Robust Rao-type tests for non-destructive one-shot device testing under step-stress model with exponential lifetimes

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Abstract One-shot devices analysis involves an extreme case of interval censoring, wherein one can only know whether the failure time is before the test time. Some kind of one-shot units do not get destroyed when tested, and then survival units can continue within the test providing extra information for inference. This not-destructiveness is a great advantage when the number of units under test are few. On the other hand, one-shot devices may last for long times under normal operating conditions and so accelerated life tests (ALTs), which increases the stress levels at which units are tested, may be needed. ALTs relate the lifetime distribution of an unit with the stress level at which it is tested via log-linear relationship, so inference results can be easily extrapolated to normal operating conditions. In particular, the step-stress model, which allows the experimenter to increase the stress level at pre-fixed times gradually during the life-testing experiment is specially advantageous for non-destructive one-shot devices. In this paper, we develop robust Rao-type test statistics based on the density power divergence (DPD) for testing linear null hypothesis for non-destructive one-shot devices under the step-stress ALTs with exponential lifetime distributions. We theoretically study their asymptotic and robustness properties, and empirically illustrates such properties through a simulation study.

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1 Introduction

One shot devices, also known as current status data in survival analysis, is an extreme case of interval censoring. One-shot devices are tested at pre-specified inspections times, when we can only know if a test unit have failed or not. In this paper we focus on non-destructive one-shot devices, which do not get destroyed when tested and, therefore, all units that did not failed before an inspection time can continue within the experiment. The non-destructiveness assumption is reasonable in many practical applications and makes best use of all units under test. For example, the proposed techniques can be applied for analyzing the effect of temperature in electronic components when instantaneous status data is not available (Guono (2001), among others.

On the other hand, many real one-shot devices have large mean lifetimes under normal operating conditions, and so accelerated life tests (ALTs) plans, which accelerate the time to failure by increasing the stress level at which units are tested, are inevitable to infer on their reliability. This acceleration process will shorten the life of devices as well as reduce the costs associated with the experiment. In particular, we assume the lifetime of one-shot devices follows an exponential distribution, which is widely used as a lifetime model in engineering and physical sciences. Step-stress ALTs apply stress to devices progressively at pre-specified times. The step-stress are specially suitable for testing non-destructive one devices, as several stress levels can applied to the same unit until the failure occurs.

Further, we assume that the lifetime distribution of the one-shot devices follows the cumulative exposure model, which relates the lifetime distribution of a device at one stress level to the distribution at preceding stress levels by assuming the residual life of that device depends only on the cumulative exposure it had experienced, with no memory of how this exposure was accumulated. In particular, if we consider a multiple step-stress ALT with k ordered stress levels, $x_1 < x_2 < \cdots < x_k$ and their corresponding times of stress change $\tau_1 < \tau_2 \cdots < \tau_k$, the cumulative distribution function is given by:

$$G_T(t) = \begin{cases} G_1(t) = 1 - e^{-\lambda_1 t}, & 0 < t < \tau_1 \\ G_2\left(t + a_1 - \tau_1\right) = 1 - e^{-\lambda_2 (t + a_1 - \tau_1)}, & \tau_1 \leq t < \tau_2 \\ \vdots & \vdots \\ G_k\left(t + a_{k-1} - \tau_{k-1}\right) = 1 - e^{-\lambda_k (t + a_{k-1} - \tau_{k-1})}, & \tau_{k-1} \leq t < \infty, \end{cases} \tag{1}$$

with

$$a_{i-1} = \frac{\sum_{l=1}^{i-1}\left(\tau_l - \tau_{l-1}\right)\lambda_l}{\lambda_i}, \quad i = 1, ..., k-1. \tag{2}$$

and

$$\lambda_i(\boldsymbol{\theta}) = \theta_0 \exp(\theta_1 x_i), \quad i = 1, .., k, \tag{3}$$

where $\boldsymbol{\theta} = (\theta_0, \theta_1) \in \mathbb{R}^+ \times \mathbb{R} = \Theta$ is the unknown parameter vector of the model. This log-linear relation in (3) is frequently assumed in accelerated life test models.

