Geodesic Growth of some 3-dimensional RACGs

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Abstract

We give explicit formulas for the geodesic growth series of a Right Angled Coxeter Group (RACG) based on a link-regular graph that is 4-clique free, i.e. without tetrahedrons.

1 Introduction

Let G be a group and X a finite (monoid) generating set of G, that is, there exists a monoid morphism $\pi\colon X^* \to G$ that is surjective. Given a word $w \in X^*$, we denote by $\ell(w)$ its length. A word $w \in X^*$ is a *geodesic* if $\ell(w) = \min_{u\in X^*}\{\ell(u) : \pi(w) = \pi(u)\}$. The *geodesic growth series* associated to (G, X) is the formal power series

$$\mathcal{G}_{(G,X)}(z) = \sum_{i=0}^{\infty} \sharp\{w \in X^n :\ w \text{ geodesic}\} z^n \in \mathbb{Z}[[z]].$$

One can similarly define the *standard growth series* associated to (G, X) as the formal power series

$$\mathcal{S}_{(G,X)}(z) = \sum_{i=0}^{\infty} \sharp\pi(X^n) z^n \in \mathbb{Z}[[z]].$$

Both the standard and geodesic growth series encode geometric information of the Cayley graph of G (with respect to the generating set X). For instance, the celebrated theorem of Gromov on groups of polynomial growth says that the sequence $\{\sharp\pi(X^n)\}_{n=0}^{\infty}$ is bounded by a polynomial on n if and only if G is virtually nilpotent. The geodesic growth has been less studied compared with standard growth, and moreover it is much more sensitive to the change of generating sets. In this paper, we focus on families of groups with some preferred generating sets.

Right angled Coxeter groups (RACGs for short) is a family of groups described in terms of their defining presentation. Giving a simplicial graph Γ with vertex set V and edge set E, one associates to Γ the right angled Coxeter group C_Γ defined by the following presentation:

$$C_\Gamma = \langle V \mid v^2 = 1\, \forall v \in V,\ uv = vu\, \forall \{u, v\} \in E\rangle.$$

One calls V is the *standard generating set* of C_Γ. We can also associate to Γ the the *right angled Artin Group* A_Γ given by the presentation:

$$A_\Gamma = \langle V \mid uv = vu\, \forall \{u, v\} \in E\rangle.$$

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One calls $V \sqcup V^{-1}$ the *standard generating set* of A_Γ.

The languages of geodesics and shortLex representatives of a RACG respect to its standard generating sets are regular [3, 4, 9], and thus the corresponding standard and geodesic growth series are rational functions. Concrete formulas for the standard growth series of Coxeter groups, proved without the use of automata theory, can be found in [10, 11]. Recently, it was shown that the growth rate of the geodesic and the standard growth function are either one or a Perron number [8],

Moreover, it is well understood how the geometry of the defining graph reflects on the standard growth of C_Γ with respect to V: it only depends on the cliques of Γ of each size (see [6, Proposition 17.4.2.] or [10, 11]). For example, in the case of RACGs based on trees, this implies that the growth only depends on the number of vertices and edges of the tree.

Geodesic growth is still a very mysterious object compared to the standard growth and it is not clear which properties of the defining graph are reflected into the geodesic growth function. In [5], Ciobanu and Kolpakov showed that there exists infinite pairs non-isomorphic RACGs based on trees with the same geodesic growth with respect to the standard generators. These examples were based on co-spectral defining graphs, but then they gave infinite pairs of non-isomorphic RACGs with con-spectral defining graph and different geodesic growth with respect to the standard generating set.

On the other hand, if the defining graph poses enough symmetry (the graph is link-regular), then the main Theorem of [1] states that the geodesic growth only depends on the number of cliques of each size and the isomorphism types of the links of the cliques. A simplicial graph is link-regular, if the number of elements of the link of a clique only depends on the size of the clique (See Definition 2.1).

For example, if Γ is a totally disconnected graph with n vertices, then it is link regular, and C_Γ is a free product of cyclic groups of order 2. It is well-known that

$$\mathcal{G}_{(C_\Gamma,V)}(z) - 1 = \frac{nz}{1-(n-1)z}.$$

Moreover, in [1] it is computed explicitly the geodesic growth series of Right Angled Coxeter Groups based on link-regular graphs with n vertices, vertices of degree l, and without triangles. For such cases, one obtains the following formula for the growth series.

$$\mathcal{G}_{(C_\Gamma,V)}(z) - 1 = \frac{nz(1+(2-l)z)}{1+(-n-l+3)z+(-2n+2+nl)z^2}.$$

Note that if $l = 0$, one recovers the formula for a totally disconnected graph.

In this paper we continue to explore the geodesic growth of RAGCs based on link-regular graphs. Our main result is to provide an explicit formula for the geodesic growth, if the graph does not contain 4-cliques.

Theorem 1.1. *Let Γ be a link-regular graph with n vertices, l-regular and let q be the link-number of an edge (which is the same for any edge), and without 4-cliques. Then,*

$$\mathcal{G}_{(C_\Gamma,V)}(z) - 1 = \frac{nz(1+(5-l-q)z+(lq-3l+6)z^2)}{1+(6-n-l-q)z+(nl+lq+qn-5n-3l-q+11)z^2+(3nl+6-nlq-6n)z^3}.$$

One can check that if one let $q = 0$, then one gets the previous formula for triangle-free link-regular graphs.

2 Definitions and notation

Let Γ be a finite simple graph. For a vertex a in Γ, we denote by $St(a)$ the set:

$$\mathrm{St}(a) = \{b \in V\Gamma \mid \{a, b\} \in E\Gamma\} \cup \{a\}.$$

Let $\sigma \subseteq V\Gamma$ be such that the vertices of σ span a complete subgraph of Γ, then σ is called a *clique*. If σ is a clique with k-vertices, then we call it a k-clique. We sometimes call 3-cliques triangles and 4-cliques tetrahedrons.

