



Pearson equations for discrete orthogonal polynomials: III—Christoffel and Geronimus transformations

Manuel Mañas^{1,2}

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Abstract

Contiguous hypergeometric relations for semiclassical discrete orthogonal polynomials are described as Christoffel and Geronimus transformations. Using the Christoffel–Geronimus–Uvarov formulas quasi-determinantal expressions for the shifted semiclassical discrete orthogonal polynomials are obtained.

Keywords Discrete orthogonal polynomials · Pearson equations · Cholesky factorization · Generalized hypergeometric functions · Contiguous relations · Christoffel transformations · Geronimus transformations · Geronimus–Uvarov transformations

Mathematics Subject Classification 42C05 · 33C45 · 33C47

1 Introduction

Discrete orthogonal polynomials constitute an important part in the theory of orthogonal polynomials and have many applications. This is well illustrated by several reputed monographs on the topic. Let us cite here [50], devoted to the study of classical discrete orthogonal polynomials and its applications, and [15] where the Riemann–Hilbert problem is the key for the study of asymptotics and further applications of these polynomials. The mentioned relevance of discrete orthogonal polynomials is also illustrated by numerous sections or chapters devoted to its discussion in excellent books on orthogonal polynomials such as [16, 38, 39, 54]. For semiclassical discrete orthogonal polynomials the weight satisfies a discrete Pearson equation, we refer the reader to [23–26] and references therein for a comprehensive account.

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✉ Manuel Mañas
manuel.manas@ucm.es

¹ Departamento de Física Teórica, Universidad Complutense de Madrid, Plaza Ciencias 1, 28040 Madrid, Spain

² Instituto de Ciencias Matemáticas (ICMAT), Campus de Cantoblanco UAM, 28049 Madrid, Spain

For the generalized Charlier and Meixner weights, Freud–Laguerre type equations for the coefficients of the three term recurrence have been discussed, see for example [21, 29–31, 52].

This paper is a sequel of [47]. There we used the Cholesky factorization of the moment matrix to study discrete orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$ on the uniform lattice, and studied semiclassical discrete orthogonal polynomials. The corresponding moments are now given in terms of generalized hypergeometric functions. We constructed a banded semi-infinite matrix Ψ , that we named as Laguerre–Freud structure matrix, that models the shifts by ± 1 in the independent variable of the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$. It was shown that the contiguous relations for the generalized hypergeometric functions are symmetries of the corresponding moment matrix, and that the 3D Nijhoff–Capel discrete lattice [37, 49] describes the corresponding contiguous shifts for the squared norms of the orthogonal polynomials. In [28] we considered the generalized Charlier, Meixner and Hahn of type I discrete orthogonal polynomials, and analyzed the Laguerre–Freud structure matrix Ψ . We got non linear recurrences for the recursion coefficients of the type

$$\gamma_{n+1} = F_1(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1} \dots), \quad \beta_{n+1} = F_2(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots),$$

for some functions F_1, F_2 . Magnus [41–44] named, attending to [32, 40], as Laguerre–Freud relations.

In this paper, we return to the hypergeometric contiguous relations and its translation into symmetries of the moment matrix given in [47], and prove that they are described as simple Christoffel and Geronimus transformations. We also show that for these discrete orthogonal polynomials we can find determinantal expressions *à la Christoffel* for the shifted orthogonal polynomials, for that aim we use the general theory of Geronimus–Uvarov perturbations.

Christoffel discussed Gaussian quadrature rules in [20], and found explicit formulas relating sequences of orthogonal polynomials corresponding to two measures dx and $p(x) dx$, with $p(x) = (x - q_1) \cdots (x - q_N)$. The so called Christoffel formula is a basic result which can be found in a number of orthogonal polynomials textbooks [19, 35, 53]. Its right inverse is called the Geronimus transformation, i.e., the elementary or canonical Geronimus transformation is a new moment linear functional \check{u} such that $(x - a)\check{u} = u$. In this case we can write $\check{u} = (x - a)^{-1}u + \xi\delta(x - a)$, where $\xi \in \mathbb{R}$ is a free parameter and $\delta(x - a)$ is the Dirac functional supported at the point $x = a$ [36, 48]. We refer to [6–8] and references therein for a recent account of the state of the art regarding these transformations. For more on Darboux, Christoffel/Geronimus and linear spectral transformations see [18, 33, 55].

1.1 Discrete orthogonal polynomials and discrete Pearson equation

Let us consider a measure $\rho = \sum_{k=0}^\infty \delta(z - k)w(k)$ with support on $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, for some weight function $w(z)$ with finite values $w(k)$ at the nodes $k \in \mathbb{N}_0$. The corresponding bilinear form is $\langle F, G \rangle = \sum_{k=0}^\infty F(k)G(k)w(k)$ and their moments are given by

$$\rho_n = \sum_{k=0}^\infty k^n w(k), \quad n \in \mathbb{N}_0. \tag{1}$$

Consequently, the moment matrix is

$$G = (G_{n,m}), \quad G_{n,m} = \rho_{n+m}, \quad n, m \in \mathbb{N}_0.$$

If the moment matrix is such that all its truncations, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{pmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{pmatrix} = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & & \rho_k \\ \rho_2 & & & & \\ \vdots & & & & \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{pmatrix}$$

are nonsingular, i. e. the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0, k \in \mathbb{N}_0$, then there exist monic polynomials

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{2}$$

with $p_0^1 = 0$, such that the following orthogonality conditions are fulfilled

$$\langle \rho, P_n(z)z^k \rangle = 0, \quad k \in \{0, \dots, n-1\}, \quad \langle \rho, P_n(z)z^n \rangle = H_n = \frac{\Delta_{n+1}}{\Delta_n} \neq 0.$$

Moreover, the set $\{P_n(z)\}_{n=0}^\infty$ is an orthogonal set of polynomials

$$\langle \rho, P_n(z)P_m(z) \rangle = \delta_{n,m} H_n, \quad n, m \in \mathbb{N}_0.$$

The second kind functions are given by

$$Q_n(z) := \sum_{k \in \mathbb{N}_0} \frac{P_n(k)w(k)}{z - k}. \tag{3}$$

In terms of the semi-infinite vector of monomials

$$\chi(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \end{pmatrix}$$

we have $G = \langle \rho, \chi \chi^\top \rangle$, and it becomes evident that the moment matrix is symmetric, $G = G^\top$. The vector of monomials χ is an eigenvector of the *shift matrix*

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

i.e., $\Lambda \chi(z) = z \chi(z)$. From here it follows immediately that $\Lambda G = G \Lambda^\top$, i.e., the Gram matrix is a Hankel matrix, as we previously said. Being the moment matrix symmetric its Borel–Gauss factorization reduces to a Cholesky factorization

$$G = S^{-1} H S^{-\top}, \tag{4}$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots \\ S_{1,0} & 1 & \dots & \dots & \dots \\ S_{2,0} & S_{2,1} & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and $H = \text{diag}(H_0, H_1, \dots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization does hold whenever the principal minors of the moment matrix, i.e., the Hankel determinants Δ_k , do not vanish.

The components $P_n(z)$ of the semi-infinite vector of polynomials

$$P(z) := S\chi(z), \tag{5}$$

are the monic orthogonal polynomials of the functional ρ . From the Cholesky factorization we get $\langle \rho, \chi\chi^\top \rangle = G = S^{-1}HS^{-\top}$ so that $S\langle \rho, \chi\chi^\top \rangle S^\top = H$. Therefore, $\langle \rho, S\chi\chi^\top S^\top \rangle = H$ and we obtain $\langle \rho, PP^\top \rangle = H$, which encodes the orthogonality of the polynomial sequence $\{P_n(z)\}_{n=0}^\infty$. The lower Hessenberg matrix

$$J = S\Lambda S^{-1} \tag{6}$$

has the vector $P(z)$ as eigenvector with eigenvalue z , that is $JP(z) = zP(z)$.

