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TÍTULO: Economic Growth Cross-Model Comparison

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Content

Abstract

This paper focuses on the topic of how an economy grows and what are the determinants that foster this development in the long run. To do this, we will work throughout the paper with 9 different economies with different characteristics in terms of elasticities, population growth rates or consumption patterns, and evaluate an empirical analysis by bringing the data to a Solow model and to an endogenous Ramsey model of economic growth. We establish the link between macroeconomic variables and estimate how changes in the assumptions can alter the rate at which we observe a country to develop by using econometric tools through Ordinary Least Squares and 2-Stage Least Squares. We will also study, from a probabilistic point of view, how the behaviour of a country has an impact on the attainment of the golden rule of savings derived by the Solow model and rely on the assumption that production functions will adopt the Cobb-Douglas specification.

1.Introduction

When one is concerned with the topic of economic growth it is frequent to resort to macroeconomic databases and statistics in order to study the long-run behaviour of unemployment, inflation or consumption. The mechanics operating behind economic movements and influencing the performance of a region are key to recognize the determinants of development and to establish the channels through which each variable is acting. The Solow model predicts two of the main features observed in current applied studies of growth and convergence. The first one is the negative relationship between income per capita and population growth rate and the second one is conditional convergence taking place across countries such that the growth rate of countries with a lower initial capital condition is greater. The aim of this paper is to acknowledge the relationship between economic growth in different regions measured by their income per capita and the role of technological advancement. In order to fully acquire the essence of the set of determinants of long-term economic prosperity, we will depart from the characterization of the steady-state or natural level of output and study potential deviations from this level, paying attention always to the overall tendency which will be established.

The exercise is divided into five sections, each of which is thoroughly introduced and justified both from a mathematical or statistical perspective and an empirical point of view, where we will propose nine different countries to assess our results. In the first part we will focus on the extended Solow growth model which will be the base of our work and from which we will derive the relevant mathematical statements and equations that lead to the following sections. The second part focuses on the practical side of the paper such that the desired correlations and relationships between the different variables will be built, and a significant segment will be devoted to dealing with the classification of each indicator and a conclusive analysis of the results obtained. Next, a probabilistic approximation will be proposed based on the theoretical part introduced in section 1 and the results stemming from section 2, which will enable us to gain intuition into the characteristics of a country when attempting to approach its golden rule of savings. Finally, we will extend our model by relaxing the assumption of the exogenous savings rate (key characteristic in the Solow model) and we will establish some relationships between the growth rate of GDP per capita and some relevant economic variables.

Methodology

The importance of this paper lies on the fact that it will enable us to understand the empirical implications of analysing economic growth by making use of different models with different assumptions and different results. Second, we study the role of the golden rule of savings for 9 different economies in terms of probability and how variables such as personal income tax or initial conditions of income have an impact in this result. In addition, we propose at the end of the paper a technique based on the results obtained throughout the exercise in order to be able to distinguish between developed and non-developed countries.

Sample size: 9 economies (Spain, Italy, Sweden, China, United States, India, Korea, Australia, and Germany)

Sample period: 1995-2017

Sample data: GDP per capita, national savings rate, personal income tax, population and population growth rate, depreciation rate.

The main methodology used across the paper is a regression process based on several techniques to analyse, for each section of the paper, the desired information to be studied. Sections for the cross-model comparison between the Solow-Swan model and the Ramsey model are explored through methods of ordinary least squares and graphical analysis to analyse the distribution of some variables such as the elasticity of GDP with respect to capital. In other cases, we make use of 2-stage least squares with the use of an instrumental variable that will be introduced in section 2 and that will enable us to isolate the effect of our variable of interest (savings rate) on the growth rate of GDP per capita. A third regression technique is introduced in section 4 to study the impact of our variables on the probability outcome of an economy achieving the golden rule. This is done through the application of a limited dependent variable model (probit model) and an applied example is proposed for the case of France to test for the marginal effects. Data is collected from the Penn World Table except for the personal income tax rate, which has been obtained from the World Bank.

The objective is to compare an economic growth country analysis between the Solow model and the Ramsey model which are two of the most frequent models economists resort to when dealing with theoretical sources of economic development. By constructing a linear regression model departing from some assumptions on the Solow model, we estimate for values of the elasticity of income with respect to capital in order to use these values to study the effects on the golden rule of savings. This golden rule is modified when we introduce the new model in section 5 because we will relax the assumption of the constant and exogenous savings rate such that households will maximize their utility as a flow variable (the discounted aggregation of utilities of all periods on an infinite time horizon). By relaxing this assumption, we will perform a new regression model in which one can easily compute the value of the elasticity above which an economy can no longer be treated as developed.

2. Solow growth model

We will base our study on a closed economy with no public sector model in which capital accumulation and population growth dynamics will be key to explain the long-term evolution of resources and savings, and to characterize the steady state such that the instantaneous rate of change of capital will be null, thus the observed level of capital per worker (to technology) will hold constant in this state. Such a rate of change will be expressed in continuous time and will depend on the rate of depreciation rate and the amount of investment:

$$\dot{K} = I - \delta K \quad \mathbf{(1)}$$

We also introduce a Cobb-Douglas production function such that we consider an economy with constant returns to scale (exponents will add up to 1) which will have labour as an input, in addition to capital, and we will bring in an auxiliary variable which will account for technological progress in our economy. It can be shown that, for this specific production function, the exponent of each of the factors accounts for the elasticity of production with respect to the input. To identify the motion of these variables, we will assume the instantaneous growth rate of population and technology is constant and equal to n and g, respectively.