Now, let consider a grid of inspection times, $t_1 < t_2 < \cdots < t_L$, including the times of stress change. The probability of a failure within the interval $(t_{j-1}, t_j]$ is given by

$$\pi_j(\boldsymbol{\theta}) = G_T(t_j) - G_T(t_{j-1}), \quad j = 1, .., L, \tag{4}$$

and the probability of survival at the end of the experiment is $\pi_{L+1}(\boldsymbol{\theta}) = 1 - G_T(t_L)$. Further, given a sample of one-shot data, $(n_1, ..., n_{L+1})$, the empirical probability vector can be defined as $\widehat{p} = (n_1/N, ..., n_{L+1}/N)$.

Classical inferential methods for one-shot are based on the maximum likelihood estimator (MLE), which is very efficient by it lacks of robustness. To overcome the robustness drawback, Balakrishnan et al. (2022a) proposed robust estimators for one-shot devices based on the popular density power divergence (DPD) under exponential lifetimes. They developed minimum DPD estimators (MDPDE) as well as Wald-type test based on them, and studied their asymptotic properties. Later, Balakrishnan et al.(2022b) extended the method and developed the restricted MDPDE for the same model, and they examined its robustness and asymptotic properties as well.

The DPD between the theoretical and empirical probability vectors is given by

$$d_\beta\left(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}\left(\boldsymbol{\theta}\right)\right) = \sum_{j=1}^{L+1} \left(\pi_j(\boldsymbol{\theta})^{1+\beta} - \left(1 + \frac{1}{\beta}\right) \widehat{p}_j \pi_j(\boldsymbol{\theta})^{\beta} + \frac{1}{\beta} \widehat{p}_j^{\beta+1} \right). \tag{5}$$

If we consider the restricted parameter space given by

$$\Theta_0 = \{\theta |\; g(\boldsymbol{\theta}) = \boldsymbol{m}^T \boldsymbol{\theta} - d = 0\},$$

with $\boldsymbol{m} = (m_0, m_1)^T \in \mathbb{R}^2$ and $d \in \mathbb{R}$, the restricted MDPDE, $\widetilde{\boldsymbol{\theta}}^{\beta}$, is naturally defined by

$$\widetilde{\boldsymbol{\theta}}^{\beta} = \arg\min\nolimits_{\boldsymbol{\theta} \in \boldsymbol{\Theta_0}} d_\beta(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})). \tag{6}$$

As Wald-type tests, Rao-type tests play a fundamental role in hypothesis testing. Indeed, each of these tests have their own important positions in the statistical literature, which are not eclipsed by the other. Classical Wald and Rao tests are based on the MLE and the restricted MLE. However, the non-robust nature of the procedures based on the MLE has motivated several researchers to look for robust generalizations of such tests. Basu et al. (2021) developed a robust generalization of the Rao test based on the DPD for general statistical models, and Jaenada et al. (2022) extended the method using the Rényi pseudistance.

In this paper, we develop Rao-type test statistics for non-destructive one-shot devices tested under step-stress ALT for testing linear null hypothesis. In Section 2 we define the Rao-type test statistics based on the restricted MDPDEs and we derive their asymptotic distribution. Section 3 theoretically analyzes the robustness properties of the tests through its IF. Finally, in Section 4 a simulation study is carried out to evaluate the performance of the proposed statistics.

2 Robust Rao-type test

Let us consider the score of the DPD loss function for the step-stress ALT model

$$\boldsymbol{U}_{\beta,N}(\boldsymbol{\theta}) = \boldsymbol{W}^T\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}^{\beta-1}(\widehat{\boldsymbol{p}} - \boldsymbol{\pi}(\boldsymbol{\theta})) \tag{7}$$

where $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ denotes a $(L+1)\times(L+1)$ diagonal matrix with diagonal entries $\pi_j(\boldsymbol{\theta})$, $j = 1, ..., L+1$, and $\boldsymbol{W}$ is a $(L+1)\times 2$ matrix with rows $\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}$, with

$$\boldsymbol{z}_j = g_T(t_j)\begin{pmatrix} \frac{t_j+a_{i-1}-\tau_{i-1}}{\theta_0} \\ (t_j + a_{i-1} - \tau_{i-1})x_i + a_{i-1}^* \end{pmatrix}, \quad j = 1, ..., L, \tag{8}$$

$$a_{i-1}^* = \frac{1}{\lambda_i}\sum_{l=1}^{i-1}\lambda_l\,(\tau_l - \tau_{l-1})\,(-x_i + x_l), \quad i = 2, .., k, \tag{9}$$

$\boldsymbol{z}_{-1} = \boldsymbol{z}_{L+1} = \boldsymbol{0}$ and i is the stress level at which the units are tested after the $j-$th inspection time. Then, the MDPDE verifies the estimating equations given by $\boldsymbol{U}_{\beta,N}(\widehat{\boldsymbol{\theta}}^{\beta}) = \boldsymbol{0}$ (see Balakrishnan et al. (2022a) for more details).