The lower Pascal matrix, built up of binomial numbers, is defined by

$$B = (B_{n,m}) = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m. \end{cases}$$

so that $\chi(z+1) = B\chi(z)$. The *dressed Pascal matrices* are the following lower unitriangular semi-infinite matrices

$$\Pi := SBS^{-1}, \quad \Pi^{-1} := SB^{-1}S^{-1},$$

which happen to be connection matrices; indeed, they satisfy

$$P(z+1) = \Pi P(z), \quad P(z-1) = \Pi^{-1} P(z).$$

The Hankel condition $\Lambda G = G\Lambda^\top$ and the Cholesky factorization lead to $\Lambda S^{-1}HS^{-\top} = S^{-1}HS^{-\top}\Lambda^\top$, or, equivalently,

$$JH = (JH)^\top = HJ^\top. \tag{7}$$

Hence, JH is symmetric, thus being Hessenberg and symmetric we deduce that J is tridiagonal. Therefore, the Jacobi matrix (6) can be written as follows

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & \dots & \dots & \dots \\ \gamma_1 & \beta_1 & 1 & & & \\ 0 & \gamma_2 & \beta_2 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

and the eigenvalue relation $JP = zP$ is a three term recurrence relation

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z), \quad n \in \mathbb{N}_0,$$

with the initial conditions $P_{-1} = 0$ and $P_0 = 1$. They completely determine the set of orthogonal polynomial sequence $\{P_n(z)\}_{n=0}^\infty$ in terms of the recursion coefficients β_n, γ_n .

Given any block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with blocks $A \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times s}, C \in \mathbb{C}^{s \times r}, D \in \mathbb{C}^{s \times s}$, being A a nonsingular matrix, we define the Schur complement $M/A := D - CA^{-1}B \in \mathbb{C}^{s \times s}$. When $s = 1$, so that $D \in \mathbb{C}$ and $B, C^T \in \mathbb{C}^r$ one can show that $M/A \in \mathbb{C}$ is a quotient of determinants $M/A = \frac{\det M}{\det A}$. These Schur complements are the building blocks of the theory of quasi-determinants that we will not treat here. For $s = 1$, using Olver's notation [51] for the last quasi determinant

$$\Theta_* \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = D - CA^{-1}B = \frac{\det \begin{pmatrix} A & B \\ C & D \end{pmatrix}}{\det A}.$$

The discrete Pearson equation for the weight is

$$\nabla(\sigma w) = \tau w, \tag{8}$$

with $\nabla f(z) = f(z) - f(z - 1)$, that is $\sigma(k)w(k) - \sigma(k - 1)w(k - 1) = \tau(k)w(k)$, for $k \in \{1, 2, \dots\}$, with $\sigma(z), \tau(z) \in \mathbb{R}[z]$ polynomials. If we write $\theta := \sigma - \tau$, the previous Pearson equation reads

$$\theta(k + 1)w(k + 1) = \sigma(k)w(k), \quad k \in \mathbb{N}_0. \tag{9}$$

In [47] it was proven that

Theorem 1 (Hypergeometric symmetry of the moment matrix) *Let the weight w be subject to a discrete Pearson equation of the type (9), where the functions θ, σ are polynomials, with $\theta(0) = 0$. Then, the corresponding moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^T. \tag{10}$$

Remark 1 This result extends to the case when θ and σ are entire functions, not necessarily polynomials, and we can ensure some meaning to $\theta(\Lambda)$ and $\sigma(\Lambda)$.

We can use the Cholesky factorization of the Gram matrix (4) and the Jacobi matrix (6) to get

Proposition 1 (Symmetry of the Jacobi matrix) *Let the weight w be subject to a discrete Pearson equation of the type (9), where the functions θ, σ are entire functions, not necessarily polynomials, with $\theta(0) = 0$. Then,*

$$\Pi^{-1}H\theta(J^T) = \sigma(J)H\Pi^T. \tag{11}$$

Moreover, the matrices $H\theta(J^T)$ and $\sigma(J)H$ are symmetric.

For a proof see [47].

In the standard discrete Pearson equation the functions θ, σ are polynomials. Let us denote their respective degrees by $N + 1 := \deg \theta(z)$ and $M := \deg \sigma(z)$. The roots of these polynomials are denoted by $\{-b_i + 1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$. Following [25] we choose

$$\theta(z) = z(z + b_1 - 1) \cdots (z + b_N - 1), \quad \sigma(z) = \eta(z + a_1) \cdots (z + a_M).$$

Notice that we have normalized θ to be a monic polynomial, while σ is not monic, where η denotes the leading coefficient of σ . Therefore, the weight is proportional to

$$w(z) = \frac{(a_1)_z \cdots (a_M)_z}{\Gamma(z + 1)(b_1)_z \cdots (b_N)_z} \eta^z, \tag{12}$$

see [25], where the Pochhammer symbol is understood as $(\alpha)_z = \frac{\Gamma(\alpha+z)}{\Gamma(\alpha)}$.

Remark 2 The 0-th moment is

$$\begin{aligned} \rho_0 &= \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_M)_k}{(b_1 + 1)_k \cdots (b_N + 1)_k} \frac{\eta^k}{k!} = {}_M F_N(a_1, \dots, a_M; b_1, \dots, b_N; \eta) \\ &= {}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right], \end{aligned}$$

is the generalized hypergeometric function, where we are using the two standard notations, see [14]. Then, according to (1), for $n \in \mathbb{N}$, the corresponding higher moments $\rho_n = \sum_{k=0}^{\infty} k^n w(k)$, are

$$\rho_n = \vartheta_\eta^n \rho_0 = \vartheta_\eta^n \left({}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right] \right), \quad \vartheta_\eta := \eta \frac{\partial}{\partial \eta}.$$

In [47] it was proven that

Theorem 2 (Laguerre–Freud structure matrix) *Let us assume that the weight w is subject to the discrete Pearson equation (9) with θ, σ polynomials such that $\theta(0) = 0, \deg \theta(z) = N + 1, \deg \sigma(z) = M$. Then, the Laguerre–Freud structure matrix*

$$\Psi := \Pi^{-1} H \theta(J^\top) = \sigma(J) H \Pi^\top = \Pi^{-1} \theta(J) H = H \sigma(J^\top) \Pi^\top \tag{13}$$

$$= \theta(J + I) \Pi^{-1} H = H \Pi^\top \sigma(J^\top - I), \tag{14}$$

has only $N + M + 2$ possibly nonzero diagonals ($N + 1$ superdiagonals and M subdiagonals)

$$\Psi = (\Lambda^\top)^M \psi^{(-M)} + \cdots + \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)} \Lambda + \cdots + \psi^{(N+1)} \Lambda^{N+1},$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\left\{ \begin{aligned} (\Lambda^\top)^M \psi^{(-M)} &= \eta (J_-)^M H, & \psi^{(-M)} &= \eta H \prod_{k=0}^{M-1} T_-^k \gamma = \eta \operatorname{diag} \left(H_0 \prod_{k=1}^M \gamma_k, H_1 \prod_{k=2}^{M+1} \gamma_k, \dots \right), \\ \psi^{(N+1)} \Lambda^{N+1} &= H (J_-^\top)^{N+1}, & \psi^{(N+1)} &= H \prod_{k=0}^N T_-^k \gamma = \operatorname{diag} \left(H_0 \prod_{k=1}^{N+1} \gamma_k, H_1 \prod_{k=2}^{N+2} \gamma_k, \dots \right). \end{aligned} \right. \tag{15}$$

The vector $P(z)$ of orthogonal polynomials fulfills the following structure equations

$$\theta(z) P(z - 1) = \Psi H^{-1} P(z), \quad \sigma(z) P(z + 1) = \Psi^\top H^{-1} P(z). \tag{16}$$

Three important relations fulfilled by the generalized hypergeometric functions are

$$(\vartheta_\eta + a_i) {}_M F_N \left[\begin{matrix} a_1 \cdots a_i \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right] = a_i {}_M F_N \left[\begin{matrix} a_1 \cdots a_i + 1 \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right], \tag{17}$$

$$(\vartheta_\eta + b_j - 1) {}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j \cdots b_N \end{matrix}; \eta \right] = (b_j - 1) {}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j - 1 \cdots b_N \end{matrix}; \eta \right], \tag{18}$$

$$\frac{d}{{d\eta}} {}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right] = \kappa {}_M F_N \left[\begin{matrix} a_1 + 1 \cdots a_M + 1 \\ b_1 + 1 \cdots b_N + 1 \end{matrix}; \eta \right],$$

$$\kappa := \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j}, \tag{19}$$

that imply

$$\eta \prod_{n=1}^M \left(\eta \frac{d}{{d\eta}} + a_n \right) u = \eta \frac{d}{{d\eta}} \prod_{n=1}^N \left(\eta \frac{d}{{d\eta}} + b_n - 1 \right) u, \quad u := {}_M F_N \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix}; \eta \right]. \tag{20}$$

In (17) and (18) we have a basic relation between contiguous generalized hypergeometric functions and its derivatives.