$$Y(K, AL) = K^{\alpha}(AL)^{1-\alpha} \quad \textbf{(2)}$$

and

$$\frac{\dot{L}}{L} = n \rightarrow L = L_0 e^{nt}$$

$$\frac{\dot{A}}{A} = g \rightarrow A = A_0 e^{gt}$$

Since we are in a closed economy with no public sector, investment will be equal to private savings, and a fixed percentage of income will be saved which we will denote as the savings rate s. This means that total investment can be expressed as the product of the savings rate and the level of production, and the latter depends at the same time on the amount of capital. We will distinguish between variables in absolute terms, in terms of per capita (small case letter), and in per capita to technology terms, which will be denoted with a Tilda on top of each symbol. We will proceed to characterize the evolution of capital per worker to technology by starting with the simple motion equation for the level of capital introduced in (1) and the production function (2):

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta$$

and

$$\tilde{k} = \frac{K}{AL} \rightarrow \ln(\tilde{k}) = \ln(K) - \ln(A) - \ln(L) \rightarrow \frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$$

$$\rightarrow \dot{\tilde{k}} = \tilde{k}\left(\frac{sY}{K} - \delta - g - n\right) = s\tilde{y} - \tilde{k}(\delta + g + n) =$$

$$s(\widetilde{k})^{\alpha} - \tilde{k}(\delta + g + n) \quad \textbf{(3)}$$

2.1 Steady State characterization

Equation (3) describes the rate of capital accumulation (in per capita to technology terms) and will depend on the current level of capital together with the depreciation rate and growth rates of population and technological progress. We will characterize the steady state or balanced growth path and develop the golden rule of savings for an economy which we will use later in the paper. This golden rule can be interpreted as the necessary condition for which consumption is being maximized, which will hold a relevant role in this study since it could be of interest to observe the link between attaining welfare from a consumer´s point of view (utility) and how pushing for technological advancement enables this.

$$\dot{\tilde{k}} = 0 \rightarrow s(\widetilde{k})^{\alpha} - \tilde{k}(\delta + g + n) = 0 \rightarrow$$

$$\tilde{k}^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{(1-\alpha)}}$$

and

$$\tilde{y}^* = \tilde{k}^{\alpha} = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{(1-\alpha)}} \quad \mathbf{(4)}$$

2.2 Building the regression model

Recalling our subject of interest for this paper, one would like to explore the extent to which technological development contributes to the prosperity of an economy, and how this variable is related to other features associated with economic growth. Our objective will be to isolate this effect and obtain the most precise outcome as possible, since it is unpleasant to get driven by other confounding factors which can end in misleading results. We will derive the expression for income per capita as a function of time and savings rate and use a simple linear regression to estimate the values of α *and* g (notice we will treat population growth and depreciation rates as exogenous variables).

$$y^{*} = \tilde{y}^{*}A = \left(\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{(1-\alpha)}}\right)A_0 e^{gt}$$

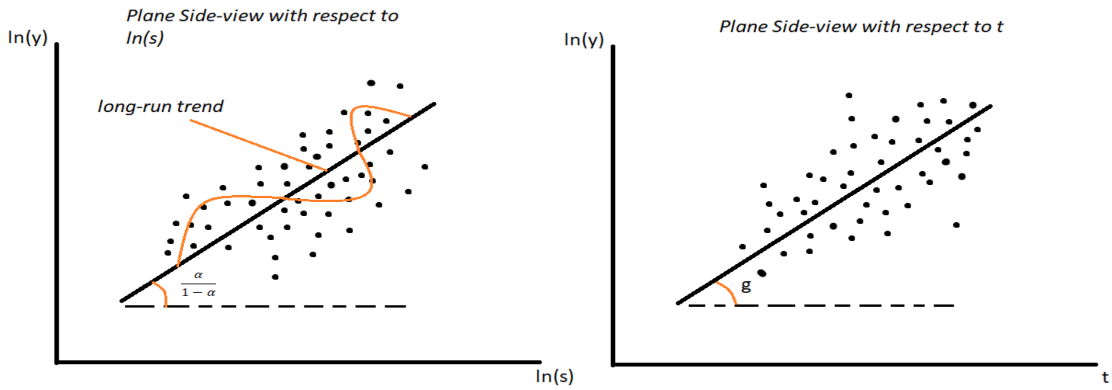
$$\rightarrow \ln(y^{*}) = \left(\frac{\alpha}{1-\alpha}\right)(\ln(s) - \ln(\delta + g + n)) + \ln(A_0) + gt$$

Now, if we rename $\beta_0 = \ln\left((A_0)(\delta + g + n)^{\frac{\alpha}{\alpha-1}}\right)$ and $\beta_1 = \frac{\alpha}{1-\alpha}$,

$$\rightarrow \ln(y^{*}) = \beta_0 + \beta_1 \ln(s) + \beta_2 t + \epsilon \quad \mathbf{(5)}$$

We will take advantage of expression (5) in our regression to approximate the values of each of the previously specified coefficients where β_1 can be interpreted as the elasticity of the long-run income per capita with respect to the savings rate and β_0 as the income per capita (in logarithms) when $t = 0$ and the savings rate is unity. The term ϵ will behave as a random variable with null expected value and will capture the deviations from the forecast obtained from our model, since we will be dealing with the evolution of the steady-state income per capita. We will take data for income per capita and savings rate in our countries and for a period of 22 years and perform the regression analysis to estimate the parameters for each economy. In fact, we are fitting a plane which could be graphically represented as a 3-dimensional plot such that we can observe how our dependent variable is behaving according to each of the regressors (figure 1). The fitted values will account for the long-run or steady-state values of y which can lie above or below for any given period.

Figure 1



One potential problem we may encounter is the fact that the value obtained for β_1 could be biased if the Gauss-Markov assumptions are not fully met, in specific there could be some omitted variable bias when establishing the correlation between savings rate and income per capita, which would be not satisfying the assumption of null expected value between our regressor and the error term. For instance, interest rates in a region can influence the current rate of savings and, at the same time, have a significant effect on income per capita through other channels such as investment (according to empirical evidence, interest rates act both through consumption and investment). This could lead to a bias in our results such that the partial effect of education on production is non-zero and the correlation between our variable s and interest rates is also remarkable.

