Distance-based approach to quantum coherence and nonclassicality

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We provide a coherence-based approach to nonclassical behavior by means of distance measures. We develop a quantitative relation between coherence and nonclassicality quantifiers, which establish the nonclassicality as the maximum quantum-coherence achievable. We compute the coherence of several representative examples and discuss whether the theory may be extended to reference observables with continuous spectra.

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I. INTRODUCTION

Coherence is the concept behind the wave nature of light and the quantum nature of physics. In quantum mechanics this is well illustrated by the Schrödinger cats as the coherent superposition of macroscopically incompatible situations. When the coherence of the superposition vanish all quantum features disappear replaced by just classical-like ignorance of the cat state. Actually, decoherence is the most popular mechanism to account for the emergence of the classical world [1].

This is a research area of fast grow in quantum and classical optics. In classical optics the interest has been motivated in recent times by the extension of interference-related phenomena to vector light [2–6]. In quantum optics this research has been prompted by the revelation of coherence as a footing for emerging quantum technologies, such as quantum information processing [7], and quantifying coherence has become a central task as expressed by resource theories [8,9].

From this understanding of coherence as the distinctive quantum feature, it seems reasonable to assume it as the basis of any approach to nonclassical behavior from first principles. In this paper we develop a quantitative relation between quantum coherence and nonclassicality. We find nonclassicality as the maximum coherence that a field state can display by varying the basis, in the same understanding that the degree of polarization is the maximum coherence between two filled modes that can be reached under unitary transformations [10–12].

The quantifier of coherence based on the *l1 norm* has been established as a good measure of coherence in spaces of finite dimension [8,9]. In this paper we express this coherence measure in terms of a Hellinger-like distance. We also define the quantifiers of all the magnitudes involved by means of this distance. In Sec. II we establish these quantifiers and derive the relation between them for finite dimensional spaces. In Sec. III we compute the coherence of some relevant states. In Sec. IV, the analysis is reproduced in infinite-dimensional spaces. In Sec. V we investigate whether the theory may be extended to reference observables with continuous spectra. Finally, in Sec. VI it is shown how these results can be replicated by using the Hilbert-Schmidt (HS) distance.

II. COHERENCE QUANTIFIED BY A HELLINGER-LIKE DISTANCE

We begin the analysis with the case of an abstract space of finite dimension N. It is worth noting that we focus on a basis-dependent approach to coherence, so we fix a given orthogonal basis $\{|j\rangle\}_{j=1,\dots,N}$ representing some physical variable or observable J as presented, for example, in Ref. [13]. The quantifiers utilized in this section are based on a suitable version of the Hellinger distance between two density matrices a and b [14], which is

$$d_H(a,b)=\sqrt{\mathrm{tr}[(\sqrt{a}-\sqrt{b})^2]}, \tag{1}$$

where throughout this paper the meaning of the square root is

$$\langle i|\sqrt{a}|j\rangle=\sqrt{\langle i|a|j\rangle}, \tag{2}$$

which is slightly different from the usual definition of square root of a matrix. We can mention a similar appearance of square roots in classical-optics coherence problems [15,16].

Accordingly, we establish the quantifier of coherence based on the Hellinger distance [17] as the distance to the closest incoherent state ρ_d,

$$\mathcal{C}_H=[d_H(\rho,\rho_d)]^2=\mathrm{tr}[(\sqrt{\rho}-\sqrt{\rho_d})^2], \tag{3}$$

where by incoherent we mean states diagonal in the reference basis so that ρ_d turns out to be the diagonal part of ρ in the same basis [13],

$$\rho_d=\sum_{j=1}^{N}\rho_{j,j}|j\rangle\langle j|, \tag{4}$$

where $\rho_{i,j}=\langle i|\rho|j\rangle$ are the matrix elements of ρ in the basis $\{|j\rangle\}$. This quantifier can be expressed as

$$\mathcal{C}_H=\sum_{j\neq k}|\rho_{j,k}|, \tag{5}$$

which coincides with the well-established quantifier of coherence C_{l_1} [8,9] . This definition relies on the idea that the coherence of any state in a given basis is essentially determined by the nondiagonal terms of its density matrix, which

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are obviously base dependent. Then, if ρ is diagonal in the basis $\{|j\rangle\}$ it is incoherent in such a basis $C_{H_{\min}} = 0$. The maximum value of $\mathcal{C}_{H_{\max}} = N - 1$ can be easily computed for pure states with $\rho_{ii} = 1/N$, those are phaselike states as we discuss around Eq. (24). This bound is actually general beyond pure states as shown in Refs. [18,19].

A useful expression for $\mathcal{C}_H$ valid for pure states is

$$\mathcal{C}_H = \left(\sum_j \sqrt{p_j} \right)^2 - 1, \tag{6}$$

where $p_j = \rho_{j,j}$ is the statistics of the basis variable J.

In line with the distance-based measures of quantumness from quantum resource theories [20–22] we utilize the distance in Eqs. (1) and (2) to define a quantifier of nonclassicality,

$$\mathcal{NC}_H = [d_H(\rho, I/N)]^2 = \mathrm{tr}[(\sqrt{\rho} - I/\sqrt{N})^2]. \tag{7}$$

As the state of reference or *classical state* we consider the maximally mixed state I/N since it has been shown in Ref. [23] that under very generic conditions the normalized identity is actually the only classical state [24]. We may invoke also the approach to nonclassicality in Ref. [25]. These two ideas merge recalling that the identity is the only matrix which is diagonal in all bases. However, as well as the previously introduced measure of coherence, $\mathcal{NC}_H$ is also a base-dependent quantity. The minimum $\mathcal{NC}_{H_{\min}} = 0$ clearly holds if and only if $\rho = I/N$ as the only classical state. On the other hand, the maximum value is $\mathcal{C}_{H_{\max}} = N - 1$ and holds again for pure phaselike states for which $\rho_{ii} = 1/N$.