For the analysis of these equations let us introduce the shift operators in the parameters $\{a_i\}_{i=1}^M$ and $\{b_j\}_{j=1}^N$. Thus, given a function $f \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix} \right]$ of these parameters we introduce the shifts ${}_i T$ and T_j as follows

$${}_i T f \left[\begin{matrix} a_1 \cdots a_i \cdots a_M \\ b_1 \cdots b_N \end{matrix} \right] = f \left[\begin{matrix} a_1 \cdots a_i + 1 \cdots a_M \\ b_1 \cdots b_N \end{matrix} \right],$$

$$T_j f \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j \cdots b_N \end{matrix} \right] = f \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j - 1 \cdots b_N \end{matrix} \right],$$

and a total shift $T = {}_1 T \cdots {}_M T T_1^{-1} \cdots T_N^{-1}$, i. e.,

$$T f \left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_N \end{matrix} \right] := f \left[\begin{matrix} a_1 + 1 \cdots a_M + 1 \\ b_1 + 1 \cdots b_N + 1 \end{matrix} \right].$$

Then, we find:

Proposition 2 (Hypergeometric relations) *The moment matrix $G = (\rho_{n+m})_{n,m \in \mathbb{N}_0}$ satisfies the following hypergeometric relations*

$$(\Lambda + a_i I)G = a_i {}_i T G, \tag{21a}$$

$$(\Lambda + (b_j - 1)I)G = (b_j - 1)T_j G, \tag{21b}$$

$$\Lambda G = \kappa B(TG)B^\top. \tag{21c}$$

Finally, from (20) we derive, in an alternative manner, the relation (10).

2 A Christoffel–Geronimus perspective

The reader familiar with Christoffel and Geronimus transformations probably noticed a remarkable similarity of those transformations with these shifts to contiguous hypergeometric parameters. The Pochhammer symbol satisfies

$$\begin{aligned}
 (\alpha + 1)_z &= \frac{\Gamma(z + \alpha + 1)}{\Gamma(\alpha + 1)} = \frac{(z + \alpha)\Gamma(z + \alpha)}{\alpha\Gamma(\alpha)} = \frac{z + \alpha}{\alpha}(\alpha)_z, \\
 \frac{1}{(\beta - 1)_z} &= \frac{\Gamma(\beta - 1)}{\Gamma(z + \beta - 1)} = \frac{(\beta - 1 + z)\Gamma(\beta)}{(\beta - 1)\Gamma(\beta + z)} = \frac{z + \beta - 1}{\beta - 1} \frac{1}{(\beta)_z}.
 \end{aligned}$$

From the explicit form of the weight (12) we get

$$\begin{cases} a_i ({}_i T w) = (z + a_i)w, & i \in \{1, \dots, M\}, \\ (b_j - 1) (T_j w) = (z + b_j - 1)w, & j \in \{1, \dots, N\}. \end{cases} \tag{22}$$

Thus, $a_i {}_i T$ and $b_j T_j$ are Christoffel transformations. Moreover, from (22) we get

$$\begin{cases} (a_i - 1)w = (z + a_i - 1)({}_i T^{-1} w), & i \in \{1, \dots, M\}, \\ b_j w = (z + b_j)(T_j^{-1} w), & j \in \{1, \dots, N\}, \end{cases} \tag{23}$$

so that the inverse transformations are

$$\begin{cases} \frac{1}{a_i - 1}({}_i T^{-1} w) = \frac{w}{z + a_i - 1}, & i \in \{1, \dots, M\}, \\ \frac{1}{b_j}(T_j^{-1} w) = \frac{w}{z + b_j}, & j \in \{1, \dots, N\}. \end{cases} \tag{24}$$

Consequently, $\frac{1}{a_i - 1}{}_i T^{-1}$ and $\frac{1}{b_j}T_j^{-1}$ are massless Geronimus transformations. As it is well known, the solutions to (23) are more general than ${}_i T^{-1} w$ and $T_j^{-1} w$, respectively. In fact, the more general solutions to (23) are given by

$${}_i T^{-1} w + i m \delta(z + a_i - 1), \quad T_j^{-1} w + m_j \delta(z + b_j),$$

for some arbitrary constants $j m$ and m_j , known as masses, respectively. For the contiguous transformations discussed here these masses are chosen to cancel. Finally, for the total shift T we have

$$\kappa T w(z) = \frac{\prod_{i=1}^M (z + a_i)}{\prod_{j=1}^N (z + b_j)} w(z)$$

that for $z = k \in \mathbb{N}_0$, using the Pearson equation (9), reads

$$\kappa T w(k) = \frac{1}{\eta}(k + 1)w(k + 1).$$

Consequently, we find

$$T^{-1} w(k) = (T^{-1} \kappa) \frac{\eta}{k} w(k - 1), \quad T^{-1} \kappa := \frac{\prod_{i=1}^M (a_i - 1)}{\prod_{j=1}^N (b_j - 1)}, \quad k \in \mathbb{N}.$$

Many of the results that follow are well known in the literature. The novelty here is the matrix approach which is original. For the statements of Theorems 3 and 5, please check [34] for a more general framework.

2.1 The Christoffel contiguous transformations

In order to apply the Cholesky factorization of the moment matrix to the previous result we introduce the following semi-infinite matrices

$${}_i\omega := ({}_iTS)(\Lambda + a_iI)S^{-1}, \quad {}_i\Omega := S({}_iTS)^{-1}, \quad i \in \{1, \dots, M\}, \tag{25a}$$

$$\omega_j := (T_jS)(\Lambda + (b_j - 1)I)S^{-1}, \quad \Omega_j := S(T_jS)^{-1}, \quad j \in \{1, \dots, N\}, \tag{25b}$$

$$\omega := (TS)B^{-1}\Lambda S^{-1}, \quad \Omega := SB(TS)^{-1}, \tag{25c}$$

that, as we immediately show, are connection matrices. The action of these matrices on the vector of orthogonal polynomials lead to the following:

Proposition 3 (Connection formulas) *The following relations among orthogonal polynomials are satisfied*

$${}_i\omega P(z) = (z + a_i){}_iTP(z), \quad {}_i\Omega {}_iTP(z) = P(z), \quad i \in \{1, \dots, M\}, \tag{26a}$$

$$\omega_j P(z) = (z + b_j - 1)T_jP(z), \quad \Omega_j T_jP(z) = P(z), \quad j \in \{1, \dots, N\}, \tag{26b}$$

$$\omega P(z) = (z - 1)TP(z - 1), \quad \Omega TP(z) = P(z + 1). \tag{26c}$$

The Cholesky factorization of the Gram matrices leads to the following expressions for these connection matrices:

Proposition 4 *Let us assume that the Cholesky factorization of the Gram matrices $G, {}_jTG, T_kG$ and TG hold. Then, we have the following expressions*

$$\begin{aligned}
 {}_i\omega &= \begin{pmatrix} \frac{a_i {}_iTH_0}{H_0} & 1 & 0 & \dots & \dots \\ 0 & \frac{a_i {}_iTH_1}{H_1} & 1 & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}, & \omega_j &= \begin{pmatrix} \frac{(b_j-1)T_jH_0}{H_0} & 1 & 0 & \dots & \dots \\ 0 & \frac{(b_j-1)T_jH_1}{H_1} & 1 & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}, \\
 \omega &= \begin{pmatrix} \kappa \frac{T H_0}{H_0} & 1 & 0 & \dots & \dots \\ 0 & \kappa \frac{T H_1}{H_1} & 1 & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}, & \Omega_j &= \begin{pmatrix} 1 & 0 & \dots & \dots & \dots \\ \frac{1}{b_j-1} \frac{H_1}{T_j H_0} & 1 & \dots & \dots & \dots \\ 0 & \frac{1}{b_j-1} \frac{H_2}{T_j H_1} & \dots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}, \\
 {}_i\Omega &= \begin{pmatrix} 1 & 0 & \dots & \dots & \dots \\ \frac{1}{a_i} \frac{H_1}{{}_iTH_0} & 1 & \dots & \dots & \dots \\ 0 & \frac{1}{a_i} \frac{H_2}{{}_iTH_1} & \dots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}, & \Omega &= \begin{pmatrix} 1 & 0 & \dots & \dots & \dots \\ \frac{H_1}{\kappa T H_0} & 1 & \dots & \dots & \dots \\ 0 & \frac{H_2}{\kappa T H_1} & \dots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \ddots \end{pmatrix}.
 \end{aligned}$$