The *link of a clique* σ, denoted by $\mathrm{Lk}(\sigma)$, is the set of vertices in $V\Gamma \setminus \sigma$ that are connected with every vertex in σ. That is,

$$\mathrm{Lk}(\sigma) = \{v \in V\Gamma \setminus \sigma : \{v\} \cup \sigma \text{ spans a clique}\}.$$

The *star of* σ, denoted by $\mathrm{St}(\sigma)$, is the set of vertices in Γ that are connected with every vertex in σ. That is,

$$\mathrm{St}(\sigma) = \{v \in V\Gamma : \{v\} \cup \sigma \text{ spans a clique}\}.$$

These sets satisfy $\sigma \cup \mathrm{Lk}(\sigma) = \mathrm{St}(\sigma)$.

Definition 2.1. A graph Γ is called *link-regular* if for any clique $\sigma \in \Gamma$, $|\mathrm{Lk}(\sigma)|$ depends on $|\sigma|$ and not on σ itself. If Γ is link-regular, we will write $|\mathrm{Lk}(i)|$ to denote $|\mathrm{Lk}(\sigma)|$ with $|\sigma| = i$.

In this paper we will consider graphs which does not contain tetrahedrons (or 4-cliques). Under this condition, a graph is link-regular if there are numbers l and q such that the graph is l-regular (there are l edges meeting at any vertex), and any edge is contained in q triangles (or q 3-cliques).

We recall the main theorem of [1]. Remember that the f-polynomial associated to Γ is the polynomial

$$f_\Gamma(z) = \sum_{n=0}^{|V|} \sharp\{\Delta \subseteq \Gamma : \ \Delta \text{ is an } n\text{-clique}\} z^n,$$

and essentially records the number of cliques of each size.

Theorem 2.2. *Let C_Γ be a RACG based on a link-regular graph Γ. The geodesic growth of G is fully determined by the f-polynomial of Γ and the size of the links of cliques in Γ, i.e. the set $\{|\mathrm{Lk}(\sigma)| : \sigma \text{ a clique in } \Gamma\}$.*

Remark 2.3. *As noted in [2], there is a relationship between the sizes of cliques and the coefficients of the f-polynomial. So one has that the geodesic growth of a RACG based on a link-regular graph Γ is fully determined by the f-polynomial of Γ.*

We remark that on a right angled Coxeter group it is easy to decide if a word is geodesic or not with respect to the standard generators. We have that a word $w \in V^*$ is **not** geodesic if and only if one can write w as $w = xayaz$ where $a \in V$, y is a word on $\mathrm{Lk}(\{a\})$ and x, z are words on V.

2.1 The double of a graph

Given a graph $\Gamma = (V, E)$, the *double graph* Γ^2 is defined as follows. The vertex set is $V\Gamma^2 = V\Gamma \sqcup V\Gamma$. Denote the vertices in the second copy as $\{a' \mid a \in V\}$. For any edge $\{a, b\}$ in Γ there are exactly four edges $\{a, b\}, \{a', b\}, \{a, b'\}, \{a', b'\}$ in Γ^2. See Figure 1 for an example.

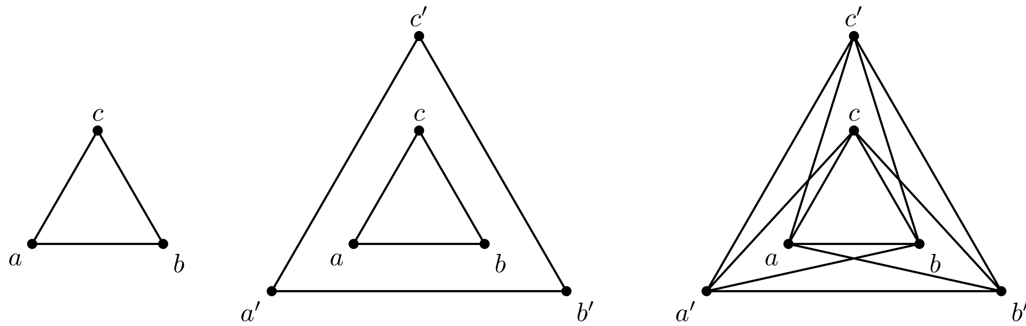

Figure 1: Construction of the double of a Graph

By definition, $|V\Gamma^2| = 2|V\Gamma|$, and $|E\Gamma^2| = 4|E\Gamma|$. One has also a projection $\rho : \Gamma^2 \longrightarrow \Gamma$, which identifies naturally the two copies of the vertices of Γ^2. The map ρ is 2-to-1 on vertices and 4-to-1 on edges.

Lemma 2.4. *If Γ is link-regular without tetrahedrons, then so is Γ^2.*

Proof. For any vertex $v \in V\Gamma^2$, consider $\rho(v)$ in Γ. Since ρ is 2-to-1 on vertices, $|\operatorname{Lk}_{\Gamma^2}(v)| = 2|\operatorname{Lk}\Gamma(\rho(v))|$.

Similarly for any edge $e = \{u, v\}$ in Γ^2 consider the edge $\rho(e)$ in Γ, and all the triangles $(\rho(u), \rho(v), c)$ over it. Now triangles over $e = \{u, v\}$ are (u, v, w) where $w \in \rho^{-1}(c)$. This means that $|\operatorname{Lk}_{\Gamma^2}(e)| = 2|\operatorname{Lk}_\Gamma(e)|$.

If there was a tetrahedron $\{x_1, x_2, x_3, x_4\}$ in Γ^2, then $\{\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)\}$ would be a tetrahedron in Γ. Indeed $\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)$ are all different, since $\{x, x'\}$ cannot be an edge in Γ^2, and all the edges of the tetrahedron in $\{x_1, x_2, x_3, x_4\}$, induce edges for a tetrahedron over $\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)$. Since there are no tetrahedrons in Γ, we conclude that there are no tetrahedrons in Γ^2. □

Remark 2.5. *An important application of the double construction is provided in [7, Lemma 2]: one has that the Cayley Graph of the RaAAG based on Γ is isomorphic as an undirected graph to the Cayley Graph of the RACG based on Γ^2.*

Using the Remark above we get the following:

Corollary 2.6. *Let $G = A_\Gamma$ be a RAAG based on a link-regular graph Γ. The geodesic growth of G is equal to the geodesic growth of C_{Γ^2} as the RACG based on Γ^2.*

3 Main Theorem

Throughout the rest of the paper, Γ will be a link-regular finite simple graph (with n vertices) which does not contain tetrahedrons (or 4-cliques). We denote by l the number of edges meeting at any vertex, and q the number of triangles containing a fixed edge.