Finally, we quantify the fluctuations of the observable defined by the basis $\{|j\rangle\}$ whose probability outcomes are the diagonal terms in ρ, $p_j = \rho_{j,j}$. We refer to this quantity as *certainty* [26,27], and we require it to be minimum when the probabilities are equally distributed, this is $p_j = 1/N$, and maximum when the probability distribution has only one therm $p_j = \delta_{j,j_0}$.

Since these fluctuations are absolutely independent on the coherence terms of ρ, we define the certainty quantifier as the distance between ρ_d and I/N,

$$\mathcal{S}_H(\rho) = [d_H(\rho_d, I/N)]^2 = \mathrm{tr}[(\sqrt{\rho_d} - I/\sqrt{N})^2], \tag{8}$$

with

$$\mathrm{tr}[(\sqrt{\rho_d} - I/\sqrt{N})^2] = 2\left(1 - \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sqrt{\rho_{jj}}\right). \tag{9}$$

This means that I/N is the incoherent state with larger indetermination on J fluctuations in the coherence basis $\{|j\rangle\}$ and, therefore, the longer the distance between ρ_d and I/N the lesser the fluctuations. As it was required, the maximum $\mathcal{S}_H(\rho) = 2(1 - 1/\sqrt{N})$ holds for the elements of the coherence basis $\{|j\rangle\}$ whereas the minimum $\mathcal{S}_{H_{\min}} = 0$ occurs when $\rho_{ii} = 1/N$. Moreover, after Eq. (9) we may relate $\mathcal{S}_H$ to the Rènyi entropy of order 1/2 [28],

$$H_{1/2} = 2\,\ln\left(\sum_{k=1}^{n} p_k^{1/2}\right). \tag{10}$$

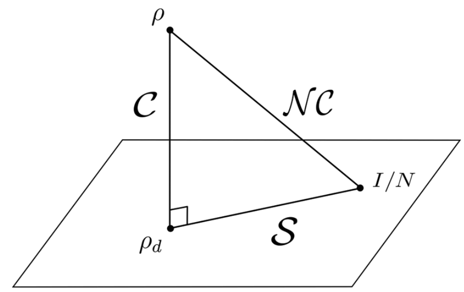

FIG. 1. Pythagoras-like theorem in finite dimension.

A. Pythagorean equation

Theorem. Given the previous definitions (3), (7), and (8), it can be established the following relation between magnitudes,

$$\textbf{Nonclassicality} = \textbf{coherence} + \textbf{certainty}, \tag{11}$$

this is

$$\mathcal{NC}_H = \mathcal{C}_H + \mathcal{S}_H. \tag{12}$$

Proof. We insert the closest incoherent state ρ_d in the definition of nonclassicality's quantifier in Eq. (7) as $\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d} + \sqrt{\rho_d} - I/\sqrt{N})^2]$ so that it equals to

$$\mathcal{NC}_H = \mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})^2] + \mathrm{tr}[(\sqrt{\rho_d} - I/\sqrt{N})^2] \\ +2\,\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})(\sqrt{\rho_d} - I/\sqrt{N})]. \tag{13}$$

As long as $\mathrm{tr}(\sqrt{\rho}\sqrt{\rho_d}) = \mathrm{tr}(\sqrt{\rho_d}^2)$, it can be readily shown that

$$\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})(\sqrt{\rho_d} - I/\sqrt{N})] = 0. \tag{14}$$

Therefore, we obtain the following Pythagoras-like equation in a finite-dimensional space:

$$\mathrm{tr}[(\sqrt{\rho} - I/\sqrt{N})^2] \\ = \mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})^2] + \mathrm{tr}[(\sqrt{\rho_d} - I/\sqrt{N})^2]. \blacksquare \tag{15}$$

The central point of this derivation is the interpretation that we can make of each term in the underlying right-triangle structure associated with this version of the Pythagorean theorem. The hypotenuse represents nonclassicality, whereas coherence and certainty are the cathetus, that are orthogonal as shown in (14). This may be illustrated with the aid of Fig. 1, where ρ_d is the orthogonal projection of ρ into the incoherence hyperplane. Note that we may obtain arbitrary Pythagoras theorems replacing I/N by any incoherent state so that the orthogonality (14) will still hold. But the choice I/N is clearly the one where hypotenuse and cathetus have a most clear physical meaning. Note that (15) and (11) adopts also the form of a duality relation between coherence and certainty in the coherence basis as already studied in Refs. [10,29]. It also worth noting that this quantum result parallels equivalent results in classical optics [11].

As a direct conclusion from the previous theorem we can see that nonclassicality $\mathcal{NC}_H$ becomes the maximum value achievable for the coherence $\mathcal{C}_H$, in agreement with Ref. [30]. In addition, the difference between them is attributed to the properties of the basis at hand. This is an interesting combination since from a more classical-like perspective $\mathcal{NC}_H$ has be seen as an intrinsic or absolute form of coherence, this is independent of any reference observable [12,31–33], see also Ref. [34].

We can draw another conclusion from this theorem recalling that the normalized identity is the only state diagonal in all bases so it is the only state without coherence in all bases. Considering this along with the fact that nonclassicality based on Hellinger distance is zero, if and only if the state is the normalized identity we can affirm that:

Nonzero coherence in, at least, one basis is necessary and sufficient condition for nonzero nonclassicality in all bases.