Proof In the one hand, observe that ${}_j\omega, \omega_k$ and ω are lower uni-Hessenberg matrices, i.e. all its superdiagonals are zero but for the first one that is Λ , while in the other hand ${}_j\Omega, \Omega_k$ and

Ω are lower unitriangular matrices. From (21) we get

$$\begin{aligned} (\Lambda + a_i I)S^{-1}HS^{-\top} &= a_i ({}_jTS)^{-1}({}_iTH)({}_iTS)^{-\top}, \\ (\Lambda + (b_j - 1)I)S^{-1}HS^{-\top} &= (b_j - 1)(T_jS)^{-1}(T_jH)(T_jS)^{-\top}, \\ B^{-1}\Lambda S^{-1}HS^{-\top} &= \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} (TS^{-1})(TH)(TS)^{-\top}B^\top, \end{aligned}$$

that can be written as follows

$${}_i\omega H = a_i({}_iTH)({}_i\Omega)^\top, \tag{27a}$$

$$\omega_j H = (b_j - 1)(T_jH)(\Omega_j)^\top, \tag{27b}$$

$$\omega H = \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} (TH)\Omega^\top. \tag{27c}$$

From these relations given that ${}_j\omega, \omega_k$ and ω are lower uni-Hessenberg matrices and $({}_j\Omega)^\top, (\Omega_k)^\top$ and Ω are upper unitriangular matrices, we conclude that ${}_j\omega, \omega_k$ and ω are upper triangular matrices with only the main diagonal and the first superdiagonal non vanishing and that ${}_j\Omega, \Omega_k$ and Ω are lower unitriangular matrices with only the first subdiagonal different from zero. The given expressions follow by identification of the coefficients in (27). \square

Let $\mathcal{Z} = \cup_{n \in \mathbb{N}_0} \mathcal{Z}_n$, with \mathcal{Z}_n being the set of zeros P_n ,

Theorem 3 (Christoffel formulas) *Whenever, $(\{-a_i\}_{i=1}^M \cup \{-b_j + 1\} \cup \{1\}_{j=1}^N) \cap \mathcal{Z} = \emptyset$, the following expressions are fulfilled*

$$\begin{aligned} {}_iTP_n(z) &= \frac{1}{z + a_i} \left(P_{n+1}(z) - \frac{P_{n+1}(-a_i)}{P_n(-a_i)} P_n(z) \right), \quad i \in \{1, \dots, M\}, \\ T_jP_n(z) &= \frac{1}{z + b_j - 1} \left(P_{n+1}(z) - \frac{P_{n+1}(-b_j + 1)}{P_n(-b_j + 1)} P_n(z) \right), \quad j \in \{1, \dots, N\}, \\ TP_n(z - 1) &= \frac{1}{z - 1} \left(P_{n+1}(z) - \frac{P_{n+1}(1)}{P_n(1)} P_n(z) \right). \end{aligned}$$

Proof From the connection formulas we obtain

$${}_i\omega P(-a_i) = 0, \quad i \in \{1, \dots, M\}, \tag{28a}$$

$$\omega_j P(-b_j + 1) = 0, \quad j \in \{1, \dots, N\}, \tag{28b}$$

$$\omega P(1) = 0, \tag{28c}$$

so that

$$a_i \frac{{}_iTH_n}{H_n} = -\frac{P_{n+1}(-a_i)}{P_n(-a_i)}, \quad i \in \{1, \dots, M\}, \tag{29a}$$

$$(b_j - 1) \frac{T_kH_n}{H_n} = -\frac{P_{n+1}(-b_j + 1)}{P_n(-b_j + 1)}, \quad j \in \{1, \dots, N\}, \tag{29b}$$

$$\frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} \frac{TH_n}{H_n} = -\frac{P_{n+1}(1)}{P_n(1)}. \tag{29c}$$

From the connection formulas we get the result. \square

Theorem 4 (Jacobi matrix and LU and UL factorization) *The following LU factorizations hold*

$$\begin{cases} J + a_i I = {}_i L {}_i U, & {}_i L := {}_i \Omega, & {}_i U := a_i ({}_i T H) {}_i \Omega^\top H^{-1}, & i \in \{1, \dots, M\}, \\ J + (b_j - 1) I = L_j U_j, & L_j := \Omega_j, & U_j = (b_j - 1) (T_j H) \Omega_j^\top H^{-1}, & j \in \{1, \dots, N\}, \\ J = LU, & L := \Omega, & U := \frac{\prod_{i=1}^N a_i}{\prod_{j=1}^M b_j} (T H) \Omega^\top H^{-1} & . \end{cases} \tag{30}$$

Moreover, the Christoffel transformed Jacobi matrices have the following UL factorizations

$${}_i T J + a_i I = {}_i U {}_i L, \quad i \in \{1, \dots, M\}, \tag{31a}$$

$$T_j J + (b_j - 1) I = U_j L_j, \quad j \in \{1, \dots, N\}, \tag{31b}$$

$$T J - I = UL. \tag{31c}$$

Proof From (21) we get

$$\begin{aligned} S(\Lambda + a_i I) S^{-1} &= a_i S({}_i T S)^{-1} ({}_i T H) ({}_i T S)^{-\top} S^\top H^{-1}, \\ S(\Lambda + (b_j - 1) I) S^{-1} &= (b_j - 1) S(T_j S)^{-1} (T_j H) (T_j S)^{-\top} S^\top H^{-1}, \\ S \Lambda S^{-1} &= \frac{\prod_{i=1}^N a_i}{\prod_{j=1}^M b_j} S B (T S^{-1}) (T H) (T S)^{-\top} B^\top S^\top H^{-1}, \end{aligned}$$

from where (30) follow. To prove (31) we write (21a) and (21b)

$$\begin{aligned} {}_j i T S(\Lambda + a_j I) ({}_i T S)^{-1} &= a_i ({}_i T H) ({}_i T S)^{-\top} S^\top H^{-1} S ({}_j T S)^{-1}, \\ T_j S(\Lambda + (b_j - 1) I) (T_j S)^{-1} &= (b_j - 1) (T_j H) (T_j S)^{-\top} S^\top H^{-1} S (T_j S)^{-1}, \end{aligned}$$

and we get (31a) and (31b). To show (31c) we write (21c) as follows

$$B^{-1} \Lambda S^{-1} H = \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} (T S)^{-1} (T H) (T S)^{-\top} B^\top S^\top,$$

and recalling that $B^{-1} \Lambda = (\Lambda - I) B^{-1}$ we obtain

$$(T S) (\Lambda - I) (T S)^{-1} (T S) B^{-1} S^{-1} H = \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} (T H) (T S)^{-\top} B^\top S^\top.$$

That is, we deduce that

$$(T J - I) \Omega^{-1} = \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j} (T H) \Omega^\top H^{-1},$$

and the third UL factorization follows. □

Remark 3 Given a symmetric tridiagonal matrix

$$\mathcal{J} = \begin{pmatrix} r_0 & s_0 & 0 & 0 & \cdots \\ s_0 & r_1 & s_1 & 0 & \cdots \\ 0 & s_1 & r_2 & s_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

its Cholesky factorization is

$$\mathcal{J} = LDL^T, \quad L = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ l_1 & 1 & 0 & \dots & \dots \\ 0 & l_2 & 1 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad D = \text{diag}(\delta_0, \delta_1, \dots),$$

with $\delta_0 = r_0, l_1 = \frac{s_0}{\delta_0}$ and

$$\delta_n = r_n - \frac{s_n^2}{\delta_{n-1}}, \quad l_{n+1} = \frac{s_n}{\delta_n}, \quad n \in \mathbb{N}.$$

Which, when iterated leads to continued fraction expressions for the Cholesky factor’s coefficients in terms of the sequences $\{r_n\}_{n=0}^\infty$ and $\{s_n\}_{n=0}^\infty$. Equating \mathcal{J} with $(J + a_j I)H, (J + (b_k - 1)I)H$ and JH (which are symmetric tridiagonal matrices) and applying the above formulas we get expressions for $({}_j\Omega, {}_jTH)$, and $(\Omega_k, T_k H)$ and (Ω, TH) , respectively. The coefficients (r_n, s_n) are $(\beta_n H_n + a_i, H_{n+1}), (\beta_n H_n + b_k - 1, H_{n+1})$ and $(\beta_n H_n, H_{n+1})$, respectively. Therefore, we get continued fraction expressions for the Ω ’s, TH ’s and ω ’s in terms of the recursion coefficients.