Notation 3.1. *We denote by G the associated RACG C_Γ and we will use the generating set V. If $w \in V^*$ is a word, we denote by E_w the set of geodesics ending in w, and by $E_w(z)$ the generating growth series of*

E_w. That is

$$E_w(z) = \sum_{n=0}^{\infty} \sharp(E_w \cap V^n) z^n.$$

We will use $\Delta\Gamma$ to denote the set of 3-cliques of Γ. In the following theorem we use the notation

$$\sum_{e \in E\Gamma} E_e(z) = \sum_{\substack{a,b \in V\Gamma \\ \{a,b\} \in E\Gamma}} E_{ab}(z),$$

$$\sum_{\Delta \in \Delta\Gamma} E_\Delta(z) = \sum_{\substack{a,b,c \in V\Gamma \\ \{a,b,c\} \in \Delta\Gamma}} E_{abc}(z).$$

Theorem 3.2. *Let Γ be a link-regular graph with n vertices, l-regular and let q be the link-number of an edge (which is the same for any edge), and without 4-cliques. Let G_Γ be the corresponding right angled Coxeter group, and $\mathcal{G}(z)$ the geodesic growth series of G_Γ with respect to the standard generators. Then, there exists polynomials p_v, p_e, p_Δ (given below) such that the following relations hold:*

$$\sum_{a \in V\Gamma} E_a(z) = \mathcal{G}(z) - 1, \tag{1}$$

$$\sum_{e \in E\Gamma} E_e(z) = [\mathcal{G}(z) - 1](1 - (n - l - 1)z) - nz, \tag{2}$$

$$\sum_{\Delta \in \Delta\Gamma} E_\Delta(z) = [\mathcal{G}(z) - 1] p_\Delta(z, n, l, q) - nz + n(l - 2q - 2)z^2, \tag{3}$$

$$\sum_{a,b,c,d \in V\Gamma} E_{abcd}(z) = [\mathcal{G}(z) - 1] - nz - n(n-1)z^2 - [n(n-1)(n-2) + n(n-l-1)]z^3, \tag{4}$$

and

$$\sum_{a,b,c,d \in V\Gamma} E_{abcd}(z) = [n + l + q - 6]z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) + p_e(n,l,q) z^2 \sum_{e \in E\Gamma} E_e(z) + p_v(n,l,q) z^3 \sum_{a \in V\Gamma} E_a(z). \tag{5}$$

Note than one can find $\mathcal{G}(z)$ by substituting the equations (1),(2),(3) and (4) into (5). Moreover, the polynomials p_v, p_t, p_Δ are given by:

$$\begin{aligned}
p_\Delta(z, n, l, q) &= 1 - (n + l - 2q - 3)z + (2(n - l - 1)(l - q - 1) - l(n - 2l + q))z^2, \\
p_e(n, l, q) &= n^2 + l^2 - 2q^2 + nl + -2nq - 2lq - 4n - 6l + 10q + 7, \\
p_v(n, l, q) &= (n - l - 1)^3 + 2l(n - 2l + q)(n - q - 2) + lq(n - 3l + 3q).
\end{aligned}$$

Remark 3.3. *Note that the equations (1), (2) and (4) are obtained by substracting to $\mathcal{G}(z)$ the generating growth series of geodesics on those of length at most $0, 1$ and 3 respectively.*

Note. *Theorem 3.2 provides a way to calculate $\mathcal{G}(z)$ using a system of linear equations. The coefficients of the system are polynomials on z (and n, l, q) of degree at most 3. Theorem 1.1 was obtained by solving this linear system of equations with the help of Sage.*

In the following example we calculate $\mathcal{G}(z)$ on a particular family. The general case can be computed similarly.

Example 3.4. Let's compute the geodesic growth of an infinite family Γ_m, defined inductively by

$$\Gamma_0 = (\{a, b, c\}, \{\{a, b\}, \{a, c\}, \{b, c\}\}) = \text{ triangle},$$

$$\Gamma_{m+1} = \Gamma_m^2 = \text{ the double of } \Gamma_m.$$

In terms of (n, l, q) we have $(n_0, l_0, q_0) = (3, 2, 1)$, and therefore, by the double construction:

$$(n_m, l_m, q_m) = (3 \cdot 2^m, 2 \cdot 2^m, 2^m).$$

Taking $2^m = k$ and substituting $3k, 2k, k$ on the polynomials of Theorem 3.2 for n, l, q respectively, we find:

$$\begin{aligned} p_\Delta(z, 3k, 2k, k) &= 2(k-1)^2 z^2 - 3(k-1)z + 1, \\ p_e(3k, 2k, k) &= 7(k-1)^2, \\ p_v(3k, 2k, k) &= (k-1)^3. \end{aligned}$$

Now substitute $3k, 2k, k$ on the other equations of Theorem 3.2 for n, l, q respectively. Use also the polynomials above, and we can find $\mathcal{G}(z)$ as a solution of the following system:

$$\begin{aligned} \sum_{a \in V\Gamma} E_a(z) &= \mathcal{G}(z) - 1, \\ \sum_{e \in E\Gamma} E_e(z) &= (\mathcal{G}(z) - 1)[1 - (k-1)z] - 3kz, \\ \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) &= (\mathcal{G}(z) - 1)[2(k-1)^2 z^2 - 3(k-1)z + 1] - 3kz - 6kz^2, \\ \sum_{a,b,c,d \in V\Gamma} E_{abcd}(z) &= 6(k-1)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) + [7(k-1)^2]z^2 \sum_{e \in E\Gamma} E_e(z) + [(k-1)^3]z^3 \sum_{a \in V\Gamma} E_a(z), \\ \mathcal{G}(z) &= 1 + 3kz + 3k(3k-1)z^2 + [3k(3k-1)(3k-2) + 3k(k-1)]z^3 + \sum_{a,b,c,d \in V\Gamma} E_{abcd}(z). \end{aligned}$$

Finally, solving for $\mathcal{G}(z)$, we find:

$$\mathcal{G}(z) = -\frac{6z^3 + (2k^2 - 7k + 11)z^2 - 3(k-2)z + 1}{(z(k-1) - 1)(2z(k-1) - 1)(3z(k-1) - 1)},$$

which agrees with the formula provided in Theorem 1.1, for $(n, l, q) = (3k, 2k, k)$.