Therefore, for a given basis, $\mathcal{C}_H \neq 0$ is sufficient condition for $\mathcal{NC}_H \neq 0$. However, $\mathcal{C}_H = 0$ in that basis does not imply $\mathcal{NC}_H = 0$ since the state may have coherence in a different basis. We illustrate this affirmation in the first example of the following section. This result agrees with previous works where nonclassical features are measured in the absence of coherence [35].

Finally, Eq. (15) allows us to arrive to the following relation between the coherence and the equivalent *purity* of the square root density matrix,

$$\mathrm{tr}(\sqrt{\rho}^2) = \mathcal{C}_H + 1. \tag{16}$$

Considering this simplification we try to find a different relation between coherence and certainty which does not involve the nonclassicality term. If we denote

$$x = \sum_{i=1}^{N} \sqrt{\rho_{ii}}, \tag{17}$$

then

$$\mathcal{C}_H = \sum_{i\neq j} |\rho_{ij}| \leqslant x^2 - 1, \quad \mathcal{S}_H = 2\left(1 - \frac{x}{\sqrt{N}}\right), \tag{18}$$

and the equality in $\mathcal{C}_H$ holds for pure states under the form (6). These quantities are combined to obtain a new relation between them, arriving at

$$1 = \frac{\mathcal{S}_H}{2} + \frac{x}{\sqrt{N}} \geqslant \frac{\mathcal{S}_H}{2} + \sqrt{\frac{\mathcal{C}_H + 1}{N}}, \tag{19}$$

where the equality holds for pure states.

III. EXAMPLES

Next we compute the $\mathcal{C}_H$ coherence of some meaningful states within the area of quantum optics.

A. Qubit

This is the case $N = 2$. In quantum optics the most famous qubit is a single photon split into two field modes, representing typically two orthogonal polarization states, or the two inner paths in a two-beam interferometer. A qubit can be fully characterized by three-dimensional real Bloch vector $\boldsymbol{s}$ with $|\boldsymbol{s}| \leqslant 1$, equivalent to the Stokes parameters if we are within a polarization context such that

$$\rho = \tfrac{1}{2}(1 + \boldsymbol{s} \cdot \sigma), \tag{20}$$

where σ are the Pauli matrices. Choosing the basis J as the eigenvectors of the σ_z matrix we have

$$\mathcal{C}_H = \sqrt{s_x^2 + s_y^2}, \quad \mathcal{S}_H = 2 - \sqrt{1 + s_z} - \sqrt{1 - s_z}, \tag{21}$$

and naturally $\mathcal{NC}_H = \mathcal{C}_H + \mathcal{S}_H$.

In order to look for maximum coherence and nonclassicality varying the basis, we equivalently vary the Bloch vector without altering its modulus.

It can be easily seen that for fixed $|\boldsymbol{s}|$, this is for fixed purity, the maximum of both $\mathcal{C}_H$ and $\mathcal{NC}_H$ holds when the projection of the Bloch vector along the direction of the basis, say $J = \sigma_z$, vanishes, this is, $s_z = 0$, giving the following maxima:

$$\mathcal{NC}_H = \mathcal{C}_H = |\boldsymbol{s}|. \tag{22}$$

This example shows again the deep equivalence between maximum coherence, nonclassicality and purity that has been already put forward in works, such as Refs. [36,37], and in Ref. [30] identifying purity as the maximal coherence which is achievable by unitary operations, being purity the most elementary resource for quantum information processing.

Through this example we can show the influence of the basis on the values of coherence and nonclassicality. To this end we choose the same state, this is, $s_z = 0$, but we compute those quantities in the basis in which ρ is diagonal where the corresponding Bloch vector reads $s'_z = |\boldsymbol{s}|$, $s'_x = s'_y = 0$ so that in contrast to (22) in this basis, we find

$$\mathcal{C}_H = 0, \quad \mathcal{NC}_H = 2 - \sqrt{1 + |\boldsymbol{s}|} - \sqrt{1 - |\boldsymbol{s}|}, \tag{23}$$

LAS where it can be seen that for $|\boldsymbol{s}| \neq 0$ we have that $\mathcal{NC}_H$ is not canceled even if $\mathcal{C}_H$ becomes zero.

This simple but meaningful example may serve well to illustrate that nonclassical features can manifest in many different particular ways. For finite-dimensional systems this includes the idea of SU(2) squeezing, but even in such a case there are many different equivalent proposals realizing this idea, such as the ones in Refs. [38–44], for example. Different definitions are reviewed and compared, for example, in Refs. [45,46]. It is the case then that some criteria may imply that every qubit is classical, as far as all qubit pure states are SU(2) coherent states with a well-behaved SU(2) P function. On the other hand, according to the criterion of qubit entropic squeezing in Ref. [47], all density matrices different from the identity would show squeezing. Furthermore, this also agrees with the general formalism of nonclassicality developed in Refs. [23,48–50]. So, this well illustrates the results already commented in Sec. II A as a good property of this formulation of coherence and nonclassicality.

B. Phase states

As already noted, the states that make $\mathcal{C}_H$ maximum should be pure states with the same value of $\rho_{ii} = 1/N$ for all i, leading to $\mathcal{S}_H = 0$ and $\mathcal{NC}_H = \mathcal{C}_H = N - 1$. The corresponding states are

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\phi_j} |j\rangle, \tag{24}$$

where ϕ_j's are arbitrary phases. With a proper phase adjustment we may say that these are finite-dimensional phase states [51,52].

