exponential form.
(2021) https://eprints.ucm.es/id/eprint/65825/ *Study I,1)*

Vectors. Dimensions 4 and 8. (2023) https://eprints.ucm.es/id/eprint/76327/ *Study I,2)*

Geometry of the time and the space. *Study I)*

Geometry of the symmetries in dimension 4=(1+[1)+"2"], and general Time-Space-Spin vectors.
(2023) https://eprints.ucm.es/id/eprint/76328/ *Study II)*

Geometry and Physics of the Elementary Fermions. (On pride of Jordan Wigner Pauli Weyl Dirac). **1**.
(2021) https://eprints.ucm.es/id/eprint/69295/ *Study III)*

Geometry and Physics of the Elementary Fermions. **2**. *Study III)*

Axial vector magnetic charge and magnetic moment. Maxwell's equations and Lorentz force law.
(2021) https://eprints.ucm.es/id/eprint/69294/ *Study IV)*

Addenda. *Study V)*

Acknowledgments: My deepest gratitude to the late Prof. Miki Wadati.

[1] W. Heisenberg. Zeitschrift für Physik **49**, 619-636 (1928).
[2] P. Jordan and E. Wigner. Zeitschrift für Physik **47**, 631-651 (1928). (*On the Pauli Exclusion Principle*).
[3] R. J. Baxter. Ann. Physics **70**, 193-228 (1972); Ann. Physics **70**, 323-337 (1972).
[4] E. H. Lieb. Phys. Rev. Lett. **18**, 692-694 (1967); Phys. Rev. Lett. **18**, 1046-1048 (1967); Phys. Rev. Lett. **19**, 108-110 (1967); Phys. Rev. **162**, 162-172 (1968).
[5] L. A. Takhtadzhan, L. D. Faddeev. Uspekhi Mat. Nauk **34**:5 (1979), 13-63. (Russian Math. Surveys **34**, 11-68 (1979) *The quantum method of the inverse problem and the Heisenberg XYZ model*).
[6] B. U. Felderhof. Physica **65**, 421-451 (1973); **66**, 279-297 (1973); **66**, 509-526 (1973).
[7] B. S. Shastry. Phys. Rev. Lett. **56**, 1529 (2334), (E) **56**,2334, 2453 (1986).
[8] E. Olmedilla, M. Wadati and Y. Akutsu. J. Phys. Soc. Jpn. **56**, 2298-2308 (1987).
[9] E. Olmedilla, M. Wadati. Phys. Rev. Lett. **60**, 1595-1598 (1989).
[10] F. A. Berezin. *The Method of Second Quantization*. Academic Press, New York, (1966).
[11] A. G. Izergin, P. P. Kulish. Theor. Math. Phys. **44**-2, 189-193 (1980).
[12] C. Fan and F. Y. Wu. Phys.Rev. **179** 560-70 (1969); Phys.Rev. B **2(3)** 723-33 (1970).
[13] C. P. Marius de Leeuw, A. Pribytok, A. L. Retore and A. Torrielli. JHEP **02** 191 (2021).
[14] W. Pauli. *Pauli lectures in Physics. Vol. 6. Selected Topics in Field Quantization.* Edited by C.P. Enz. The MIT Press (1977), page 4.