2.2 The Geronimus contiguous transformations

From Proposition 3 we get the following connections formulas

$$\begin{aligned} ({}_i T^{-1} \omega)_i T^{-1} P(z) &= (z + a_i - 1)P(z), \quad ({}_i T^{-1} \Omega) P(z) := {}_i T^{-1} P(z), \quad i \in \{1, \dots, M\}, \\ (T_j^{-1} \omega_j) T_j^{-1} P(z) &= (z + b_j)P(z), \quad (T_j^{-1} \Omega_j) P(z) = T_j^{-1} P(z), \quad j \in \{1, \dots, N\}, \\ (T^{-1} \omega) T^{-1} P(z) &= (z - 1)P(z - 1), \quad (T^{-1} \Omega) P(z) = T^{-1} P(z + 1). \end{aligned}$$

From these connections formulas we do not get Christoffel type formulas as for the Christoffel transformations. We need to use associated second kind functions, see (3).

Proposition 5 *For the second kind functions $Q_n(z)$, the following relations hold*

$$(a_i - 1)({}_i T^{-1} \Omega) Q(z) = (z + a_i - 1)({}_i T^{-1} Q(z)) - \begin{pmatrix} {}_i T^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}, \quad i \in \{1, \dots, M\}, \tag{32a}$$

$$b_j(T_j^{-1} \Omega_j) Q(z) = (z + b_j)(T_j^{-1} Q(z)) - \begin{pmatrix} T_j^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}, \quad j \in \{1, \dots, N\}, \tag{32b}$$

$$(T^{-1} \Omega)(\Upsilon Q(z - 1) - P(z - 1)) = zT^{-1} Q(z) - T^{-1} P(z) - \begin{pmatrix} T^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}, \tag{32c}$$

with $\Upsilon := \eta \frac{\prod_{i=1}^M (a_i - 1)}{\prod_{j=1}^N (b_j - 1)} = \eta T^{-1} \kappa$.

Proof Let us compute

$$\begin{aligned}
 & (z + a_i - 1)({}_i T^{-1} Q(z)) - (a_i - 1)({}_i T^{-1} \Omega) Q(z) \\
 &= (z + a_i - 1) \sum_{k=0}^{\infty} \frac{({}_i T^{-1} P(k))({}_i T^{-1} w(k))}{z - k} \\
 &\quad - \sum_{k=0}^{\infty} \frac{j T^{-1} P(k)}{z - k} ({}_j T^{-1} w(k))(k + a_j - 1) \\
 &= \sum_{k=0}^{\infty} ({}_i T^{-1} P(k))({}_i T^{-1} w(k)) = \begin{pmatrix} {}_i T^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}.
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 & (z + b_j)({}_j T^{-1} Q(z)) - b_j({}_j T^{-1} \Omega_j) Q(z) \\
 &= (z + b_j) \sum_{k=0}^{\infty} \frac{({}_j T^{-1} P(k))({}_j T^{-1} w(k))}{z - k} - \sum_{k=0}^{\infty} \frac{{}_j T^{-1} P(k)}{z - k} ({}_j T^{-1} w(k))(k + b_j) \\
 &= \sum_{k=0}^{\infty} ({}_j T^{-1} P(k))({}_j T^{-1} w(k)) = \begin{pmatrix} {}_j T^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}.
 \end{aligned}$$

Finally, we prove the last equation. In the one hand, we have $T^{-1} Q(z) = \sum_{k=0}^{\infty} (T^{-1} P(k)) \frac{T^{-1} w(k)}{z - k}$. On the other hand, we find

$$\begin{aligned}
 (T^{-1} \Omega) \Upsilon Q(z - 1) &= (T^{-1} \Omega) \sum_{k=0}^{\infty} P(k) \frac{\Upsilon w(k)}{z - 1 - k} = (T^{-1} \Omega) \sum_{k=1}^{\infty} P(k - 1) \frac{\Upsilon w(k - 1)}{z - k} \\
 &= \sum_{k=1}^{\infty} (T^{-1} P(k)) \frac{k T^{-1} w(k)}{z - k} = \sum_{k=0}^{\infty} (T^{-1} P(k)) \frac{k T^{-1} w(k)}{z - k} \\
 &= \sum_{k=0}^{\infty} (T^{-1} P(k)) \left(\frac{z}{z - k} - 1 \right) T^{-1} w(k) \\
 &= z \sum_{k=0}^{\infty} (T^{-1} P(k)) \frac{T^{-1} w(k)}{z - k} - \sum_{k=0}^{\infty} (T^{-1} P(k)) T^{-1} w(k),
 \end{aligned}$$

so that

$$(T^{-1} \Omega) \Upsilon Q(z - 1) = T^{-1} Q(z) - \begin{pmatrix} T^{-1} H_0 \\ 0 \\ \vdots \end{pmatrix}, \tag{33}$$

and using $(T^{-1} \Omega) P(z - 1) = T^{-1} P(z)$ we get the announced result. □

Observe that, as far $-a_i + 1, -b_j \notin \mathbb{N}_0$, the discrete support of ρ_z , from (32a) and (32b) we obtain

$$(a_j - 1)(jT^{-1}j\Omega)Q(-a_j + 1) = - \begin{pmatrix} jT^{-1}H_0 \\ 0 \\ \vdots \end{pmatrix}, \quad b_j(T_j^{-1}\Omega_j)Q(-b_j) = - \begin{pmatrix} T_j^{-1}H_0 \\ 0 \\ \vdots \end{pmatrix},$$

so that

$$(iT^{-1}i\Omega)_{n,n-1} = -\frac{Q_n(-a_i + 1)}{Q_{n-1}(-a_i + 1)}, \quad n > 1, \quad iT^{-1}H_0 = -(a_i - 1)Q_0(-a_i + 1),$$

$$(T_j^{-1}\Omega_j)_{n,n-1} = -\frac{Q_n(-b_j)}{Q_{n-1}(-b_j)}, \quad n > 1, \quad T_j^{-1}H_0 = -b_jQ_0(-b_j).$$

Why we write (32c) instead of the equivalent Eq. (33)? Because (32c) is prepared for the limit $z \rightarrow 0$. Notice that $z = 0$ belongs to the support \mathbb{N}_0 of ρ_z , and $\lim_{z \rightarrow 0} zT^{-1}Q(z)$ does not necessarily vanish. Observe that $T^{-1}Q(z)$ is meromorphic with simple poles at \mathbb{N}_0 , in fact

$$\text{Res}(zT^{-1}Q(z), 0) = T^{-1}P(0)T^{-1}w(0) = T^{-1}P(0) = (T^{-1}\Omega)P(-1),$$

where we have used that $w(0) = 1$ does not depend on the parameters a_i, b_j and, consequently, $T^{-1}w(0) = 1$. Hence, $\lim_{z \rightarrow 0}(zT^{-1}Q(z) - T^{-1}P(z)) = 0$. Therefore, from (32c) we obtain that

$$(T^{-1}\Omega)(\Upsilon Q(-1) - P(-1)) = - \begin{pmatrix} T^{-1}H_0 \\ 0 \\ \vdots \end{pmatrix}$$

and, consequently, we deduce

$$(T^{-1}\Omega)_{n,n-1} = \frac{\Upsilon Q_n(-1) - P_n(-1)}{\Upsilon Q_{n-1}(-1) - P_{n-1}(-1)}, \quad T^{-1}H_0 = P_n(-1) - \Upsilon Q_0(-1).$$