C. Rotated number states

Beam splitting is a traditional form of creating coherence after incoherent states in classical and quantum optics, being the basis of interferometry. So let us examine the coherence gained when incoherent number states $|n\rangle|m\rangle$ illuminate a lossless beam splitter. In particular, we focus on the optimum case of a 50% beam splitters in the sense of providing maximum coherence. Since we consider energy-conserving processes and a finite number of photons $NT = n+m$ for all practical purposes the system is described by finite-dimensional spaces of dimension $N = NT + 1$, being isomorphic to an spin $s = N/2$. These states includes SU(2) coherent states as the case $m = 0$ [53], and the Holland-Burnett states of maximum SU(2) squeezing and maximum interferometric resolution as the twin photon states $|n\rangle|n\rangle$ [54].

The 50% beam splitter induces a suitable mode transformation from the input modes a, b to the output modes a_1, a_2, some phases irrelevant for our purposes,

$$a_1 = \frac{1}{\sqrt{2}}(a+b), \quad a_2 = \frac{1}{\sqrt{2}}(a-b), \tag{25}$$

such that the input state in modes a, b,

$$|n\rangle|m\rangle = \frac{1}{\sqrt{n!m!}} a^{\dagger n} b^{\dagger m}|0,0\rangle \tag{26}$$

transforms into the following state in the output modes a_1, a_2,

$$|n\rangle|m\rangle = \frac{1}{\sqrt{2^{m+n}n!m!}}(a_1^\dagger + a_2^\dagger)^n (a_1^\dagger - a_2^\dagger)^m |0,0\rangle, \tag{27}$$

leading to

$$|n\rangle|m\rangle = \sum_{j=0}^{n+m} c_j |j\rangle, \tag{28}$$

where $|j\rangle$'s are photon-number states on the modes a_1, a_2, $|j\rangle = |j\rangle_1 |n+m-j\rangle_2$, and omitting an irrelevant phase,

$$c_j = \frac{\sqrt{n!m!}}{\sqrt{2^{m+n}}} \sum_{k=0}^{j} \frac{(-1)^k \sqrt{j!(n+m-j)!}}{k!(j-k)!(n-k)!(m+k-j)!}. \tag{29}$$

We start comparing the coherence of the SU(2) coherent states $m = 0$ and the twin states $n = m$, by using Eq. (6) with $p_j = |c_j|^2$.

It can be seen in Fig. 2 how the total amount of coherence increases in both cases with the total number of photons. The SU(2) coherent states are also more coherent when the total number of photons is low and less coherent than the twin states when the energy of the states increases.

For fixed $n+m$ there is a general trend in which coherence tends to be maximum around equal splitting of the photons between the input modes $n \simeq m \simeq NT/2$, curiously except the exact equality $n = m$ that shows a clear coherence dip as displayed in Fig. 3.

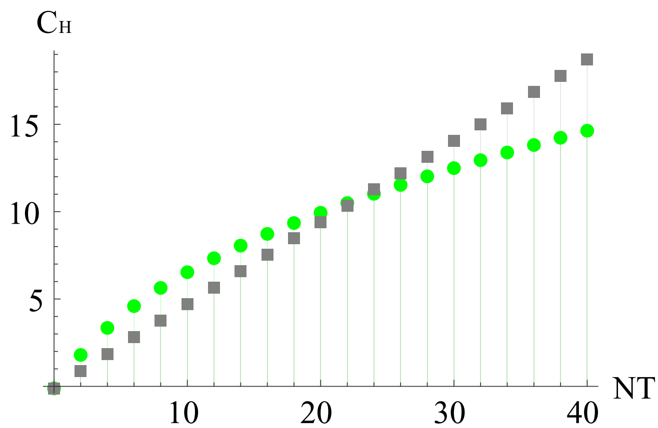

FIG. 2. Coherence based on Hellinger distance of SU(2) coherent states (green circles) and twin states (gray squares) as a function of the total number of photons.

IV. INFINITE DIMENSION: NUMERABLE BASIS

Now we extend the previous analysis to a Hilbert space of infinite dimension. We choose a numerable basis, $\{|n\rangle\}_{n=0,1,\ldots\infty}$, representing, for example, the number of photons. The case of observables with continuous bases is examined separately below. The translation to this area of the finite-dimensional analysis made above finds a major difficulty. This is that there can be no physical state proportional to the identity. This is to say that in infinite dimension there are no classical states.

As discussed after Eq. (15), we may expect that the Pythagorean theorem will still hold replacing the identity by any incoherent state, but the point is the physical interpretation of the terms. Because of this, in this context we replace the identity by an incoherent physical state ρ_T as close as desired to have a uniform distribution in the coherence basis $\{|n\rangle\}$; this is approaching to be a maximally mixed state. This can be the case of a thermal-like state in the limit when the analog of the

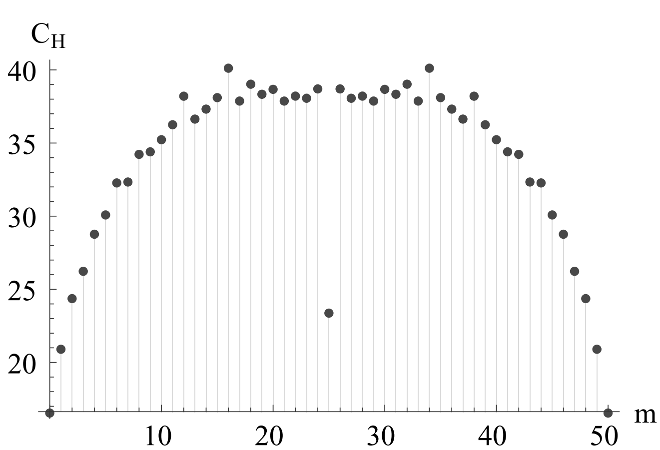

FIG. 3. Coherence of rotated number states $|n\rangle|m\rangle$ as a function of m for the total number of photons $NT = n+m = 50$ showing a general trend in which coherence increases when $n \simeq m \simeq NT/2$, except the exact equality.