We define Rao-type test statistics for testing linear null hypothesis

$$\mathrm{H}_0 : \boldsymbol{m}^T\boldsymbol{\theta} = d, \tag{10}$$

as

Definition 1 The Rao-type statistics, based on the restricted to the linear null hypothesis (10) MDPDE, $\widetilde{\boldsymbol{\theta}}^{\beta}$, for testing (10) is given by

$$\boldsymbol{R}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta}) = N\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta})^T\boldsymbol{Q}_{\beta}(\widetilde{\boldsymbol{\theta}}^{\beta})\left[\boldsymbol{Q}_{\beta}(\widetilde{\boldsymbol{\theta}}^{\beta})^T\boldsymbol{K}_{\beta}(\widetilde{\boldsymbol{\theta}}^{\beta})\boldsymbol{Q}_{\beta}(\widetilde{\boldsymbol{\theta}}^{\beta})\right]^{-1}\boldsymbol{Q}_{\beta}(\widetilde{\boldsymbol{\theta}}^{\beta})^T\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta}), \tag{11}$$

where

$$\begin{aligned} \boldsymbol{K}_{\beta}(\boldsymbol{\theta}_0) &= \boldsymbol{W}^T\left(D_{\boldsymbol{\pi}(\boldsymbol{\theta}_0)}^{2\beta-1} - \boldsymbol{\pi}(\boldsymbol{\theta}_0)^{\beta}\boldsymbol{\pi}(\boldsymbol{\theta}_0)^{\beta T}\right)\boldsymbol{W} \\ \boldsymbol{Q}_{\beta}(\boldsymbol{\theta}_0) &= \boldsymbol{J}_{\beta}(\boldsymbol{\theta}_0)^{-1}\boldsymbol{m}(\boldsymbol{m}^T\boldsymbol{J}_{\beta}(\boldsymbol{\theta}_0)^{-1}\boldsymbol{m})^{-1}, \text{with} \quad \boldsymbol{J}_{\beta}(\boldsymbol{\theta}_0) = \boldsymbol{W}^T D_{\boldsymbol{\pi}(\boldsymbol{\theta}_0)}^{\beta-1}\boldsymbol{W}. \end{aligned} \tag{12}$$

and $\boldsymbol{U}_{\beta,N}(\boldsymbol{\theta})$, is defined in (7).

Here, the matrix $\boldsymbol{Q}_{\beta}(\boldsymbol{\theta})$ depends on the null hypothesis trough $\boldsymbol{m}$, and the term d is only used to obtain the restricted MDPDE. The proposed testing procedure can be easily extended to more general composite null hypothesis defined by a set of restrictions. Before presenting the asymptotic distribution of the Rao-type test statistics, $\boldsymbol{R}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta})$, we shall establish the asymptotic distribution of the score $\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta})$.

Theorem 1 *The asymptotic distribution of the score $\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta})$ for the step-stress ALT model under exponential lifetimes, is given by*

$$\sqrt{N}\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{\theta}}^{\beta}) \xrightarrow[N\to\infty]{L} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{K}_{\beta}(\boldsymbol{\theta})\right)$$

where the variance-covariance matrix $\boldsymbol{K}_{\beta}(\boldsymbol{\theta})$ is defined in (12).

Proof See proof in Balakrishnan et al. (2022b). □

Now, the following results states the asymptotic distribution of the Rao-type test statistics

Theorem 2 *The asymptotic distribution of the Rao-type test statistics defined in (11) under the linear null hypothesis (10) is a chi-square with 1 degree of freedom.*

Proof The proof can be found in Balakrishnan et al. (2022b). □

Based on Theorem 2, for any $\beta \geq 0$ and $\boldsymbol{m} \in \mathbb{R}^2$, the critical region with significance level α for the hypothesis test with null hypothesis (10) is given by

$$\mathcal{R}_{\alpha} = \{(n_1, ..., n_{L+1}) \text{ s.t. } \boldsymbol{R}_{\beta,N}(\boldsymbol{\theta}) > \chi^2_{1,\alpha}\} \tag{13}$$

where $\chi^2_{1,\alpha}$ denotes the lower α-quantile of a chi-square with 1 degree of freedom.