In order to account for this problem, we will introduce an instrumental variable z which will have to satisfy both an inclusion restriction (non-zero correlation with the savings rate) and an exclusion restriction (null correlation with the error term of the original model). In this case, we are using the yearly personal income tax as our instrumental regressor assuming that all the partial effect that this variable is having on the current level of income per capita is attained through household consumption or savings rate, and it is weakly linked to other potential confounders. We obtain the "clean" part of our variable s by regressing it with all the exogenous variables (including our instrument) and replacing it in the original model:

$$\ln(y^*) = \beta_0 + \beta_1 \ln(s) + \beta_2 t + \epsilon$$

$$\rightarrow \widehat{\ln(s)} = \pi_0 + \pi_1 z + \pi_2 t + v \quad \mathbf{(6)}$$

$$\rightarrow \ln(y^*) = \beta_0 + \beta_1 \widehat{\ln(s)} + \beta_2 t + \varepsilon \quad \mathbf{(7)}$$

with a variance for each of the coefficients of

$$Var(\beta) = \frac{\sigma_u^2}{n\sigma_x^2(1 - R^2)}$$

Once we have executed our linear regression for each country by means of two-stage least squares in order to account for our instrumental variable, the values of α and g can be easily computed by rearranging expressions which were renamed in (5):

$$\alpha = \frac{\beta_1}{1 + \beta_1}$$

$$g = \beta_2$$

2.3 Golden rule

Consumption per capita (to technology) can be expressed as the fraction of income that is not saved and can also be characterized in the steady state. We will determine the value of the savings rate that maximizes consumption and establish the golden rule for which an economy is saving at an effective ratio such that household expenditure is optimal. Our goal is to find, among all the different time paths, the one in which income per worker to technology is such that no other will lead to a ´better´ resource allocation from the point of view of consumption. The extreme values for $\tilde{c}$ are zero when the savings rate is unity (in this case β_0 in equation (5) becomes zero), and zero when the savings rate is null because this leads to a steady state equal to zero according to equation (4). The simple expression (8) shows that, the closer an economy is, in theory, to approximate its savings rate to the parameter α, the greater the level of consumption per capita to technology the country can achieve. We could delve deeper into this result and enter the field of policy economics such that one could study the mechanisms through which a country can control for the savings rate and hence manage to behave according to this golden rule, but we will limit ourselves to empirical data and devote a future section of the paper to this matter.

$$\frac{d}{ds}\tilde{c} = \frac{d}{ds}(1-s)\tilde{y} = \frac{d}{ds}(1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\rightarrow (1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{1}{\delta+n+g}\right)\left(\frac{\alpha}{1-\alpha}\right) - \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} = 0$$

finally,

$$s = \alpha \quad \mathbf{(8)}$$

3. Parameter estimation

3.1 Model Criteria

For this section of the paper, we will have to perform the regression analysis and determine the degree of precision and accuracy to which our model is fitting the data for each of the countries. In order to do this, we will proceed by establishing some criteria to improve the selected model for each country and determine the one which best adapts to the observations of each of the regions.

1) Hausman test

   The idea is to avoid dropping one of the main assumptions of multiple linear regression, which in this case refers to the null correlation between the explanatory variable and the error term that enables the researcher to obtain an unbiased estimation. If the savings rate is proven to behave as an endogenous regressor, then an improved version of the regression would be to resort to two-stage least squares, and this is what we will examine at this stage. To do this, we will save the residuals of equation (6), include them in the first stage equation (5) and test for the significance of this residuals in this new model. The null hypothesis for this test will be obtaining consistent estimates in OLS:

$$\widehat{\ln(s)} = \pi_0 + \pi_1 z + \pi_2 t + v$$

$$\rightarrow \quad \ln(y^*) = \beta_0 + \beta_1 \ln(s) + \beta_2 t + \beta_3 v + \varepsilon$$

with

$$H_0: \beta_3 = 0$$
$$H_1: \beta_3 \neq 0$$

2) Hypothesis test

This part refers to having an inspection on the overall significance of each of the coefficients which will help us understand what the most important determinants for each country are in order to explain the evolution of the steady state income per capita. One should notice that the variance, which has been previously defined in the paper, depends not only on the amount of dispersion of its associated regressor, but also on the degree of correlation with the rest of the variables in the model. With respect to this statement, it is noteworthy to mention that the amount of uncertainty we are subject to when resorting to the use of the instrumental variable in the first step, is much greater than with OLS. This can lead to high p-values of our coefficients, which means that we will assume that each variable has at least some magnitude of relevance, despite the low significance of our parameters in some cases, for the sake of the model, and we will stick with OLS whenever the Hausman test allows us to.

3) Evaluation criteria

Once we have specified the model and checked for the inclusion of personal income tax as an instrumental variable, we will assess the nature and properties of our regression output through three main indicators. The fundamental approach with which one recognizes the effectiveness of the model to fit the data is by computing the R-squared which describes the percentage of variation in our dependent variable (income per capita) that is explained by the model, which in practice is no other thing than computing the correlation coefficient between the fitted values and the original data. We will follow the standard technique which is obtaining the ratio between the explained sum of squares and the total sum of squares as in (9). We will also explore the normality of the residuals by resorting to the Jarque-Bera statistic which will consider the asymmetry and kurtosis coefficients, and, under the null, the distribution is normal (10). We will summarize this information by providing both the p-value for this test and for an heteroskedasticity analysis supported by a White test (11) to detect any non-linear forms of non-constant variance in the error term. Recall that we expect to avoid the latter since we are already working with logarithmic transformations which usually smooth the observed data.

$$R^2 = \frac{SSE}{SST} = \frac{\sum(\hat{y} - \bar{y})^2}{\sum(y - \bar{y})^2} \quad (9)$$

$$Jarque\ bera = n\left(\frac{AC^2}{6} + \frac{(KC-3)^2}{24}\right) \sim \chi_2{}^2 \quad (10)$$

$$\varepsilon^2 = \delta_0 + \delta_1\hat{y} + \delta_2\hat{y}^2 + error \quad (11)$$
$$H_0: \delta_1 = \delta_2 = 0$$
$$H_1: H_0\ not\ verified$$

where $\hat{y}$ in (9) and (11) accounts for the fitted values of $\ln(y)$ in our structural equation.