temperature tends to infinity,

$$\rho_T = (1-\xi)\sum_{n=0}^{\infty} \xi^n |n\rangle\langle n| \quad \text{with } \xi \to 1. \tag{30}$$

A. Pythagorean equation

So let us derive the infinite-dimensional version of the Pythagorean theorem (11) illustrated in Fig. 1. As the key point of the derivation we have the orthogonality condition,

$$\mathrm{tr}[(\sqrt{\rho}-\sqrt{\rho_d})(\sqrt{\rho_d}-\sqrt{\rho_T})] = 0 \tag{31}$$

for the same meaning of ρ_d as the diagonal part of ρ in the number basis. The above relation holds for any ρ_T diagonal in the number basis since

$$\mathrm{tr}(\sqrt{\rho}\sqrt{\rho_d}) = \mathrm{tr}(\sqrt{\rho_d}\sqrt{\rho_d}),$$
$$\mathrm{tr}(\sqrt{\rho}\sqrt{\rho_T}) = \mathrm{tr}(\sqrt{\rho_T}\sqrt{\rho_d}). \tag{32}$$

Therefore, we readily get this new version of Pythagoras theorem in an infinite-dimension Hilbert space,

$$\mathrm{tr}[(\sqrt{\rho}-\sqrt{\rho_T})^2] = \mathrm{tr}[(\sqrt{\rho}-\sqrt{\rho_d})^2] + \mathrm{tr}[(\sqrt{\rho_d}-\sqrt{\rho_T})^2],$$

which has the same interpretation as in the finite-dimension scenario,

$$\textbf{Nonclassicality} = \textbf{coherence} + \textbf{certainty}, \tag{33}$$

as far as we consider the above-mentioned limit for ρ_T in Eq. (30). Let us compute the certainty and simplify this expression as follows:

$$\mathcal{S}_H = \mathrm{tr}(\sqrt{\rho_d}^2) + \mathrm{tr}(\sqrt{\rho_T}^2) - 2\,\mathrm{tr}(\sqrt{\rho_d}\sqrt{\rho_T}), \tag{34}$$

with

$$\mathrm{tr}(\sqrt{\rho_d}^2) = \mathrm{tr}(\sqrt{\rho_T}^2) = 1, \tag{35}$$

whereas for the third term we have

$$\mathrm{tr}(\sqrt{\rho_d}\sqrt{\rho_T}) = \sqrt{1-\xi}\sum_{n=0}^{\infty} \xi^{n/2}\sqrt{p_n}, \tag{36}$$

where $p_n = \langle n|\rho|n\rangle$. In order to proceed with the $\xi \to 1$ limit we will consider that the following quantity is finite, which is the key ingredient of coherence as shown in Eq. (6),

$$\sum_{n=0}^{\infty} \sqrt{p_n} < \infty. \tag{37}$$

This is actually satisfied by all the cases to be considered in this paper. In such a case, when $\xi \to 1$ we get

$$\mathrm{tr}(\sqrt{\rho_d}\sqrt{\rho_T}) \to 0, \tag{38}$$

so that $\mathcal{S}_H = 2$ and

$$\mathcal{NC}_H = C_H + 2. \tag{39}$$

Roughly speaking, $\mathcal{S}_H = 2$ means that as $\xi \to 1$, the distance between *physical* state ρ_d and ρ_T tends to be maximum. Therefore, it is worth noting that in this infinite-dimensional case we get that, in the conditions specified above, coherence equals nonclassicality.

B. Examples

1. Number states

As the elements of the coherence basis they are incoherent having maximum certainty,

$$\mathcal{C}_H = 0, \quad \mathcal{S}_H = \mathcal{NC}_H = 2. \tag{40}$$

2. Phase states

In the case of the normalizable Susskind-Glogower phase states [55],

$$|\xi\rangle = \sqrt{1-|\xi|^2}\sum_{n=0}^{\infty} \xi^n |n\rangle, \tag{41}$$

the coherence becomes

$$\mathcal{C}_H = \frac{1+|\xi|}{1-|\xi|} - 1 = \frac{2|\xi|}{1-|\xi|}, \tag{42}$$

so that $\mathcal{C}_H \to \infty$ as $|\xi| \to 1$. In terms of the mean number of photons,

$$\bar{n} = \frac{|\xi|^2}{1-|\xi|^2}, \quad |\xi|^2 = \frac{\bar{n}}{1+\bar{n}}, \tag{43}$$

we have

$$\mathcal{C}_H = 2(\bar{n} + \sqrt{\bar{n}+\bar{n}^2}), \tag{44}$$

that for large enough $\bar{n} \gg 1$ the coherence scales linearly with the mean number of photons as

$$\mathcal{C}_H \simeq 4\bar{n}. \tag{45}$$

3. Two-mode squeezed vacuum

The results for the phase states can be easily translated to the case of the two-mode squeezed vacuum because of their form similarity,

$$|\xi\rangle = \sqrt{1-|\xi|^2}\sum_{n=0}^{\infty} \xi^n |n, n\rangle \tag{46}$$

made just of twin-photon states we will obtain the same expression for the coherence as in (42), which in this case means the more squeezing the more coherence.

4. Squeezed coherent states

We compute the coherence in the photon-number basis of pure displaced squeezed vacuum states with displacement or coherent amplitude R and squeezing parameter r.