Now using Theorem (1.1), and Corollary (2.6) we get the following:

Corollary 3.5. *Let Γ be a graph as in the hypothesis of Theorem* (1.1). *One can find the geodesic growth series $\mathcal{A}(z)$ for right angled Artin group based on Γ by substituting $2n, 2l, 2q$ for n, q, l respectively in the formula of Theorem* (1.1) *and we get:*

$$\mathcal{A}(z) - 1 = \frac{2nz[1 + (5 - 2l - 2q)z + (4lq - 6l + 6)z^2)]}{1 + (6 - 2n - 2l - 2q)z + (4nl + 4lq + 4qn - 10n - 6l - 2q + 11)z^2 + (12nl + 6 - 8nlq - 12n)z^3}.$$

4 Proof of the main theorem

Throughout this section Γ is a link-regular graph without tetrahedrons. The graph Γ has n vertices, the link of each vertex has l vertices, and the link of each edge has q vertices. Let $G = G_\Gamma$ be the associated RAGC.

Note that there is 1 geodesic word of length 0, n geodesic words of length 1, $n(n-1)$ geodesics of length 2. A word of length 3 is geodesic in G if all its 3 letters are different or if it is of the form aba with $b \notin \mathrm{St}(a)$. Thus there are $n(n-1)(n-2) + n(n-l-1)$ geodesic words of length 3.

With Notation 3.1, one can write the geodesic growth series $\mathcal{G}(z)$ in any of the following forms:

$$\mathcal{G}(z) = 1 + \sum_{a \in V\Gamma} E_a(z), \tag{6}$$

$$\mathcal{G}(z) = 1 + nz + \sum_{a,b \in V\Gamma} E_{ab}(z), \tag{7}$$

$$\mathcal{G}(z) = 1 + nz + n(n-1)z^2 + \sum_{a,b,c \in V\Gamma} E_{abc}(z), \tag{8}$$

$$\mathcal{G}(z) = 1 + nz + n(n-1)z^2 + [n(n-1)(n-2) + n(n-l-1)]z^3 + \sum_{a,b,c,d \in V\Gamma} E_{abcd}(z). \tag{9}$$

We get (1) and (4) of Theorem 3.2 from equations (6) and (9), respectively.

We will derive (2) of the Theorem 3.2 from (7) by expanding $\sum_{a,b \in V\Gamma} E_{ab}(z)$. Given a word $ab \in V^*$, we distinguish three cases: $a = b$, $b \in \mathrm{Lk}(a)$ and when $b \notin \mathrm{St}(a)$. We can describe these cases geometrically as in the Figure 2 (omitting the case $a = b$).

Figure 2: Configurations of 2 generators.

The case $a = b$ is impossible, since no geodesic ends with aa. In the case when $b \notin \mathrm{St}(a)$ we can write $E_{ab} = E_a \cdot b$. Hence $E_{ab}(z) = E_a(z) \cdot z$ and we have $n - l - 1$ choices for b.

$$\begin{aligned}
\sum_{a,b \in V\Gamma} E_{ab}(z) &= \sum_{a \in V\Gamma} \left(\sum_{b \in \mathrm{Lk}(a)} E_{ab}(z) + \sum_{b \notin \mathrm{St}(a)} E_{ab}(z) \right) \\
&= \sum_{a \in V\Gamma} \left(\sum_{b \in \mathrm{Lk}(a)} E_{ab}(z) \right) + \sum_{a \in V\Gamma} (n - l - a) z E_a(z) \\
&= \sum_{e \in E\Gamma} E_e(z) + (n - l - 1) z \sum_{a \in V\Gamma} E_a(z).
\end{aligned} \tag{10}$$

So, from (7), (6), and (10) we get:

$$\sum_{e \in E\Gamma} E_e(z) = (1 - (n - l - 1)z)[\mathcal{G}(z) - 1] - nz$$

which appears in the main theorem as the equation (2).

We can work similarly to get (3) of the main theorem from the equation (8). The generators of the geodesics abc, lie in one of the following disjoint cases:

(I) $\{a, b\} \subset \mathrm{St}(c)$	(II) $\{a, b\} \not\subset \mathrm{St}(c)$
(I.0) $a = b$	(II.0) $a = b$
(I.1) $a \in \mathrm{Lk}(b)$	(II.1) $a \in \mathrm{Lk}(b)$
(I.2) $a \notin \mathrm{St}(b)$	(II.2) $a \notin \mathrm{St}(b)$

We can express these cases using the configurations of generators as in Figure 3.

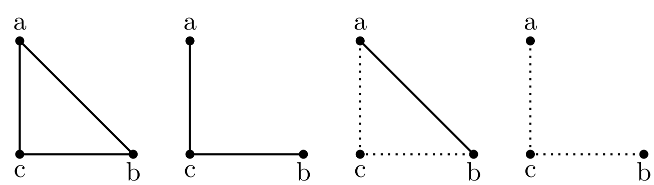

Figure 3: Configurations of 3 generators where $a \neq b$. Dashed edges might or might not appear in the configuration. Two vertices connected by a dashed edge might be the same vertex in Γ. In this case, no two dashed edges can be edges of the configuration simultaneously.

We can express $\sum_{a,b,c \in V\Gamma} E_{abc}(z)$ as a sum over the 6 disjoint subcases given above. For convenience, we will write $\sum_X$, where X is a case, to denote the summation over all triples $a, b, c \in V\Gamma$ satisfying the hypothesis of case X.

In (I.0) and (II.0) there are no geodesics, so $\sum_{(\mathrm{I}.0)} E_{abc}(z) = \sum_{(\mathrm{II}.0)} E_{abc}(z) = 0$.

In (I.1) the generators a, b, c form a triangle so we get $\sum_{(\mathrm{I}.1)} E_{abc}(z) = \sum_{\Delta \in \Delta\Gamma} E_\Delta(z)$.