3.2 Endogeneity and hypothesis tables

Table 1 Regression results and test analysis

Country	β_0	Jarque-Bera p-value	Endogeneity p-value	R-squared	α	g	White test
Spain	7.22786	0.56355	0.048117	0.922302	0.44	2.7%	0.00194
Italy	9.35627	0.34558	0.012234	0.911099	0.22	1.6%	0.228990
Sweden	8.91999	0.44389	0.022301	0.984664	0.39	1.7%	0.203633
China	8.34836	0.69149	-	0.992222	0.26	6.3%	0.200511
USA	10.3537	0.00902	0.164059	0.969511	0.156	1.62%	0.075339
Australia	10.5602	0.30619	0.525728	0.972588	0.29	1.6%	0.319200
India	3.28447	0.83243	0.0132965	0.995	0.56	8.0%	0.0868
Germany	9.29069	0.35693	0.607391	0.995	0.26	2.2%	0.2212
Korea	8.28806	0.06389	0.0014257	0.975920	0.13	4%	0.00748

Table 2 Exclusion restriction Instrumental variable

	Spain	Italy	Korea	Australia	India
Exclusion coefficient (absolute value)	0.007288	0.00168	0.000539	0.139433	0.156382
Ratio	29.8	271.8	50.1	0.846	18.9

3.3 Observations on procedure, results and diagnosis

One should notice that, for the cases in which instrumental variable intervention were needed, the R-squared coefficient is not reliable because we are working under a non-null correlation between the explanatory variable and the error term, so we will not be concerned for low values of this coefficient, and we will not make use of it for F-tests computation for joint rejection. In addition, it is very difficult to prove the exclusion restriction of our instrumental regressor in equation (7) since we are only using one, so in order to hold a general notion of the implications behind the usage of our variables, we will regress our dependent variable $\left(\ln(y^*)\right)$ against the logarithmic savings rate $\left(\ln(s)\right)$ and the personal income tax rate in order to account for the amount of correlation between our dependent and instrumental variable when holding constant the variation in savings rate. Although in many cases we notice there is a relatively weak correlation between the personal income tax and our savings regressor, if we compare it with the partial effect observed on GDP per capita by the savings rate, it becomes reasonable to assume a correlation of 30-40% is enough when we already observe these values for the coefficients associated to β_1 for each of the economies in table 1. However, we come across some countries in which the endogeneity test returns such a high p-value that we directly apply ordinary least squares, with the advantage of obtaining coefficients with lower variances and, accordingly, whose null hypothesis are more likely to be rejected.

For the Spanish case, we obtain not only a relatively large value for g, but also a relatively small significance associated to it. The Hausman test clearly points to the use of our instrumental variable, indicating a strong correlation between the regressors and the unique errors in the model. There seems not to be any problem associated with heteroscedasticity or normality of the residuals and one notices two anomalies with respect to the other countries: it holds one of the lowest values for β_0 and a very high α value. The coefficient obtained in our exclusion test shows a very low correlation between the personal income tax and the logarithmic GDP per capita when holding constant the variation in savings. The particularity for this economy lies in its large values of yearly unemployment rate, which has proven to be very persistent and stable in the last 22 years, especially since the 2008 crisis. This implies minor effects in the Spanish economy arising from the parameter L in equation (2), which could partially reflect why this obtained elasticity of savings with respect to production is so large in this country. In fact, a recent study of potential growth of the Spanish economy carried out by the bank of Spain suggests this α parameter for Spain is between 0.45 and 0.55, which proves to be very close to our predicted value.

For Italy, the Hausman test still indicates at 10% significance that we are better off by providing two-stage least squares instead of OLS. At this same significance level, we cannot reject the nulls for both the white test and the one for normality, which means there is no issues in these fields. The value for constant is, like the Spanish case, positive and significant, but it is larger than the one we reported for Spain, which could indicate, for instance, a higher value for n. The elasticity with respect to capital (α) is around 0.2 for the 22-year period studied above and the value for technological growth is found to be among the lowest for these countries. According to recent research and World Economic Forum expert surveys, Italy has been suffering from a significant stagnation in which the country has not fully adapted to new technological advances which has been costing the Italian economy between 13% to 16% in productivity growth. This is mainly justified, experts point out, by the lack of innovation and transparency in firm and government managing, translating into many cases of nepotism and lack of accomplishment, which could explain the negative values obtained for both the elasticity and the growth in technology during these last 22 years.

The constant for Sweden is very similar to the one for Italy and, once again, the statistical ratio tests for normality of residuals show that the model is not having any problem related to this issue as well as for heteroscedasticity test. Analogous to Italy and Spain, we face an endogeneity test which enough evidence to reject the null coming from Hausman, so we will stick with 2SLS for this country´s model. The R-squared is above 90% which means the data is being fitted not that accurately and some percentage of the variation in the steady state level of income per capita can be explained by our model, but one should keep in mind that working with 2SLS stains our R-squared and we shouldn't take this result as explicit as in OLS. The value of α is around 0.4 which means that the value for the elasticity of y with respect to s in equation (5) is around 2/3 (recall we are working with the steady state).

For USA, we don´t find enough evidence to reject the null on the endogeneity test, which means we will provide the results according to OLS. Both at a significance level of 5% and 10% we encounter no heteroscedasticity nor non-normality problems associated to the model, and the R-squared is among the highest within all countries. For this economy we find a relatively low value of α and a very significant figure for the estimated growth of technology of around 0.64%. The case of Australia is analogous to the one for USA, providing the same results for normality and non-constant variance tests, and almost the same value for the constant. The value for α is almost double than the one for USA and it holds similar amount for g in relation to the USA.