Theorem 5 For $n \in \mathbb{N}_0$, the Geronimus transformed orthogonal polynomials we have the Christoffel–Geronimus expressions

$$iT^{-1}P_n(z) = P_n(z) - \frac{Q_n(-a_i + 1)}{Q_{n-1}(-a_i + 1)}P_{n-1}(z), \quad i \in \{1, \dots, M\},$$

$$T_j^{-1}P_n(z) = P_n(z) - \frac{Q_n(-b_j)}{Q_{n-1}(-b_j)}P_{n-1}(z), \quad j \in \{1, \dots, N\},$$

$$T^{-1}P_n(z) = P_n(z - 1) - \frac{\Upsilon Q_n(-1) - P_n(-1)}{\Upsilon Q_{n-1}(-1) - P_{n-1}(-1)}P_{n-1}(z - 1).$$

From Theorem 4 we get

Theorem 6 (Jacobi matrix and UL and LU factorization) *The Jacobi matrix has following UL factorizations*

$$\begin{cases} J + a_i I = iU_i L, \quad iL := iT^{-1}i\Omega, \quad iU := a_i H(iT^{-1}i\Omega)^\top (iT^{-1}H)^{-1}, & i \in \{1, \dots, M\}, \\ J + (b_j - 1)I = U_j L_j, \quad L_j := T_j^{-1}\Omega_j, \quad U_j = (b_j - 1)H(T_j^{-1}\Omega_j)^\top (T_j^{-1}H)^{-1}, & j \in \{1, \dots, N\}, \\ J - I = UL, \quad L := T^{-1}\Omega, \quad U := \prod_{i=1}^M \frac{a_i}{\prod_{j=1}^N b_j} H(T^{-1}\Omega)^\top (T^{-1}H)^{-1}. \end{cases}$$

The Geronimus transformed Jacobi matrices have the following LU factorizations

$$\begin{cases} {}_i T^{-1} J + a_i I = {}_i L {}_i U, & i \in \{1, \dots, M\}, \\ T_j^{-1} J + (b_j - 1)I = L_j U_j, & j \in \{1, \dots, N\}, \\ T^{-1} J = LU. \end{cases}$$

2.2.1 An example: the Meixner polynomials

The Meixner polynomials correspond to the choice

$$w(z) = \frac{(a)_z}{\Gamma(z + 1)} \eta^z, \quad \theta = z, \quad \sigma = \eta(z + a).$$

The zero-order moment is

$$\rho_0 = {}_1 F_0 \left[\begin{matrix} a \\ - \end{matrix}; \eta \right] = \frac{1}{(1 - \eta)^a},$$

while the moments are given by

$$\rho_n = \frac{1}{(1 - \eta)^a} \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} (a)_m \frac{\eta^m}{(1 - \eta)^m}, \tag{34}$$

with $\begin{Bmatrix} n \\ m \end{Bmatrix}$ the Stirling numbers of the second kind. The recursion coefficients [23] are

$$\beta_n = \frac{n + (n + a)\eta}{1 - \eta}, \quad \gamma_n = \frac{n(n + a - 1)\eta}{(1 - \eta)^2}. \tag{35}$$

The monic orthogonal polynomials are expressed in terms of the Gaussian hypergeometric function as follows

$$P_n(x) = (a)_n \frac{\eta^n}{(\eta - 1)^n} M_n(x; a, \eta), \quad M_n(x; a, \eta) := {}_2 F_1 \left[\begin{matrix} -n, -x \\ a \end{matrix}; \frac{\eta - 1}{\eta} \right]. \tag{36}$$

Moreover, as we know that [38, Theorem 6.1.1]

$$\sum_{k=0}^{\infty} M_n^2(k; a, \eta) w(k) = \frac{n!(1 - \eta)^{-a}}{\eta^n (a)_n},$$

we find

$$H_n = \sum_{k=0}^{\infty} P_n^2(k) w(k) = (a)_n^2 \frac{\eta^{2n}}{(\eta - 1)^{2n}} \frac{n!(1 - \eta)^{-a}}{\eta^n (a)_n} = n!(a)_n \frac{\eta^n}{(1 - \eta)^{2n+a}}.$$

Observe that, as $\gamma_n = \frac{H_n}{H_{n-1}}$ and $\beta_n = \vartheta_\eta \log H_n$, we recover the previous expressions (35).

For the Laguerre–Freud structure matrix (15) we have [47]

$$\Psi = \begin{pmatrix} \beta_0 H_0 & H_1 & 0 & \dots & \dots & \dots \\ \eta H_1 & (\beta_1 - 1)H_1 & H_2 & \dots & \dots & \dots \\ 0 & \eta H_2 & (\beta_2 - 2)H_2 & H_3 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

The connection formulas (16) are, in this case,

$$zP(z - 1) = \Psi H^{-1}P(z), \quad (z + a)P(z + 1) = \Psi^\top H^{-1}P(z).$$

In this case we only have one shift, that is $\hat{w} := {}_1Tw = \frac{(a+1)_z}{\Gamma(z+1)}\eta^z$, so that $a\hat{w} = (z + a)w$.

$$\omega = \begin{pmatrix} \frac{a\hat{H}_0}{H_0} & 1 & 0 & \dots & \dots \\ 0 & \frac{a\hat{H}_1}{H_1} & 1 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 & \dots & \dots \\ \frac{1}{a} \frac{H_1}{H_0} & 1 & \dots & \dots \\ 0 & \frac{1}{a} \frac{H_2}{H_1} & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Hence, Theorem 3 reads

$$\hat{P}_n(z) = \frac{1}{z + a} \left(P_{n+1}(z) - \frac{P_{n+1}(-a)}{P_n(-a)} P_n(z) \right).$$

Notice that,

$$P_n(-a) = (a)_n \frac{\eta^n}{(\eta - 1)^{n-1}} F_0 \left[\begin{matrix} -n, & \eta - 1 \\ - & \eta \end{matrix} \right] = \frac{(a)_n}{(\eta - 1)^n},$$

so that

$$\hat{P}_n(z) = \frac{1}{z + a} \left(P_{n+1}(z) - \frac{a + n}{\eta - 1} P_n(z) \right).$$

In terms of Meixner orthogonal polynomials (36) we get for the Gauss hypergeometric function the following contiguous relation

$$(z + a)(\eta - 1)_2 F_1 \left[\begin{matrix} -n, & -x \\ a + 1 & \eta \end{matrix} \right] = a\eta_2 F_1 \left[\begin{matrix} -n - 1, & -x \\ a & \eta \end{matrix} \right] - a_2 F_1 \left[\begin{matrix} -n, & -x \\ a & \eta \end{matrix} \right].$$

2.3 Christoffel–Geronimus–Uvarov transformation and shifts in z

Here we follow [6–8] adapted to the scalar case. If we denote $P_n^{(\pm)}(z) = P_n(z \pm 1)$, we notice that $\{P_n^{(\pm)}(z)\}_{n=0}^\infty$ is a sequence of monic orthogonal polynomials

$$\sum_{k=\mp 1}^\infty P_n^{(\pm)}(k) P_m^{(\pm)}(k) w^{(\pm)}(k) = \delta_{n,m} H_n,$$

with $w^{(\pm)}(k) := w(k \pm 1)$. The two perturbed functionals $\rho^{(\pm)} := \sum_{k=\mp 1}^\infty \delta(z - k)w^{(\pm)}(z)$ satisfy

$$\theta(z + 1)\rho^{(+)} = \sigma(z)\rho, \quad \sigma(z - 1)\rho^{(-)} = \theta(z)\rho. \tag{37}$$