In (I.2) we have $E_{abc}(z) = E_{acb}(z) = E_{ac}(z) \cdot z$, for a fixed b. Starting by fixing the edge $e = \{a, c\}$ we get $l - 1 - q$ choices for b, so in this case we have

$$\begin{aligned}\sum_{(\mathrm{I}.2)} E_{abc}(z) &= \sum_{a \in V} \sum_{c \in \mathrm{Lk}(a)} \sum_{\substack{b \in \mathrm{Lk}(c) \\ b \notin \mathrm{Lk}(a)}} E_{abc}(z) \\ &= \sum_{a \in V} \sum_{c \in \mathrm{Lk}(a)} (l - 1 - q) z E_{ac}(z) = (l - 1 - q) z \sum_{e \in E\Gamma} E_e(z).\end{aligned}$$

In (II.1) we count by first fixing the edge $\{a, b\}$, and then letting c be any vertex different to a and b that does not form a triangle with $\{a,b\}$. One has $n-2-q$ choices for c. Arguing as above, we get

$$\sum_{\text{(II.1)}} E_{abc}(z) = (n-q-2)z \sum_{e\in E\Gamma} E_e(z).$$

In (II.2) we count by first fixing the vertex a and then considering the choices when $c=a, c\in \mathrm{Lk}(a)$, and $c\notin \mathrm{St}(a)$. In this case, $b\notin \mathrm{St}(a)\cup\{c\}$ and moreover, when $c\in \mathrm{Lk}(a)$ then b is not linked to c.

For $c=a$ we have $(n-l-1)$ possible choices for b; for $c\in \mathrm{Lk}(a)$ we have $l\cdot(n-2l+q)$ possible choices for c,b; finally for $c\notin \mathrm{St}(a)$ we get $(n-l-1)(n-l-2)$ possible choices for c,b.

$$\begin{aligned}\sum_{\text{(II.2)}} E_{abc}(z) &= \sum_{a\in V}\left(\sum_{c=a}\sum_{b\notin \mathrm{Lk}(a)} E_{abc}(z) + \sum_{c\in \mathrm{Lk}(a)}\sum_{\substack{b\notin \mathrm{Lk}(a)\\ b\notin \mathrm{Lk}(c)}} E_{abc}(z) + \sum_{c\notin \mathrm{St}(a)}\sum_{\substack{b\notin \mathrm{Lk}(a)\\ b\neq c}} E_{abc}(z)\right)\\ &= \sum_{a\in V}\big((n-l-1)z^2E_a(z) + l(n-2l+q)z^2E_a(z) + (n-l-1)(n-l-2)z^2E_a(z)\big)\\ &= [(n-l-1)^2 + l(n-2l+q)]z^2 \sum_{a\in V\Gamma} E_a(z).\end{aligned}$$

Now summing everything up we get:

$$\begin{aligned}\sum_{a,b,c\in V\Gamma} E_{abc}(z) = &\sum_{\Delta\in\Delta\Gamma} E_\Delta(z)\\ &+ (n+l-2q-3)z\sum_{e\in E\Gamma} E_e(z)\\ &+ [(n-l-1)^2 + l(n-2l+q)]z^2\sum_{a\in V\Gamma} E_a(z)\end{aligned}$$

Substituting (2),(7),(8), we get

$$\begin{aligned}\mathcal{G}(z) - (1+nz+n(n-1)z^2) = &\sum_{\Delta\in\Delta\Gamma} E_\Delta(z)\\ &+ (n+l-2q-3)\cdot z\cdot((1-(n-l-1)z)[\mathcal{G}(z)-1] - nz)\\ &+ [(n-l-1)^2 + l(n-2l+q)]\cdot z^2\cdot(\mathcal{G}(z)-1)\end{aligned}$$

And one gets a formula for $\sum_{\Delta\in\Delta\Gamma} E_\Delta(z)$:

$$\begin{aligned}\sum_{\Delta\in\Delta\Gamma} E_\Delta(z) =& (\mathcal{G}(z)-1)[1-(n+l-2q-3)z + (2(n-l-1)(l-q-1) - l(n-2l+q))z^2]\\ &- nz + n(l-2q-2)z^2\end{aligned}$$

which appears in the main theorem as Equation (3).

To finish the proof, we need to show that (5) holds. As in the previous cases, we proceed to rewrite $\sum_{a,b,c,d\in V\Gamma} E_{abcd}(z)$ depending on different cases for the word $abcd$.

We distinguish the following eight disjoint cases:

- (I) $\{a,b,c\} \subset \mathrm{St}(d)$
  - (I.1) $\{a,b\} \subset \mathrm{St}(c)$
    - (I.1.0) $a = b$
    - (I.1.1) $a \in \mathrm{Lk}(b)$
    - (I.1.2) $a \notin \mathrm{St}(b)$
  - (I.2) $\{a,b\} \not\subset \mathrm{St}(c)$
    - (I.2.0) $a = b$
    - (I.2.1) $a \in \mathrm{Lk}(b)$
    - (I.2.2) $a \notin \mathrm{St}(b)$
- (II) $\{a,b,c\} \not\subset \mathrm{St}(d)$
  - (II.1) $\{a,b\} \subset \mathrm{St}(c)$
    - (II.1.0) $a = b$
    - (II.1.1) $a \in \mathrm{Lk}(b)$
    - (II.1.2) $a \notin \mathrm{St}(b)$
  - (II.2) $\{a,b\} \not\subset \mathrm{St}(c)$
    - (II.2.0) $a = b$
    - (II.2.1) $a \in \mathrm{Lk}(b)$
    - (II.2.2) $a \notin \mathrm{St}(b)$

We can express them geometrically as configurations of 4 points as in Figure 4.