In Fig. 4 it is shown how the larger the displacement R, the larger the coherence, almost in a linear way. In Fig. 5 it can be seen how coherence raises with the compression parameter, r, then squeezed coherent states have more coherence than coherent states.

This behavior can be understood recalling that for large displacements R and not too large squeezing the photon-number distribution of squeezed coherent states can be well approximated by a continuous Gaussian distribution. In such a case, after the suitable generalization of Eq. (6) to this situation, the coherence can be readily computed to give

$$\mathcal{C}_H \simeq 2\sqrt{2\pi\,\Delta^2 n} - 1, \tag{47}$$

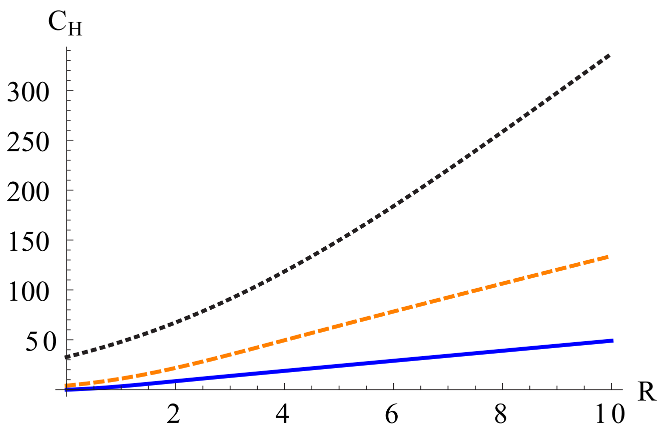

FIG. 4. Coherence of squeezed coherent states with compression parameter $r = 0.0001$ blue-solid line, $r = 1$ orange-dashed line, and $r = 2$ black-dotted line as a function of the displacement R.

where Δn is the number uncertainty, which in these conditions might be approximated on the form $\Delta^2 n \simeq \bar{n} e^{2r}$. So we see that coherence increases with the photon-number variance, and that squeezing can always increase fluctuations via super-Poissonian number statistics.

Finally, we fix the mean photon number in order to study the optimum distribution of energy between squeezing and displacement. In Fig. 6 we observe an optimum distribution of this energy when around 30% is utilized to squeeze the state. In this case, the optimum configuration supposes an important improvement in the total amount of coherence. Roughly speaking, such optimum configuration agrees with the limit in which squeezed coherent states become suitable approximations of normalized phase states as states that tend to be optimum regarding metrology [56].

5. Displaced-number states

We examine the contribution of the displacement to the coherence in the case of displaced-number states $D(\alpha)|n_0\rangle$, where $D(\alpha)|n_0\rangle$ where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. We find that the general trend of coherence is to grow with $|\alpha|$ as in the squeezed coherent state case. More specifically, in Fig. 7 it is shown how this growth is softer for states the coherent state $n_0 = 0$.

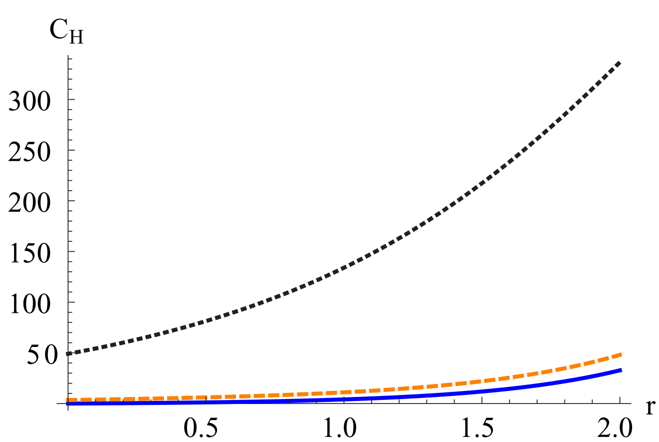

FIG. 5. Coherence of squeezed coherent states with coherent displacement $R = 0$ blue-solid line, $R = 1$ orange-dashed line, and $R = 10$ black-dotted line as a function of the squeeze parameter R.

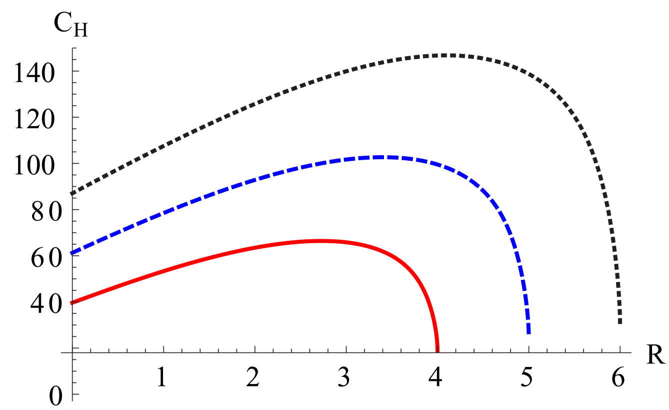

FIG. 6. Coherence of squeezed coherent states as a function of the displacement R for determined mean number of photons, $\bar{n} = 16$ red-solid line, $\bar{n} = 25$ blue-dashed line, and $\bar{n} = 36$ black-dotted line.

V. CONTINUOUS BASES

In this section we attempt to extend the previous analysis to continuous bases both in finite and in infinite-dimensional spaces.