When running the regression using the data from India, we obtain a high significance on both coefficients and the weak instrument test is also of importance at 5% significance level, indicating the relevance of our instrumental regressor in this context. This was the only economy in which there seems to be some issue regarding the normal behaviour and homoscedasticity of the error term. This could be because this economy is subject to a more dynamic process of economic growth in which the variance of GDP per capita for the selected sample is much greater than for other economies. Like other countries, India presents a positive and significant value for β_0, but it is among the lowest in the table, which makes sense if we account for the large value of g obtained for India and the fact that it is a country where population growth is very significant. Although these two countries hold very high values of the elasticity of capital with respect to GDP per capita, Germany faces a smaller value of g, which is, for instance, one percentage point larger than the value we already obtained in Italy. Bangalore (India) was ranked as the world´s most dynamic city and many other Indian regions are included in this top 20 index in terms of R&D, connectivity, and corporate activity, among others. Moreover, the country is one of the world's leaders in communication and information technology investment, defence and space technologies and development functioning, which explains the large value obtained for g in table 1 and the high returns on capital investment captured in the output regression.

China and Korea display similar values of the constant β_0 and both regressions report very large values for the growth rate of technological advancement. However, we do notice this value is around 2 percentual points greater for the case of China when compared to Korea. We shall remember that neither of these countries are included among the top 20 countries in the world in terms of income per capita due to their large population values, which will be covered by the value of the constant in our model (equation 5). However, in aggregate GDP terms, they are both part of the top 10 countries in the world. When we observe table 1, one notices that the country which stands out the most in terms of technological advancement and dynamic development is India. With respect to the elasticity of capital, Korea holds an average alpha value of 0.13, which lies among the lowest in the data whereas China presents a figure similar to other economies such as Australia or Germany. As we noted earlier, there is some empirical (negative) correlation between this value for alpha and the degree of development a country has achieved such that we could, according to these results, include Korea among the list of developed economies (see section 5 for further detail on the distribution of alpha).

We had to apply a two step least squares with our instrumental variable for the case of Korea since the p-value for the endogeneity test rejected the null for OLS, but the value of the R-squared remains high enough and all coefficients showed up as very significant. For the case of China, there was no sufficient data for values of personal income tax throughout the desired timeline, so a regression of ordinary least squares was directly applied providing one of the largest fit in the table.

4. Probability approximation

In this section, we will construct a probabilistic model based on the previous results of tables 1 and 2 such that our aim will be to establish the link between the Golden rule of savings introduced in (8), and the set of variables (instrumental, explanatory and dependent) we have been dealing with so far. Recall one approach to study whether an economy is approaching its Golden rule of savings is to compare the relative proximity of its savings rate to the elasticity of income with respect to capital. In order to account for this issue under a probability framework, we will transform our data into some binary results in which we will denote by 1 those countries being close to its respective Golden rule, and 0 otherwise. This last definition of being close to this rule will be set as those regions in which the difference between its savings rate for a given year and its value for α must be ± 0.2. The probability of lying between these two bounds will be explained as a function of g, y and our previously defined instrumental variable (yearly average personal income tax). We will not, however, use a linear probability model in which each of these regressors directly influence the desired probability because this could lead to some economies holding a probability higher than one, and the marginal effect on the probability of approaching the Golden rule will be constant for each of the explanatory variables (not very realistic).

We will employ a probit model which will make use of the cumulative normal distribution function as the associated probability, and the input variable will be constructed as a linear function of our regressors whose coefficients will be obtained by the maximum likelihood estimator approach. The advantage of using a probit model is that the probability obtained for any sample will lie between 0 and 1 and the marginal effect of any of the explanatory variables is smaller the more we approach one extreme (e.g., a country holding a probability of less than

10% cannot change this likelihood by a lot when one of its regressors increases/decreases). This idea of the marginal effect will be useful for us later in order to study the relative effect that each of our explanatory variables is having on the output probability of being close to the savings rate golden rule (12).

$$p(y|x) = G(\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k)$$

where

$$G(z) = \Phi(z) = \int_{-\infty}^{z} \varphi(z)\, dz$$

and a marginal effect

$$\frac{\partial p(y|x)}{\partial x_i} = \frac{\partial G(z)}{\partial z} \times \frac{\partial z}{\partial x_i} = \varphi(z) \times \beta_i \quad \mathbf{(12)}$$

Table 3 shows the output regression when applying the probit model to our estimated parameters together with the amount of correctly predicted cases, which in this case is 80% that can be interpreted as an R-squared measure such that our model is right most of the time in fitting the data. The "coefficient" column will not be of much interest since it is reporting the marginal effect on the latent variable, not on the probability itself, but we do notice the significance of each of our explanatory variables due to the large values obtained in the Z-statistic. We will focus on the "Slope" column which is directly computing the partial effect described in equation (12) for the mean candidate in our sample, that is, with the average values obtained for each of the explanatory variables.

Table 3 Probit model parameter result

	coefficient	std. error	Z-tail value	Slope
constant	−16.3160	3.69240	−4.419	-
g	0.374412	0.108271	3.458	0.146486
$\ln(y)$	2.14164	0.424346	5.047	0.737898
PIT	−0.131828	0.0319359	−4.128	−0.0515765
Number of cases 'correctly predicted'		96	80.0%	-

One should notice how the values obtained for the coefficients of the latent variable ("coefficient" column) correspond to the value of β_i in equation (12), which means that the sign of the marginal effect of each of the regressors on the observed probability can be directly inferred from these results, independent of the specific case were dealing with. We arrive to an interesting result such that technological growth and the current level of GDP per capita are prone to make an economy reach its golden rule of savings whereas the personal income tax yields the opposite effect. This means that richer countries already benefit from higher chances of achieving this level of consumption per capita ceteris paribus, and in fact this effect seems to be not only very significant, but also very large: the average country will increase this likelihood by roughly 70 probability points for every point increase in $\ln(y)$.