3 Influence function analysis

We evaluate the robustness of the Rao-type test statistics through its IF. The robustness of an estimator or test statistic is widely analyzed using the concept of Influence Function (IF), which intuitively describes the effect of an infinitesimal contamination of the model on the estimate. The IF of the restricted MDPDE for the step-stress model with one-shot devices was established in Balakrishnan et al.(2022b), and the boundedness of the function was discussing there, concluding that the IF is always bounded for positive values of the tuning parameter.

The IF is computed as the Gateaux derivative of the functional defining the Rao-type test statistics (14) at a direction $\Delta_{\boldsymbol{x}}$. The functional associated to the Rao-type test statistic in terms of the statistical functional associated to the restricted MDPDE, $\widetilde{\boldsymbol{T}}_\beta$, is given by

$$\begin{aligned}\boldsymbol{R}_{\beta,N}(\widetilde{\boldsymbol{T}}_\beta(G)) =& N\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{T}}_\beta(G))^T\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\left[\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T\boldsymbol{K}_\beta(\boldsymbol{\theta}_0)\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\right]^{-1}\\ &\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T\boldsymbol{U}_{\beta,N}(\widetilde{\boldsymbol{T}}_\beta(G)).\end{aligned} \tag{14}$$

We define $G_\varepsilon = (1-\varepsilon)G + \varepsilon\Delta_{\boldsymbol{x}}$ the contaminated version of the distribution G. Taking derivatives in (14) and evaluating at $\varepsilon = 0$ we obtain, by the consistency of the restricted MDPDE under the null hypothesis, that the first order IF vanishes so it is inadequate to evaluate the robustness properties of the proposed Rao-type tests.

The second-order influence function is computed as the second-order derivative of the functional associated to the Rao-type test statistic, $\widetilde{\boldsymbol{T}}_\beta(G_\varepsilon)$, evaluated at $\varepsilon = 0$. Therefore, simple calculations yield the expression of the IF of the Rao-type test statistics,

$$\begin{aligned}\mathrm{IF}^{(2)}\left(\boldsymbol{x}, \boldsymbol{R}_{\beta,N}(\widetilde{\boldsymbol{T}}_\beta), G\right) =& 2\mathrm{IF}\left(\boldsymbol{x}, \widetilde{\boldsymbol{T}}_\beta, G\right)^T\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\left[\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T\boldsymbol{K}_\beta(\boldsymbol{\theta}_0)\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\right]^{-1}\\ &\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T\mathrm{IF}\left(\boldsymbol{x}, \widetilde{\boldsymbol{T}}_\beta, G\right).\end{aligned}$$

Since $\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\left[\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T\boldsymbol{K}_\beta(\boldsymbol{\theta}_0)\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)\right]^{-1}\boldsymbol{Q}_\beta(\boldsymbol{\theta}_0)^T$ are typically assumed to be bounded, the boundedness of the second order influence function of the Rao-type test statistics under the null hypothesis is determined by the boundedness of the IF of the restricted MDPDEs. Then, Rao-type statistics based on the restricted MDPDEs are robust for positives values of β, but lack of robustness for $\beta = 0$, corresponding to the MLE.

4 Simulation study

In this section we empirically examine the performance of the Rao-type test statistics based on the restricted MDPDE for the step-stress model with exponential lifetime distributions, $\widetilde{\boldsymbol{\theta}}^\beta$, under different contamination scenarios. We consider a 2-step stress ALT experiment with $L = 11$ inspection times and a total of $N = 180$ one-shot devices. We consider two stress levels $x_1 = 35$, switched at $\tau_1 = 25$ to $x_2 = 45$. The experiment ends at $\tau_2 = 70$. During the experiment, inspection is performed at a grid of inspection times containing the times of stress change, $\mathrm{IT} = (10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70)$.

In our context, we must consider "outlying intervals" rather than "outlying devices", so we introduce contamination by increasing (or decreasing) the probability of failure in (4) for (at least) one interval. In our simulations the probability of failure in the third interval is switched as

$$\tilde{\pi}_3(\boldsymbol{\theta}) = G_{\boldsymbol{\theta}}(IT_3) - G_{\tilde{\boldsymbol{\theta}}}(IT_2) \tag{15}$$

where $\tilde{\boldsymbol{\theta}} = (\widetilde{\theta}_0, \widetilde{\theta}_1)$ is a contaminated parameter with $\tilde{\theta}_0 \leq \theta_0$ and $\tilde{\theta}_1 \leq \theta_1$. Of course, the probability vector is normalized after introducing contamination.