Indeed, using the Pearson equation (9) and that $\theta(0) = 0$ we get

$$\begin{aligned} \theta(z + 1)\rho^{(+)} &= \sum_{k=-1}^\infty \delta(z - k)\theta(z + 1)w(z + 1) = \sum_{k=0}^\infty \delta(z - k)\sigma(z)w(z) = \sigma(z)\rho, \\ \sigma(z - 1)\rho^{(-)} &= \sum_{k=1}^\infty \delta(z - k)\sigma(z - 1)w(z - 1) = \sum_{k=0}^\infty \delta(z - k)\theta(z)w(z) = \theta(z)\rho. \end{aligned}$$

Consequently, the Pearson equation could be understood as describing a perturbation of the functional, a perturbation of Geronimus–Uvarov type (a composition of a Geronimus and a Christoffel perturbation). If fact, for the $\rho^{(+)}$ perturbation, if $\sigma = 1$ we have a Geronimus transformation and for $\theta = 1$ we have a Christoffel transformation. The reverse occurs for the $\rho^{(-)}$ perturbation, if $\theta = 1$ we have a Geronimus transformation and for $\sigma = 1$ we have a Christoffel transformation. These interpretations, together with (16), allow to find explicit expressions for the shifted polynomials in terms of Christoffel type formulas that involve the evaluation of the polynomials and the second kind functions at the zeros of σ and θ .

Attending to (37) and following [6–8] adapted to the scalar case, we have the interpretation

$$W_G^{(+)} = \theta(z + 1), \quad W_C^{(+)} = \sigma(z), \quad W_G^{(-)} = \sigma(z - 1), \quad W_C^{(-)} = \theta(z).$$

The corresponding perturbed Gram matrices are

$$\begin{aligned} G^{(\pm)} &= \langle \rho^{(\pm)}, \chi \chi^\top \rangle = \sum_{k=\mp 1}^{\infty} \chi(k) \chi(k)^\top w^{(\pm)}(k) = \sum_{k=\mp 1}^{\infty} \chi(k) \chi(k)^\top w(k \pm 1) \\ &= \sum_{k=0}^{\infty} \chi(k \mp 1) \chi(k \mp 1)^\top w(k) = B^{\mp 1} \left(\sum_{k=0}^{\infty} \chi(k) \chi(k)^\top w(k) \right) B^{\mp \top} = B^{\mp 1} G B^{\mp \top}. \end{aligned}$$

We have

$$\rho^{(\pm)} = \sum_{k=-\mp 1}^{\infty} \delta(z - k) w^{(\pm)}(k) = \sum_{k=-\mp 1}^{\infty} \delta(z - k) w(k \pm 1),$$

and also, using Pearson equation (9)

$$\begin{aligned} \frac{\sigma(z)}{\theta(z + 1)} \rho &= \sum_{k=0}^{\infty} \delta(z - k) \frac{\sigma(k)}{\theta(k + 1)} w(k) = \sum_{k=0}^{\infty} \delta(z - k) w(k + 1), \\ \frac{\theta(z)}{\sigma(z - 1)} \rho &= \sum_{k=0}^{\infty} \delta(z - k) \frac{\theta(k)}{\sigma(k - 1)} w(k) = \sum_{k=1}^{\infty} \delta(z - k) w(k - 1). \end{aligned}$$

Consequently, we can write

$$\rho^{(+)} = \frac{\sigma(z)}{\theta(z + 1)} \rho + \delta(z + 1) w(0), \quad \rho^{(-)} = \frac{\theta(z)}{\sigma(z - 1)} \rho.$$

Hence, for the (+) perturbation we need a Geronimus mass $\delta(z + 1) w(0)$, while for the (−) perturbation there is no mass at all.

The Cholesky factorizations for the corresponding perturbed Gram matrices $G^{(\pm)}$ give

$$G^{(\pm)} = (S^{(\pm)})^{-1} H^{(\pm)} (S^{(\pm)})^{-\top} = B^{\mp 1} S^{-1} H S^{-\top} B^{\mp \top},$$

and from the uniqueness of such factorization we get $S^{(\pm)} = S B^{\pm 1} = \Pi^{\pm 1} S$ and $H^{(\pm)} = H$. The resolvent matrices, see Definition 2 in [8], of these two Geronimus–Uvarov perturbations are

$$\omega^{(\pm)} = S^{(\pm)} W_C^{(\pm)}(\Lambda) S^{-1} = H^{(\pm)} (S^{(\pm)})^{-\top} W_G^{(\pm)}(\Lambda^\top) S^\top H^{-1}.$$

That is,

$$\begin{aligned} \omega^{(\pm)} &= SB^{\pm 1}W_C^{(\pm)}(\Lambda)S^{-1} = SB^{\pm 1}SS^{-1}W_C^{(\pm)}(\Lambda)S^{-1} = \Pi^{\pm 1}W_C^{(\pm)}(J) \\ &= H^{(\pm)}\left(SW_G^{(\pm)}(\Lambda)(S^{(\pm)})^{-1}\right)^{\top}H^{-1} \\ &= H(SW_G^{(\pm)}(\Lambda)B^{\mp 1}S^{-1})^{\top}H^{-1} = H(W_G^{(\pm)}(J)\Pi^{\mp 1})^{\top}H^{-1}. \end{aligned}$$

Hence, recalling iii) in [8, Proposition 3], formulas (5) and (6) we get

$$\begin{aligned} \omega^{(+)} &= \Pi\sigma(J) = H\Pi^{-\top}\theta(J^{\top} + I)H^{-1} = H\theta(J^{\top})\Pi^{-\top}H^{-1} = \Psi^{\top}H^{-1}, \\ \omega^{(-)} &= \Pi^{-1}\theta(J) = H\Pi^{\top}\sigma(J^{\top} - I)H^{-1} = H\sigma(J^{\top})\Pi^{\top}H^{-1} = \Psi H^{-1}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \sigma(z)P(z + 1) &= \omega^{(+)}P(z) = \Pi\sigma(J)P(z), \\ (\omega^{(+)})^{\top}H^{-1}P(z + 1) &= H^{-1}\theta(J + I)\Pi^{-1}HH^{-1}P(z + 1) = \theta(z + 1)H^{-1}P(z), \\ \theta(z)P(z - 1) &= \omega^{(-)}P(z) = \Pi^{-1}\theta(J)P(z), \\ (\omega^{(-)})^{\top}H^{-1}P(z - 1) &= H^{-1}\sigma(J - I)\Pi HH^{-1}P(z - 1) = \sigma(z - 1)H^{-1}P(z). \end{aligned}$$

These equations recover (16) from this perturbation perspective. More interesting are the results in [8] regarding Geronimus–Uvarov perturbations and the second kind functions. The new perturbed second kind functions are

$$\begin{aligned} Q^{(\pm)}(z) &= \left\langle \rho_{\zeta}^{(\pm)}, \frac{P^{(\pm)}(\zeta)}{z - \zeta} \right\rangle = \sum_{k=\mp 1}^{\infty} \frac{P^{(\pm)}(k)w^{(\pm)}(k)}{z - k} = \sum_{k=\mp 1}^{\infty} \frac{P(k \pm 1)w(k \pm 1)}{z - k} \\ &= \sum_{k=\mp 1}^{\infty} \frac{P(k \pm 1)w(k \pm 1)}{z \pm 1 - (k \pm 1)} = \sum_{k=0}^{\infty} \frac{P(k)w(k)}{z \pm 1 - k} \\ &= Q(z \pm 1). \end{aligned}$$

According to the Proof of [8, Proposition 4] we have

$$Q^{(\pm)}(z)W_G^{(\pm)}(z) - \omega^{(\pm)}Q = \left\langle \rho_{\zeta}^{(\pm)}, P(\zeta) \frac{W_G^{(\pm)}(z) - W_G^{(\pm)}(\zeta)}{z - \zeta} \right\rangle,$$

and we get the following relations

$$\begin{aligned} Q(z + 1)\theta(z + 1) - \Psi^{\top}H^{-1}Q(z) &= \left\langle \rho_{\zeta}^{(+)}, P(\zeta) \frac{\theta(z + 1) - \theta(\zeta + 1)}{z - \zeta} \right\rangle \\ &= \sum_{k=0}^{\infty} P(k) \frac{\theta(z + 1) - \theta(k)}{z + 1 - k} w(k), \\ Q(z - 1)\sigma(z - 1) - \Psi H^{-1}Q(z) &= \left\langle \rho_{\zeta}^{(-)}, P(\zeta) \frac{\sigma(z - 1) - \sigma(\zeta - 1)}{z - \zeta} \right\rangle \\ &= \sum_{k=0}^{\infty} P(k) \frac{\sigma(z - 1) - \sigma(k)}{z - 1 - k} w(k). \end{aligned}$$

Finally, we collect these results together.