4.1 Associated GDP per capita growth and applied example

Recall we are working with the logarithmic transformation of our variable y to account for heteroscedastic problems and because it arises naturally from the model depicted in (5). We have previously established the positive relation between this value and the probability of an economy lying within a close interval to its golden rule of savings/consumption. In order to describe changes in the absolute value of GDP per capita we will depart from the results we obtained in table 3, in which a point increase in $\ln(y)$ yields a large increase in the observed probability for the average economy. In terms of absolute value GDP per capita and applying properties of logarithms:

$$\ln(y_2) - \ln(y_1) = 1$$

$$\rightarrow \ln\left(\frac{y_2}{y_1}\right) = 1 \rightarrow y_2 = y_1 e$$

$$\rightarrow y_2 - y_1 = (e-1)y_1$$

or in percentage points

$$\%\Delta y = (e-1) \times 100 \quad \mathbf{(13)}$$

According to expression (13), a percentage growth of roughly 171.8% (more than double) yields a rise of one unit in our regressor $\ln(y)$ which translates into a large increase in the observed probability. This result is not achievable, which explains why there is such a high probability increase for the average country when this regressor increases by one. Taking France as an example, with an average estimated g of around 4% per year, a GDP per capita of 34000 euros and an average personal income tax of 30%, the estimated probability is above 90%, meaning that we would expect this economy to be within ±0.2 in its golden rule of savings with a very high probability. This is mainly due to the high GDP per capita of this economy, which represents a very large value for the latent variable, and if we compute the marginal effect of each regressor for this economy according to equation (12), we obtain negligible results for all of them since the associated value of $\varphi(z)$ is less than 0.1%. Since our model assigns a positive outcome on those samples holding a probability of more than 50%, one would conclude that the value for $(\alpha - s)$ for France will be ±0.2, bearing in mind the model is correct approximately 80% of the times.

5. Endogenizing the savings rate

Throughout the paper we have been assuming an exogenous nature for the value of s such that the behaviour of households has not been characterized by any optimizing condition and consumers consume and save a constant proportion of their income each period. This could become a more attractive assumption if we observed some smooth and average pattern for savings in the period taken for the study, but this constraint does not allow for any decision-making taking place by the side of consumers and subtracts reliability since it simplifies the dynamics of households which may differ significantly from region to region.

We will relax this assumption by considering the Ramsey-Cass-Koopmans model in which consumers seek to maximize their total utility throughout the periods and with an infinite time horizon, not just in a specific period. In the real world there is a trade-off between the amount of consumption in the present and in the future. We denote (14) as the utility function for a representative household in which σ accounts for the relative risk aversion and represents the degree of concavity of the utility function with respect to consumption.

c

$$U = \frac{C^{1-\sigma} - 1}{1-\sigma} \quad \mathbf{(14)}$$

The simplest form of the budget constraint will contemplate the fact that consumers will receive some income w for their participation in the labour market and some interest rate r for the possession of financial assets. We denote $\dot{V}$ as the change in continuous time of household assets and will account for savings such that:

$$Vr + wN = C + \dot{V}$$

$$\rightarrow \frac{Vr}{N} + w = c + \frac{\dot{V}}{N}$$

And considering a similar decomposition we did in (3) to express it in per capita terms,

$$\frac{\dot{V}}{V} = \frac{\dot{v}}{v} + n \quad \rightarrow \dot{v}N + nV = \dot{V}$$

$$\rightarrow vr + w = c + \dot{v} + nv$$

$$\rightarrow \dot{v} = w - c + (r-n)v \quad \mathbf{(15)}$$

Now we will set the representative household problem in which the objective is to maximise the sum in continuous time of its consumption on an infinite horizon subject to the budget constraint (15). This problem assumes a rational behaviour of consumers, but we will take advantage of the results to test empirically the information we can obtain from this development that we couldn't when we were facing the same problem in section 2 with an exogenous investment/savings rate. We introduce a constant discount factor ρ which accounts for the fact that consumers will gain more utility from consuming today than for future values of units of consumption:

$$Max \int_0^{\infty} (e^{-\rho t}) \frac{c^{1-\sigma} - 1}{1-\sigma}\, dt$$

$$s.t \quad \dot{v} = w - c + (r-n)v$$

And we will solve the problem by introducing the Hamiltonian in current value and obtaining the first order conditions to describe the dynamics of consumption per capita. In this case the control variable will be the consumption per capita, which will be the factor which consumers can decide on:

$$H_{cv} = \frac{c^{1-\sigma} - 1}{1-\sigma} + m(w - c + (r-n)v) \rightarrow$$

$$1) \ \frac{d}{dc}H_{cv} = c^{-\sigma} - m = 0 \ \rightarrow m = c^{-\sigma}$$

$$2) \ \dot{m} = -\frac{d}{dv}H_{cv} + m\rho \ \rightarrow \dot{m} = m(\rho + n - r)$$

and finally, we obtain the so-called Euler equation of growth dynamics of consumption per capita in equation (16). The result describes the motion or growth rate of consumption per capita at any point in time such that the path is maximizing the total utility throughout the periods. The volatility of this growth rate will depend on the value of σ such that a consumption pattern with higher risk aversion will result in a lower value of such a growth rate. This is the first condition differential equation to obtain the steady state of the model. Together with this condition and the budget constraint in (15), we will take the assumption that the capital per capita k will be equal the asset value v since each share is exchanged for one unit of capital in the financial market.