We introduce contamination in two different ways: decreasing the value of the first parameter θ_0, (first scenario of contamination) and decreasing the value of the second parameter θ_1 (second scenario of contamination). In both scenarios, the mean lifetime is decreased for the outlying cell. Moreover, we consider the linear hypothesis test defined by

$$\mathrm{H}_0 : \theta_1 = 0.03 \quad \text{vs} \quad \mathrm{H}_1 : \theta_1 \neq 0.03.$$

We evaluate the empirical significance level and power of the Rao-type test statistics under increasing contamination rate on the third cell. We compute the empirical level generating data with true parameter value $\boldsymbol{\theta} = (0.003, 0.03)^T$ (satisfying the null hypothesis) and the empirical power generating data with true parameter value $\boldsymbol{\theta} = (0.003, 0.06)^T$ (violating H_0) over $R = 1000$ repetitions. The contamination rate on the third cell is introduced in one of the model parameters, either θ_0 or θ_1, and correspondingly the contamination rate is calculated as $\varepsilon = 1 - \widetilde{\theta}_i/\theta_i$, $i = 0, 1$ where $\widetilde{\theta}_i$ denotes the contaminated parameter. It is interesting to note that, for $\beta = 0$, the associated Rao-type test statistic is not based on the MLE but on the restricted MLE under the constrain defined by the null hypothesis.

Figures 1 and 2 show the empirical level and power under increasing contamination rates for a $\alpha = 5\%$ significance level. As expected, the Rao test based on the restricted MLE is the most efficient in the absence of contamination, but it performance gets worse when there is data contamination. However, the Rao-type test statistics based on restricted MDPDEs with positive values of β keep competitive in the absence of contamination, and outperform the Rao-type statistics based on the restricted MLE when increasing the contamination rate. Furthermore, greater values of β produce more robust statistics.

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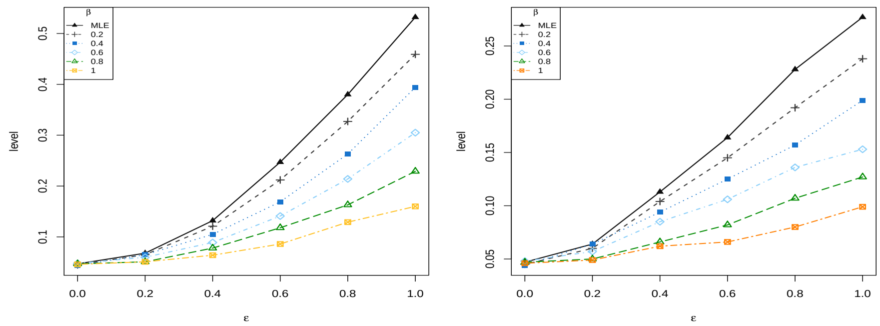
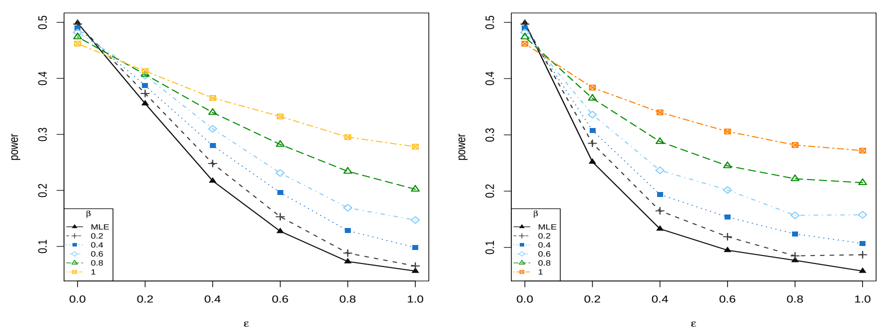


(a) θ_0-contaminated cell (b) θ_1-contaminated cell

Fig. 1: Empirical significance level against contamination cell proportion in $R = 1000$ replications



(a) θ_0-contaminated cell (b) θ_1-contaminated cell

Fig. 2: Empirical power against contamination cell proportion in $R = 1000$ replications

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