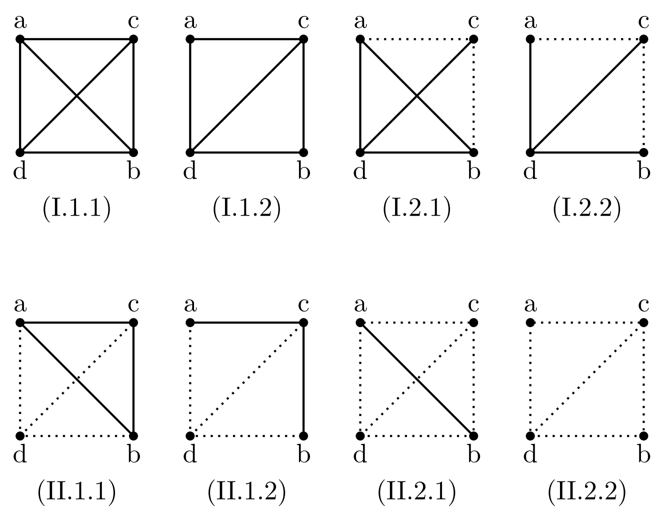

Figure 4: Configurations of 4 generators. Dashed edges represent pairs of vertices that might or might not be in the star of each other (including the possibility of being equal). In cases (I.2) at least one dashed edge is not an edge in Γ (i.e. the vertices are different and do not commute). In cases (II) at least one of the dashed edges incident to d is not an edge in Γ. Moreover, in cases (II.2), at least one of a or b does not form an edge with c.

As above, we will write $\sum_X$, where X is one of these 8 cases, to denote the summation over all quadruples $a,b,c,d \in V\Gamma$ satisfying the hypothesis of case X.

In (I.1.0), (I.2.0), (II.1.0) and (II.2.0) the summation is equal to zero as there is no geodesic $abcd$ with $a = b$.

In (I.1.1), since there are no 4-cliques, $\sum_{((I.1.1))} E_{abcd}(z) = 0$.

In (I.1.2), fixing the triangle acd, we have $q-1$ choices for b. Also, for a,b,c,d in this case, $E_{abcd}(z) =$

$E_{acd}(z) \cdot z$. Therefore

$$\sum_{(\text{I.1.2})} E_{abcd}(z) = (q-1)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z).$$

In (I.2.1), fixing the triangle abd, we have $l-2$ choices for c. Also, for a, b, c, d in this case, we have $E_{abcd}(z) = E_{abd}(z) \cdot z$. Therefore

$$\sum_{(\text{I.2.1})} E_{abcd}(z) = (l-2)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z).$$

In (I.2.2), we start by fixing the edge ad. As, $c, d \in \text{Lk}(d)$, we have $E_{abcd}(z) = E_{ad}(z) \cdot z^2$. We need to count the possibilities for b and c. We have the following disjoint subcases for c:

(1) $c = a$: Here we have $l-1-q$ choices for b.

(2) $c \in \text{Lk}(a)$: Here we have q choices for c. Note that as $\{a, b\} \not\subset \text{St}(c)$, we get that $b \notin \text{St}(c)$. Therefore, b is the link of d but b is not in the link of the edge $\{a, d\}$ and neither in the link of the edge $\{c, d\}$. Note that the $\text{Lk}(\{a, d\}) \cap \text{Lk}(\{c, d\})$ is empty, as if not, Γ will contain a tetrahedron. Therefore, there are $l-2q$ choices for b in this subcase.

(3) $c \notin \text{St}(a)$: we have $l-1-q$ choices for c and $l-2-q$ choices for b, since on top we have that $b \neq c$, as there is no geodesic of the form $accd$.

At the end one gets

$$\begin{aligned}
\sum_{(\text{I.2.2})} E_{abcd}(z) &= \sum_{a \in V} \sum_{d \in \text{Lk}(a)} \left(\sum_{c=a} E_{abcd}(z) + \sum_{c \in \text{Lk}(a)} E_{abcd}(z) + \sum_{c \notin \text{St}(a)} E_{abcd}(z) \right) \\
&= \sum_{a \in V} \sum_{d \in \text{Lk}(a)} \big((l-1-q)z^2 E_{ad}(z) + q(l-2q)z^2 E_{ad}(z) + (l-1-q)(l-2-q)z^2 E_{ad}(z) \big) \\
&= [(l-1-q)^2 + q(l-2q)]z^2 \sum_{e \in E\Gamma} E_e(z).
\end{aligned}$$

We now consider the case (II).

In (II.1.1), we count by fixing the triangle abc. We have $E_{abcd}(z) = E_{abc}(z) \cdot z$. Since Γ does not have tetrahedrons, there is no condition on d except that $d \neq a, b, c$. We have $n-3$ choices for d, so we get

$$\sum_{(\text{II.1.1})} E_{abcd}(z) = (n-3)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z).$$

In (II.1.2), we count by first fixing the edge $\{a, c\}$ and then considering the different cases for d: $d \in \{a, c\}$, $d \in \text{Lk}(\{a, c\})$ or $d \notin \text{St}(\{a, c\})$. As c commutes with b, we have $E_{abcd}(z) = E_{ac}(z) \cdot z^2$. We now count the choices for b, d:

(1) $d \in \{a,c\}$: The case $d=c$ is impossible. The case $d=a$, we have that $b \in \mathrm{Lk}(c) \setminus \mathrm{St}(a)$, and this gives $l-1-q$ choices for b.

(2) $d \in \mathrm{Lk}(\{a,c\})$: there are q choices for d. In this case we have that $b \in \mathrm{Lk}(c)$. Since $\{a,b,c\} \not\subset \mathrm{St}(d)$ and $d \in \mathrm{Lk}(\{a,c\})$, we get $b \notin \mathrm{Lk}(d)$. Also from the hypothesis, we get $b \notin \mathrm{Lk}(a)$. Therefore b is in $\mathrm{Lk}(c) \setminus (\mathrm{Lk}(\{a,d\}) \cup \mathrm{Lk}(\{c,d\}))$. Note that the links of two edges in a triangle are disjoint as Γ has no tetrahedrons. Thus, there are $l-2q$ choices for b.

(3) $d \notin \mathrm{St}(\{a,c\})$: We subdivide this case into two subcases:

   (3.1) $d \in \mathrm{St}(c)$: In this case, d is in $\mathrm{St}(c) \setminus \mathrm{St}(\{a,c\})$ and there are $(l+1)-(q+2) = l-q-1$ possibilities for d.
   Note that in this case $b \neq d$ since otherwise $abcd$ is not geodesic. Thus we have that b is in $\mathrm{St}(c) \setminus (\mathrm{St}(\{a,c\}) \cup \{d\})$, and we have $l-2-q$ possibilities for b.