$$\frac{\dot{c}}{c} = \frac{r - \rho - n}{\sigma} \quad \mathbf{(16)}$$

5.1 Building the regression model

The profit maximization objective set by firms will be such that each factor price will be constant and equal to the marginal productivity of each of the inputs, and in equilibrium the salary received by workers will match the payments made by firms. Introducing these results in equation (15) we obtain the second condition we will take advantage of to describe the steady state situation in which per capita variables hold a null growth rate. The system of non-linear differential equations (17) and (18) show that there is an optimum level $c > 0, k > 0$ which will describe this steady state, and capital accumulation will be inversely proportional to the growth rate of consumption per capita.

$$\dot{k} = k^{\alpha} - c - (\delta + n)k \quad \mathbf{(17)}$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[\,\alpha k^{1-\alpha} - (n + \delta + \rho)] \quad \mathbf{(18)}$$

If we set these results equal to zero to account for the steady state and we introduce these results in the production function but omitting the existence of technological advancement, which is a simplifying assumption we made when we develop the maximizing behaviour of firms, we do observe some different results from what we deducted in the previous section in the Solow model. Income per capita will be partly explained by the value of the elasticity of production with respect to capital and will also be a function of population growth rate and rate of depreciation. If we compute the expression for the growth rate of income per capita at any point in time making use of equation (17) and applying the same mechanics as we did for the previous Solow model (except we will not hold constant the value of savings) we arrive to our final expression (19).

This result, like the model introduced in the first section, describes the motion of GDP per capita of an economy according to the Ramsey model at any point in time such that if it arrives to the steady state, production per capita will not grow since we have assumed no technological growth at this stage. However, we do infer some interesting information on the dynamics of economic growth, which will behave as a function of capital per capita, the ratio of consumption to the stock of capital, the combination of depreciation and population growth rates, and the value of the elasticity of the production function (in this case Cobb-Douglas) with respect to capital, the latter being the only variable which economic growth will depend positively on. This expression can account for one of the conclusions we reached earlier in the paper such that economies with higher values of α are usually those which are less economically developed and, thus, will benefit from greater growth rates of income per capita, as the Solow model predicts in its conditional growth convergence framework. Second, an economy in which the ratio of consumption to capital is too high, this effect will negatively impact the evolution of growth in the region, which means that, although consumption is positively correlated with income, it should be controlled in relation to the stock of capital. If an economy attains high levels of consumption, but this value holds relatively large with respect to the accumulation of capital, then the economy will not grow as much, or could behave as a barrier. We will use this expression to perform another ordinary least square estimate for the time period from 1995 and 2017 and for the 8 economies studied in this paper in aggregate terms to explore the extent to which this relationship holds true.

This result is analogous to the one we would have obtained by setting the planning problem such that the maximization of utility of households would be subject to the constraint derived from a direct resource allocation restriction. Similar to the results accounting for equation (19) in our research, Koopman introduces an approach to improve the model by allowing for an endogenous nature of the economy´s population growth rate. The key point in this analysis would be to realize how population dynamics will behave as a function of public health, cultural and religious factors. One could also argue that this could lead to a reverse causality such that more developed economies (measured with income per capita) can attain longer life expectancies and better living conditions, thus, leading to our variable “n” to also depend on the dependent variable in our model.

$$\frac{\dot{y}}{y} = \alpha \left[k^{\alpha-1} - \frac{c}{k} - (\delta + n) \right] \quad \textbf{(19)}$$

We run two separate regressions accounting for capital and consumption, and one to explore the relationship with the parameter alpha. This is due to the apparent relationship between the growth rate of income per capita and the value of the elasticity with respect to capital, since it seems to exist a positive correlation which is consistent with what we defined earlier in conditional convergence, but it is also included in the capital per capita term in expression (19). As this value increases, the term in the denominator would increase, but we would be assuming a value of capital per worker greater than one for this to take place, and since we are measuring in units of millions of dollars, this could not always hold true. To simplify things, we consider the two ordinary least squares estimates separately in tables 4 and 5, with its respective graphical representation.

We observe negative values associated to the coefficients in table 4, as we expected from expression (19), and all regressors are very significant. The R-squared is above 80% and the F test for the joint significance of the parameters presents a very low value for the p-value. For each additional amount of consumption per units of capital there is a decrease in the average growth rate of GDP per capita of 21 percentual points, which seems to be a very big decrease, but we should recall that this ratio is always between 0 and 1. This is another example of the output gap such that if we find an economy with an overaccumulation of capital above the steady state, then expression (18) will become negative and start a process of decreasing consumption per capita levels until the new steady state is reached.

Table 4: Output Regression on Consumption and capital per worker

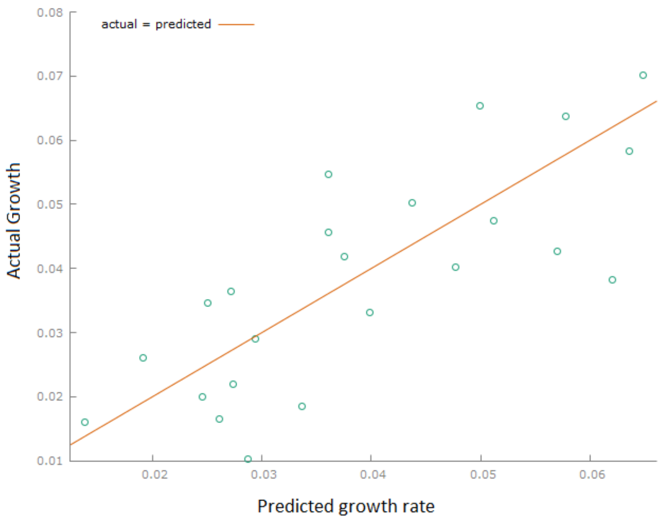
Model 10: OLS, using observations 1-24
Dependent variable: growthrate

	coefficient	std. error	t-ratio	p-value	
const	0.122943	0.0139306	8.825	1.65e-08	***
kL	-2.88538e-07	2.97823e-08	-9.688	3.36e-09	***
ck	-0.217857	0.0588500	-3.702	0.0013	***