Proposition 6 *The following holds*

$$\begin{aligned} \theta(z)Q(z) - \Psi^\top H^{-1}Q(z-1) &= \sum_{k=0}^\infty P(k) \frac{\theta(z) - \theta(k)}{z-k} w(k), \\ \sigma(z)Q(z) - \Psi H^{-1}Q(z+1) &= \sum_{k=0}^\infty P(k) \frac{\sigma(z) - \sigma(k)}{z-k} w(k). \end{aligned}$$

If $\theta(z) = z^{N+1} + \theta_N z^N + \dots + \theta_1 z$ and $\sigma(z) = \eta z^M + \sigma_{M-1} z^{M-1} + \dots + \sigma_0$, we have for each of the polynomials in the Pearson equation

$$\frac{\theta(z) - \theta(k)}{z-k} = (\chi(k))^\top \begin{pmatrix} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix}, \quad \frac{\sigma(z) - \sigma(k)}{z-k} = (\chi(k))^\top \begin{pmatrix} M_\sigma \chi^{[M]}(z) \\ 0 \end{pmatrix},$$

where we have used the matrices

$$\begin{aligned} M_\theta &= \begin{pmatrix} 0 & \theta_1 & \dots & \theta_N & 1 \\ \theta_1 & & & & 0 \\ \vdots & & & & \vdots \\ \theta_N & & & & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{(N+2) \times (N+2)}, \\ M_\sigma &= \begin{pmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_{M-1} & \eta \\ \sigma_1 & & & & 0 \\ \vdots & & & & \vdots \\ \sigma_{M-1} & & & & \vdots \\ \eta & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{(M+1) \times (M+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^\infty P(k) \frac{\theta(z) - \theta(k)}{z-k} w(k) &= S \sum_{k=0}^\infty \chi(k) (\chi(k))^\top w(k) \begin{pmatrix} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix} = SG \begin{pmatrix} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix} \\ &= HS^{-\top} \begin{pmatrix} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix} = \begin{pmatrix} (HS^{-\top})^{[N+1]} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} H^{[N]} (S^{[N+1]})^{-\top} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix}. \end{aligned}$$

So that, the previous proposition may be recast as follows

Proposition 7 *The following relations are satisfied*

$$\begin{aligned} \theta(z)Q(z) - \Psi^\top H^{-1}Q(z-1) &= \begin{pmatrix} H^{[N+1]}(S^{[N+1]})^{-\top} M_\theta \chi^{[N+1]}(z) \\ 0 \end{pmatrix}, \\ \sigma(z)Q(z) - \Psi H^{-1}Q(z+1) &= \begin{pmatrix} H^{[M]}(S^{[M]})^{-\top} M_\sigma \chi^{[M]}(z) \\ 0 \end{pmatrix}, \end{aligned}$$

and, in particular, we have

$$\begin{aligned} \theta(z)Q_n(z) &= \sum_{m=n-N-1}^{n+M} \frac{Q_m(z-1)}{H_m} \Psi_{m,n}, \quad n > N+1, \\ \sigma(z)Q_n(z) &= \sum_{m=n-M}^{n+N+1} \Psi_{n,m} \frac{Q_m(z+1)}{H_m}, \quad n > M. \end{aligned}$$

From (3), if $k \in \mathbb{N}_0$ is not a zero of P_n , we see that $Q_n(z)$ is a meromorphic function with simple poles located at $z \in \mathbb{N}_0$, with residues at these poles given by $\text{Res}(Q_n, k) = P_n(k)w(k)$.

Thus, we get

$$\begin{aligned} \theta(k)P_n(k)w(k) &= (1 - \delta_{k,0}) \sum_{m=n-N-1}^{n+M} \frac{P_m(k-1)w(k-1)}{H_m} \Psi_{m,n}, \quad n > N, \\ \sigma(k)P_n(k)w(k) &= \sum_{m=n-M}^{n+N+1} \Psi_{n,m} \frac{P_m(k+1)w(k+1)}{H_m}, \quad n > M, \end{aligned}$$

which are in disguise (16) evaluated at $k \in \mathbb{N}_0$, i.e.

$$\theta(k)P_n(k-1) = \sum_{m=n-M}^{n+N} \Psi_{n,m} \frac{P_m(k)}{H_m}, \quad \sigma(k)P_n(k+1) = \sum_{m=n-N}^{n+M} \frac{P_m(k)}{H_m} \Psi_{m,n}.$$

Finally, we have

Theorem 7 *Assume that*

$$\theta(z) = z \prod_{k=1}^N (z + b_k - 1), \quad \sigma(z) = \eta \prod_{k=1}^M (z + a_k),$$

with b 's all different and a 's all different. Then, in terms of quasi-determinants (in this case quotients of determinants), for $n \geq M$

$$\begin{aligned} &\theta(z)P_n(z-1) \\ &= \Theta_* \begin{bmatrix} P_{n-M}(0) & P_{n-M}(-b_1+1) & \cdots & P_{n-M}(-b_N+1) & Q_{n-M}(-a_1+1) & \cdots & Q_{n-M}(-a_M+1) & P_{n-M}(z) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{n+N+1}(0) & P_{n+N}(-b_1+1) & \cdots & P_{n+N+1}(-b_N+1) & Q_{n+N+1}(-a_1+1) & \cdots & Q_{n+N+1}(-a_M+1) & P_{n+N+1}(z) \end{bmatrix} \end{aligned}$$

and, for $n \geq N+1$

$$\begin{aligned} &\frac{\sigma(z)}{\eta} P_n(z+1) \\ &= \Theta_* \begin{bmatrix} P_{n-N-1}(-a_1) & \cdots & P_{n-N-1}(-a_M) & Q_{n-N-1}(-1) - P_{n-N-1}(-1) & Q_{n-N-1}(-b_1) & \cdots & Q_{n-N-1}(-b_N) & P_{n-N}(z) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ P_{n+M}(-a_1) & \cdots & P_{n+M}(-a_M) & Q_{n+M}(-1) - P_{n+M}(-1) & Q_{n+M}(-b_1) & \cdots & Q_{n+M}(-b_N) & P_{n+M}(z) \end{bmatrix} \end{aligned}$$

Conclusions and outlook

Adler and van Moerbeke have thoroughly used the Gauss–Borel factorization of the moment matrix in their studies of integrable systems and orthogonal polynomials [1–3]. Our Madrid group extended and applied it in different contexts, namely CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials and multivariate orthogonal, see [4–12]. For a general overview see [45].

Recently [47] we extended those ideas to the discrete scenario, and studied the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description a new banded matrix is required, the Laguerre–Freud structure matrix that encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled generalized hypergeometric functions determining the moments of the weight described for the squared norms of the orthogonal polynomials a discrete Toda hierarchy known as Nijhoff–Capel equation, see [49]. In [28] these ideas are applied to generalized Charlier, Meixner, and Hahn orthogonal polynomials extending the results of [23, 29–31, 52].

In this paper we have seen how the contiguous relations could be understood as Christoffel and Geronimus transformations. Moreover, we also used the Geronimus–Uvarov transformations to give determinantal expressions for the shifted discrete orthogonal polynomials.

For the future, we will study cases involving more general hypergeometric functions for the corresponding first moments, and extend these techniques to multiple discrete orthogonal polynomials [13] and its relations with the transformations presented in [17] and quadrilateral lattices [22, 46].

We are also working on illustrative examples of Christoffel and Geronimus transformations for classical discrete orthogonal polynomials. This would be interesting as Geronimus transformations can complete the approach in [27] for Christoffel transformations. The relevance of the LU and UL factorizations in both cases would represent an interesting addition to the matrix approach and results of this paper.

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