   (3.2) $d \notin \mathrm{St}(c)$: As $d \in V \setminus \mathrm{St}(c)$, we have $n-l-1$ choices for d. As $b \in \mathrm{St}(c) \setminus \mathrm{St}(\{a,c\})$, we have $l-1-q$ choices for b.

Ultimately, we get

$$\begin{aligned}
\sum_{\text{(II.1.2)}} E_{abcd}(z) &= \sum_{a\in V}\sum_{c\in \mathrm{Lk}(a)} \left(\sum_{d\in\{a,c\}} E_{abcd}(z) + \sum_{d\in \mathrm{Lk}(\{a,c\})} E_{abcd}(z) + \sum_{d\notin \mathrm{St}(\{a,c\})} E_{abcd}(z) \right) \\
&= \sum_{a\in V}\sum_{c\in \mathrm{Lk}(a)} \big(1\cdot(l-1-q)z^2E_{ac}(z) + q(l-2q)z^2E_{ac}(z) + (l-q-1)(n-q-3)z^2E_{ac}(z)\big) \\
&= [(l-1-q)(n-q-2) + q(l-2q)]z^2 \sum_{e\in E\Gamma} E_e(z).
\end{aligned}$$

In (II.2.1), we start by fixing the edge $\{a,b\}$ and then considering the different cases for d: $d \in \{a,b\}$, $d \in \mathrm{Lk}(\{a,b\})$ or $d \notin \mathrm{St}(\{a,b\})$. We will use that $E_{abcd}(z) = E_{ab}(z)\cdot z^2$. We count the choices for c and d.

(1) $d \in \{a,b\}$: If $d=a$ we can take for c any vertex outside $\mathrm{St}(a)$ and we have $n-l-1$ choices for c. If $d=b$, we can take for c any vertex outside $\mathrm{St}(b)$ and we have $n-l-1$ choices for c.

(2) $d \in \mathrm{Lk}(\{a,b\})$: Here we have q choices for d. Since $\{a,b,c\} \not\subset \mathrm{St}(d)$, c can not be in $\mathrm{Lk}(\{d\})$. Since $\{a,b\} \not\subset \mathrm{St}(c)$, c is not in $\mathrm{Lk}(\{a,b\})$. Since a,b,d form a triangle, and we do not have tetrahedrons, these two links are disjoint. There are $n-q-l$ choices for c.

(3) $d \notin \mathrm{St}(\{a,b\})$: Here both c and d are not in $\mathrm{St}(\{a,b\})$ and moreover $c \neq d$ to have $abcd$ a geodesic. We have $d \in V \setminus \mathrm{St}(\{a,b\})$ that gives $n-q-2$ choices for d and we have $c \in V \setminus (\mathrm{St}(\{a,b\}) \cup \{d\})$ that gives $n-q-3$ for c.

Ultimately, we get

$$\sum_{\text{(II.2.1)}} E_{abcd}(z) = [2(n-l-1) + q(n-q-l) + (n-q-2)(n-q-3)]z^2 \sum_{e\in E\Gamma} E_e(z).$$

In (II.2.2), we first fix a and then we consider different cases for d: $d=a, d\in \mathrm{Lk}(a)$ or $d \notin \mathrm{St}(a)$. Note that we have $E_{abcd}(z) = E_a(z)\cdot z^3$. We count the choices for b,c,d.

(1) $d = a$: Here we have 1 choice for d. We split this case into the following disjoint subcases:

(1.1) $c = a$: this is impossible, since $abaa$ is not a geodesic.

(1.2) $c \in \mathrm{Lk}(a)$: Here we have l choices for c. In this case, b can be any vertex of $V \setminus (\mathrm{St}(a) \cup \mathrm{St}(c))$. As $|\mathrm{St}(a) \cap \mathrm{St}(c)| = q + 2$, we have $n - 2(l+1) + (q+2) = n - 2l + q$ possibilities for b.

(1.3) $c \notin \mathrm{St}(a)$: Here $b \notin \mathrm{St}(a) \cup \{c\}$. So we have $n - l - 1$ choices for c, and $n - l - 2$ for b.

Accounting for (1.1), (1.2) and (1.3), for a given vertex $a \in V$ we have

$$\sum_{\substack{c,b,d \in (\mathrm{II}.2.2) \\ d=a}} E_{abcd}(z) = [l(n - 2l + q) + (n - l - 1)(n - l - 2)] z^3 E_a(z)$$

(2) $d \in \mathrm{Lk}(a)$: Here we have l choices for d. We divide now the analysis into the following disjoint subcases:

(2.1) $c \in \{a, d\}$: The case $c = d$ is impossible. In the case $c = a$, we have one choice for c and b can be any vertex of $V \setminus (\mathrm{St}(a) \cup \mathrm{St}(d))$. As $|\mathrm{St}(a) \cap \mathrm{St}(d)| = q + 2$, we have $n - 2l + q$ possibilities for b.

(2.2) $c \in \mathrm{Lk}(\{a, d\})$: which gives us q choices for c. We have a triangle $\{a, c, d\}$ in Γ and by hypothesis of case (II.2.2), $b \notin \mathrm{St}(a) \cup \mathrm{St}(c) \cup \mathrm{St}(d) = \mathrm{Lk}(a) \cup \mathrm{Lk}(c) \cup \mathrm{Lk}(d)$. We have that $\mathrm{Lk}(x) \cap \mathrm{Lk}(y)$ has q elements, for $x \neq y$, $x, y \in \{a, c, d\}$. As Γ has no tetrahedrons, $\mathrm{Lk}(\{a, b, c\}) = \mathrm{Lk}(a) \cap \mathrm{Lk}(c) \cap \mathrm{Lk}(d)$ is empty. Using the inclusion-exclusion principle, we have $n - 3l + 3q$ choices for b.

(2.3) $c \notin \mathrm{St}(\{a, d\})$: we subdivide this case into the following disjoint subcases:

(2.3.1) $c = a$: This is impossible since $c \notin \mathrm{St}(\{a, d\})$.