Mean dependent var	0.028408	S.D. dependent var	0.021513
Sum squared resid	0.001835	S.E. of regression	0.009349
R-squared	0.827591	Adjusted R-squared	0.811171
F(2, 21)	50.40180	P-value(F)	9.64e-09
Log-likelihood	79.68880	Akaike criterion	-153.3776
Schwarz criterion	-149.8434	Hannan-Quinn	-152.4400

Source: Own elaboration

Figure 2: Fitted vs Actual plot



Source: Own elaboration

Table 5 was generated according to the Ramsey test which provided the result that there was some cubic and squared relation inside the ordinary least square estimate when considering the value of alpha. We observe how each of the coefficients is significant and, again, we obtain an R-squared higher than 80%. This would mean that the positive relation between the elasticity and the growth rate of income per capita is diffused by this polynomic term, in which we obtain a value of alpha for which the growth rate begins to rise rapidly very close to one third, which is associated to a range of countries in which India (part of our sample) is included. For countries with a relatively high value of α, which could be many developing countries, the growth rate coming from this parameter will be high (we have also to add the growth rate stemming from table 4). The estimated value for Spain ($\alpha = 0.44$) is an average growth rate of 2.9% according to the results in table 5 and recall we did obtain a value for g in table 1 very close to 2.7%, and we would still have to account for the rate of growth coming from the consumption path per capital and the relative value of capital per worker.

Table 5: Output Regression on elasticity of capital

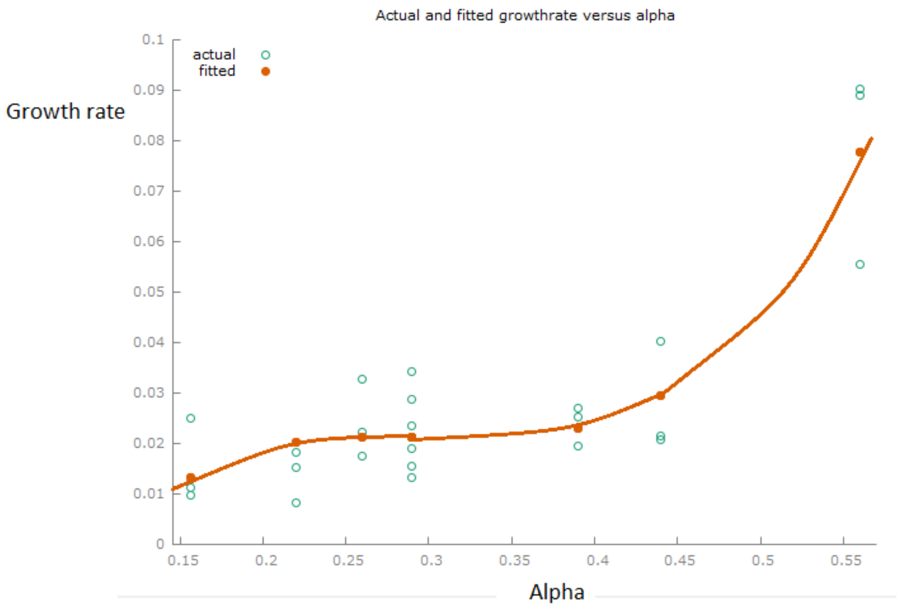
```
Model 11: OLS, using observations 1-24
Dependent variable: growthrate

               coefficient   std. error   t-ratio   p-value
  ---------------------------------------------------------
  const        -0.0603239    0.0422470    -1.428    0.1688
  alpha         0.839523     0.408910      2.053    0.0534   *
  sq_alpha     -2.85822      1.22934      -2.325    0.0307   **
  alpha_cube    3.21350      1.14762       2.800    0.0111   **

Mean dependent var   0.028408   S.D. dependent var   0.021513
Sum squared resid    0.001872   S.E. of regression   0.009675
R-squared            0.824122   Adjusted R-squared   0.797740
F(3, 20)             31.23838   P-value(F)           9.60e-08
Log-likelihood       79.44972   Akaike criterion    -150.8994
Schwarz criterion   -146.1872   Hannan-Quinn        -149.6493
```

Source: Own elaboration

Figure 3: Plot of Growth rate vs alpha



Source: Own elaboration

Conclusions

The results arising from the second section of the paper suggest that the more economically dynamic countries such as India or USA are benefiting from higher growth rates of technology or productivity whereas other economies such as Italy have stagnated to some degree over the past 15 years, reporting low values for g. Moreover, the elasticity of production with respect to capital accumulation has no clear relation with the growth rate of technological advancement or any other observed characteristic in our sample, although there is slight evidence of this elasticity being positively correlated with average unemployment in the region. The correlation coefficient between the average unemployment rate and the value for $\frac{\alpha}{1-\alpha}$ is roughly 0.6, which means that if we run a simple linear regression analysis between these two, we would obtain an R-squared of 36% (the value of the correlation squared in SLR). Another interesting outcome is the one coming from the brief probability approximation part in which we study the link between the golden rule of savings and its potential determinants.

Evidence shows that the personal income tax is inversely proportional to the probability of approaching to the golden rule whereas the growth rate of technological improvement contributes positively to the latter. However, the most remarkable result is the fact that the main determinant is the departure GDP per capita conditions, such that this measure has the biggest impact on the golden rule on savings in percentage points. This result could point to the existing inequality and divergence taking place among countries in which richer countries already belong to the sub-group of economies which are more likely to benefit from better consumption per capita patterns associated to this golden rule (although consumption per capita has proven to be negatively correlated with the growth rate of GDP per capita according to the literature).

When endogenizing the savings rate following the Ramsey model, we observe an inverse relation between the growth rate of income per capita and levels of capital per capita and consumption per unit of capital. The empirical relation between economic growth and the value for α is not as clear, although we empirically obtain that there is some polynomial behavior and that the value above which an economy begins to have a significant growth rate of income (in per capita terms) is very close to one third.

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