By a suitable generalization of the preceding analyses we may consider as coherence with respect to any basis $|\phi\rangle$, even if it is continuous or nonorthogonal, the contribution of the nondiagonal terms of ρ, this is an expression of the form

$$\mathcal{C}_H = \mathrm{tr}(\sqrt{\rho}^2) - 1 \tag{48}$$

taken from Eq. (16). The question to be addressed next is whether such a definition of coherence has the same geometrical meaning we have found above in the case of discrete orthogonal basis. To discuss this we focus on whether there is a proper definition of ρ_d as the incoherent state closest to ρ. First we consider normalized nonorthogonal bases in

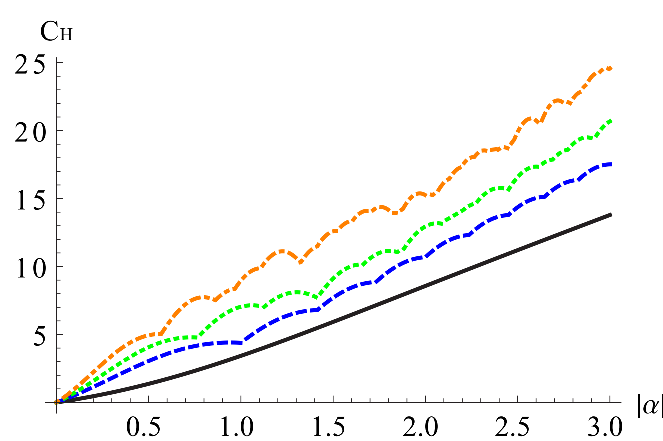

FIG. 7. Coherence for displaced number states $D(\alpha)|n_0\rangle$ as a function of the displacement $|\alpha|$ for $n_0 = 0$ in the solid black line, this is a coherent state $n_0 = 1$ in the dashed blue line, $n_0 = 2$ in the dotted green line, and $n_0 = 4$ in the dashed-dot orange line.

finite-dimensional spaces and then orthogonal nonnormalized ones in infinite-dimensional spaces.

A. Finite-dimensional space

For definiteness we use as the basis the set of finite-dimensional phase states,

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ij\phi} |j\rangle, \tag{49}$$

where the $|j\rangle$ refers to some orthonormal numberlike basis. In this scenario we may consider

$$\rho_d = \frac{N}{2\pi} \int d\phi \langle\phi|\rho|\phi\rangle |\phi\rangle\langle\phi|, \tag{50}$$

to be the incoherent state of reference, no longer diagonal as we will see in the following. In addition, it is necessary to determine the meaning of the square root suitable for this continuous framework. Thus, we define $\sqrt{\rho_d}$ as

$$\sqrt{\rho_d} = \sqrt{\frac{N}{2\pi}} \int d\phi \sqrt{\langle\phi|\rho|\phi\rangle} |\phi\rangle\langle\phi|, \tag{51}$$

so that $C = 0$ if $\rho = \rho_d$.

After these definitions it turns out that $\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})^2]$ does not reproduce Eq. (48) nor the Pythagorean theorem holds due to

$$\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})(\sqrt{\rho_d} - I/\sqrt{N})] \neq 0, \tag{52}$$

as it can be easily checked, for example, for the qubit state. We may ascribe this behavior to the lack of orthogonality of the phase states

$$\langle\phi'|\phi\rangle = \frac{1}{N} \sum_{j}^{N} e^{ij(\phi-\phi')} \neq 0, \tag{53}$$

which makes ρ_d nondiagonal, meaning

$$\langle\phi'|\rho_d|\phi\rangle \neq 0, \quad \phi \neq \phi'. \tag{54}$$

B. Infinite-dimensional space

Let us consider next the case of a continuous basis made of unnormalizable orthogonal states, such as the quadrature or position eigenstates $|x\rangle$, where x can take any real value. Although they are orthogonal in the sense of $\langle x'|x\rangle = \delta(x - x')$ there is the difficulty of $|x\rangle$ being not normalizable. As a consequence, any state diagonal in the $|x\rangle$ basis is not physical since its trace diverges, in particular, this is the case of the following definition of ρ_d:

$$\rho_d = \int_{-\infty}^{\infty} dx \langle x|\rho|x\rangle |x\rangle\langle x|. \tag{55}$$

As we have performed above with ρ_T we can try to avoid this via some kind of regularization in some proper limit. To this end we replace $|x\rangle$ by some normalizable states, for example, displaced-squeezed states $|x\rangle_\Delta$ with quadrature-coordinate wave function,

$$\langle x'|x\rangle_\Delta = \frac{1}{(2\pi\Delta^2)^{1/4}} \exp\left[-\frac{(x-x')^2}{4\Delta^2}\right], \tag{56}$$

so that we can define a truly unit-trace ρ_d as

$$\rho_d = \int_{-\infty}^{\infty} dx \langle x|\rho|x\rangle |x\rangle_{\Delta\,\Delta}\langle x|, \tag{57}$$

in the spirit of considering afterwards the limit $\Delta \to 0$. With this definition it can be easily seen that

$$\lim_{\Delta\to 0} \langle x|\rho_d|x\rangle = \langle x|\rho|x\rangle, \tag{58}$$

simply by invoking the Gaussian representation of the Dirac-δ function,

$$\lim_{\Delta\to 0} |\langle x'|x\rangle_\Delta|^2 = \delta(x - x'). \tag{59}$$

Now we try to recover an expression for the coherence in Eq. (48) as a suitable trace distance, this is in terms of $\mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})^2]$. To this end we introduce $\sqrt{\rho}$ and $\sqrt{\rho_d}$ as

$$\sqrt{\rho} = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \sqrt{\langle x|\rho|x'\rangle} |x\rangle\langle x'|, \tag{60}$$

and

$$\sqrt{\rho_d} = \int_{-\infty}^{\infty} dx \sqrt{\langle x|\rho|x\rangle} |x\rangle_{\Delta\,\Delta}\langle x|. \tag{61}$$

respectively.