(2.3.2) $c \in \mathrm{Lk}(a)$: In this case, necessarily, $c \notin \mathrm{St}(d)$. Here we get $l - 1 - q$ choices for c. Also $b \notin \mathrm{Lk}(a) \cup \mathrm{Lk}(c)$ and we have $n - 2l + q$ choices for b.

(2.3.3) $c \notin \mathrm{St}(a)$: We do now again, three subcases:

(2.3.3.1) $c = d$: This is impossible since $c \notin \mathrm{St}(\{a, d\})$.

(2.3.3.2) $c \in \mathrm{Lk}(d)$: In this case $c \in \mathrm{Lk}(d) \setminus \mathrm{St}(a)$ here we get $l - 1 - q$ choices for c. Also $b \notin \mathrm{Lk}(a) \cup \mathrm{Lk}(d)$ and we have $n - 2l + q$ choices for b.

(2.3.3.3) $c \notin \mathrm{St}(d)$: Here one has $|V \setminus (\mathrm{St}(a) \cup \mathrm{St}(d))| = n - 2l + q$ choices for c. Note that since a, d span an edge and b is not star of a in (II.2.2) we have that b can not be equal to d neither to a. We subdivide this case into the following disjoint subcases:

(2.3.3.3.1) $b = d$: impossible.

(2.3.3.3.2) $b \in \mathrm{Lk}(d)$: then $b \in \mathrm{Lk}(d) \setminus \mathrm{St}(a)$ and we have $l - 1 - q$ choices for b.

(2.3.3.3.3) $b \notin \mathrm{St}(d)$: here $b \neq c$ to get a geodesic, and $b \notin \mathrm{St}(a) \cup \mathrm{St}(d)$. We have $n - 2l + q - 1$ choices for b.

Accounting for (2.1), (2.2) and (2.3), for a given vertex $a \in V$ we have

$$\sum_{\substack{c,b,d \in (\mathrm{II}.2.2) \\ d \in \mathrm{Lk}(a)}} E_{abcd}(z) = [lq(n - 3l + 3q) + l(n - 2l + q)(n + l - 2q - 3)] z^3 E_a(z).$$

(3) $d \notin \mathrm{St}(a)$:

(3.1) $b = d$: here we have $n-1-l$ choices for d, 1 choice for b. As $abcb$ is a geodesic, c does not belong to $\mathrm{St}(b)$, and we have $n-l-1$ choices for c.

(3.2) $b \neq d$: here we split into these following disjoint cases:

(3.2.1) $c = a$: here we get 1 choice for c, and b, d are any pair of different vertices of $V \setminus \mathrm{St}(a)$, thus we have $(n-l-1)(n-l-2)$ possibilities for b and d.

(3.2.2) $c \in \mathrm{Lk}(a)$: here we get l choices for c. The hypothesis of case II.2, $\{a, b\} \notin \mathrm{St}(c)$ implies that $b \notin \mathrm{St}(c)$. We consider the following disjoint subcases:

(3.2.2.1) $d = c$: which is impossible since $abcd$ is geodesic.

(3.2.2.2) $d \in \mathrm{Lk}(c)$: here get $|\mathrm{Lk}(c) \setminus \mathrm{St}(a)| = l-1-q$ choices for d. We have $|V \setminus (\mathrm{St}(a) \cup \mathrm{St}(c))| = n-2l+q$ for b.

(3.2.2.3) $d \notin \mathrm{St}(c)$: here get $|V \setminus (\mathrm{St}(a) \cup \mathrm{St}(c))| = n-2l+q$ choices for d and $|V \setminus (\mathrm{St}(a) \cup \mathrm{St}(c) \cup \{b\})| = n-2l+q-1$ choices for b.

(3.2.3) $c \notin \mathrm{St}(a)$: here for b, c, d we get $(n-l-1)(n-l-2)(n-l-3)$ choices as b, c, d can be any vertex outside of $\mathrm{St}(a)$, $b \neq c$, $c \neq d$ because $abcd$ is geodesic, and $b \neq d$ by hypothesis.

Accounting for (3.1) and (3.2), we get

$$\sum_{\substack{c,b,d \in (\mathrm{II}.2.2) \\ d \notin \mathrm{St}(a)}} E_{abcd}(z) = [(n-1-l)^2 + (n-l-1)(n-l-2)^2 + l(n-2l+q)(n-l-2)] z^3 E_a(z)$$

Ultimately, in case (II.2.2), we get

$$\sum_{(\mathrm{II}.2.2)} E_{abcd}(z) = [(n-l-1)^3 + 2l(n-2l+q)(n-q-2) + lq(n-3l+3q)] z^3 \sum_{v \in V\Gamma} E_v(z).$$

We finally collect all these cases together

$$\begin{aligned}
\sum_{a,b,c,d \in V} E_{abcd}(z) = {} & (q-1)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) \\
& + (l-2)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) \\
& + [(l-1-q)^2 + q(l-2q)] z^2 \sum_{e \in E\Gamma} E_e(z) \\
& + (n-3)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) \\
& + [(l-1-q)(n-q-2) + q(l-2q)] z^2 \sum_{e \in E\Gamma} E_e(z) \\
& + [2(n-l-1) + q(n-q-l) + (n-q-2)(n-q-3)] z^2 \sum_{e \in E\Gamma} E_e(z) \\
& + [(n-l-1)^3 + 2l(n-2l+q)(n-q-2) + lq(n-3l+3q)] z^3 \sum_{v \in V\Gamma} E_v(z)
\end{aligned}$$

$$
\begin{aligned}
=&(n+q+l-6)z\sum_{\Delta\in\Delta\Gamma}E_{\Delta}(z)\\
+&(l^2+ln+n^2-2lq-2nq-2q^2-6l-4n+10q+7)z^2\sum_{e\in E\Gamma}E_e(z)\\
+&[(n-l-1)^3+2l(n-2l+q)(n-q-2)+lq(n-3l+3q)]z^3\sum_{v\in V\Gamma}E_v(z)
\end{aligned}
$$

Now, summing everything up, we get the equation (5) of the main theorem.

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