It can be seen that $\mathrm{tr}(\sqrt{\rho_d}^2)$ vanish in the limit $\Delta \to 0$ since

$$\lim_{\Delta\to 0} |_\Delta\langle x'|x\rangle_\Delta|^2 = 2\sqrt{\pi}\Delta\delta(x - x') \to 0. \tag{62}$$

Similarly, taking into account Eq. (56), in the limit $\Delta \to 0$ we may consider that $\langle x'|x\rangle_\Delta$ and $_\Delta\langle x|x''\rangle$ are so peaked functions so that they act as Dirac-δ functions,

$$\lim_{\Delta\to 0} \langle x'|x\rangle_{\Delta\,\Delta}\langle x|x''\rangle = 2\sqrt{2\pi}\Delta\delta(x - x')\delta(x - x''), \tag{63}$$

and, therefore, in this limit,

$$\mathrm{tr}(\sqrt{\rho_d}\sqrt{\rho}) \to 2\sqrt{2\pi}\Delta \int dx \langle x|\rho|x\rangle \to 0. \tag{64}$$

All this together it emerges that

$$\lim_{\Delta\to 0} \mathrm{tr}[(\sqrt{\rho} - \sqrt{\rho_d})^2] = \mathrm{tr}(\sqrt{\rho}^2), \tag{65}$$

so Eq. (48) is essentially recovered. In view of this we wonder whether Pythagorean relation in Eq. (33) also holds. Note that in this continuous case $\sqrt{\rho_T}$ is defined as

$$\sqrt{\rho_T} = \int dx \sqrt{\langle x|\rho_T|x\rangle} |x\rangle_{\Delta\,\Delta}\langle x|. \tag{66}$$

We answer this question in the affirmative since the limits (62) and (63) ensure the orthogonality condition in Eq. (31).

VI. COHERENCE QUANTIFIED BY THE HILBERT-SCHMIDT DISTANCE

The previous results can be reproduced by using the Hilbert-Schmidt distance to quantify all the magnitudes involved [21,57],

$$d_{\mathrm{HS}}(a, b) = \sqrt{\mathrm{tr}[(a - b)^2]}. \tag{67}$$

We consider this scenario since the coherence based on the Hilbert-Schmidt distance is widely utilized [8,9],

$$\mathcal{C}_{\rm HS}(\rho) = [d_{\rm HS}(\rho, \rho_d)]^2 = {\rm tr}[(\rho - \rho_d)^2] = \sum_{j\neq k} |\rho_{j,k}|^2, \quad (68)$$

with ρ_d defined in Eq. (4). This distance allows us to recover an equivalent Pythagoras-like equation in a finite-dimensional space,

$$\begin{gathered} {\rm tr}[(\rho - I/N)^2] = {\rm tr}[(\rho - \rho_d)^2] + {\rm tr}[(\rho_d - I/N)^2], \\ \mathcal{N}\mathcal{C}_{\rm HS} = \mathcal{C}_{\rm HS} + \mathcal{S}_{\rm HS}. \end{gathered} \quad (69)$$

The states making these quantities extremal are the same as for Hellinger quantifiers. The only difference is the maximum value of $\mathcal{N}\mathcal{C}_{\rm HS}$ and $\mathcal{C}_{\rm HS}$ which becomes $1 - 1/N$ so in this case the coherence, nonclassicality, and certainty are bounded by 1 in finite-dimensional spaces.

Furthermore, in infinite-dimensional spaces with numerable bases we also arrive at an equivalent Pythagoras-like equation by means of the same classical reference ρ_T introduced in Eq. (30),

$$ {\rm tr}[(\rho - \rho_T)^2] = {\rm tr}[(\rho - \rho_d)^2] + {\rm tr}[(\rho_d - \rho_T)^2]. \quad (70)$$

We complete the analysis of the Hilbert-Schmidt scenario with the translation into a continuous basis.

In the case of finite-dimensional spaces and nonorthogonal bases, the difficulties caused by the definition of the closest incoherent state [see Eqs. (53) and (54)] remain, so it is also impossible to find a suitable geometrical formulation of coherence and nonclassicality in such a continuous framework.

Finally we consider an infinite-dimensional space and continuous bases made of unnormalizable orthogonal states where the very same definition of the incoherent state proposed in Eq.(57) can be utilized. As a result of the limits in Eqs. (62) and (63) we arrive at

$$\lim_{\Delta\to 0} {\rm tr}[(\rho - \rho_d)^2] = {\rm tr}(\rho^2), \quad (71)$$

which is a base-independent quantity. Thus, in this case of Hilbert-Schmidt distance the translation into the continuous basis is not possible neither for infinite- nor for finite-dimensional spaces.

VII. CONCLUSIONS

We have carried out a study of coherence based on distance measurements. We have developed a relation among coherence, certainty, and nonclassicality which establishes the latter as the upper bound of the coherence, ascribing the difference to the basis at hand. Moreover, we find nonzero coherence with respect to, at least, one basis as a necessary and sufficient condition for finding some nonclassicality in any basis. This relation can be extended to infinite-dimensional systems through a discrete basis as well as a continuous orthogonal basis. We conclude that there is no straightforward expansion of the formalism to the case of continuous nonorthogonal bases. All these conclusions are shared by the analyses made with both Heillinger-like and Hilbert-Schmidt distances, except from the case of continuous nonorthogonal basis since there is no proper expansion to continuous basis in any case when using the Hilbert-Schmidt distance.

The examples examined show the increase in coherence with squeezing on squeezed coherent states and with the displacement on coherent and number states. Also we find an interesting and abrupt reduction in the coherence of twin states.

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